2 Matrices :

- Eigen values and vectors
- Cayley - Hamilton theorem (without proof)
- Similar matrices, diagonalisable matrix
- Derogatory and non-derogatory matrices
- Functions of square matrix

- May 2019

1. Find the Eigen values of $2A^3 + 5A^2 - 3A$ where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 8 & 2 & 0 \\ 8 & 8 & -1 \end{bmatrix}$$

2. Is the matrix $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ diagonalizable ?

5-63

If so find the diagonal form and the transforming matrix

3. Find the Eigen values and the Eigen vectors of the matrix $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

4. Find 4^A where $A = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$

5-79

- Dec 2018

1. Find the eigen values of adj. A and $A^2 - 2A + I$ where $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

2. Is the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ diagonalisable?

5-61

If so, find the diagonal form and the transformation matrix

3. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, find A^{50}

5-87

4. Show that the matrix $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ is derogatory.

5-90

- May 2018

1. 5-35. If $A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$, then find the eigen values of $4A^{-1} + 3A + 2I$

2. 5-91 Show that the matrix $A = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}$ is derogatory.

3. 5-68 Show that the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonalisable. Find the transforming matrix M and the diagonal form D

- Dec 2017

1. Find the eigen values of the adjoint of

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

2. 5-93 Show that $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ is non-derogatory

3 Is the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ diagonalisable?

5-65

If so find the diagonal form and the transforming matrix.

- May 2017

1. If $A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$, then find the eigen values of $6A^{-1} + A^3 + 2I$

5-38

2. Show that the matrix $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ is derogatory

5-90

3. Show that the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ is diagonalisable. Find the transforming matrix and the diagonal matrix.

5-63

- Nov 16

1 Find the eigen values and eigen vectors of the matrix

5-24 $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

2. Show that the matrix $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ is non-derogatory.

3. If $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, find A^{50}

5 - 78

- May 16

1. Find the eigen values of $A^2 + 2I$, where $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 3 & 5 & 3 \end{bmatrix}$

and I is the identity matrix of order 3.

2. Is the following matrix Derogatory? Justify

5 - 90

$$\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

3. Is the following matrix diagonalizable? If yes, find the transforming matrix and the diagonal matrix

5 - 63

$$\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

Dec

- ~~May~~ 15

1. State Cayley-Hamilton theorem and verify the same for $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

2. Find the Eigen values and the eigen vectors of the matrix $\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$
5-22

3. Show that the matrix $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonalizable. Find the transforming matrix and the diagonal matrix.
5-68

- May 15

1. If $A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$ then find the eigen values of $4A^{-1} + 3A + 2I$
5-35

2. Show that $A = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}$ is derogatory.
5-91

3. Show that the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ is
5-61
diagonalisable. Find the transforming matrix and the diagonal matrix.

- May 14

1. If $A = \begin{bmatrix} x & 4x \\ 2 & y \end{bmatrix}$ has eigen values 5 and -1
5-4
then find values of x and y.

2. Determine whether matrix A is derogatory
5-24 $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

3. Show that the matrix A is diagonalizable, find
5-68 its diagonal form and transforming matrix, if
$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$

- Dec 14

1. If $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}$. Find the ~~[illegible]~~ eigen
5-35
values of $A^3 + 5A + 8I$.

2. Show that the matrix $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & 6 & -4 \end{bmatrix}$ is
5-90

derogatory

3. Show that the following ~~~~ matrix is
5-68. diagonalizable. Find the transforming matrix and the Diagonal matrix.

$$\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

1.

→ Since A is a triangular matrix, the eigen values are its diagonal elements i.e; 1, 2 and -1.

Now the eigen values of $2A^3$ are $2(1)^3, 2(2)^3, 2(-1)^3$.
The eigenvalues of $5A^2$ are $5(1)^2, 5(2)^2, 5(-1)^2$.
The eigenvalues of $3A$ are $3(1), 3(2), 3(-1)$

Hence the eigenvalues of $2A^3 + 5A^2 - 3A$ are

$2(1)^3 + 5(1)^2 - 3(1)$, $2(2)^3 + 5(2)^2 - 3(2)$ and
$2(-1)^3 + 5(-1)^2 - 3(-1)$ i.e 4, 30 and 6

2.

→ The characteristic equation of A is,

$$\begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0.$$

$\therefore \lambda^3 - 6\lambda^2 + (-11+14+8)\lambda - (-88+80+14) = 0$.
$\therefore \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$
$\therefore (\lambda-1)(\lambda^2 - 5\lambda + 6) = 0$
$\therefore (\lambda-1)(\lambda-2)(\lambda-3) = 0$.
$\therefore \lambda = 1, 2, 3$

Since all the eigen values are distinct the matrix A is diagonalisable.

i) For $\lambda = 1$, $[A - \lambda I] X = 0$ gives

$$\begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ $4x_1 - 4x_2 - 2x = 0$; $3x_1 - 4x_2 + 0x_3 = 0$

By Cramer's rule

$$\frac{x_1}{\begin{vmatrix} -4 & -2 \\ -4 & 0 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 4 & -2 \\ 3 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 4 & -4 \\ 3 & -4 \end{vmatrix}}$$

$$\therefore \frac{x_1}{-8} = \frac{x_2}{-6} = \frac{x_3}{-4} = t.$$

$\therefore$ $x_1 = -8t$, $x_2 = -6t$, $x_3 = -4t$

$$\therefore X_1 = \begin{bmatrix} -8t \\ -6t \\ -4t \end{bmatrix} = -2t \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

$$\therefore X_1 = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

$\therefore$ Corresponding to eigenvalue 1, the eigenvector is $[4 \; 3 \; 2]'$

ii) For $\lambda = 2$, $[A - \lambda I] X = 0$ gives

$$\begin{bmatrix} 6 & -8 & -2 \\ 4 & -5 & -2 \\ 3 & -4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore 4x_1 - 5x_2 - 2x_3 = 0 \; ; \; 3x_1 - 4x_2 - x_3 = 0$

By Cramer's Rule,

$$\frac{x_1}{\begin{vmatrix} -5 & -2 \\ -4 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 4 & -2 \\ 3 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 4 & -5 \\ 3 & -4 \end{vmatrix}}$$

$$\therefore \frac{x_1}{-3} = \frac{-x_2}{2} = \frac{x_3}{-1} = t.$$

$\therefore x_1 = -3t, \quad x_2 = -2t, \quad x_3 = -t$

$$\therefore X_2 = \begin{bmatrix} -3t \\ -2t \\ -t \end{bmatrix} = -t \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\therefore X_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$\therefore$ Corresponding to eigenvalue 2, the eigenvector is $[3 \;\; 2 \;\; 1]'$

for $\lambda = 3$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ $4x_1 - 6x_2 - 2x_3 = 0$; $3x_1 - 4x_2 - 2x_3 = 0$

By Cramer's rule,

$$\frac{x_1}{\begin{vmatrix} -6 & -2 \\ -4 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 4 & -2 \\ 3 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 4 & -6 \\ 3 & -4 \end{vmatrix}}$$

$$\therefore \frac{x_1}{4} = \frac{x_2}{2} = \frac{x_3}{2} = t.$$

$\therefore$ $x_1 = 4t$, $x_2 = 2t$, $x_3 = 2t$.

$$\therefore X_3 = \begin{bmatrix} 4t \\ 2t \\ 2t \end{bmatrix} = 2t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

$$\therefore X_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ Corresponding to eigenvalue 3, eigenvector is $[2 \; 1 \; 1]$.

$\therefore M = [X_1 \ X_2 \ X_3] = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

Since $M^{-1}AM = D$, the matrix $A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

will be diagonalised to the diagonal matrix

$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ by the transforming matrix

$M = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

3

→ The characteristic equation is,

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$\therefore \lambda^3 - 7\lambda^2 + (4+3+4)\lambda - (8-2-1) = 0$

$\therefore \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

$\therefore (\lambda-1)(\lambda^2 - 6\lambda + 5) = 0$

$\therefore (\lambda-1)(\lambda-1)(\lambda-5) = 0$

$\therefore \lambda = 1, 1, 5$

Hence 1, 1 and 5 are the eigen values.

For $\lambda = 1$, $[A - \lambda I] X = 0$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - R_1$ and $R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad x_1 + 2x_2 + x_3 = 0.$

i) Let $x_2 = 0$

$\therefore \quad x_1 = -x_3$

Let $x_1 = 1$

$\therefore \quad x_3 = -1.$

$$\therefore \quad X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

ii) Let $x_3 = 0$

$\therefore \quad x_1 = -2x_2$

Let $x_1 = 2$

$\therefore \quad x_2 = -1$

$$\therefore \quad X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

Hence corresponding to $\lambda = 1$, the eigenvectors are

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

for $\lambda = 5$, $[A - \lambda I] X = 0$

$$\therefore \begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$\therefore$ $x_1 - 2x_2 + x_3 = 0$; $x_1 + 2x_2 - 3x_3 = 0$.

By cramer's rule,

$$\frac{x_1}{\begin{vmatrix} -2 & 1 \\ 2 & -3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix}}$$

$$\therefore \frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4} = t.$$

$\therefore$ $x_1 = 4t$, $x_2 = 4t$, $x_3 = 4t$

$$\therefore X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4t \\ 4t \\ 4t \end{bmatrix} = 4t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\therefore X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ Hence corresponding to eigenvalue 5, the eigen vector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

4.

→

$$A = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$$

$$|A - \lambda I| = 0.$$

$$\begin{vmatrix} 3/2 - \lambda & 1/2 \\ 1/2 & 3/2 - \lambda \end{vmatrix} = 0.$$

$$\left(3/2 - \lambda\right)^2 - 1/4 = 0.$$

$$\frac{9}{4} - 3\lambda + \lambda^2 - \frac{1}{4} = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 1)(\lambda - 2) = 0$$

$$\lambda = 1, 2$$

For $\lambda = 1$, $[A - \lambda I] X = 0$.

$$\therefore \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore x_1 + x_2 = 0.$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

for $\lambda = 2$, $[A - \lambda I] X = 0$

$$\therefore \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore x_1 = x_2$$

$$\therefore X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad ; \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\therefore M^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\because M^{-1} A M = D$$

$$\therefore A = M D M^{-1}$$

$$\therefore 4^A = M \, 4^D \, M^{-1}$$

$$\therefore 4^A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\therefore 4^A = \frac{1}{2} \begin{bmatrix} 4 & 16 \\ -4 & 16 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 20 & 12 \\ 12 & 20 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$$

5.

→ $\lambda = 2, 4, 3$

$A^2 - 2A + I$

$4 - 4 + \cancel{2}1, \; 16 - 8 + \cancel{4}1, \; 9 - 6 + \cancel{3}1$

$\cancel{2}, \cancel{12}, \cancel{6}$

$1, 9, 4$

6.

→ $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

$|A - \lambda I| = 0.$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0.$$

$\lambda^3 - 18\lambda^2 + (5 + 20 + 20)\lambda - (40 - 60 + 20) = 0.$

$\lambda^3 - 18\lambda^2 + 45\lambda = 0.$

$\lambda(\lambda^2 - 18\lambda + 45) = 0.$

$\lambda(\lambda - 3)(\lambda - 15) = 0.$

$\lambda = 0, 3, 15.$

For $\lambda = 0$;

$$\frac{x_1}{5} = \frac{-x_2}{-10} = \frac{x_3}{10} = \frac{1}{5}$$

$$X_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

For $\lambda = 3$.
$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}$$

$$\frac{x_1}{-16} = \frac{-x_2}{8} = \frac{x_3}{16} = \frac{-1}{8}$$

$$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

For $\lambda = 15$;
$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}$$

$$\frac{x_1}{40} = \frac{-x_2}{40} = \frac{x_3}{20} = \frac{1}{20}$$

$$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$\because M^{-1}AM = D$. diagonalizable.

7
→

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$|A - \lambda I| = 0.$$

$$\lambda^3 - \lambda^2 + (-1)\lambda - (-1) = 0.$$

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0.$$

$$\lambda^2(\lambda - 1) - 1(\lambda - 1) = 0.$$

$$(\lambda^2 - 1)(\lambda - 1) = 0.$$

$$(\lambda + 1)(\lambda - 1)(\lambda - 1) = 0.$$

$$\lambda = -1, 1, 1$$

$$A^{50} = aA^2 + bA + cI$$
$$\lambda^{50} = a\lambda^2 + b\lambda + c$$
$$\lambda = 1, -1$$
$$1 = a + b + c \quad -1$$
$$1 = a - b + c. \quad -2$$
$$50\lambda^{49} = 2a\lambda + b$$
$$50 = 2a + b \quad -3$$

$a = 25$, $b = 0$, $c = -24$.

$$A^{50} = 25A^2 + -24I$$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A^{50} = \begin{bmatrix} 25-24 & 0 & 0 \\ 25 & 25-24 & 0 \\ 25 & 0 & 25-24 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

8. $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$

$\rightarrow$ $|A - \lambda I| = 0$.

$$\begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix} = 0.$$

$\lambda^3 - 5\lambda^2 + (-4-2+14)\lambda - (-20-12+36) = 0$

$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$.

$(\lambda - 1)(\lambda^2 - 4\lambda + 4) = 0$

$(\lambda - 1)(\lambda - 2)(\lambda - 2) = 0$.

$\lambda = 1, 2, 2$.

$f_1(\lambda) = \lambda - 1$

$= A - I \neq 0$.

$f_2(\lambda) = \lambda - 2I \neq 0$

$f_3(\lambda) = (\lambda-1)(\lambda-2) = \lambda^2 - 3\lambda + 2$

$= A^2 - 3A + 2$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ derogatory

9.

→ $\lambda = 1, 4.$

$4A^{-1} + 3A + 2I$

$4(1) + 3 + 2 \quad , \quad 1 + 12 + \cancel{8}2$

$9, \cancel{2}\cancel{1}\ 15$

10. $A = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}$

→ $|A - \lambda I| = 0.$

$$\begin{vmatrix} 7-\lambda & 4 & -1 \\ 4 & 7-\lambda & -1 \\ -4 & -4 & 4-\lambda \end{vmatrix} = 0.$$

$\lambda^3 - 18\lambda^2 + (24 + 24 + 33)\lambda - (168 - 48 - 12) = 0.$

$\lambda^3 - 18\lambda^2 + 81\lambda - 108 = 0$

$(\lambda - 3)(\lambda^2 - 15\lambda + 36) = 0.$

$(\lambda - 3)(\lambda - 3)(\lambda - 12) = 0.$

$\lambda = 3, 3, 12.$

$A^2 - 6A + 9I \neq 0 \qquad A^2 - 15A + 36I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

11 $A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

$\rightarrow$ $|A| = 8-2 = 6.$

$$A^c = \begin{bmatrix} +4 & -0 & +2 \\ -0 & +3 & -0 \\ +2 & -0 & +4 \end{bmatrix}$$

$$(A^c)' = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

$|A - \lambda I| = 0.$

$$\begin{vmatrix} 4-\lambda & 0 & 2 \\ 0 & 3-\lambda & 0 \\ 2 & 0 & 4-\lambda \end{vmatrix} = 0.$$

$\lambda^3 - 11\lambda^2 + (12+12+12)\lambda - (48-12) = 0.$

$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

$(\lambda-2)(\lambda^2 - 9\lambda + 18) = 0.$

$(\lambda-2)(\lambda-3)(\lambda-6) = 0.$

$\lambda = 2, 3, 6.$

12. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$.

$\rightarrow$ $|A - \lambda I| = 0$.

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 3-\lambda & 4 \\ 3 & 4 & 5-\lambda \end{vmatrix} = 0.$$

$\lambda^3 - 9\lambda^2 + (-\cancel{5}-4-1)\lambda - (-1+4-3) = 0.$

$\lambda^3 - 9\lambda^2 - 6\lambda = 0$

$\lambda(\lambda^2 - 9\lambda - 6) = 0$

$\lambda/$ $\therefore$ derogatory

13. $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

$\rightarrow$ $|A - \lambda I| = 0$.

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0.$$

$\lambda^3 - 5\lambda^2 + (2+2+3)\lambda - (4-1) = 0.$

$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0.$

$(\lambda-1)(\lambda^2 - 4\lambda + 3) = 0.$ $\therefore \lambda = 1, 1, 3$

1 $\qquad\qquad$ 3

$$\cancel{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}} \qquad \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\frac{x_1}{2} = \frac{+x_2}{+2} = \frac{x_3}{0} = \frac{1}{2}$$

~~$am = 2$~~

~~$Gm = 1$~~

~~$am \neq Gm$~~

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$am = Gm$

$x_1 + x_2 + x_3 = 0$

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

diagonalisable.

14 $A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$

$\lambda = 2, 3$

$6A^{-1} + A^3 + 2I$

$6\left(\frac{1}{2}\right) + (2)^3 + 2 \quad , \quad 6\left(\frac{1}{3}\right) + (3)^3 + 2$

$3 + 8 + 2 \quad , \quad 2 + 27 + 2$

$13 \quad , \quad 31$

15. $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$

$\rightarrow \quad |A - \lambda I| = 0$

$\begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix} = 0$

$\lambda^3 +$

5-90.

16

17.

→ $\lambda = 2, 2, 2.$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\cancel{x_2 = 0}$

$\cancel{x_3 = 0}$

$$\frac{x_1}{1} = \frac{-x_2}{0} = \frac{x_3}{0} = 1$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

18 $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

→ $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0.$$

$$\lambda^3 - 2\lambda^2 + (-4-5+4)\lambda - (-8-4+6) = 0$$

$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda-1)(\lambda^2-\lambda-6) = 0.$$

$$(\lambda-1)(\lambda-3)(\lambda+2) = 0.$$

$$\lambda = 1, 3, -2$$

$\because$ distinct $\rightarrow$ derogatory

$\lambda = 1 \rightarrow A - I \neq 0$

$A - 3I \neq 0$

$A + 2I \neq 0.$

$\lambda^2 - 4\lambda + 3 = A^2 - 4A + 3I \neq 0$

$\lambda^2 - \lambda - 6 = A^2 - A - 6I \neq 0.$

$\lambda^2 + \lambda - 2 = A^2 + A - 2I \neq 0.$

$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

$A^3 - 2A^2 - 5A + 6I = 0.$

degree's are equal $\rightarrow$ non-derogatory

19. $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$; $A^{50} = ?$

$|A - \lambda I| = 0.$

$(2-\lambda)^2 - 1 = 0.$

$4 - 4\lambda + \lambda^2 - 1 = 0.$

$\lambda^2 - 4\lambda + 3 = 0.$

$(\lambda - 1)(\lambda - 3) = 0.$

$\lambda = 1, 3$

distinct $\rightarrow$ diagonalisable.

$D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

1 $\qquad\qquad\qquad$ 3

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $\quad M^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$M^{-1}AM = D.$$

$$A = MDM^{-1}$$

$$A^{50} = MD^{50}M^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3^{50} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 3^{50} \\ -1 & 3^{50} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1+3^{50} & 3^{50}-1 \\ 3^{50}-1 & 1+3^{50} \end{bmatrix}.$$

$$= \frac{1}{2} \begin{bmatrix} 1+3^{50} & -1+3^{50} \\ -1+3^{50} & 1+3^{50} \end{bmatrix}.$$

20. $\lambda = 1, -2, 3.$

$A^2 + 2I$

$1+2, \ 4+2, \ 9+2$

$3, 6, 11$

21.

→

22.

→

23.

→ Cayley Hamilton theorem states that every square matrix satisfies its characteristic equation.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda) - 6 = 0$$

$$2 - 3\lambda + \lambda^2 - 6 = 0$$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$(\lambda-4)(\lambda+1) = 0 \quad ; \quad \lambda = 4, -1$$

$$A^2 - 3A - 4I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ $CE(\lambda) = CE(A) = 0$.

24.
→ $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$

$|A - \lambda I| = 0.$

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix} = 0.$$

$\lambda^3 - 5\lambda^2 + (4-2+6)\lambda - (16-12) = 0.$

$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0.$

$(\lambda-1)(\lambda^2 - 4\lambda + 4) = 0.$

$(\lambda-1)(\lambda-2)^2 = 0.$

$\lambda = 1, 2, 2.$

1

$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{bmatrix}$

$\frac{x_1}{4} = \frac{-x_2}{-1} = \frac{x_3}{-3}$

$\begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$

2

$\begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ +1 & +5 & +4 \end{bmatrix}$

$\frac{x_1}{-6} = \frac{+x_2}{-2} = \frac{x_3}{4} = \frac{-1}{2}$

$\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$

25
$\rightarrow$

26
$\rightarrow$

27
$\rightarrow$

28.
$\rightarrow$

29.
$\rightarrow$

$$x + y = 4$$
$$xy - 8x = -5$$

$$\cancel{x(4-y) - 8x = -5}$$

$$\cancel{4x - xy - 8x = -5}$$

$$x(4-x) - 8x = -5$$

$$4x - x^2 - 8x = -5$$

$$-x^2 - 4x = -5$$

$$x^2 + 4x - 5 = 0$$

$$(x+5)(x-1)$$
$$x = -5, 1$$
$$y = 9, 3$$

30.
→

31.
→

32.
→ $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}$

$\lambda = -1, 3, -2.$

$A^3 + 5A + 8I$

$-1 - 5 + 8, \quad 27 + 15 + 8, \quad -8 - 10 + 8$

$2, 50, -10$

33.
→

34.
→

35

$\rightarrow$

$$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

$$|A - \lambda I| = 0.$$

$$\begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0.$$

$$\lambda^3 - \lambda^2 + (-11+1+5)\lambda - (99-32-64) = 0.$$

$$\lambda^3 - \lambda^2 - 5\lambda - 3 = 0$$

$$(\lambda+1)(\lambda^2 - 2\lambda - 3) = 0$$

$$(\lambda+1)(\lambda+1)(\lambda-3) = 0.$$

$$\lambda = -1, -1, 3$$

-1

$$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}$$

$$2x_1 - x_2 - x_3 = 0$$

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

3

$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}$$

$$\frac{x_1}{-32} = \frac{+x_2}{-32} = \frac{x_3}{-64} = \frac{-1}{32}$$

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}$$

$\therefore M^{-1} A M = D \rightarrow$ diagonalisable.