Applied Mathematics-IV

Chapter 1 : Calculus of Variation

Q. 1 **Find the external of $\int_{x_2}^{x_1} (16\,y^2 - (y'')^2 + x^2)\,dx$**

May 2014

Ans. :

$$\int_{x_1}^{x_2} f\,dx = \int_{x_1}^{x_2} (16\,y^2 - y''^2 + x^2)\,dx$$

$$\therefore\ f = 16\,y^2 - y''^2 + x^2$$

$$\frac{\partial f}{\partial y} = 32y\ ;\ \frac{\partial f}{\partial y'} = 0\ ;\ \frac{\partial f}{\partial y''} = -2y'' = -2\frac{d^2y}{dx^2}\ ;$$

The necessary condition for the given functional to be extremum is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0$$

$$\therefore\ 32y - \frac{d}{dx}(0) + \frac{d^2}{dx^2}\left(-2\frac{d^2y}{dx^2}\right) = 0$$

$$\therefore\ 32y - 2\frac{d^4y}{dx^4} = 0$$

$$\frac{d^4y}{dx^4} - 16y = 0$$

This is L.D.E.

The auxillary equation is $D^4 - 16 = 0$

$$\therefore\ D^4 = 16$$

$$\therefore\ D^2 = 4 \text{ or } D^2 = -4 = 4\,i^2$$

$$\therefore\ D = \pm 2 \text{ or } D = \pm 2i = 0 \pm 2i$$

$\therefore$ Complimentary function is $y = c_1e^{2x} + c_2e^{-2x} + e^{0x}(c_3\cos 2x + c_4\sin 2x)$

Hence the solution is $y = c_1e^{2x} + c_2e^{-2x} + c_3\cos 2x + c_4\sin 2x$

Q. 2 **Find the curve $y = f(x)$ for which $\int_{x_1}^{x_2} y\sqrt{1+(y')^2}\,dx$ is minimum subject to the constraint $\int_{x_1}^{x_2} \sqrt{1+(y')^2}\,dx = l$.**

May 2014

Ans. :

Let $\int_{x_1}^{x_2} F\,dx = \int_{x_1}^{x_2} y\sqrt{1+y'^2}\,dx$ and $\int_{x_1}^{x_2} G\,dx = \int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx = 1$

Let $F = y\sqrt{1+y'^2}$ and $G = \sqrt{1+y'^2}$

$\therefore$ Langrangian function $H = F + \lambda G$

$$\therefore\ H = (y+\lambda)\sqrt{1+y'^2}$$

$$\therefore\ \frac{\partial H}{\partial y'} = (y+\lambda) \times \frac{1}{2\sqrt{1+y'^2}} \cdot 2y'$$

And, $\int_{x_1}^{x_2} H\,dx = \int_{x_1}^{x_2} (y+\lambda)\sqrt{1+y'^2}\,dx$

Since, the above integral does not contain 'x' explicitly, we use reduced form of Euler's equation,

$$H - y'\frac{\partial H}{\partial y'} = c$$

$$\therefore\ (y+\lambda)\sqrt{1+y'^2} - y' \times \frac{(y+\lambda)}{\sqrt{1+y'^2}} \cdot y' = c$$

$$\therefore\ (y+\lambda)\left[\sqrt{1+y'^2} - y' \cdot \frac{1}{\sqrt{1+y'^2}}\right] = c$$

$$\therefore\ (y+\lambda)\left[\frac{1+\cancel{y'^2}-\cancel{y'^2}}{\sqrt{1+y'^2}}\right] = c$$

$$\therefore\ y+\lambda = c\sqrt{1+y'^2}$$

On squaring both sides, $(y+\lambda)^2 = c^2(1+y'^2)$

$$\therefore\ (y+\lambda)^2 = c^2 + c^2y'^2$$

$$\frac{(y+\lambda)^2 - c^2}{c^2} = y'^2$$

$$\therefore \quad y' = \frac{dy}{dx} = \frac{\sqrt{(y+\lambda)^2 - c^2}}{c}$$

$$\therefore \frac{c}{\sqrt{(y+\lambda)^2 - c^2}}\, dy = dx$$

On integration,

$$c \cosh^{-1}\left(\frac{y+\lambda}{c}\right) = x + c_1$$

$$\therefore \cosh^{-1}\left(\frac{y+\lambda}{c}\right) = \frac{x+c_1}{c}$$

$$\therefore \frac{y+\lambda}{c} = \cosh\left(\frac{x+c_1}{c}\right)$$

$$\therefore y = c \cosh\left(\frac{x+c_1}{c}\right) - \lambda,$$ which is the required curve.

Q. 3 Find the plane curve of fixed perimeter and maximum area. **May 2014, Dec. 2014**

Ans. :

Let C be a closed curved of perimeter 's'.

Length of curve is given by, $s = \int_{x_1}^{x_1} \sqrt{1+y'^2}\, dx = 1$

By Green's theorem, are enclosed by curve C is

$$A = \frac{1}{2}\int_c x\, dy - y dx = \frac{1}{2}\int_c \left(x\frac{dy}{dx} - y\right) dx$$

$$= \int_{x_1}^{x_1} \frac{1}{2}(xy' - y)\, dx$$

Let $F = \frac{1}{2}(xy'-y)$ and $G = \sqrt{1+y'^2}$

$\therefore$ Langrangian function $H = F + \lambda G$

$$\therefore \quad H = \frac{1}{2}(xy'-y) + \lambda\sqrt{1+y'^2}$$

$$\therefore \frac{\partial H}{\partial y} = \frac{1}{2}(0-1) + 0 = \frac{-1}{0}\ ; \text{ and} \qquad \ldots(1)$$

$$\frac{\partial H}{\partial y} = \frac{1}{2}(x-0) + \lambda \times \frac{1}{2\sqrt{1+y'^2}} \cdot 2y'$$

$$\frac{\partial H}{\partial y} = \frac{x}{2} + \frac{\lambda y'}{\sqrt{1+y'^2}} \qquad \ldots(2)$$

Now, $$\int_{x_1}^{x_1} H\, dx = \int_{x_1}^{x_1} (F+\lambda G)\, dx$$

$$= \int_{x_1}^{x_1} \frac{1}{2}(xy'-y) + \lambda\sqrt{1+y'^2}\, dx$$

By Euler's equation, the condition for maximum or minimum is,

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$$

$$\therefore \frac{-1}{2} - \frac{d}{dx}\left(\frac{x}{2} + \frac{\lambda y'}{\sqrt{1+y'^2}}\right) = 0$$

(from Equations (1) and (2))

$$\therefore \frac{-1}{2} - \frac{1}{2} - \frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\therefore -1 = \frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right)$$

$$\therefore d\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) = -dx$$

On integration,

$$\frac{\lambda y'}{\sqrt{1+y'^2}} = -x + c_1$$

$$\therefore \lambda y' = -(x-c_1)\sqrt{1+y'^2}$$

On squaring

$$\therefore \lambda^2 y'^2 = (x-c_1)^2(1+y'^2)$$

$$\therefore \lambda^2 y'^2 = (x-c_1)^2 + y'^2(x-c_1)^2$$

$$\therefore \lambda^2 y'^2 - y'^2(x-c_1)^2 = (x-c_1)^2$$

$$\therefore y'^2[\lambda^2 - (x-c_1)^2] = (x-c_1)^2$$

$$\therefore y'^2 = \frac{(x-c_1)^2}{\lambda^2-(x-c_1)^2}$$

$$\therefore y'^2 = \frac{dy}{dx} = \frac{(x-c_1)^2}{\sqrt{\lambda^2-(x-c_1)^2}}\, dx$$

$$\therefore dy = \frac{1}{-2} \times \frac{-2(x-c_1)}{\sqrt{\lambda^2-(x-c_1)^2}}\, dx$$

On integration,

$$y = \frac{1}{-2} \times 2\sqrt{\lambda^2-(x-c_1)^2} + c_2$$

$$\therefore y - c_2 = -\sqrt{\lambda^2(x-c_1)^2}$$

On squaring ,

$$\therefore (y-c_2)^2 = \lambda^2 - (x-c_1)^2$$

$$\therefore (y-c_2)^2 + (x-c_1)^2 = \lambda^2\ ;$$ which is a circle

Hence, circles are the plane curves of fixed perimeter and maximum area.

Q. 4 Using Rayleigh-Ritz method, find an appropriate solution for the extremal of the functional $I[y(x)] = \int_0^1 \left[xy + \frac{1}{2}(y')^2\right] dx$ subject to $y(0) = y(1) = 0$. May 2014

Ans. :

$$I[y(x)] = \int_0^1 \left[xy + \frac{1}{2}y'^2\right] dx \quad ...(1)$$

Let the approximate solution be

$$y(x) = c_0 + c_1 x + c_2 x^2 \quad ...(2)$$

put $x = 0$,

$$y(0) = c_0 + 0 + 0 \quad [\because y(0) = 0]$$

$$\therefore 0 = c_0 \quad ...(3)$$

put $x = 1$ in Equation (2)

$$y(1) = c_0 + c_1 + c_2 \quad [\because y(1) = 0]$$

$$\therefore 0 = 0 + c_1 + c_2 \quad \text{[from 3 Equation]}$$

$$\therefore c_2 = -c_1 \quad ...(4)$$

Substituting Equations (3) and (4) in Equation (2),

$$Y = 0 + c_1 x - c_1 x^2 \quad ...(5)$$

Differentiate w.r.t. 'x'

$$y' = c_1 - 2c_1 x \quad ...(6)$$

Substituting Equations (5) and (6) in Equation (1),

$$I = \int_0^1 \left[x(c_1 x - c_2 x^2) + \frac{1}{2}(c_1 - 2c_1 x)^2\right] dx$$

$$= \int_0^1 \left[c_1 x^2 - c_2 x^3 + \frac{1}{2}(c_1^2 - 4c_1^2 x + 4c_1^2 x^2)\right] dx$$

$$= \int_0^1 \left[c_1 x^2 - c_2 x^3 + \frac{1}{2}c_1^2 - 2c_1^2 x + 2c_1^2 x^2\right] dx$$

$$= \left[c_1 \frac{x^3}{3} - c_1 \frac{x^4}{4} + \frac{1}{2}c_1^2 x - 2c_1^2 \frac{x^2}{2} + 2c_1^2 \frac{x^3}{3}\right]_0^1$$

$$= \left[c_1 \frac{1}{3} - c_1 \frac{1}{4} + \frac{1}{2}c_1^2 - c_1^2 + 2c_1^2 \frac{1}{3}\right] - [0 - 0 + 0 - 0 + 0]$$

$$= \frac{c_1}{3} - \frac{c_1}{4} + \frac{1}{2}c_1^2 - c_1^2 + \frac{2}{3}c_1^2$$

$$\therefore I = \frac{c_1}{12} + \frac{1}{6}c_1^2$$

For maximum or minimum, $\frac{dI}{dc_1} = 0$

$$\therefore \frac{dI}{dc_1} = \frac{1}{12} + \frac{1}{6} \times 2c_1 = 0$$

$$\therefore \frac{1}{3}c_1 = \frac{-1}{12}$$

$$\therefore c_1 = \frac{-1}{4}$$

Hence, from Equation (5), the approximate solution is

$$y = \frac{-1}{4}x + \frac{1}{4}x^2$$

$$\therefore y = \frac{x}{4}(x - 1)$$

Q. 5 Find the extermal of the function. Dec. 2014

$$\int_{x_1}^{x_2} [y^2 - y'^2 - 2y \cosh x]\, dx$$

Ans. :

$$\int_{x_1}^{x_2} f\, dx = \int_{x_1}^{x_2} [y^2 - y'^2 - 2y \cosh x]\, dx$$

$$\therefore f = y^2 - y'^2 - 2y \cosh x$$

$$\frac{\partial f}{\partial y} = 2y - 2\cosh x$$

$$\frac{\partial f}{\partial y'} = -2y' = -2\frac{dy}{dx}$$

The necessary condition for the given functional to be extremum is

$$\frac{dy}{dx} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$2y - 2\cosh x - \frac{d}{dx} - 2\left(\frac{dy}{dx}\right) = 0$$

$$y - \cosh x + \frac{d^2y}{dx^2} = 0$$

$$\frac{d^2y}{dx^2} + y = \cosh x$$

This is L.D.E

The auxillary equation is $D^2 + 1 = 0$

$$D^2 = -1$$

$$D = \pm i$$

$\therefore$ Complimentory function is

$$y = e^{0x}(c_1 \cos x + c_2 \sin x) = c_1 \cos x + c_2 \sin x$$

Hence the solution is

$$y = c_1 \cos x + c_2 \sin x$$

Q. 6 **Solve the boundary value problem.**

$$I = \int_0^1 \left(2xy - y^2 - y'^2\right) dx \text{ given } y(0) = y(1) = 0$$

by Rayleigh-Ritz method. Dec. 2014

Ans. :

$$I = \int_0^1 (2xy - y^2 - y'^2)\, dx \quad \ldots(1)$$

Let the approximate solution be

$$y(x) = c_0 + c_1 x + c_2 x^2 \quad \ldots(2)$$

$$\text{put } x = 0$$

$$y(0) = c_0 + 0 + 0 \quad [\because y(0) = 0]$$

$$\therefore 0 = c_0 \quad \ldots(3)$$

$$\text{put } x = 1 \text{ in Equation (2)}$$

$$y(1) = c_0 + c_1 + c_2 \quad [\because y(1) = 0]$$

$$0 = 0 + c_1 + c_2 \quad \text{(from Equation (3))}$$

$$c_2 = -c_1 \quad \ldots(4)$$

Substituting Equations (3) and (4) in Equation (2). ,

$$y = 0 + c_1 x - c_1 x^2 \quad \ldots(5)$$

Differentiating w.r.t. x

$$y' = c_1 - 2c_1 x \quad \ldots(6)$$

Substituting Equations (5) and (6) in Equation (1)

$$I = \int_0^1 2x(c_1 x - c_1 x^2) - (c_1 x - c_1 x^2)^2 - (c_1 x - 2c_1 x)^2\, dx$$

$$I = \int_0^1 2c_1 x^2 - 2c_1 x^3 - c_1^2 x^2 + 2c_1^2 x^3 - c_1^2 x^4 - c_1^2 + 4c_1^2 x^2 - 4c_1^2 x\, dx$$

$$I = \int_0^1 2c_1 x^2 - 2c_1 x^3 - c_1^2 [x^2 - 2x^3 + x^4 + 4x + 4x^2]\, dx$$

$$I = \int_0^1 2c_1 x^2 - 2c_1 x^3 - c_1^2 [5x^2 - 2x^3 + x^4 + 4x]\, dx$$

$$= \left[2c_1 \frac{x^3}{3} - \frac{2c_1 x^4}{4} - c_1^2 \frac{5x^3}{3} + c_1^2 \frac{2x^4}{4} - c_1^2 \frac{x^5}{5} - c_1^2\, 4\frac{x^2}{2}\right]_0^1$$

$$= \left[\frac{2c_1}{3} - \frac{c_1}{2} - \frac{c_1^2 5}{3} + \frac{c_1^2}{2} - \frac{c_1^2}{5} - 2c_1^2\right] - [0 - 0 - 0 + 0 - 0 - 0]$$

$$= \frac{3c_1}{6} - c_1^2 \left[\frac{5}{3} - \frac{1}{2} + \frac{1}{5} + 2\right] = \frac{c_1}{2} - c_1^2$$

$$\left[\frac{50 - 15 + 6 + 60}{30}\right] = \frac{c_1}{2} - c_1^2 \left[\frac{101}{30}\right]$$

For maximum or minimum, $\frac{dI}{dc_1} = 0$

$$\frac{dI}{dc_1} = \frac{1}{2} - 2c_1 \left[\frac{101}{30}\right] = 0$$

$$\left[\frac{101}{15}\right] c_1 = \frac{1}{2}$$

$$\therefore c_1 = \frac{1}{2} \times \frac{15}{101} = \frac{15}{202}$$

Hence from Equation (5), the approximate solution is

$$y = \frac{15}{202}x - \frac{15}{202}x^2 = \frac{15}{202}[x - x^2]$$

Q. 7 **Find the extremal of** $\int_{x_1}^{x_2} \frac{1 + y^2}{y'^2}\, dx$**.**

May 2015

Ans. :

$$\int_{x_1}^{x_2} F\, dx = \int_{x_1}^{x_2} \frac{1 + y^2}{y'^2}\, dx$$

$$\therefore F = \frac{1 + y^2}{y'^2} = (1 + y^2)\, y'^{-2}$$

$$\therefore \frac{\partial F}{\partial y'} = (1 + y^2) - 2y'^{-2}$$

Since F does not contain x explicitly, the necessary condition for the given functional to be extremum, is

$$F - y' \frac{\partial F}{\partial y'} = c$$

$$\therefore \frac{1 + y^2}{y'^2} - y' \times (1 + y^2) \frac{-2}{y'^3} = c$$

$$\therefore \frac{1 + y^2}{y'^2} + \frac{2(1 + y^2)}{y'^2} = c$$

$$\therefore \frac{3(1 + y^2)}{y'^2} = c$$

$$\therefore \frac{3}{c}(1 + y^2) = y'^2$$

$$\therefore y' = \sqrt{\frac{3}{c}(1 + y^2)}$$

$$\therefore \frac{dy}{dx} = \sqrt{\frac{3}{c}}\, \sqrt{1 + y^2}$$

$$\therefore \frac{dy}{\sqrt{1+y^2}} = c_1\,dx \text{ where, } c_1 = \sqrt{\frac{3}{c}}$$

On integration, $\sinh^{-1} y = c_1 x + c_2$

Hence the solution is $y = \sinh(c_1 x + c_2)$

Q. 8 Find the curve on which the functional $\int_0^1 (y'^2 + 12\,xy)\,dx$ with $y(0) = 0$ and $y(1) = 1$ can be extremised. **May 2015**

Ans. :

$$\int_{x_1}^{x_2} F\,dx = \int_0^1 (y'^2 + 12\,xy)\,dx$$

$$\therefore F = y'^2 + 12xy$$

$$\therefore \frac{\partial F}{\partial y} = 0 + 12x = 12x; \text{ and } \frac{\partial F}{\partial y'} = 2y';$$

The necessary condition for the given functional to be extremum is $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$$\therefore 12x - \frac{d}{dx}(2y') = 0$$

$$\therefore 6x - \frac{d}{dx}\left(\frac{dy}{dx}\right) = 0 \qquad \text{(Dividing by 2)}$$

$$\therefore \frac{d^2y}{dx^2} = 6x$$

On integration, $\frac{dy}{dx} = 6 \times \frac{x^2}{2} + c_1$

$$\therefore \frac{dy}{dx} = 3x^2 + c_1$$

Again on integration, $y = 3 \times \frac{x^2}{3} + c_1 x + c_2$ ...(1)

Given $y(0) = 0$

Put $x = 0$ and $y = 0$ in Equation (1)

$$\therefore 0 = 0 + 0 + c_2$$

$$\therefore c_2 = 0$$

Given $y(1) = 1$

Put $x = 1$ and $y = 1$ in Equation (1)

$$\therefore 1 = 1^3 + c_1(1) + 0$$

$$\therefore c_1 = 0$$

Hence, from Equation (1) the required curve is $y = x^3$

Q. 9 Using Rayleigh – Ritz method, find an approximate solution for the extremal of the functional $I(y) = \int_0^1 (y'^2 - 2y - 2xy)\,dx$ subject to $y(0) = 2$, $y(1) = 1$. **May 2015**

Ans. :

$$I[y(x)] = \int_0^1 (y'^2 - 2y - 2xy)\,dx \quad \text{...(1)}$$

Let the approximate solution be $y(x) = c_0 + c_1 x + c_2 x^2$ ...(2)

Put $x = 0$, $y(0) = c_0 + 0 + 0$

$$\therefore 2 = c_0 \quad \text{...(3) } [\because y(0) = 2]$$

Put $x = 1$ in Equation (2), $y(1) = c_0 + c_1 + c_2$

$$\therefore 1 = 2 + c_1 + c_2$$

[From Equation (3) and given $y(1) = 1$]

$$\therefore -1 = c_1 + c_2 \quad \text{...(4)}$$

Substituting Equation (3) in Equation (2), $y = 2 + c_1 x + c_1 x^2$ ...(5)

Differentiating w.r.t. 'x', $y' = c_1 + 2c_2 x$ ...(6)

Substituting Equation (5) and Equation (6) in Equation (1),

$$I = \int_0^1 [(c_1 + 2c_2 x)^2 - 2(2 + c_1 x + c_2 x^2) - 2x(2 + c_1 x + c_2 x^2)]\,dx$$

$$= \int_0^1 \left[c_1^2 + 4c_1c_2 x + 4c_2^2 x^2 - 4 - 2c_1 x - 2c_2 x^2 - 4x - 2c_1 x^2 - 2c_2 x^3\right] dx$$

$$= \left[c_1^2 x + 4c_1c_2 \times \frac{x^2}{2} + 4c_2^2 \times \frac{x^3}{3} - 4x - 2c_1 \times \frac{x^2}{2} - 2c_2 \times \frac{x^3}{3} - 4 \times \frac{x^2}{2} - 2c_1 \times \frac{x^3}{3} - 2c_2 \times \frac{x^4}{4}\right]_0^1$$

$$= \left[c_1^2 + 2c_1c_2 + \frac{4}{3}c_2^2 - 4 - c_1 - \frac{2}{3}c_2 - 2 - \frac{2}{3}c_1 - \frac{1}{2}c_2\right]$$

$$- [0 + 0 + 0 - 0 - 0 - 0 - 0 - 0 - 0]$$

$$= c_1^2 + 2c_1c_2 + \frac{4}{3}c_2^2 - 6 - \frac{7}{6}c_2 - \frac{5}{3}c_1$$

$$= c_1^2 + 2c_1(-1 - c_1) + \frac{4}{3}(-1 - c_1)^2 - 6 - \frac{7}{6}(-1 - c_1)$$

$$- \frac{5}{3}c_1 \text{ (from Equation (4))}$$

$$= c_1^2 - 2c_1 - 2c_1^2 + \frac{4}{3}(1 + 2c_1 + c_1^2) - 6 + \frac{7}{6} + \frac{7}{6}c_1 - \frac{5}{3}c_1$$

$$= -c_1^2 - 2c_1 + \frac{4}{3} + \frac{8}{3}c_1 + \frac{4}{3}c_1^2 - \frac{29}{6} - \frac{1}{2}c_1$$

$$\therefore I = \frac{1}{3}c_1^2 + \frac{1}{6}c_1 - \frac{7}{2}$$

For maximum or minimum, $\frac{dI}{dc_1} = 0$

$$\therefore \frac{1}{3} \times 2c_1 + \frac{1}{6} = 0$$

$$\therefore \frac{2}{3} c_1 = \frac{-1}{6}$$

$$c_1 = \frac{-1}{4}$$

From Equation (4), $c_2 = -c_1 - 1 = \frac{1}{4} - 1 = \frac{-3}{4}$

Hence, from Equation (5) the approximate solution is $y = 2 - \frac{1}{4}x - \frac{3}{4}x^2$

Q. 10 **Find the extremal of $\int_{x_0}^{x_1} (2xy - y''^2)\, dx$.** May 2015

Ans.:

$$\int_{x_1}^{x_2} f\, dx = \int_{x_0}^{x_1} (2xy - y''^2)\, dx$$

$$\therefore f = 2xy - y''^2$$

$$\therefore \frac{\partial f}{\partial y} = 2x\,; \quad \frac{\partial f}{\partial y'} = 0; \quad \frac{\partial f}{\partial y''} - 2y'' = -2\frac{d^2y}{dx^2}$$

The necessary condition for the given functional to be extremum is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0$$

$$\therefore 2x - \frac{d}{dx}(0) + \frac{d^2}{dx^2}\left(-2\frac{d^2y}{dx^2}\right) = 0$$

$$\therefore 2x - 2\frac{d^4y}{dx^4} = 0$$

$$\therefore \frac{d^4y}{dx^4} = x$$

On integration, $\frac{d^3y}{dx^3} = \frac{x^2}{2} + c_1$

Again on integration, $\frac{d^2y}{dx^2} = \frac{1}{2} \times \frac{x^3}{3} + c_1 x + c_2$

Again on integration, $\frac{dy}{dx} = \frac{1}{6} \times \frac{x^4}{4} + c_1 \frac{x^2}{2} + c_2 x + c_3$

Again on integration, $y = \frac{1}{24} \times \frac{x^5}{4} + \frac{1}{2} c_1 \times \frac{x^3}{3} + c_2 \times \frac{x^2}{2} + c_3 x + c_4$

Hence the solution is $y = \frac{x^5}{5!} + k_1 x^3 + k_2 x^2 + k_3 x + k_4$

Q. 11 **Find the extremal of**

$$\int_{x_1}^{x_2} \left(y^2 - y'^2 - 2\,y \cosh x \right) dx$$

Dec. 2015

Ans.:

$$\int_{x_1}^{x_2} (y^2 - y'^2 - 2y \cosh x)\, dx$$

$$F(x, y, y') = y^2 - y'^2 - 2y \cosh x$$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$(2y - 2\cosh x) - \frac{d}{dx}(-2y') = 0$$

$$2y - 2\cosh x + 2y'' = 0$$

$$y'' + y = \frac{2}{2} \cosh x$$

$$y'' + y = \cosh x \qquad \text{...(1)}$$

$$D^2 + 1 = 0 \Rightarrow D = \pm i$$

$$\text{C.F.} = C_1 \cos x + C_2 \sin x$$

$$\text{P.I.} = \frac{1}{(D^2+1)}\left[\frac{e^x + e^{-x}}{2}\right]$$

$$= \frac{1}{2}\frac{1}{(D^2+1)} e^x + \frac{1}{2}\frac{1}{(D^2+1)} e^{-x}$$

$$= \frac{1}{2}\frac{1}{(1^2+1)} e^x + \frac{1}{2}\frac{1}{((-1)^2+1)} e^{-x}$$

$$= \frac{1}{4} e^x + \frac{1}{4} e^{-x} = \frac{1}{2}\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{1}{2} \cosh x$$

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{2} \cosh x$$

Q. 12 **Solve the boundary value problem $\int_0^1 (2xy + y^2 - y'^2)\, dx$, $0 \le x \le 1$, $y(0) = 0$, $y(1) = 0$ by Rayleigh – Ritz Method.** Dec. 2015

Ans.: $\int_0^1 (2xy + y^2 - y'^2)\, dx$, $0 \le x \le 1$, $y(0) = 0$, $y(1) = 0$

Assume the trial solution, $y(x) = C_0 + C_1 x + C_2 x^2$

$y(0) = 0 \Rightarrow C_0 = 0$

$$y(1) = C_1 + C_1 = 0 \Rightarrow C_2 = -C_1$$

$$y(x) = C_1 x + C_2 x^2 \Rightarrow y'(x) = C_1(1-2x)$$

$$I = \int_0^1 \left(2x\,(C_1 x - C_1 x^2) + C_1^2\,(x-x^2)^2 - C_1^2 (1-2x)^2\right) dx$$

$$= \int_0^1 \left(2C_1 x^2 - 2C_1 x^3 + C_1^2 (x^2 - 2x^3 + x^4) - C_1^2 (1 - 4x^2 - 4x)\right) dx$$

$$= \int_0^1 \left(2C_1 x^2 - 2C_1 x^3 + C_1^2 x^2 - 2C_1^2 x^3 + C_1^2 x^4 - C_1^2 - 4C_1^2 x^2 + 4C_1^2 x\right) dx$$

$$= 2C_1\left(\frac{x^3}{3}\right)_0^1 - 2C_1\left(\frac{x^4}{4}\right)_0^1 + C_1^2\left(\frac{x^3}{3}\right)_0^1 - 2C_1^2\left(\frac{x^4}{4}\right)_0^1 + C_1^2\left(\frac{x^5}{5}\right)_0^1 - C_1^2(x)_0^1 - 4C_1^2\left(\frac{x^3}{3}\right)_0^1 + 4C_1^2\left(\frac{x^2}{2}\right)_0^1$$

$$= \frac{2C_1}{3} - \frac{2C_1}{4} + \frac{C_1^2}{3} - \frac{2C_1^2}{4} + \frac{C_1^2}{5} - C_1^2 - \frac{4C_1^2}{3} + \frac{4C_1^2}{2}$$

$$= \left(\frac{8-6}{12}\right) C_1 + \left(\cancel{\frac{1}{3}} - \frac{1}{2} + \frac{1}{5} - \cancel{1} - \cancel{\frac{4}{3}} + \cancel{2}\right) C_1^2$$

$$= \frac{C_1}{6} + \left(\frac{2-5}{10}\right) C_1^2$$

$$I = \frac{C_1}{6} - \frac{3}{10} C_1^2$$

$$\frac{dI}{dq} = \frac{1}{6} - \frac{6}{10} C_1 = 0\ ;$$

$$C_1 = \frac{1}{6} \times \frac{10}{6} \Rightarrow C_1 = \frac{5}{18}$$

$$y(x) = \frac{5}{18}\, x\,(1-x)$$

Q. 13 **Find the external of the functional** $\int_0^1 [y'^2 + 12xy]\,dx$ **subject to** $y(0) = 0$ **and** $y(1) = 1$. **May 2016**

Ans. :

$$\int_0^1 (y'^2 + 12xy)\,dx$$

$$f(x, y, y') = y'^2 + 12xy$$

Eulers equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0\ ;$$

$$(12x) - \frac{d}{dx}(2y') = 0$$

$$12x - 2y'' = 0\ ; \qquad y'' = 6x$$

$$y' = 3x^2 + C\ ; \qquad y = x^3 + Cx + C_1$$

$$y(0) = 0 + 0 + C_1 = 0 \Rightarrow C_1 = 0\ ; \quad y = x^3 + Cx$$

$$y(1) = 1 + C = 1 \Rightarrow C = 0\ ; \quad y = x^3$$

Q. 14 **Find the extremal that minimises the integral.** $\int_{x_0}^{x_1} (16y^2 - y''^2)\,dx$ **May 2016**

Ans. :

$$\int_{x_0}^{x_1} (16\,y^2 - y''^2)\,dx$$

$$f(x, y, y', y'') = 16\,y^2 - y''^2$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0$$

$$32y - \frac{d}{dx}(0) + \frac{d^2}{dx^2}(2y'') = 0$$

$$32y + y^{V} = 0 \Rightarrow y^{V} + 16y = 0 \Rightarrow (D^4 + 16)\,y = 0$$

$$D^4 + 16 = 0$$

$$(D^2)^2 - 16\,i^2 = 0$$

$$(D^2 + 4i)(D^2 - 4i) = 0$$

$$D = (-4i)^{1/2} \qquad D = (4i)^{1/2}$$

$$= \left\{4\left(\cos\frac{\pi}{2} - i \sin\frac{\pi}{2}\right)\right\}^{\frac{1}{2}} = \left\{4\left(\cos\left(2k\pi + \frac{\pi}{2}\right) - i \sin\left(2k\pi + \frac{\pi}{2}\right)\right)\right\}^{\frac{1}{2}}$$

$$= 2\left(\cos\left(\frac{4k\pi + \pi}{4}\right) - i \sin\left(\frac{4k\pi + \pi}{4}\right)\right)$$

$$K = 0, 1$$

$$D = 2\left(\cos\frac{\pi}{4} - i \sin\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$$

$$= 2\left(\cos\frac{5\pi}{4} - i \sin\frac{5\pi}{4}\right) = 2\left(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

$$D = 2\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)^{1/2} = 2\left(\cos\left(\frac{4n\pi + \pi}{2}\right) + i \sin\left(\frac{4n\pi + \pi}{2}\right)\right)^{1/2}$$

$$= 2\left(\cos\left(\frac{4n\pi + \pi}{4}\right) + i \sin\left(\frac{4n\pi + \pi}{4}\right)\right)$$

$$n = 0, 1$$

$$D = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

$$= 2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = 2\left(\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$$

$$D = \sqrt{2} - i\sqrt{2}, \sqrt{2} + i\sqrt{2}, -\sqrt{2} + i\sqrt{2}, -\sqrt{2} - i\sqrt{2}$$

$$C.F = e^{\sqrt{2}x}\{C_1 \cos\sqrt{2}x + C_2 \sin\sqrt{2}x\} + e^{-\sqrt{2}x}\{C_3 \cos\sqrt{2}x + C_4 \sin\sqrt{2}x\}$$

$$y = e^{\sqrt{2}x}\{C_1 \cos\sqrt{2}x + C_2 \sin\sqrt{2}x\} + e^{-\sqrt{2}x}\{C_3 \cos\sqrt{2}x + C_4 \sin\sqrt{2}x\}$$

Q. 15 Show that a closed curve 'C' of given fixed length (perimeter) which encloses maximum area is a circle. May 2016

Ans. : Let the parametric equation of the curve be

$$x = x(t) \text{ and } y = y(t)$$

The area enclosed by the curve

$$I = \frac{1}{2}\int_{t_1}^{t_2} (x\dot{y} - \dot{x}y)\, dt$$

Now the perimeter of the curve is

$$= \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$$

Consider $$H = \frac{1}{2}(x\dot{y} - y\dot{x}) + \lambda\sqrt{\dot{x}^2 + \dot{y}^2}$$

Where λ is the Lagrangian multiplex

$$\frac{\partial H}{\partial x} - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{x}}\right) = 0, \quad \frac{\partial H}{\partial y} - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{y}}\right) = 0$$

$$\frac{1}{2}\dot{y} - \frac{d}{dt}\left(-\frac{1}{2}y + \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = 0 \quad \ldots(1)$$

$$-\frac{1}{2}\dot{x} - \frac{d}{dt}\left(\frac{1}{2}x + \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = 0 \quad \ldots(2)$$

Integrating (1) and (2) w.r.to t

$$y - \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = C_1 \text{ and } x + \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = C_2$$

$$(x - C_2)^2 + (y - C_1)^2 = \lambda^2 \frac{(\dot{x}^2 + \dot{y}^2)}{(\dot{x}^2 + \dot{y}^2)} = \lambda^2$$

Which is the equation of a circle

Q. 16 Using the Rayleigh- Ritz method find an approximate solution for the external of the functional $\int_0^1 \left\{xy + \frac{1}{2}y'^2\right\} dx$ subject to $y(0) = y(1) = 0$ May 2016

Ans. :

$$\int_0^1 \left(xy + \frac{1}{2}y'^2\right) dx \text{ given that } y(0) = y(1) = 0$$

$$F(x, y, y') = xy + \frac{1}{2}y'^2; \qquad \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$(x) - \frac{d}{dx}(y') = 0$$

$$x - y'' = 0 \quad \Rightarrow y'' = x$$

$$\Rightarrow y' = \frac{x^2}{2} + C_1$$

$$\Rightarrow y = \frac{x^3}{6} + C_1 x + C_2$$

$$y(0) = 0 + 0 + C_2 = 0 \Rightarrow C_2 = 0$$

$$y = \frac{x^3}{6} + C_1 x$$

$$y(1) = \frac{1}{6} + C_1 = 0 \Rightarrow C_1 = \frac{-1}{6}$$

$$y(x) = \frac{x^3}{6} - \frac{x}{6}$$

Q. 17 Find the extremall of $\int_{x_0}^{x_1} (x + y')\, y'\, dx$. Dec. 2016

Ans. :

Here $f(x, y, y') = (x + y')\, y'$

Eulers lagrang equation is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \quad \ldots(1)$$

$$\because \; f(x, y, y') = xy' + (y')^2 \quad \therefore \frac{\partial f}{\partial y'} = x + 2y'$$

$$\frac{\partial f}{\partial y} = 0$$

$\therefore$ from (1)

$$0 - \frac{d}{dx}(x + 2y') = 0 \qquad \therefore 1 + 2y'' = 0$$

$$y'' = -\frac{1}{2} \qquad \therefore y' = -\frac{1}{2}x + C_1$$

$$y = -\frac{1}{2}\frac{x^2}{2} + C_1 x + C_2 \therefore\ y = -\frac{x^2}{4} + C_1 x + C_2$$

Q. 18 **Express (6, 11, 6) as linear combination of $v_1 = (2, 1, 4)$, $v_2 = (1, -1, 3)$, $v_3 = (3, 2, 5)$.** **Dec. 2016**

Ans. :

$$(6, 11, 6) = a_1 v_1 + a_2 v_2 + a_3 v_3$$

$$(6, 11, 6) = a_1 (2, 1, 4) + a_2 (1, -1, 3) + a_3 (3, 2, 5)$$

$$(6, 11, 6) = (2a_1, a_1, 4a_1) + (a_2, -a_2, 3a_2) + (3a_3, 2a_3, 5a_3)$$

$$(6, 11, 6) = (2a_1 + a_2 + 3a_3, a_1 - a_2 + 2a_3, 4a_1 + 3a_2 + 5a_3)$$

$$2a_1 + a_2 + 3a_3 = 6$$

$$a_1 - a_2 + 2a_3 = 11$$

$$4a_1 + 3a_2 + 5a_3 = 6$$

$$a_1 = 4, \quad a_2 = -5, \quad a_3 = 1$$

$$\therefore\ (6, 11, 6) = 4v_1 - 5v_2 + v_3$$

Q. 19 **Find the curve $y = f(x)$ for which $\int_0^{\pi} (y'^2 - y^2)\,dx$ is extremum if $\int_0^{\pi} y\,dx = 1$.** **Dec. 2016**

Ans. :

$$\int_0^{\pi} (y'^2 - y^2)\,dx \text{ if } \int_0^{\pi} y\,dx = 1$$

$$f(x, y, y') = y'^2 - y^2 \text{ and } g(x, y, y') = y$$

Let $\quad H = f + \lambda g$

$$= (y'^2 - y^2) + \lambda y$$

Where λ is the Lagrange's multiplier.

The Eulers equation for H

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right)$$

$$(-2y + \lambda) - \frac{d}{dx}(2y') = 0$$

$$\therefore\ -2y + \lambda - 2y'' = 0$$

$$y'' + y = \frac{\lambda}{2}$$

$$\therefore\ (D^2 + 1)\,y = \frac{\lambda}{2}$$

A.E. $\quad \ldots 0 \Rightarrow D = \pm i$

$$\text{C.F.} = e^{0x}\,(C_1 \cos x + C_2 \sin x)$$

$$= C_1 \cos x + C_2 \sin x$$

$$\text{P.I.} = \frac{1}{(D^2+1)}\frac{\lambda}{2} e^{0x} = \frac{\lambda}{2}\frac{1}{(D^2+1)} e^{0x}$$

$$= \frac{\lambda}{2}\frac{1}{(1)} e^{0x} = \frac{\lambda}{2}$$

$$\text{G.S.} = \text{C.F.} + \text{P.I.}$$

$$y = C_1 \cos x + C_2 \sin x + \frac{\lambda}{2}$$

Q. 20 **Using Rayleigh-Ritz method, solve the boundary value problem**

$$I = \int_0^1 (y'^2 - y^2 - 2xy)\,dx\ ; \quad 0 \le x \le 1, \text{ given}$$

$y(0) = y(1) = 0$ **Dec. 2016**

Ans. :

$$I = \int_0^1 (y'^2 - y^2 - 2xy)\,dx \text{ given } y(0) = y(1) = 0$$

$$F(x, y, y') = y'^2 - y^2 - 2xy$$

By Eulers equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$(-2y - 2x) - \frac{d}{dx}(2y') = 0$$

$$\therefore\ -2y - 2x - 2y'' = 0$$

$$y'' + y = -x$$

$$\therefore\ (D^2 + 1)\,y = -x$$

$$D^2 + 1 = 0 \Rightarrow D = \pm i$$

$$\text{C.F.}\ \ y = C_1 \cos x + C_2 \sin x$$

$$\text{P.I.} = \frac{1}{(D^2+1)}(-x)$$

$$= -(1 + D^2)^{-1} x = -(1 - D^2)\,x = -x$$

$$\text{G.S.} = \text{C.F.} + \text{P.I.}$$

$$\therefore\ y = C_1 \cos x + C_2 \sin x - x$$

$$y(0) = C_1 = 0 \Rightarrow C_1 = 0$$

$$\therefore\ y = C_2 \sin x - x$$

$$y(1) = C_2 \sin 1 - 1 = 0 \Rightarrow C_2 = \frac{1}{\sin 1}$$

$$y = (\operatorname{cosec} 1) \sin x - x$$

Q. 21 **Find the extremal of the function $\int_0^{\pi/2} (2xy + y^2 - y'^2)\,dx$; with y (0) = 0, y(π/2) = 0.** Dec. 2016

Ans. :

$$\int_0^{\pi/2} (2xy + y^2 - y'^2)\,dx$$

$$f(x, y, y') = 2xy + y^2 - y'^2$$

Eulers equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$$(2x + 2y) - \frac{d}{dx}(2y') = 0$$

$$2x + 2y - 2y'' = 0 \qquad \therefore \quad y'' - y = x$$

$$(D^2 - 1)y = x$$

A.E. $D^2 - 1 = 0 \Rightarrow D = \pm 1$

$$\therefore \text{ C.F.} = C_1 e^x + C_2 e^{-x}$$

$$\text{P.I.} = \frac{1}{(D^2 - 1)} x = \frac{1}{-(1 - D^2)} x = -(1 - D^2)^{-1} x$$

$$= -(1 + D^2)x = -x$$

$$\text{G.S.} = \text{C.F.} + \text{P.I.} \qquad \therefore \quad y = C_1 e^x + C_2 e^{-x} - x$$

$$y(0) = C_1 + C_2 = 0 \Rightarrow C_1 + C_2 = 0 \qquad \text{...(i)}$$

$$y(\pi/2) = C_1 e^{\frac{\pi}{2}} + C_2 e^{-\frac{\pi}{2}} = \frac{\pi}{2}$$

$$C_1 e^{\frac{\pi}{2}} + C_2 e^{-\frac{\pi}{2}} = \frac{\pi}{2} \qquad \text{...(ii)}$$

$$C_1 e^{\frac{\pi}{2}} - C_1 e^{-\frac{\pi}{2}} = \frac{\pi}{2} \qquad \therefore \quad C_1 \left(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}\right) = \frac{\pi}{2}$$

$$2 C_1 \sinh\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \Rightarrow C_1 = \frac{\pi}{4 \sinh\left(\frac{\pi}{2}\right)} = -C_2$$

$$y = \frac{\pi}{4 \sinh \frac{\pi}{2}} e^x - \frac{\pi}{4 \sinh \frac{\pi}{2}} e^{-x} - x$$

Chapter 2 : Linear Algebra Vector Spaces

Q. 1 **State Cauchy-schwartz Inequality and hence show that $(x^2 + y^2 + z^2)^{1/2} \geq \frac{1}{13}(3x + 4y + 12z)$, x, y, z are positive.** May 2014

Ans. :

Cauchy Schwartz inequality states that "If 'u' and 'v' are vectors in a real inner product space then

$$|(\vec{u}, \vec{v})| \leq \|\vec{u}\| \|\vec{v}\| \qquad \text{...(1)}$$

Let $u = (3, 4, 12)$, $v = (x, y, z)$,

where x, y, z are positive real numbers.

$$\|u\| = \sqrt{u_1^2 + u_2^2 + u_3^2} = \sqrt{3^2 + 4^2 + 12^2} = 13 \text{ and ,}$$

$$\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{x^2 + y^2 + z^2}$$

$$= (x^2 + y^2 + z^2)^{1/2}$$

$$\text{Also } (u, v) = 3x + 4y + 12z$$

Substituting in Equation (1) ,

$$(3x + 4y + 12z) \leq 13 \times (x^2 + y^2 + z^2)^{1/2}$$

$$\therefore \frac{1}{13}(3x + 4y + 12z) \leq (x^2 + y^2 + z^2)^{1/2}$$

$$\therefore (x^2 + y^2 + z^2)^{1/2} \geq \frac{1}{13}(3x + 4y + 12z)$$

Q. 2 **Find an orthonormal basis of the following subspace of R^3, S = {[1, 2, 0], [0, 3, 1]}.** May 2014

Ans. :

Let u = (1, 2, 0), v = (0, 3, 1)

Let w = $(w_1\ w_2\ w_3)$ be a vector which is orthogonal to both u and v.

$$\therefore \langle u, v \rangle = 0 \Rightarrow 1w_1 + 2w_2 + 0w_3 = 0 \text{ and}$$

$$\therefore \langle v, w \rangle = 0 \Rightarrow 0w_1 + 3w_2 + 1\,w_3 = 0$$

By Crammer's Rule,

$$\frac{w_1}{\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix}} = \frac{-w_2}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}} = \frac{w_3}{\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix}}$$

$$\therefore \frac{w_1}{2} = \frac{-w_2}{1} = \frac{w_3}{3}$$

$$\text{Let } \frac{w_1}{2} = \frac{w_2}{-1} = \frac{w_3}{3} = t$$

$$w_1 = 2t\,;\ w_2 = -t;\ w_3 = 3t\,;$$

$$\therefore w = (2t, -t, 3t)$$

$\therefore$ For t = 1, w = (2 − 1 3)

Now, $\|w\| = \sqrt{w_1^2 + w_2^2 + w_3^2} = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}$ and

$\therefore$ Orthonormal basis $= \frac{w}{\|w\|} = \left(\frac{2}{\sqrt{14}}\ \frac{-1}{\sqrt{14}}\ \frac{3}{\sqrt{14}}\right)$

Hence, Orthogonal basis of the subspace S = {[1, 2, 0] [0, 3, 1] } is $\left(\frac{2}{\sqrt{14}}\ \frac{-1}{\sqrt{14}}\ \frac{3}{\sqrt{14}}\right)$

Q. 3 Find the singular value decomposition of the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$.

May 2014

Ans. :

$$\text{Given } A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\therefore A^1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Step 1 :

$$\text{Let } B = AA^1 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$\therefore$ Characteristic equation is $|B - \lambda I| = 0$, where λ is eigen value of B.

$$\therefore \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} = 0$$

On solving, $\lambda^2 - 6\lambda + 8 = 0$

$\therefore$ Eigen values (λ) are 4,2

Case I : $\lambda = 4$

$$\therefore [B - \lambda I] X = 0$$

$$\therefore \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 + R_1$;

$$\therefore \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore -x_1 + x_2 = 0$$

$$\text{put } x_1 = 1$$

$$\therefore x_2 = 1$$

$$\therefore \text{Eigen vector } X_1 = [1\ 1]'$$

Case 2 : $\lambda = 2$

$$\therefore [B - \lambda I] X = 0$$

$$\therefore \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 - R_1$;

$$\therefore \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore x_1 + x_2 = 0$$

$$\text{put } x_1 = 1$$

$$\therefore x_2 = -1$$

$$\therefore \text{Eigen vector } X_2 = [1\ -1]'$$

Using Gram Schmidt method, we orthonormalize the above eigen vectors

$$\text{Let } v_1 = X_1 = [1\ 1]$$

$$\therefore |v_1|^2 = (1)^2 + (1)^2 = 2 \text{ and}$$

$$\frac{x_1}{\begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 2 \\ 0 & 0 \end{vmatrix}}$$

$$\therefore \frac{x_1}{-4} = \frac{-x_2}{4} = \frac{x_3}{0}$$

$$\therefore \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{0}$$

$$\therefore X_1 = [1\ 1\ 0]'$$

Case 3 : $\lambda_2 = 2$

$$\therefore [C - \lambda I] X_2 = 0$$

$$\begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore 0x_1 + 2x_2 + 0x_3 = 0 \text{ and } 2x_1 + 0x_2 + 0x_3 = 0$$

Using Crammers rule,

$$\frac{x_1}{\begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 0 & 0 \\ 2 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix}}$$

$$\therefore \frac{x_1}{0} = \frac{-x_2}{0} = \frac{x_3}{-4}$$

$$\therefore \frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{1}$$

$$\therefore X_2 = [0\ 0\ 1]'$$

Case 3 : $\lambda_3 = 0$

$$\therefore [C - \lambda I] X_3 = 0$$

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore 2x_1 + 2x_2 + 0\,x_3 = 0 \text{ and } 0\,x_1 + 0\,x_2 + 2\,x_3 = 0$$

Using Crammers rule,

$$\frac{x_1}{\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & 2 \\ 0 & 0 \end{vmatrix}}$$

$$\therefore \frac{x_1}{4} = \frac{-x_2}{4} = \frac{x_3}{0}$$

$$\therefore \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0}$$

$$\therefore X_3 = [1 -1\ 0]'$$

Using Gram Schmidt method, we orthonormalize the above eigen vectors

$$\text{Let } v_1 = X_1 = [1\ 1\ 0]$$

$$\therefore |v_1|^2 = (1)^2 + (1)^2 + (0)^2 = 2$$

Q. 4 **Verify Cauchy-Schwartz inequality for the vectors.**

$u = (-4, 2, 1)$ and $v = (8, -4, -2)$ Dec. 2014

Ans. :

Cauchy's Schwarz inequality is given as

$$|\langle u, v\rangle| \le \|u\| \cdot \|v\| \qquad \ldots(1)$$

$$\langle u, v\rangle = (-32 - 8 - 2) = -42$$

$$\|u\| = \sqrt{21}; \quad \|v\| = \sqrt{84}$$

$$\langle u, v\rangle^2 \le \|u\|^2 \|v\|^2$$

$$\text{L.H.S } \langle u, v\rangle^2 = (-42)^2 = 1764$$

$$\|u\|^2 = 21; \quad \|v\|^2 = 84$$

$$\text{R.H.S} = \|u\|^2 \|v\|^2 = 1764$$

$$\text{L.H.S} = \text{R.H.S}$$

Hence (1) is satisfied.

Q. 5 (I) **Show that the set $W = \{(1, x) \mid x \in R\}$ is a subspace of R^2 under operations $[1, x] + [1, y] = [1, x + y]$; $k[1, x] = [1, kx]$; k is any scalar.**

(II) **Is the set $W = \{[a, 1, 1] \mid a \in R\}$ a subspace of R^3 under the usual addition and scalar multiplication ?** Dec. 2014

Ans. :

(I) Now, we shall check for all the properties

For $u, v \in V$

1. $u = (1, x), \quad v = (1, x')$

$$u + v = (1, x + x') \in V$$

2. $k(1, x) = (1, kx) \in V$

3. $(u + v) = (1, x + x') = (1, x') + (1, x) = v + u$

4. $u + (v + w') = (1, x) + ((1, x') + (1, x)) = (1, x) + (1, x' + x) = (1 + x + (x' + x))$

$= (1 + ((x + x') + x)) = (1 + (x + x')) + (1, x) = (u + v) + w$

5. There exist an element $\bar{0} = (1, 0) \in v$ such that

$$(u + \bar{0}) = (1, x) + (1, 0) = (1, x) = u$$

6. There exist as element $-u = -(1, x)$ such that

$u + (-u) = (1, x) + (-(1, x)) = (1, x - x) = (1, 0) = \bar{0}$ vector of V

7. $\alpha\left(\overline{u + v}\right) = \alpha(1, x + x') = (1, \alpha(x + x')) = (1, \alpha x + \alpha x')$

$$= (1, \alpha x) + (1, \alpha x') = \alpha u + \alpha v$$

8. $(\alpha + \beta)\vec{u} = (\alpha + \beta)(1, x) = (1, (\alpha + \beta)x) = (1, \alpha x + \beta x)$

$$= (1, \alpha x) + (1, \beta x) = \alpha u + \beta v$$

9. $\alpha\beta\vec{u} = \alpha\beta(1, x) = (1, \alpha\beta x) = \alpha(1, \beta x)$

10. $1 \cdot \vec{u} = 1 \cdot (1, x) = (1, x) = \vec{u}$

Hence all the axioms of the vector spaces are satisfied. Hence V is a vector space respect to given operation.

(II)

Let $W = \{(a, 1, 1) \mid a \in R\}$

Here W is a non empty subset of R^3

Let $u, V \in W$ then

$u = (a, 1, 1), \quad V = (b, 1, 1)$ for some $a, b \in R$

Now $u + v = (a + b, 2, 2) \in W$, since R is closed under addition.

Thus for every $u, v \in W$, $u + v \in W$

Q. 6 **Construct an orthonormal basis of R^2 by applying Gram schmidt orthogonalization to $S = \{[3, 1], [2, 2]\}$** Dec. 2014

Ans. :

Step 1 : Let $v_1 = u_1 = (3, 1)$

Step 2 : $v_2 = u_2 - \frac{(u_2, v_1)}{\|v_1\|^2} v_1 = (2,2) - \frac{(2,2)(3,1)}{(\sqrt{9+1})^2}(3,1)$

$$= (2,2) - \frac{(2,2)(3,1)}{10}(3,1) = (2,2) - \frac{(6,2)}{10}(3,1)$$

$$= (2,2) - \left(\frac{18}{10}, \frac{2}{10}\right) = (2,2) - \left(\frac{9}{5} - \frac{1}{5}\right)$$

$$= \left(2 - \frac{9}{5}, 2 - \frac{1}{5}\right) = \left(\frac{1}{5}, \frac{9}{5}\right)$$

The vectors v_1, v_2 form an orthogonal basis for R^2

$$W_1 = \frac{v_1}{\|v_1\|} = \frac{(3,1)}{\sqrt{10}} = \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$$

$$W_2 = \frac{v_2}{\|v_2\|} = \frac{(2,2)}{\sqrt{8}} = \left(\frac{2}{\sqrt{8}}, \frac{2}{\sqrt{8}}\right)$$

The vectors w_1, w_2 form an orthogonal basis of R^2

Q. 7 **Show that the set V of positive real numbers with operations.** Dec. 2014

Addition : x + y = xy

scalar multiplication : $kx = x^k$.

is a vector space

where x, y are any two real numbers and k is any scalar.

Ans. :

Let us verify that V is a vector space.

(1) Since the product of two positive real number is again positive real number, for any $\vec{x}, \vec{y} \in R^+$, $x + y \in R^+$

(2) For any real number k and $x \in R^+$, x^k is also a positive number. Hence $k \cdot x \in R^+$.

(3) For any $x, y, z \in R^+$

$x + y = x \cdot y$

$= y \cdot x$ (Multiplication in R is commutation)

$= y + x$ (By define of +)

$\therefore x + y = y + x$

(4) $(x+y)+z = (x \cdot y)+z$ (By the definition of +)

$= (x \cdot y) \cdot z$ (By the definition of +)

$= x \cdot (y \cdot z)$ (By associativity) $= x + (y \cdot z) = x + (y + z)$

Thus $(x+y)+z = (x+(y+z))$

(5) $1 \in R^+$ such that for any $x \in R^+$

$x + 1 = x \cdot 1 = x$

$1 + x = 1 \cdot x = x$

Thus 1 is zero element in R^+

(6) For any $x \in R^+$; $\left(\frac{1}{x}\right) \in R^+$; $x > 0$

$$x + \frac{1}{x} = x \cdot \frac{1}{x} = 1$$

Thus each element in R^+ has negative element

(7) For any $x, y \in R^+$ and any scalar k,

$$k(x+y) = k(x \cdot y) = (xy)^k = x^k - y^k = x^k + y^k$$

$$= kx + ky = kx + ky$$

(8) For any $x \in R^+$ and for any scalars k and l consider

$$(k+l)x = x^{k+l} = x^k x^l = x^k + x^l = kx + lx$$

$$(k+l)x = kx + lx$$

(9) For any $x \in R^+$ and any scalar k and l consider

$$(k\,l)\,x = x^{kl} = (x^l)^k = k\,x^l = k\,(lx)$$

$$(kl)\,x = k(lx)$$

(10) For any $x \in R^+$

$$1 \cdot x = x^1 = x$$

$$1 \cdot x = x$$

Thus R^+ is a vector space with 1 as zero element and negative of any x as $\left(\frac{1}{x}\right)$.

Q. 8 **Find the unit vector orthogonal to both [1, 1, 0] and [0, 1, 1].** May 2015

Ans. :

Let u = (1, 1, 0), v = (0, 1, 1)

Let $w = (w_1 \; w_2 \; w_3)$ be a vector which is orthogonal to both u andv.

$\therefore (u, w) = 0 \Rightarrow 1w_1 + 1w_2 + 0w_3 = 0$ and

$\therefore (u, w) = 0 \Rightarrow 0w_1 + 1w_2 + 1w_3 = 0$

By Crammer's Rule. $\dfrac{w_1}{\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}} = \dfrac{-w_2}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}} = \dfrac{w_3}{\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}}$

$$\therefore \frac{w_1}{1} = \frac{-w_2}{1} = \frac{w_3}{1}$$

Let $\frac{w_1}{1} = \frac{w_2}{-1} = \frac{w_3}{1} = t$

$w_1 = t;\; w_2 = -t;\; w_3 = t;$

$\therefore w = (t \; -t \; t)$

$\therefore |w| = \sqrt{t^2 + (-t)^2 + t^2} = \sqrt{3t^2} = t\sqrt{3}$

$\therefore$ Unit vector orthogonal to both 'u' and 'v' is $\frac{\pm w}{|w|}$

$$= \frac{\pm 1}{t\sqrt{3}}(t \ -t \ \ t)$$

$$= \pm\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Hence, Unit vector orthogonal to both 'u' and 'v' $= \pm\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

Q. 9 Find the vector orthogonal to both [– 6, 4, 2] and v = [3, 1, 5]. May 2015

Ans. :

$u = [-6, 4, 2], v = [3, 1, 5]$

Let $w = (w_1 \ w_2 \ w_3)$ be a vector which is orthogonal to both u and v.

$\therefore (u, w) = 0 \Rightarrow -6 w_1 + 4 w_2 + 2 w_3 = 0$ and

$\therefore (v, w) = 0 \Rightarrow 3w_1 + 1w_2 + 5 w_3 = 0$

By crammer's rule,

$$\frac{w_1}{\begin{vmatrix} 4 & 2 \\ 1 & 5 \end{vmatrix}} = \frac{-w_2}{\begin{vmatrix} -6 & 2 \\ 3 & 5 \end{vmatrix}} = \frac{w_3}{\begin{vmatrix} -6 & 4 \\ 3 & 1 \end{vmatrix}}$$

$$\therefore \frac{w_1}{18} = \frac{-w_2}{-36} = \frac{w_3}{-18}$$

$$\therefore \frac{w_1}{1} = \frac{w_2}{2} = \frac{w_3}{-1}$$

Let $\frac{w_1}{1} = \frac{w_2}{2} = \frac{w_3}{-1} = t$

$w_1 = t; \ w_2 = 2t; \ w_3 = -t;$

$\therefore w = [t \ \ 2t \ \ -t]$

$\therefore$ For $t = 1, w = [1 \ \ 2 \ \ -1]$

Hence, a vactor orthogonal to both u and v is $w = [1 \ \ 2 \ \ -1]$

Q. 10 Show that the set W = {[x, y, z], y = x + z } is a subspace of R^n under the usual addition and scalar multiplication. May 2015

Ans. :

Let $\bar{u}$ and $\bar{v}$ be any two vectors in W and let 'k' be any scalar.

If W is non-empty subset of V then W is sub-space if

(i) $\bar{u} + \bar{v}$ is in W (ii) $k\,\bar{u}$ is in W.

Let $\bar{u} = (x_1, y_1, z_1)$ and $\bar{v} = (x_2, y_2, z_2)$ be any two vectors belonging to set W such that

$y_1 = x_1 + z_1$ ...(1)

and $y_2 = x_2 + z_2$ ...(2)

(a) $\bar{u} + \bar{v} = (x_1, y_1, z_1) + (x_2, y_2, z_2)$

$\therefore \bar{u} + \bar{v} = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

Consider, $y_1 + y_2 = (x_1 + z_1) + (x_2 + z_2)$

(From Equations (1) and (2))

$\therefore y_1 + y_2 = (x_1 + x_2) + (z_1 + z_1)$

$\bar{u} + \bar{v}$ is in W.

(b) $k\,\bar{u} = k\,(x_1, y_1, z_1)$

$k\,\bar{u} = k\,(kx_1, ky_1, kz_1)$

Consider, $k\,y_1 = k\,(x_1 + z_1)$ (From Equation (1))

$\therefore k\,y_1 = kx_1 + kz_1$

$\therefore k\,\bar{u}$ is in W.

Q. 11 If $f(a) = \int_c \frac{3z^2 + 7z + 1}{z - a}\,dz$, where c is a circle $|z| = 2$, find the values of

(i) f(– 3) (ii) f(i)
(iii) f′ (1 – i). May 2015

Ans. : The circle $|z| = 2$ has centre (0, 0) and radius 2

$z_0 = i$; $1 - i$ lies inside while $z_0 = -3$ lies outside the circle $|z| = 2$

Let $\phi(z) = 3z^2 + 7z + 1$

We assume $z_0 = a$ lies inside the circle $|z| = 2$

" $z_0 = a$" is a simple pole.

$$\therefore \ f(a) = \int_c \frac{3z^2 + 7z + 1}{z - a}\,dz = 2\pi i \phi\,(a)$$

(Cauchy's integral formula)

$\therefore \ f(a) = 2\pi i\,(3a^2 + 7a + 1)$ ...(1)

$\therefore \ f'(a) = 2\pi i\,(6a + 7)$ ...(2)

From Equation (1),

$f(i) = 2\pi i\,(3\,i^2 + 7i + 1)$

$f(i) = 2\pi i\,(-2 + 7\,i)$

From Equation (2),

$f'(1 - i) = 2\pi i\,[6\,(1 - i) + 7]\ f'(1 - i)$

$\therefore \ f'(1 - t) = 2\pi i\,(13 - 6i)$

Since $z_0 = -3$ lies outside the circle

$\int_c g(z)\,dz$ (Cauchy's Integral theorem)

$\therefore\ f(-3) = 0$

Q. 12 **Find an orthonormal basis for the subspaces of R^3 by applying Gram-Schmidt process where**

S = {(1, 2, 0) (0,3,1) } Dec. 2015

Ans. : $S = \{(1,2,0)\ (0,3,1)\}$, $V_1 = u_1 = (1,2,0)$

$$V_2 = U_2 - \frac{(u_2 \cdot V_1)}{\|V_1\|^2} V_1$$

$$= (0,3,1) - \frac{(0,3,1)\begin{bmatrix}1\\2\\0\end{bmatrix}}{\left(\sqrt{1^2+2^2+0^2}\right)^2}(1,2,0)$$

$$= (0,3,1) - \frac{6}{5}(1,2,0)$$

$$= (0,3,1) - \left(\frac{6}{5}, \frac{12}{5}, 0\right) = \left(\frac{-6}{5}, \frac{3}{5}, 1\right)$$

Q. 13 **State and prove Cauchy-Schwartz Inequality. Verify the inequality for vectors u = (– 4, 2, 1) and v = (8, – 4, – 2)** Dec. 2015

Ans. : Cauchy – Schwartz Inequality :

Let u and v are two vectors them

$$|\langle u, v\rangle| \le \|u\| \cdot \|v\|$$

Proof : Let u and v are two vectors then

$$|\langle u, v\rangle| \le \|u\|\,\|v\| \cos\theta$$

$$\because\ |\cos\theta| \le 1$$

$$\Rightarrow\ |\langle u \cdot v\rangle| \le \|u\|\ \|v\|$$

$$u = (-4, 2, 1),\quad v = (8, -4, -2)$$

$$u \cdot v = (-4, 2, 1)\begin{bmatrix}8\\-4\\-2\end{bmatrix} = -4\times 8 + 2\times(-4) + 1(-2)$$

$$= -32 - 8 - 2 = -42$$

$$|\langle u \cdot v\rangle| = |-42| = 42$$

$$\|u\| = \sqrt{(-4)^2 + (2)^2 + 1^2} = \sqrt{16+4+1} = \sqrt{21}$$

$$\|v\| = \sqrt{(8)^2 + (-4)^2 + (-2)^2} = \sqrt{64+16+4} = \sqrt{84}$$

$$\|u\|\|v\| = \sqrt{21}\,\sqrt{84} = \sqrt{21}\,\sqrt{4\times 21} = 2\times 21 = 42$$

$$\therefore\ |\langle u \cdot v\rangle| = \|u\|\|v\|$$

$\Rightarrow$ Cauchy Schwartz inequality in verified.

Q. 14 **If W = (α ; $\alpha \in R^n$ and $a_1 \ge 0$) a subset of V = R^n with $\alpha = (a_1, a_2, \ldots\ldots, a_n)$ in R^n ($n \ge 3$). Show that W is not a subspace of V by giving suitable counter example.** Dec. 2015

Ans. :

$$W = (\alpha : \alpha \in \mathbb{R}^n \text{ and } a_1 \ge 0)$$

Where $\alpha = (a_1, a_2, a_3 \ldots\ldots a_n) \in R^n$

Let $\alpha_1 = (a_1, a_2, a_3, \ldots\ldots, a_n)$ and

$$\alpha_2 = (b, b_1, b_2, \ldots\ldots, b_n) \in W$$

$$\Rightarrow\ a_1, b_1 \ge 0$$

Let $\alpha_1 + \alpha_2 = (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3, \ldots\ldots, a_n + b_n) \in W$

as $\alpha_1 + b_1 \ge 0$

Let, K is – 1 then

$$K\alpha_1 = (-1)\alpha_1 = (-1)(a_1, a_2, a_3, \ldots\ldots, a_n)$$

$$= (-a_1, -a_2, -a_3, \ldots\ldots, -a_n) \notin W$$

$\Rightarrow$ W is not a subspace.

Q. 15 **Verify Cauchy - Schwartz inequality for u = (1, 2, 1) and v = (3, 0, 4) also find the angle between u and v .** May 2016

Ans. :

$$u = (1,2,1),\ v = (3,0,4)$$

$$\langle u, v\rangle = (1,2,1)\begin{bmatrix}3\\0\\4\end{bmatrix} = 1\times 3 + 2\times 0 + 1\times 4 = 7$$

$$\|\langle u, v\rangle\| = 7$$

$$\|u\| = \sqrt{1^2+2^2+1^2} = \sqrt{6}$$

$$\|v\| = \sqrt{3^2+0^2+4^2} = \sqrt{25} = 5$$

$$\|\langle u, v\rangle\| \le \|u\|\,\|v\|$$

$$7 \le 5\sqrt{6}$$

$\therefore$ Cauchy Schwartz Inequality is satisfied.

The angle between these vectors.

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\,\|v\|} = \frac{7}{5\sqrt{6}} \Rightarrow \theta = \cos^{-1}\left(\frac{7}{5\sqrt{6}}\right)$$

Q. 16 Find an orthonormal basis for subspace of R^3 by applying Gram-Schmidt process where S {(1,1,1), (0,1,1) (0, 0, 1)} May 2016

Ans. :

$$S = \{(1, 1, 1), (0, 1, 1) (0, 0, 1)\}$$

Gram - Schmitt orthogonalisation process

$$u_1 = (1, 1, 1), u_2 = (0, 1, 1), u_3 = (0, 0, 1)$$

$$v_1 = u_1 = (1, 1, 1)$$

$$v_2 = u_2 - \frac{< u_2 \cdot u_1 >}{\|v_1\|^2} v_1$$

$$= (0, 1, 1) - \frac{(0, 1, 1) \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\left(\sqrt{1^2+1^2+1^2}\right)^2}(1, 1, 1)$$

$$= (0, 1, 1) - \frac{(1+1)}{3}(1, 1, 1) = (0, 1, 1) - \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$= \left(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$v_3 = u_3 - \frac{< u_3 \cdot v_1 >}{\|v_1\|^2} v_1 - \frac{< u_3 \cdot v_2 >}{\|v_2\|^2} v_2$$

$$= (0, 0, 1) - \frac{(0, 0, 1) \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\left(\sqrt{1^2+1^2+1^2}\right)^2}(1, 1, 1)$$

$$- \frac{(0, 0, 1) \cdot \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}}{\left(\sqrt{\left(\frac{-2}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2}\right)^2}\left(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= (0, 0, 1) - \frac{1}{3}(1, 1, 1) - \frac{\frac{1}{3}}{\frac{2}{3}}\left(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= (0, 0, 1) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) - \frac{1}{2}\left(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= (0, 0, 1) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) - \left(\frac{-1}{3}, \frac{1}{6}, \frac{1}{6}\right)$$

$$= \left(0, \frac{-1}{2}, \frac{1}{2}\right)$$

$$v_1 = (1, 1, 1), v_2 = \left(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right) v_3 = \left(0, \frac{-1}{2}, \frac{1}{2}\right)$$

So the orthogonal basis $\delta = \left\{ (1,1,1) \left(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(0, \frac{-1}{2}, \frac{1}{2}\right)\right\}$

Q. 17 Prove that W = {(x, y) | x = 3y} subspace of R^2 is W_1 = {(a, 1, 1) | a in R} subspace of R^3 ? May 2016

Ans. :

$$W = \{(x, y) \mid x = 3y\}$$

Let $(x_1, y_1), (x_2, y_2) \in W \Rightarrow x_1 = 3y_1$ and $x_2 = 3y_2$

$$\Rightarrow \quad x_1 + x_2 = 3y_1 + 3y_2 ;$$

$$\Rightarrow \quad (x_1 + x_2) = 3(y_1 + y_2)$$

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$\because (x_1 + x_2) = 3(y_1 + y_2) \Rightarrow (x_1 + x_2, y_1 + y_2) \in W$$

Let k is a scalar and $(x, y) \in W \Rightarrow x = 3y \Rightarrow kx = 3\,ky$

$$\therefore \quad k(x, y) = (kx, ky) \in W$$

$\Rightarrow$ W is a subspace

$W_1 = \{(a, 1, 1) \mid a \in R\}$ let $(a, 1, 1)$ and $(b, 1, 1) \in W$

$(a_1, 1, 1) + (b_1, 1, 1) = (a_1 + b_1, 2, 2) \notin W \Rightarrow$ It is not subspace

Q. 18 (i) Determine the function that gives shortest distance between two given points.

(ii) Express any vector (a, b,c) in R^3 as a linear combination of v_1, v_2, v_3 where v_1, v_2, v_3 are in R^3. May 2016

Ans. :

(i) Assume two distinct points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ lie in the xy – plane. If y = f(x) is the equation of any plane curve C in xy - plane and passing through the points P_1 and P_2 then the

$$\text{length of the curve} = \int_{x_1}^{x_2} \sqrt{1+(y')^2}\, dx$$

Now the variational problem is to find the curve so that the length is minimum. By Eulers equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$0 - \frac{d}{dx}\left(\frac{1}{2\sqrt{1+(y')^2}}\, 2y'\right) = 0$$

Then

$$-\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = 0$$

$$\frac{y'}{\sqrt{1+(y')^2}} = C$$

$$y' = C\sqrt{1+(y')^2} \Rightarrow (y')^2 = C^2\left(1+(y')^2\right)$$

$$(1 - C^2)(y')^2 = C^2$$

$$(y')^2 = \frac{C^2}{1-C^2} \Rightarrow y' = \sqrt{\frac{C^2}{1-C^2}} \left(\text{Let} \sqrt{\frac{C^2}{1-C^2}} = k\right)$$

$$y' = k$$

$y = kx + C$

Which is the equation of straight line connecting (x_1, y_1) (x_2, y_2)

(ii) The linear expression of (a, b, c) in terms of v_1, v_2, v_3 is

$$(a, b, c) = k_1 v_1 + k_2 v_2 + k_3 v_3$$

Q. 19 **Construct an orthonormal basis of R^3 using Gram Schmidt process to** **Dec. 2016**

$S = \{(3, 0, 4), (-1, 0, 7), (2, 9, 11)\}$

Ans. : Given basis is

$$S = \{(3, 0, 4)\ (-1, 0, 7)\ (2, 9, 11)\}$$

$$u_1 = (3, 0, 4) \quad u_2 = (-1, 0, 7), \quad u_3 = (2, 9, 11)$$

Let $V_1 = (3, 0, 4) = u_1$

$$V_2 = u_2 - \frac{\langle u_2, u_1 \rangle}{\| V_1 \|^2} V_1$$

$$= (-1, 0, 7) - \frac{(-1, 0, 7)\begin{pmatrix}3\\0\\4\end{pmatrix}}{(\sqrt{3^2+0^2+4^2})^2} (3, 0, 4)$$

$$= (-1, 0, 7) - \frac{(-3+28)}{25} (3, 0, 4)$$

$$= (-1, 0, 7) - (3, 0, 4) = (-4, 0, 3)$$

$$V_3 = u_3 - \frac{\langle u_3 \cdot V_1 \rangle}{\| V_1 \|^2} V_1 - \frac{\langle u_3 \cdot V_2 \rangle}{\| V_2 \|^2} V_2$$

$$= (2, 9, 11) - \frac{(2, 9, 11)\begin{pmatrix}3\\0\\4\end{pmatrix}}{(\sqrt{3^2+0^2+4^2})^2} (3, 0, 4) - \frac{(2, 9, 11)\begin{pmatrix}-4\\0\\3\end{pmatrix}}{\sqrt{(-4)^2+0^2+3^2}} (-4, 0, 3)$$

$$= (2, 9, 11) - \frac{(6+44)}{25} (3, 0, 4) - \frac{(-8+33)}{(\sqrt{25})^2} (-4, 0, 3)$$

$$= (2, 9, 11) - 2\,(3, 0, 4) - (-4, 0, 3)$$

$$= (2, 9, 11) - (6, 0, 8) - (-4, 0, 3)$$

$$= (-4, 9, 3) - (-4, 0, 3) = (0, 9, 0)$$

So orthonormal basis is

$\{(3, 0, 4)\ (-4, 0, 3)\ (0, 9, 0)\}$

Q. 20 **(i) Check whether $W = \{(x, y, z) \mid y = x + z,\ x, y, z$ are in $R\}$ is a subspace of R^3 with usual addition and usual multiplication.** **Dec. 2016**

(ii) Find the unit vector in R^3 orthogonal to both $u = (1, 0, 1)$ and $v = (0, 1, 1)$

Ans. :

(i) $W = \{(x, y, z) \mid y + x + z,\ x, y, z \in \mathbb{R}\}$

(a) let $(x_1, y_1, z_1),\ (x_2, y_2, z_2) \in W$

$\Rightarrow y_1 = x_1 + z_1$ and $y_2 = x_2 + z_2$

$\Rightarrow (y_1 + y_2) = (x_1 + x_2) + (z_1 + z_2)$

$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$\because\ y_1 + y_2 = (x_1 + x_2) + (z_1 + z_2)$

$\Rightarrow (x_1 + x_2, y_1 + y_2, z_1 + z_2) \in W$

(b) Let K is a scaler and $(x, y, z) \in W$

If $(x, y, z) \in W \Rightarrow y = x + z$

$\Rightarrow ky = kx + kz$

$\Rightarrow (kx, ky, kz) \in W$

$\Rightarrow k\,(x, y, z) \in W$

$\Rightarrow$ W is a subspace

(ii) $u = (1, 0, 1),\ V = (0, 1, 1)$

Let that vector is (l, m, n)

$$(1, 0, 1)\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \Rightarrow l + 0m + n = 0$$

$$(0, 1, 1)\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \Rightarrow 0l + m + n = 0$$

$$\frac{l}{(0-1)} = \frac{-m}{(1-0)} = \frac{n}{(1-0)}$$

$$\therefore \frac{l}{-1} = \frac{-m}{1} = \frac{n}{1}$$

So that orthogonal vector is $= (-1, -1, 1)$

To make it unit vector $= \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

Chapter 3 : Linear Algebra Matrix Theory

Q. 1 **Prove that Eigen values of a hermitian matrix are real.** **May 2014**

Ans. :

Let A be a Hermitian matrix.

$$\therefore A = A^{\theta} \qquad \ldots(1)$$

Let λ be eigen value and X be corresponding eigen vector of matrix A.

$$\therefore AX = \lambda X \quad ...(2)$$

Pre-multiplying by X^θ,

$$\therefore X^\theta AX = X^\theta (\lambda X)$$

$$\therefore X^\theta AX = \lambda (X^\theta X) \quad ...(3)$$

Taking transpose conjugate

$$\therefore (X^\theta AX)^\theta = (\lambda X^\theta X)^\theta$$

$$\therefore X^\theta A^\theta (X^\theta)^\theta = X^\theta (X^\theta)^\theta \lambda^\theta$$

$$\therefore X A X^\theta = X^\theta X \bar{\lambda} \quad ...(4) \text{ (From Equation 1)}$$

From Equations (3) and (4)

$$\therefore \lambda X^\theta X = \bar{\lambda} X^\theta X$$

$$\therefore X^\theta X(\lambda - \bar{\lambda}) = 0$$

$$\therefore X^\theta = 0 \text{ or } X = 0 \text{ or } \lambda = \bar{\lambda}$$

But, eigen vector is non-zero.

$$\therefore X \neq 0 \text{ and so } X^\theta \neq 0$$

$$\therefore \lambda = \bar{\lambda}$$

$\therefore \lambda$ is real.

Hence, eigen values of Hermitian matrix are real.

Q. 2 Find eigen values and eigen vectors of the matrix $A = \begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix}$ **May 2014**

Ans. :

Let λ be eigen value and X be corresponding eigen vector of matrix A.

The characteristics equation is $|A - \lambda I| = 0$

$$\therefore \begin{bmatrix} -2-\lambda & 5 & 4 \\ 5 & 7-\lambda & 5 \\ 4 & 5 & -2-\lambda \end{bmatrix} I = 0$$

On solving, ,

λ^3 – (sum of diagonal elements) λ^2 + (sum of the minors of diagonal elements) $\lambda - |A| = 0$

$$\therefore \lambda^3 - (-2 + y - 2)\lambda^2 + (-39 - 12 - 39)\lambda - 216 = 0$$

$$\therefore \lambda^3 - 3\lambda^2 - 90\lambda - 216 = 0$$

$\therefore$ Eigen values (λ) are 12, – 3, – 6

Case 1 :

$$\lambda = 12$$

$$\therefore |A - \lambda I| X = 0$$

$$\begin{bmatrix} -14 & 5 & 4 \\ 5 & -5 & 5 \\ 4 & 5 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore -14 x_1 + 5 x_2 + 4 x_3 = 0 \text{ and } 5 x_1 - 5 x_2 + 5 x_3 = 0$$

Using Crammer's rule,

$$\frac{x}{\begin{vmatrix} 5 & 4 \\ -5 & 5 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -14 & 4 \\ -5 & 5 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -14 & 5 \\ -5 & -5 \end{vmatrix}}$$

$$\therefore \frac{x_1}{45} = \frac{-x_2}{-90} = \frac{x_3}{45}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$\therefore X_1 = [1\ 2\ 1]$$

If k is non-zero scalar then k X_1 is also eigen vector.

Case 2 :

$$\lambda = -3$$

$$[A - \lambda I] X = 0$$

$$\therefore \begin{bmatrix} 1 & 5 & 4 \\ 5 & 10 & 5 \\ 4 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore 1x_1 + 5 x_2 + 4 x_3 = 0 \text{ and } 5 x_1 + 10 x_2 + 5 x_3 = 0$$

Using Crammer's rule,

$$\frac{x_1}{\begin{vmatrix} 5 & 4 \\ 10 & 5 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 4 \\ 5 & 5 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 5 \\ 5 & 10 \end{vmatrix}}$$

$$\therefore \frac{x_1}{-15} = \frac{-x_2}{-15} = \frac{x_3}{-15}$$

$$\therefore \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{-15}$$

$$X_2 = [1 - 1\ 1]$$

If k is non-zero scalar then k X_2 is also eigen vector.

Case 3 : $\lambda = -6$

$$\therefore [A - \lambda I]X = 0$$

$$\begin{bmatrix} 4 & 5 & 4 \\ 5 & 13 & 5 \\ 4 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore 4x_1 + 5 x_2 + 4 x_3 = 0 \text{ and } 5 x_1 + 13 x_2 + 5 x_3 = 0$$

Using Crammer's rule,

$$\frac{x_1}{\begin{vmatrix} 5 & 4 \\ 13 & 5 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 4 & 4 \\ 5 & 5 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 4 & 5 \\ 5 & 13 \end{vmatrix}}$$

$$\therefore \frac{x_1}{-27} = \frac{-x_2}{0} = \frac{x_3}{27}$$

$$\therefore \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$$X_3 = [-1\ 0\ 1]$$

If k is non-If k is non-zero scalar then k X_3 is also eigen vector.

Hence,

Eigen values (λ) are 12, – 3, – 6,

Eigen vectors are [1 2 1], [1 – 1 1], [– 1 0 1]

Q. 3 **State Cayley-Hamilton Theorern, hence deduce that $A^8 = 625\ I$, where $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.**

May 2014

Ans. :

Let λ be eigen value and X be corresponding eigen vector of matrix A.

Characteristic equation is $|A - \lambda I| = 0$

$$\begin{bmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix} = 0$$

On solving,

λ^2 – (sum of diagonal elements) $\lambda + |A| = 0$

$$\therefore \lambda^2 - (1-1)\lambda - 5 = 0$$

$$\therefore \lambda^2 - 5 = 0$$

Cayley Hamilton Theorem states that the characteristic equation is satisfied by A,

$$\therefore A^2 - 5I = 0$$

$$\therefore A^2 = 5I \quad \ldots(1)$$

Now, $A^4 = A^2 \times A^2$

$= 5I \times 5I$ (from Equation (1))

$= 25I \ (I \times I = I)$ ...(2)

$A^8 = A^4 \times A^4$

$= 25I \times 25I$ (from Equation 2) $= 625I$

$$A^8 = 625I$$

Q. 4 **Using calculus of Residues, prove that $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - n\theta)\, d\theta = \frac{2\pi}{n!}$.**

May 2014

Ans. :

$$\text{Let } I = \int_c \frac{e^z}{z^{n+1}}\, dz$$

Consider a circle $|z| = 1$ which has centre (0, 0) and radius 1.

Here, $z_0 = 0$ lies inside the circle

$z_0 = 0$ is a pole of order n + 1

$\therefore R_1$ = Residue of f(z) at "z = 0"

$$= \frac{1}{(N-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^N \times f(z)$$

$$= \frac{1}{(n+1-1)!} \lim_{z \to 0} \frac{d^{n+1-1}}{dz^{n+1-1}} \left[z^{n+1} \times \frac{e^z}{z^{n+1}} \right]$$

$$= \frac{1}{n!} \lim_{z \to 0} \frac{d^n}{dz^n} e^z = \frac{1}{n!} \lim_{z \to 0} e^z$$

$$= \frac{1}{n!} e^0 = \frac{1}{n!}$$

By Cauchy's Residue theorem

$$\int_c f(z)\, dz = 2\pi i (R_1 + R_2 + \ldots\ldots)$$

$$\int_c \frac{e^z}{z^{n+1}}\, dz = 2\pi i \cdot \frac{1}{n!}$$

Now, Put $z = e^{i\theta}$

$$\therefore dz = e^{i\theta} \cdot i\, d\theta$$

Limits of integration for complete circle is 0 to 2π

$$\therefore \int_0^{2\pi} \frac{e^{e^{i\theta}}}{(e^{i\theta})^{n+1}} \times e^{i\theta} \cdot i d\theta = \frac{2\pi i}{n!}$$

$$\therefore \int_0^{2\pi} \frac{e^{\cos\theta + i\sin\theta}}{(e^{i\theta})^n}\, d\theta = \frac{2\pi}{n!} \text{ (Dividing by 'i')}$$

$$\therefore \int_0^{2\pi} \frac{e^{\cos\theta}\, e^{i\sin\theta}}{e^{in\theta}}\, d\theta = \frac{2\pi}{n!}$$

$$\therefore \int_0^{2\pi} e^{\cos\theta} e^{i\sin\theta} e^{-in\theta} d\theta = \frac{2\pi}{n!}$$

$$\therefore \int_0^{2\pi} e^{\cos\theta} e^{i(\sin\theta - n\theta)} d\theta = \frac{2\pi}{n!}$$

$$\therefore \int_0^{2\pi} e^{\cos\theta} [\cos(\sin\theta - n\theta) + i\sin(\sin\theta - n\theta)]\, d\theta = \frac{2\pi}{n!} + 0i$$

Equating real parts

$$\therefore \int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - n\theta)\, d\theta = \frac{2\pi}{n!}$$

Q. 5 **If $A = \begin{bmatrix} \frac{\pi}{2} & \frac{2\pi}{2} \\ \pi & \pi \end{bmatrix}$, find SinA.**

May 2014

Ans. :

(i) Let λ be eigen value of matrix A.

Characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} \pi/2 - \lambda & 3\pi/2 \\ \pi & \pi - \lambda \end{vmatrix} = 0$$

On solving ,

$$\lambda^2 - (\text{sum of diagonal elements})\lambda + |A| = 0$$

$$\lambda^2 - \left(\frac{\pi}{2} + \pi\right)\lambda + \left(\frac{\pi^2}{2} - \frac{3\pi^2}{2}\right) = 0$$

$$\therefore \lambda^2 - \frac{3\pi}{2}\lambda - \pi^2 = 0$$

$$2\lambda^2 - 3\pi\lambda - 2\pi^2 = 0$$

$$2\lambda^2 - 4\pi\lambda + \pi\lambda - 2\pi^2 = 0$$

$$(2\lambda + \pi)(\lambda - 2\pi) = 0$$

$\therefore$ Eigen values (λ) are $\frac{-\pi}{2} \cdot 2\pi$

Since A is order of 2×2. Let

$$\sin A = aA + bI \quad \ldots(1)$$

(where a, b are constants)

We assume Equation (1) is satisfied by λ.

$$\therefore \sin\lambda = a\lambda + b \quad \ldots(2)$$

When $\lambda = 2\pi$,

$$\sin 2\pi = a \times 2\pi + b$$

$$\therefore 0 = 2\pi a + b$$

$$\therefore b = -2\pi a \quad \ldots(3)$$

when $\lambda = \frac{-\pi}{2}$

$$\sin\left(\frac{-\pi}{2}\right) = a\left(\frac{-\pi}{2}\right) + b$$

$$\therefore -1 = \frac{-\pi}{2}a - 2\pi a \text{ (from Equation 3)}$$

$$\therefore -1 = \frac{-5\pi}{2}a$$

$$\therefore a = \frac{2}{5\pi} \quad \ldots(4)$$

$\therefore$ From Equation (3), $b = -2\pi \times \frac{2}{5\pi} = \frac{-4}{5}$

$\therefore$ From Equation (1), $\sin A = \frac{2}{5\pi} A - \frac{4}{5} I = \frac{1}{5}\left\{\frac{2}{\pi} A - 4I\right\}$

$$= \frac{1}{5}\left\{\frac{2}{\pi}\begin{bmatrix} \pi/2 & 3\pi/2 \\ \pi & \pi \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\left\{\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right\}$$

$$\therefore \sin A = \frac{1}{5}\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}$$

Q. 6 **Show that the matrix $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ is Derogatory.**

May 2014

Ans. :

Let λ be eigen value of matrix A.

Characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix} = 0$$

On solving,

λ^3 – (sum of diagonal elements) λ^2 + (sum of the minors of diagonal elements) $\lambda - |A| = 0$

$$\therefore \lambda^3 - (5 + 4 - 4)\lambda^2 + (-4 - 2 + 14)\lambda - 4 = 0$$

$$\therefore \lambda^3 - 5\lambda^2 + 8\lambda^3 - 4 = 0$$

$\therefore$ Eigen values (λ) are 1, 2, 2

$$\text{Let } f(x) = (x-1)(x-2) = x^2 - 3x + 2$$

$$\text{Now, } A^2 = A \times A = \begin{vmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{vmatrix} \times \begin{vmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{vmatrix}$$

$$A^2 = \begin{vmatrix} 13 & -18 & -18 \\ -3 & 10 & 6 \\ 9 & -18 & -14 \end{vmatrix}$$

$$\therefore A^2 - 3A + 2I = \begin{vmatrix} 13 & -18 & -18 \\ -3 & 10 & 6 \\ 9 & -18 & -14 \end{vmatrix} - 3\begin{vmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{vmatrix} + 2\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$\therefore f(x) = x^2 - 3x + 2$ annihilates A

$\therefore$ f(x) is a minimal polynomial.

Degree of f(x) = 2 and order of A = 3

So degree of f(x) < order of A

$\therefore$ Matrix A is derogatory.

Q. 7 Is the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ **diagonalizable. If so find diagonal form and transforming matrix.** May 2014

Ans. :

Let λ be eigen value of matrix A.

Characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

On solving,

λ^3 – (sum of diagonal elements) λ^2 + (sum of the minors of diagonal elements) $\lambda - |A| = 0$

$$\therefore \lambda^3 - (2+2+1)\lambda^2 + (2+2+3)\lambda - (3) = 0$$

$$\therefore \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

Eigen values (λ) are 1, 1, 3

Case 1 : $\lambda = 1$

$$\therefore [A - \lambda I]\, X = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - R_1$;

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{...(1)}$$

Number of unknowns (n) = 3

Rank (r) = number of non-zero rows = 1

Algebraic multiplicity (A.M.) = No. of times "$\lambda = 1$" is repeated = 2

Geometric multiplicity (G.M.) = $n - r = 3 - 1 = 2$

$\therefore$ A.M. = G.M. for "$\lambda - 1$"

Now from Equation (1)

$$\therefore x_1 + x_2 + x_3 = 0 \quad \text{...(2)}$$

Put $x_1 = 0$; $x_2 = 1$;

$$\therefore 0 + 1 + x_3 = 0$$

$$\therefore x_1 = -1$$

Eigen vector $X_1 = [0\ 1\ -1]'$

If k is non-zero scalar then k X_1 is also an eigen vector.

Put $x_1 = 1$; $x_2 = 0$; in Equation (2)

$$\therefore 1 + 0 + x_3 = 0$$

$$\therefore x_3 = -1$$

$\therefore$ Eigen vector $X_2 = [1\ 0\ -1]'$

If k non-zero scalar then k X_2 is also an eigen vector.

Case 2 : $\lambda = 3$

$$\therefore [A - \lambda I]\, X = 0$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - R_1$;

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 + R_2$; $R_2/2$;

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{...(3)}$$

Rank (r) = 2

A.M. = No. of times "$\lambda = 3$" is repeated = 1

G.M. = $n - r = 3 - 2 = 1$

$$\therefore \text{A.M.} = \text{G.M for "}\lambda = 3\text{"}$$

Now, from Equation (3)

$$\therefore x_3 = 0 \text{ and}$$

$$-x_1 + x_2 + x_3 = 0$$

$$\therefore -x_1 + x_2 + 0 = 0$$

$$-x_1 = x_2 \quad \ldots(4)$$

$$\text{Put } x_2 = 1;$$

From Equation (4), $x_1 = 1$

$$\therefore \text{Eigen vector } X_3 = [1\ 1\ 0]'$$

If k is non-zero scalar then k X_3 is also an eigen vector.

Since, A.M. = G.M. for all eigen values, matrix A is diagonalizable.

$$\therefore M^{-1} AM = D$$

$\therefore$ Matrix A is diagonalized to diagonal matrix D by the transforming matrix M, where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ and } M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Q. 8 Find eigen values and eigen vectors of

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

Dec. 2014

Ans. :

$$\text{Let } A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

The characteristic Equation is $|A - \lambda I| = 0$

$$\begin{bmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 3 & 3 & 4-\lambda \end{bmatrix} = 0$$

On solving,

$$(2-\lambda)[(3-\lambda)(4-\lambda)-6] - 1[2(4-\lambda)-6] + 1[6-3(3-\lambda)] = 0$$

$$(2-\lambda)[12 - 3\lambda - 4\lambda + \lambda^2 - 6] - [8 - 2\lambda - 6] + [6 - 9 + 3\lambda] = 0$$

$$(2-\lambda)[6 - 7\lambda + \lambda^2] - 2 + 2\lambda - 3 + 3\lambda = 0$$

$$12 - 14\lambda + 2\lambda^2 - 6\lambda + 7\lambda^2 - \lambda^3 - 5 + 5\lambda = 0$$

$$-\lambda^3 + 9\lambda^2 - 15\lambda + 7 = 0$$

$$\lambda = -1, -1, -7$$

Case I : $\lambda = -1$

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 + x_3 = 0 \text{ and } 2x_1 + 2x_2 + 2x_3 = 0$$

Using Crammer's rule

$$\frac{x_1}{\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}}$$

$$\frac{x_1}{0} = \frac{-x_2}{0} = \frac{x_3}{0}$$

$$X_1 = [0\,0\,0] \quad \text{Similarly } X_2 = [0\,0\,0]$$

Case 2 : $\lambda = -7$

$$\begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-5x_1 + x_2 + x_3 = 0 \text{ and } 2x_1 - 4x_2 + 2x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} 1 & 1 \\ -4 & 2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -5 & 1 \\ 2 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -5 & 1 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x_1}{6} = \frac{-x_2}{-12} = \frac{x_3}{18}$$

$$X_1 = [6\ 12\ 18]$$

$$\text{Hence eigen vectors are} \begin{bmatrix} 0 & 0 & 6 \\ 0 & 0 & 12 \\ 0 & 0 & 18 \end{bmatrix}$$

Q. 9 Verify Cayley Hamilton theorem for

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \textbf{ hence find } A^{-2}.$$

Dec. 2014

Ans. :

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$\therefore (1-\lambda)[(-1-\lambda)^2]-2[2(-1-\lambda)] = 0$$

$$\therefore (1-\lambda)(1+\lambda)^2+4(1+\lambda) = 0$$

$$\therefore (1+\lambda)[(1-\lambda^2)+4] = 0$$

$$\therefore (1+\lambda)[5-\lambda^2] = 0$$

$$\therefore 5-\lambda^2+5\lambda-\lambda^3 = 0$$

$$\therefore \lambda^3+\lambda^2-5\lambda-5 = 0$$

Replacing λ by [A]

$$\therefore \quad A^2 = A\cdot A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad A^3 = A^2\cdot A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\therefore A^3+A^2-5A-5I = \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5+5-5-5 & 10+0-10+0 & 0 \\ 10+0-10+0 & -5+5+5-5 & 0 \\ 0 & 0 & -1+1+5-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore$ [A] satisfies its own characteristics equation. $\therefore$ Cayley Hamilton theorem is verified.

$$A^2+A-5I-5A^{-1} = 0$$

$$\therefore A^{-1} = \frac{1}{5}[A^2+A-5I]$$

$$= \frac{1}{5}\left\{\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$A^{-2} = \frac{1}{5}[A+I-5A^{-1}]$$

$$= \frac{1}{5}\left\{\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

Q. 10 Show that the matrix $A = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}$ is derogatory. **Dec. 2014**

Ans. :

The characteristic equation of A is

$$\begin{bmatrix} 7-\lambda & 4 & -1 \\ 4 & 7-\lambda & -1 \\ -4 & -4 & 4-\lambda \end{bmatrix} = 0$$

$$\therefore (7-\lambda)[(7-\lambda)(4-\lambda)-4]-4[4(4-\lambda)-4]-1[-16+4(7-\lambda)] = 0$$

$$\therefore (7-\lambda)[(24-11\lambda+\lambda^2]-4[12-4\lambda]-[12-4\lambda)] = 0$$

$$\therefore \lambda^3+18\lambda^2+81\lambda-108 = 0$$

$$\therefore (\lambda-3)(\lambda^2-15\lambda+36) = 0$$

$$\therefore (\lambda-3)(\lambda-12)(\lambda-3) = 0$$

Hence, the roots of $|A-\lambda I| = 0$ are 3, 3, 12.

Let us now find the minimal polynomial of A. Characteristic root of A is also a root of the minimal polynomial of A. So if f (x) is the minimal polynomial of A then x − 3 and x − 12 are the factors of f(x). Let us see whether $(x-3)(x-12) = x^2+15x+36$ annihilates A.

$$\text{Now, } A^2 = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}^2 = \begin{bmatrix} 69 & 60 & -15 \\ 60 & 69 & -15 \\ -60 & -60 & 24 \end{bmatrix}$$

$$\therefore A^2-15A+36I = \begin{bmatrix} 69 & 60 & -15 \\ 60 & 69 & -15 \\ -60 & -60 & 24 \end{bmatrix} - 15\begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix} + 36\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \quad f(x) = x^2+15x+36 \text{ annihilates A.}$$

Thus, f (x) is the monic polynomial of lowest degree that annihilates A. Hence, f (x) is the minimal polynomial of A. Since its degree is less than the order of A, A is derogatory.

Q. 11 Show that the matrix $A = \begin{bmatrix} -9 & -4 & -4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonable. Also find diagonal form and diagonalising matrix. **Dec. 2014**

Ans. :

Let λ be eigen value and X be corresponding eigen vector of matrix A.

Characteristic equation is $|A - \lambda I| = 0$.

$$\therefore \begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0$$

On solving, ,

λ^2 – (sum of diagonal elements) λ^2 + (sum of the minors of diagonal elements) $\lambda - |A| = 0$.

$$\therefore \lambda^3 - (-9+3+7)\lambda^2 + (-11+1+5)\lambda - (3) = 0$$

$$\therefore \lambda^3 - \lambda^2 - 5\lambda - 3 = 0$$

$\therefore$ Eigen values (λ) are 3, – 1, – 1.

Case 1 : $\lambda = -1$

$$\therefore [A - \lambda I] X = 0$$

$$\therefore \begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - R_1$; $R_3 - 2R_1$; $R_1/4$;

$$\therefore \begin{bmatrix} -2 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad ...(1)$$

Number of unknowns (n) = 3.

Rank (r) = number of non-zero rows = 1.

Here, algebraic multiplicity (A.M.) = No. of times "$\lambda = -1$" is repeated = 2.

Geometric multiplicity (G.M.) = n – r = 3 – 1 = 2.

$\therefore$ A.M. = G.M. for "$\lambda = -1$".

From Equation (1),

$$-2x_1 + x_2 + x_3 = 0 \quad ...(2)$$

$$\text{Put } x_1 = 0; \quad x_2 = 1$$

$$\therefore 0 + 1 + x_3 = 0$$

$$\therefore x_3 = -1$$

$\therefore$ Eigen vector x_1 = [0 1 – 1].

If k is non-zero scalar then k x_1 is also an eigen vector.

Put $x_1 = 1$; $x_2 = 0$ in Equation (2)

$$\therefore -2 + 0 + x_3 = 0$$

$$\therefore x_3 = 2$$

$\therefore$ Eigen vector x_2 = [1 0 2].

If k is non-zero scalar then k X_2 is also an eigen vector.

Case 2 : $\lambda = 3$

$$\therefore [A - \lambda I] X = 0$$

$$\therefore \begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 - R_2$; $R_3 - R_2$; $R_2/4$;

$$\therefore \begin{bmatrix} -4 & 4 & 0 \\ -2 & 0 & 1 \\ -8 & 8 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - 2R_1$; $R_1/4$;

$$\therefore \begin{bmatrix} -1 & 1 & 0 \\ -2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad ...(3)$$

Here n = 3 and r = 1.

A.M. = No. of times "$\lambda = 3$" is repeated = 1.

G.M. = n – r = 3 – 2 = 1

$\therefore$ A.M. = G.M. for "$\lambda = 3$".

Expanding Equation (3),

$$-2x_1 + x_3 = 0 \quad ...(4)$$

$$\text{And } -x_1 + x_2 = 0 \quad ...(5)$$

$$\text{Put } x_1 = 1$$

From Equation (4),

$$-2(1) + x_3 = 0$$

$$\therefore x_3 = 2$$

From Equation (5),

$$-1 + x_2 = 0$$

$$\therefore x_2 = 1$$

$\therefore$ Eigen vector x_3 = [1 1 2].

If k is non-zero scalar then k x_3 is also an eigen vector.

Since, A.M. = G.M. for all eigen values, matrix A is diagonalizable.

$$\therefore M^{-1}AM = D$$

So the given matrix A is diagonalized to diagonal matrix D by the transforming matrix M, where

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\text{and } M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 2 & 2 \end{bmatrix}$$

Q. 12 If λ is an Eigen value of the matrix A with corresponding Eigen vector X, prove that λ^n is an Eigen value of A^n with corresponding Eigen vector X. May 2015

Ans. :

Since, λ is eigen value and X is corresponding eigen vector,

$AX = \lambda X$ ...(1)

Pre multiply by A,

$$\therefore A(AX) = A(\lambda X)$$

$$\therefore A^2X = \lambda(AX)$$

$$\therefore A^2X = \lambda(\lambda X) \quad \text{(From Equation (1))}$$

$$\therefore A^2X = \lambda^2X \quad ...(2)$$

$\therefore \lambda^2$ is eigen value and X is corresponding eigen vector of A^2

Pre multiply Equation (2) by A.

$$\therefore A(A^2X) = A(\lambda^2X)$$

$$A^3X = \lambda^2(AX)$$

$$\therefore A^3X = \lambda^2(\lambda X) \quad \text{(From Equation (1))}$$

$$\therefore A^3X = \lambda^3X$$

$\therefore \lambda^3$ is eigen value and X is corresponding eigen vector of A^3.

And so on ....

In general,

$$\therefore A^nX = \lambda^nX$$

λ^n is eigen value and X is corresponding eigen vector of A^n.

Q. 13 Find the Eigen values and Eigen vectors for the matrix $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ May 2015

Ans. :

Let λ be eigen value and X be corresponding eigen vector.

The characteristic equation is $|A - \lambda I| = 0$

$$\therefore \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix} = 0$$

On solving ,

λ^3 – (sum of diagonal elements) λ^2 + (sum of minors of diagonal elements) $\lambda - |A| = 0$

$$\therefore \lambda^3 - (2+3+2)\lambda^2 + (4+3+4)\lambda - 5 = 0$$

$$\therefore \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

$\therefore$ Eigen values (λ) are 5, 1, 1

Case 1 : $\lambda = 5$

$$\therefore [A - \lambda I]\ X = 0$$

$$\therefore \begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore -3x_1 + 2x_2 + 1x_3 = 0$ and $1x_1 - 2x_2 + 1x_3 = 0$

Using Crammer's rule.

$$\frac{x_1}{\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & 2 \\ 1 & -2 \end{vmatrix}}$$

$$\therefore \frac{x_1}{4} = \frac{-x_2}{-4} = \frac{x_3}{4}$$

$$\therefore \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$\therefore$ Eigen vector $X_1 = [1\ 1\ 1]$

Case 2 : $\lambda = 1$

$$\therefore [A - \lambda I]\ X = 0$$

$$\therefore \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - R_2$; $R_2 - R_1$;

$$\therefore \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore 1x_1 + 2x_2 + 1x_3 = 0 \quad ...(1)$$

n = number of unknowns = 3

r = rank = number of non-zero rows = 1

n – r = 3 – 1 = 2

Put x_1 = s and x_2 = t in Equation (1)

$$\therefore s + 2t + x_3 = 0$$

$$\therefore x_3 = -s - 2t$$

$\therefore$ Eigen Vector

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ t \\ -s-2t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore$ Eigen vector X_2 = [1 0 –1] and

X_3 = [0 1 2]

Q. 14 **If A $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ find A^{50}**

May 2015

Ans. : Let λ be eigen value and X be corresponding eigen vector of matrix A. Characteristic equation is $|A - \lambda I| = 0$.

$$\therefore \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

On solving λ^2 – (sum of diagonal elements) $\lambda + |A| = 0$

$$\therefore \lambda^2 - (2+2)\lambda + 3 = 0$$

$$\therefore \lambda^2 - 4\lambda + 3 = 0$$

$\therefore$ Eigen values (λ) are 3, 1

Since, A is 2×2 matrix, let $A^{50} = a_1 A + a_0 I$ ...(1)

We assume, eigen value λ satisfies Equation (1)

$$\therefore \lambda^{50} = a_1 \lambda + a_0 \quad ...(2)$$

Put $\lambda = 1$

$$\therefore 1^{50} = a_1(1) + a_0$$

$$\therefore 1 = a_1 + a_0 \quad ...(3)$$

Put λ = 3 in Equation (2)

$$\therefore 3^{50} = a_1(3)\, a_0$$

$$\therefore 3^{50} = 3a_1 + a_0 \quad ...(4)$$

Subtract Equation (3) from Equation (4),

$$\therefore 3^{50} - 1 = 2a_1$$

$$\therefore a_1 = \frac{3^{50}-1}{2}$$

From Equation (3), $a_0 = 1 - a_1 = 1 - \frac{3^{50}-1}{2} = \frac{2-(3^{50}-1)}{2}$

$$\therefore a_0 = \frac{3-3^{50}}{2}$$

Hence, from Equation (1), $A^{50} = \left(\frac{3^{50}-1}{2}\right) A + \left(\frac{3-3^{50}}{2}\right) I$

$$= \frac{1}{2}\left\{(30^{50}-1)\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + (3-3^{50})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{2}$$

$$\left\{\begin{bmatrix} 2\times 3^{50}-2 & 3^{50}-1 \\ 3^{50}-1 & 2\times 3^{50}-2 \end{bmatrix} + \begin{bmatrix} 3-3^{50} & 0 \\ 0 & 3-3^{50} \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 2\times 3^{50}-2+3-3^{50} & 3^{50}-1 \\ 3^{50}-1 & 2\times 3^{50}-2+3-3^{50} \end{bmatrix}$$

$$\therefore A^{50} = \frac{1}{2}\begin{bmatrix} 3^{50}+1 & 3^{50}-1 \\ 3^{50}-1 & 3^{50}+1 \end{bmatrix}$$

Q. 15 **Show that the matrix A = $\begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}$ is derogatory and find its minimal polynomial.**

May 2015

Ans. :

Let λ be eigen value of matrix A.

Characteristic equation is $|A - \lambda I| = 0$

$$\therefore \begin{bmatrix} 7-\lambda & 4 & -1 \\ 4 & 7-\lambda & -1 \\ -4 & -4 & 4-\lambda \end{bmatrix} = 0$$

On solving

λ^3 – (sum of diagonal elements) λ^2 + (sum of the minors of diagonal elements) $\lambda - |A| = 0$

$$\therefore \lambda^3 - (7+7+4)\lambda^2 + (24+24+33)\lambda - 108 = 0$$

$$\therefore \lambda^3 - 18\lambda^2 + 81\lambda - 108 = 0$$

$\therefore$ Eigen values (λ) are 3, 3, 12

Let $f(x) = (x-3)(x-12) = x^2 - 15x + 36$

$$\text{Now, } A^2 = A \times A = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}\begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 69 & 60 & -15 \\ 60 & 69 & -15 \\ -60 & -60 & 24 \end{bmatrix}$$

$$\therefore A^2 - 15A + 36I = \begin{bmatrix} 69 & 60 & -15 \\ 60 & 69 & -15 \\ -60 & -60 & 24 \end{bmatrix} - 15\begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix} + 36\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore f(x) = x^2 - 15x + 36$ annihilates A.

$\therefore f(x) = x^2 - 15x + 36$ is a minimal polynomial

Degree of f(x) = 2 and Order of A = 3.

$\therefore$ Degree of f(x) < Order of A

Hence, matrix A is derogatory

Q. 16 Show that the following matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$ **is diagonalizable. Also find the diagonal form and a diagonalising matrix.**

May 2015

Ans. :

Let λ be eigen value and X be corresponding eigen vector.
Characteristic equation is $|A - \lambda I| = 0$

$$\therefore \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} = 0$$

On solving, ,

λ^3 – (sum of diagonal elements) λ^2 + (sum of the minors of diagonal elements) $\lambda - |A| = 0$

$$\therefore \lambda^3 - (6+3+3)\lambda^2 + (8+14+14)\lambda - 32 = 0$$

$$\therefore \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

∴ Eigen values (λ) are 8, 2, 2

Case 1 : $\lambda = 2$

$$\therefore |A - \lambda I|X = 0$$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 + R_2 ; R_2 + \frac{1}{2}R_1 ; \frac{1}{2}R_1$;

$$\therefore \begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore 2x_1 - 1x_2 + 1x_3 = 0 \quad \text{...(1)}$$

n = number of unknowns = 3

r = rank = number of non-zero rows = 1

n – r = 3 – 1 = 2

Algebraic multiplicity (A.M.) = Number of times = "λ = 2" is repeated = 2

Geometric multiplicity (G.M.) = n – r = 2

∴ A.M. = G.M. for "λ= 2"

Put x_1 = s and $x_2 = t$ in Equation (1)

$$\therefore 2s - 1t + x_3 = 0$$

$$\therefore x_3 = -2s + t$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ t \\ -2s+t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

∴ Eigen vector $X_1 = [1 \ 0 \ -2]$ and

$$X_2 = [0 \ 1 \ 1]$$

Case 2 : $\lambda = 8$

$$\therefore [A - \lambda I]X = 0$$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore R_3 + R_1 ; R_2 - R_2 ; \frac{-1}{2}R_1 ;$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore R_3 - R_2 ; \frac{-1}{2}R_2 ;$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore 1x_1 + 1x_2 - 1x_3 = 0 \quad \text{...(2)}$$

$$\text{and } 1x_2 + 1x_3 = 0 \quad \text{...(3)}$$

n = number of unknowns = 3

r = rank = number of non-zero rows = 2

n – r = 3 – 2 = 1

Algebraic multiplicity (A.M.) = Number of times "λ = 8" is repeated = 1

Geometric multiplicity (G.M.) = n – r = 1

∴ A.M = G.M. for "λ = 8"

Put $x_2 = t$ in Equation (3)

$$\therefore t + x_3 = 0$$

$$\therefore x_3 = -t$$

From Equation (2), $x_1 + t - (-t) = 0$

$$\therefore x_1 = -2t$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ -t \end{bmatrix} = -t\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

∴ Eigen vector $X_3 = [2 \ -1, \ 1]$

Since, A.M. = G.M. for all eigen values, matrix A is diagonalizable

$$\therefore \; M^{-1} AM = D$$

$\therefore$ Matrix A is diagonalized to diagonal matrix D by the transforming matrix M, where

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} \text{ and }$$

$$M = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ -2 & 1 & 1 \end{bmatrix}$$

Q. 17 Show that eigen values of unitary matrix are of unit modulus. Dec. 2015

Ans. :

Let A be an unitary matrix and λ is the eigen value of A and corresponding eigen vector is X

$$\Rightarrow \quad AX = \lambda X \quad \text{...(i)}$$

$$\Rightarrow \quad (AX)^{\theta} = (\lambda X)^{\theta}$$

$$\Rightarrow \quad X^{\theta} A^{\theta} = \bar{\lambda} X^{\theta} \quad \text{...(ii)}$$

From (i) and (ii)

$$(X^{\theta} A^{\theta})(AX) = (\bar{\lambda} X^{\theta}) \times (\lambda X)$$

$$X^{\theta} (A^{\theta} A) X = \bar{\lambda} \lambda X^{\theta} X$$

$$X^{\theta} X = \bar{\lambda} \lambda X^{\theta} X$$

(as $A^{\theta} A = I$ since A is unitary)

$$\Rightarrow \quad \bar{\lambda} \lambda = 1$$

$$\Rightarrow \quad |\lambda|^2 = 1 \Rightarrow |\lambda| = 1$$

$\therefore$ eigen value of unitary matrix is of unit modulus.

Q. 18 Find the eigen value and Eigen vectors of the matrix A^3 where $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$ Dec. 2015

Ans. :

$$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$$

Characteristics equation of A

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - |A| = 0$$

$$S_1 = 4 + 3 - 2 = 5$$

$$S_2 = \begin{vmatrix} 3 & 2 \\ -5 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix}$$

$$= (-6+10) + (-8+6) + (12-6) = 4 - 2 + 6 = 8$$

$$|A| = 4(-6+10) - 6(-2+2) + 6(-5+3)$$

$$= 4(4) - 6(0) + 6(-2) = 16 - 12 = 4$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\lambda = 2, 2, 1$$

eigen vector $\lambda = 2$ $(A - 2I)X = 0$

$$\begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2 \quad \begin{bmatrix} 1 & 1 & 2 \\ 2 & 6 & 6 \\ -1 & -5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - 2R_1$, $R_3 + R_1$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 4 & 2 \\ 0 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow R_3 + R_2 \begin{bmatrix} 1 & 1 & 2 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho = 2, n = 3 \Rightarrow$ Number of Linearly independent eigen vector $= 3 - 2 = 1$

$$x_1 + x_2 + x_3 = 0$$

$$0x_1 + 4x_2 + 2x_3 = 0$$

$$\frac{x_1}{(2-4)} = \frac{-x_2}{(2-0)} = \frac{x_3}{(4-0)} \Rightarrow \frac{x_1}{-2} = -\frac{x_2}{2} = \frac{x_3}{4}$$

$$\therefore \text{ eigen vector is } = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$$\text{for } \lambda = 1 \quad (A - I)X = 0 \Rightarrow \begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 + 2x_3 = 0$$

$$-x_1 - 5x_2 - 3x_3 = 0$$

$$\Rightarrow \frac{x_1}{(-6+10)} = \frac{-x_2}{(-3+2)} = \frac{x_3}{(-5+2)}$$

$$\Rightarrow \frac{x_1}{4} = \frac{-x_2}{-1} = \frac{x_3}{-3} \Rightarrow X_2 = \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$$

$\therefore$ Eigen values of A^3 are 2^3, 2^3, $1^3 = 8, 8, 1$

And corresponding eigen vector is $\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$

Q. 19 **Verify Cayley-Hamilton Theorem and find the value of A^{64} for the maxim $A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$.** **Dec. 2015**

Ans. :

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

Characteristic equation,

$$|A - \lambda I| = \begin{bmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix} = 0$$

$\Rightarrow -(1-\lambda)(1+\lambda) - 4 = 0$

$\Rightarrow -(1-\lambda^2) - 4 = 0 \Rightarrow \lambda^2 - 5 = 0$

$$A^2 - 5I = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}^2 - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Cayley Hamilton theorem is verified.

$$\Rightarrow A^2 = 5I$$

$$\Rightarrow (A^2)^{32} = (5I)^{32}$$

$$\Rightarrow A^{64} = 5^{32}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5^{32} & 0 \\ 0 & 5^{32} \end{bmatrix}$$

Q. 20 **Show that a closed curve 'C' of given fixed length (perimeter) which encloses maximum area is a circle.** **Dec. 2015**

Ans. : Area inclosed by the curve

$$\text{Area} = \frac{1}{2}\int_C (x\,dy - y\,dx) = \frac{1}{2}\int_C (xy' - y)\,dx \quad \ldots(1)$$

Length of the curve,

$$S = \int_C \sqrt{1+(y')^2}\,dx$$

$$F(x, y, y') = \frac{1}{2}(xy' - y) + \lambda\sqrt{1+(y')^2}$$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$-\frac{1}{2} - \frac{d}{dx}\left(\frac{1}{2}x + \frac{\lambda 2y'}{2\sqrt{1+(y')^2}}\right) = 0$$

$$-\frac{1}{2} - \frac{d}{dx}\left(\frac{1}{2}x + \frac{\lambda y'}{\sqrt{1+(y')^2}}\right) = 0$$

$$-\frac{1}{2} - \frac{1}{2} - \frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+(y')^2}}\right) = 0$$

$$\frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+(y')^2}}\right) = -1$$

$$\frac{\lambda y'}{\sqrt{1+(y')^2}} = -x + C_1$$

We solve this equation of y'

$$(\lambda y')^2 = (x - C_1)^2(1 + (y')^2)$$

$$\lambda^2 (y')^2 = (x - C_1)^2 (x - C_1)^2 (y')^2$$

$$(y')^2[\lambda^2 - (x - C_1)^2] = (x - C_1)^2$$

$$(y')^2 = \frac{(x - C_1)^2}{[\lambda^2 - (x - C_1)^2]} \Rightarrow y' = \frac{(x - C_1)}{\pm\sqrt{\lambda^2 (x - C_1)^2}}$$

$$\frac{dy}{dx} = \frac{(x - C_1)}{\pm\sqrt{\lambda^2 - (x - C_1)^2}}$$

$$y = \frac{1}{+-}\int \frac{(x - C_1)}{\sqrt{\lambda^2 - (x - C_1)^2}}\,dx + C_2$$

$$y = \pm\sqrt{\lambda^2 - (x - C_1)^2} + C_2$$

$$(y - C_2)^2 = \lambda^2 - (x - C_1)^2$$

$(x - C_1)^2 + (y - C_2)^2 = \lambda^2$ which is the equation of a circle.

Q. 21 **If $A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$ then find e^A and 4^A with help of Modal matrix.** **Dec. 2015**

Ans. :

$$A = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$$

$$|A - \lambda I| = \begin{bmatrix} 3/2 - \lambda & 1/2 \\ 1/2 & 3/2 - \lambda \end{bmatrix} = 0$$

$$\Rightarrow \left(\frac{3}{2} - \lambda\right)^2 - \frac{1}{4} = 0$$

$$\Rightarrow \frac{9}{4} - 3\lambda + \lambda^2 - 1/4 = 0 \Rightarrow \lambda^2 - 3\lambda + 2 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - \lambda + 2 = 0 \Rightarrow \lambda(\lambda - 2) - 1(\lambda - 2) = 0$$

$$\Rightarrow (\lambda - 1)(\lambda - 2) = 0 \Rightarrow \lambda = 1, 2$$

1) Consider e^{λ}

$$e^{\lambda} = \phi(\lambda)(\lambda^2 - 3\lambda + 2) + a\lambda + b$$

Put $\lambda = 1$ $\qquad e = a + b$ ...(i)

Put $\lambda = 2$ $\qquad e^2 = 2a + b$ ...(ii)

Equation (ii) – (i)

$$e^2 - e = a \text{ from (i) } b = e - a = e - e^2 + e = 2e - e^2$$

$$e^{\lambda} = \phi(\lambda)(\lambda^2 - 3\lambda + 2) + (e^2 - e)\lambda + (2e - e^2)$$

$$e^{A} = \phi(A)(A^2 - 3A + 2I) + (e^2 - e)A + (2e - e^2)I$$

$$e^{A} = (e^2 - e)\begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix} + (2e - e^2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^2+e}{2} & \frac{e^2-e}{2} \\ \frac{e^2-e}{2} & \frac{e^2+e}{2} \end{bmatrix}$$

(i) Consider 4^{λ}

$$4^{\lambda} = \phi(\lambda)(\lambda^2 - 3\lambda + 2) + a\lambda + b$$

Put $\lambda = 1$ $\qquad 4 = a + b$ ...(iii)

Put $\lambda = 2$ $\qquad 4^2 = 2a + b$ ...(iv)

Equation (iv) – (iii)

$$12 = a$$

From Equation (iii) $b = 4 - a = 4 - 12 = -8$

$$4^{\lambda} = \phi(\lambda)(\lambda^2 - 3\lambda + 2) + 12\lambda - 8$$

$$4^{A} = 0 + 12A - 8I$$

$$= 12\begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix} - 8\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 6 \\ 6 & 18 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$$

Q. 22 **Show that the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ is similar to diagonal matrix. Find the diagonalising matrix and diagonal form.**

Dec. 2015

Ans. :

$$A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

Characteristic equation,

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

$$S_1 = 8 - 3 + 1 = 6$$

$$S_2 = \begin{bmatrix} -3 & -2 \\ -4 & 1 \end{bmatrix} + \begin{bmatrix} 8 & -2 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 8 & -8 \\ 4 & -3 \end{bmatrix}$$

$$= (-3 - 8) + (8 + 6) + (-24 + 32)$$

$$= -11 + 14 + 8 = 11$$

$$|A| = 8(-3 - 8) + 8(4 + 6) - 2(-16 + 9)$$

$$= 8(-11) + 8(10) - 2(-7) = -88 + 80 + 14 = 6$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\lambda = 1, 2, 3$$

Since all eigen values are district hence,

algebraic multiplicity = Geometric multiplicity

for each eigen values. Hence the matrix is diagonalisable.

(i) for $\lambda = 1$

$$(A - I)X = 0$$

$$\begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$7x_1 - 8x_2 - 2x_3 = 0$$
$$4x_1 - 4x_2 - 2x_3 = 0$$

$$\Rightarrow \frac{x_1}{(16 - 8)} = \frac{x_2}{(-14 + 8)} = \frac{x_3}{(-28 + 32)}$$

$$\Rightarrow \frac{x_1}{8} = \frac{-x_2}{-6} = \frac{x_3}{4}$$

$$X_1 = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

(ii) for $\lambda = 2$

$$(A - 2I)X = 0 \Rightarrow \begin{bmatrix} 6 & -8 & -2 \\ 4 & -5 & -2 \\ 3 & -4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{matrix} 6x_1 - 8x_2 - 2x_3 = 0 \\ 4x_1 - 5x_2 - 2x_3 = 0 \end{matrix} \Rightarrow \frac{x_1}{16 - 10} = \frac{-x_2}{-12 + 8} = \frac{x_3}{-30 + 32}$$

$$\Rightarrow \frac{x_1}{6} = \frac{-x_2}{-4} = \frac{x_3}{2} \Rightarrow X_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

(iii) for $\lambda = 3$

$$(A - 3I)X = 0 \Rightarrow \begin{bmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{matrix} 5x_1 - 8x_2 - 2x_3 = 0 \\ 4x_1 - 6x_2 - 2x_3 = 0 \end{matrix} \Rightarrow \frac{x_1}{16 - 12} = \frac{-x_2}{-10 + 8} = \frac{x_3}{-30 + 32}$$

$$\Rightarrow \quad \frac{x_1}{4} = \frac{-x_2}{-2} = \frac{x_3}{2} \Rightarrow X_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ Diagonalising matrix

$$M = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$M^{-1}AM = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Q. 23 If λ and X are eigen values and eigen vectors of A then prove that $\frac{1}{\lambda}$ and X are eigen values and eigen vectors of A^{-1}, provided A is non singular matrix. May 2016

Ans. : Let λ an d X are the eigen value and corresponding eigen vector of a matrix A

$$\Rightarrow \quad AX = \lambda X$$

Premultiply it by A^{-1}

$$\Rightarrow \quad A^{-1}AX = A^{-1}\lambda X$$

$$\Rightarrow \quad X = \lambda A^{-1}X$$

$\because$ A is non singular so λ can not be zero

$$\Rightarrow \quad A^{-1}X = \frac{1}{\lambda}X \Rightarrow \frac{1}{\lambda} \text{ is the eigen value of } A^{-1}$$

Q. 24 Find eigen values and eigen vectors of A^3

Where $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

May 2016

Ans. :

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

Characteristic equation,

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

$$S_1 = 2+3+4 = 9$$

$$S_2 = \begin{bmatrix} 3 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$

$$= (12-6) + (8-3) + (6-2) = 6+5+4 = 15$$

$$|A| = 2(12-6) - 1(8-6) + 1(6-9)$$

$$= 2(6) - 1(2) + 1(-3) = 12-5 = 7$$

$$\lambda^3 - 9\lambda^2 + 15\lambda - 7 = 0$$

$$\lambda = 7, 1, 1$$

eigen value of A^3 are $7^3, 1^3, 1^3 = 343, 1, 1$

(i) Eigen vector for $\lambda = 7$

$$(A - 7I)X = 0$$

$$\begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -5x_1 + x_2 + x_3 = 0 \\ 2x_1 - 4x_2 + 2x_3 = 0 \end{matrix}$$

$$\frac{x_1}{(2+4)} = \frac{-x_2}{(-10-2)} = \frac{x_3}{(20-2)}$$

$$\Rightarrow \quad \frac{x_1}{6} = \frac{-x_2}{-12} = \frac{x_3}{18}$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(ii) Eigen vector for $\lambda = 1$

$$(A - I)X = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow x_1 + x_2 + x_3 = 0$$

$$X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad X_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore \text{Eigen vector's of } A^3 \text{ are } \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Q. 25 Verify Cayley-Hamilton Theorem for :

$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and hence find A^{-1}

May 2016

Ans. :

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Characteristic equation ,

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

$$S_1 = 2+2+2 = 6$$

$$S_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= (4-1) + (4-1) + (4-1) = 3+3+3 = 9$$

$$|A| = 2(4-1) + 1(-2+1) + 1(1-2) = 6-1-1 = 4$$

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

$$\Rightarrow \quad \frac{x_1}{4} = \frac{-x_2}{-2} = \frac{x_3}{2} \Rightarrow X_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ Diagonalising matrix

$$M = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$M^{-1} AM = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Q. 23 **If λ and X are eigen values and eigen vectors of A then prove that $\frac{1}{\lambda}$ and X are eigen values and eigen vectors of A^{-1}, provided A is non singular matrix.** May 2016

Ans. : Let λ an d X are the eigen value and corresponding eigen vector of a matrix A

$$\Rightarrow \quad AX = \lambda X$$

Premultiply it by A^{-1}

$$\Rightarrow \quad A^{-1}AX = A^{-1}\lambda X$$

$$\Rightarrow \quad X = \lambda A^{-1}X$$

$\because$ A is non singular so λ can not be zero

$$\Rightarrow \quad A^{-1}X = \frac{1}{\lambda} X \Rightarrow \frac{1}{\lambda} \text{ is the eigen value of } A^{-1}$$

Q. 24 **Find eigen values and eigen vectors of A^3**

Where $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$ May 2016

Ans. :

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

Characteristic equation,

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

$$S_1 = 2+3+4 = 9$$

$$S_2 = \begin{bmatrix} 3 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$

$$= (12-6) + (8-3) + (6-2) = 6+5+4 = 15$$

$$|A| = 2(12-6) - 1(8-6) + 1(6-9)$$

$$= 2(6) - 1(2) + 1(-3) = 12 - 5 = 7$$

$$\lambda^3 - 9\lambda^2 + 15\lambda - 7 = 0$$

$$\lambda = 7, 1, 1$$

eigen value of A^3 are $7^3, 1^3, 1^3 = 343, 1, 1$

(i) Eigen vector for $\lambda = 7$

$$(A - 7I)X = 0$$

$$\begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -5x_1 + x_2 + x_3 = 0 \\ 2x_1 - 4x_2 + 2x_3 = 0 \end{matrix}$$

$$\frac{x_1}{(2+4)} = \frac{-x_2}{(-10-2)} = \frac{x_3}{(20-2)}$$

$$\Rightarrow \quad \frac{x_1}{6} = \frac{-x_2}{-12} = \frac{x_3}{18}$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(ii) Eigen vector for $\lambda = 1$

$$(A - I)X = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow x_1 + x_2 + x_3 = 0$$

$$X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad X_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ Eigen vector's of A^3 are $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Q. 25 **Verify Cayley-Hamilton Theorem for :**

$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and hence find A^{-1} May 2016

Ans. :

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Characteristic equation ,

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

$$S_1 = 2+2+2 = 6$$

$$S_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= (4-1) + (4-1) + (4-1) = 3+3+3 = 9$$

$$|A| = 2(4-1) + 1(-2+1) + 1(1-2) = 6 - 1 - 1 = 4$$

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

$$A^3-6A^2+9A-4I = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}^3 - 6$$

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}^2 + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$+9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Cayley Hamilton theorem is verified

$$\therefore A^3-6A^2+9A-4I = 0$$

Premultiply it by A^{-1}

$$A^{-1}A^3-6A^{-1}A^2+9A^{-1}A-4A^{-1}=0$$

$$A^2-6A+9I = 4A^{-1} \Rightarrow A^{-1} = \frac{1}{4}[A^2-6A+9I]$$

$$\Rightarrow A^{-1} = \frac{1}{4}\left\{\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & \frac{-1}{4} \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ \frac{-1}{4} & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

Q. 26 Find A^{50} where. $A = \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}$

May 2016

Ans. :

$$A = \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}$$

Characteristic equation of A

$$|A-\lambda I| = 0 \Rightarrow \begin{bmatrix} 2-\lambda & 3 \\ -3 & -4-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)(-4-\lambda)+9 = 0$$

$$-8+4\lambda-2\lambda+\lambda^2+9 = 0$$

$$\lambda^2+2\lambda+1 = 0 \Rightarrow (\lambda+1)^2 = 0 \Rightarrow \lambda = -1, -1$$

Consider λ^{50}

$$\therefore \lambda^{50} = \phi(\lambda)(\lambda^2+2\lambda+1)+a\lambda+b \quad \text{...(i)}$$

Put $\lambda = -1$

$$(-1)^{50} = 0-a+b$$

$$\Rightarrow \quad 1 = (-a+b) \quad \text{...(ii)}$$

Differentiate (i) w.r.to λ

$$50\lambda^{49} = \phi(\lambda)(2\lambda+2)+\phi'(\lambda)(\lambda^2+2\lambda+1)+a$$

Put $\lambda = -1$

$$50(-1)^{49} = 0+0+a \Rightarrow a = -50 \text{ put in equation (ii)}$$

$$50+b = 1 \Rightarrow b = -49$$

$$\therefore \lambda^{50} = \phi(\lambda)(\lambda^2+2\lambda+1)+(-50)\lambda-49$$

$$\lambda^{50} = \phi(\lambda)(\lambda^2+2\lambda+1)-50\lambda-49$$

Replace λ by A

$$A^{50} = \phi(A)(A^2+2A+I)-50A-49I$$

$$A^{50} = -50A-49I = -50\begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix} - 49\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -100 & -150 \\ 150 & 200 \end{bmatrix} - \begin{bmatrix} 49 & 0 \\ 0 & 49 \end{bmatrix}$$

$$= \begin{bmatrix} -149 & -150 \\ 150 & 151 \end{bmatrix}$$

Q. 27 Prove that A is diagonizable matrix. Also find diagonal form and transforming matrix where

$$A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$$

May 2016

Ans. :

$$A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$$

Characteristic equation,

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

$$S_1 = 1+4-3 = 2$$

$$S_2 = \begin{bmatrix} 4 & 2 \\ -6 & -3 \end{bmatrix} + \begin{bmatrix} 1 & -4 \\ 0 & -3 \end{bmatrix} + \begin{bmatrix} 1 & -6 \\ 0 & 4 \end{bmatrix}$$

$$= (-12+12)+(-3)+(4) = 0-3+4 = 1$$

$$|A| = 1(0) = 0$$

$$\lambda^3-2\lambda^2+\lambda = 0$$

$$\lambda(\lambda^2-2\lambda+1) = 0 \Rightarrow \lambda(\lambda-1)^2 = 0 \Rightarrow 0, 1, 1$$

(i)for $\lambda = 0$ (A. M. = 1)

$$(A-0I)X = 0 \Rightarrow \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} x_1-6x_2-4x_3 = 0 \\ 0x_1+4x_2+2x_3 = 0 \end{array}\right\} \Rightarrow \frac{x_1}{\begin{bmatrix} -6 & -4 \\ 4 & 2 \end{bmatrix}} = \frac{x_2}{\begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix}} = \frac{x_3}{\begin{bmatrix} 1 & -6 \\ 0 & 4 \end{bmatrix}}$$

$$\Rightarrow \quad \frac{x_1}{(4)} = -\frac{x_2}{2} = \frac{x_3}{4}$$

$$X_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \text{ and A. M.} = \text{G. M.} = 1 \text{ for } \lambda = 0$$

(ii) for $\lambda = 1$ (A. M. = 2)

$$(A - I) \quad X = 0 \Rightarrow \begin{bmatrix} 0 & -6 & -4 \\ 0 & 3 & 2 \\ 0 & -6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 + \frac{1}{2} R_1$

$R_3 - R_1$

$$\begin{bmatrix} 0 & -6 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rho = 1, \; n = 3$$

$$\text{G. M} = 3 - 1 = 2$$

$$\text{So A. M.} = \text{G. M.} = 2 \text{ for } \lambda = 1$$

Hence the matrix is diagonalisable.

The corresponding eigen vector's for $\lambda = 1$

$$0\,x_1 - 6x_2 - 4\,x_3 = 0$$

$$X_2 = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}; \qquad X_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \text{Diagonlising matrix is } = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 2 & 0 \\ 2 & -3 & 0 \end{bmatrix} = M$$

$$\therefore M^{-1} AM = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q. 28 If f(x) is an algebraic polynomial in x and λ is an eigen value and X is the corresponding eigen vector of a square matrix A then f(λ) is eigen value and X is the corresponding an eigen value and X is the corresponding eigen vector of f(A). **Dec. 2016**

Ans. : Given that f(x) is an algebraic polymial in x

Let $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots + a_n$

Then $f(A) = a_0 A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots\ldots + a_n I$

$\because$ λ is the eigen value of A and corresponding eigen vector X then

$$AX = \lambda X \qquad \ldots(i)$$

To prove that λ^n is the eigen value of A^n.

Permultiplying (i) by A

$$A\,AX = A\,\lambda\,X$$

$$\therefore A^2X = \lambda\,(AX)$$

$$\Rightarrow A^2X = \lambda\,(\lambda X) \Rightarrow A^2X = \lambda^2 X$$

$\therefore$ λ^2 is the eigen value of A^2

Consider that λ^K is the eigen value of A^K

$$\Rightarrow A^K X = \lambda^K X$$

Premultiply it by A

$$\Rightarrow AA^K X = A\,\lambda^K X$$

$$\Rightarrow A^{K+1} X = \lambda^K AX$$

$$\Rightarrow A^{K+1} X = \lambda^K \lambda X$$

$$\Rightarrow A^{K+1} X = \lambda^{K+1} X$$

$\Rightarrow$ A^{K+1} is the eigen value of A^{K+1}

$\Rightarrow$ λn is the eigen vlaueof A^n

$$f(A)X = (a_0 A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots\ldots + a_n) X$$

$$= a_0 A^n X + a_1 A^{n-1} X + a_2 A^{n-2} X + \ldots\ldots + a_n X$$

$$= a_0 \lambda^n X + a_1 \lambda^{n-1} X + a_2 \lambda^{n-2} X + \ldots\ldots + a_n X$$

$$= (a_0 \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots\ldots + a_n) X$$

$$f(A)X = f(\lambda)X \qquad \Rightarrow \quad f(A)X = f(\lambda)X$$

$\Rightarrow f(\lambda)$ is the eigen value of f (A)

Q. 29 Verify Cayley Hamilton theorem for $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ **and hence, find the matrix represented by** $A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - 1$. **Dec. 2016**

Ans. :

$$A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$$

Characteristic equation

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - |A| = 0$$

$$S_1 = 3 - 3 + 7 = 7$$

$$S_2 = \begin{vmatrix} -3 & -4 \\ 5 & 7 \end{vmatrix} + \begin{vmatrix} 3 & 5 \\ 3 & 7 \end{vmatrix} + \begin{vmatrix} 3 & 10 \\ -2 & -3 \end{vmatrix}$$

$$= (-21+20)+(21-15)+(-9+20)$$

$$= -1+6+11 = 16$$

$$|A| = 3(-21+20)-10(-14+12)+5(-10+9)$$

$$= 3(-1)-10(-2)+5(-1)$$

$$= -3+20-5 = 12$$

$$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$$

$$A^3 - 7A^2 + 16A - 12I$$

$$\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}^3 - 7\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}^2$$

$$+16\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix} - 12\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 15 & -10 \\ -52 & -157 & -118 \\ 92 & 270 & 208 \end{bmatrix} - 7\begin{bmatrix} 4 & 25 & 10 \\ -12 & -31 & -26 \\ 20 & 50 & 44 \end{bmatrix}$$

$$+16\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix} - 12\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Cayley Hamilton theorem is verified

$$\lambda^6 - 6\lambda^5 + 9\lambda^4 + 4\lambda^3 - 12\lambda^2 + 2\lambda - 1 = (\lambda^3 + \lambda^2)(\lambda^3 - 7\lambda^2 + 16\lambda - 12) + (2\lambda - 1)$$

$$\therefore A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I = (A^3 + A^2)(A^3 - 7A^2 + 16A - 12) + (2A - I)$$

$$= 2A - I = 2\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 20 & 10 \\ -4 & -6 & -8 \\ 6 & 10 & 14 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 20 & 10 \\ -4 & -7 & -8 \\ 6 & 10 & 13 \end{bmatrix}$$

Q. 30 Find the orthogonal matrix P that diagonalises $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$

Dec. 2016

Ans. :

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ \dots \end{bmatrix}$$

Characteristic equation,

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

$$S_1 = 4+4+4 = 12$$

$$S_2 = \begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix}$$

$$= 12+12+12 = 36$$

$$|A| = 4(12) - 2(4) + 2(-4)$$

$$= 48 - 8 - 8 = 32$$

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$$\lambda = 8, 2, 2$$

Eigen vector for $\lambda = 8$

$$(A - 8I)X = 0$$

$$\begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} -4x_1 + 2x_2 + 2x_3 = 0 \\ 2x_1 - 4x_2 + 2x_3 = 0 \end{array}$$

$$\frac{x_1}{(4+8)} = \frac{-x_2}{(-8-4)} = \frac{x_3}{(16-4)}$$

$$\frac{x_1}{12} = \frac{-x_2}{-12} = \frac{x_3}{12}$$

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ Normalised form of it } = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Eigen vector for $\lambda = 2$

$$(A - 2I)X = 0$$

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow 2x_1 + 2x_2 + 2x_3 = 0$$

$$X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad X_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Normalized form of it

$$X_2 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \qquad X_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$\therefore \text{orthogonal diagonalising matrix is} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} = P$$

$$P^T AP = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Q. 31 **If $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, prove that $A^{50} - 5A^{49} = \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix}$**

Dec. 2016

Ans. :

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

Characteristic equation is

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$$

$$\Rightarrow 3 - 3\lambda - \lambda + \lambda^2 - 8 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + \lambda - 5 = 0$$

$$\Rightarrow \lambda(\lambda - 5) + 1(\lambda - 5) = 0$$

$$\Rightarrow (\lambda + 1)(\lambda - 5) = 0$$

$$\Rightarrow \lambda = -1, 5$$

Let $\lambda^{50} - 5\lambda^{49} = \phi(\lambda)(\lambda^2 - 4\lambda - 5) + a\lambda + b$

Put $\lambda = -1$

$$(-1)^{50} - 5(-1)^{49} = 0 - a + b$$

$$1 + 5 = -a + b \Rightarrow -a + b = 6 \quad \ldots(i)$$

Put $\lambda = 5$ in equation (i)

$$5^{50} - 5 \cdot 5^{49} = 0 + 5a + b \Rightarrow 5a + b = 0 \quad \ldots(ii)$$

(ii) – (i)

$$5a + b = 0$$

$$\mp a \pm b = \_6$$

$$6a = -6 \Rightarrow a = -1,\ b = 5$$

$$\lambda^{50} - 5\lambda^{49} = \phi(\lambda)(\lambda^2 - 4\lambda - 5) - \lambda + 5$$

$$A^{50} - 5A^{49} = 0 - A + 5I$$

$$= -\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -4 \\ -2 & -3 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix}$$

Chapter 4 : Probability

Q. 1 **The daily consumption of electric power (in millions of kwh) is r.v. X with PDF $f(x) = k\,xe^{-x/3}$, $x > 0$. Find k and the probability that on a given day the electricity consumption is more than expected electricity consumption.**

May 2014

Ans. :

For any probability density function,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\therefore \int_0^{\infty} k\,x\,e^{-x/3}\,dx = 0$$

$$\therefore k\left[\frac{x\,e^{-x/3}}{-1/3} - 1 \cdot \frac{e^{-x/3}}{(-1/3)^2}\right]_0^{\infty} = 1$$

$$\therefore k\left[(0 \cdot 0) - \left(0 - \frac{e^0}{(1/9)}\right)\right] = 1$$

$$\therefore k \times 9 = 1$$

$$\therefore k = \frac{1}{9}$$

$$\therefore f(x) = \frac{1}{9} x\,e^{-x/3}$$

Expected electric consumption = E(x)

$$= \int_{-\infty}^{\infty} x\,f(x)\,dx = \int_{-\infty}^{\infty} x\frac{1}{9}x\,e^{-x/3}\,dx$$

$$= \frac{1}{9}\int_0^{\infty} x^2\,e^{-x/3}\,dx$$

$$= \frac{1}{9}\left[x^2\frac{e^{-x/3}}{-1/3} - 2x\frac{e^{-x/3}}{(-1/3)^2} + 2\frac{e^{-x/3}}{(-1/3)^3}\right]_0^{\infty}$$

$$= \frac{1}{9}\left\{(0 - 0 + 0) - \left[0 - 0 + 2\frac{e^0}{(-1/27)}\right]\right\}$$

$$= \frac{1}{9}\left\{2 \times \frac{27}{1}\right\} = 6$$

P (consumption is more than the expected value)

$$= P(X > 6) = \int_6^{\infty} \frac{1}{9} x e^{-2x/3} dx = \frac{1}{9}\left[x\frac{e^{-x/3}}{-1/3} - 1\frac{e^{-x/3}}{(-1/3)^2}\right]_6^{\infty}$$

$$= \frac{1}{9}\left[(0-0) - \left[6\cdot\frac{e^{-2}}{-1/3} - \frac{e^{-2}}{(1/9)}\right]\right]$$

$$= \frac{1}{9} \times (6\times 3 e^{-2} + 9 e^{-2}) = \frac{1}{9}\times 27 e^{-2} = 3 e^{-2} = 0.406$$

Q. 2 Find the moment generating function of Poisson distribution and hence find mean and variance. May 2014

Ans. :

For poission distribution $P(X = x) = \frac{e^{-m} m^x}{X!}$ where m is the poission parameter by definition

$$\text{Moment} = E(e^{tx}) = \sum_{X=0}^{\infty} P_t e^{tx}$$

$$= \sum_{X=0}^{\infty} \frac{e^{-m} m^x}{X!} e^{tx} = e^{-x} \sum_{X=0}^{\infty} \frac{(m e^t)^x}{X!}$$

$$= e^{-m} \cdot e^{-me^t} \qquad \left[\because \sum_{X=0}^{\infty} \frac{a^x}{x!} = e^a \right]$$

$$= e^{-m+me^t} = e^{-m(1-e^t)}$$

$$\text{Now } \mu_r' = \left[\frac{d^r}{dt^r} M_o(t) \right]_{t=0} = \left[\frac{d^r}{dt^r} e^{-m(1-e^t)} \right]_{t=0}$$

$$\therefore \mu_1' = \left[\frac{d}{dt} e^{-m(1-e^t)} \right]_{t=0}$$

$$= [e^{-m(1-e^t)} \cdot -m(0-e^t)]_{t=0}$$

$$= e^{-m(1-1)} \cdot (-m)(0-1) = e^{o} \cdot m = m$$

$$\mu_2' = \left[\frac{d^2}{dt^2} e^{-m(1-e^t)} \right]_{t=0} = \left[\frac{d}{dt} e^{-m(1-e^t)} \cdot me^t \right]_{t=0}$$

$$= \{ m \cdot [e^{-m(1-e^t)} \cdot e^t \cdot e^{-m(1-e^t)} - m \cdot (0-e^t)] \}_{t=0}$$

$$= m \cdot [e^{-m(1-1)} \cdot 1 + 1 \cdot e^{-m(1-1)} \cdot m \cdot 1]$$

$$= m[1+m] = m + m^2$$

$$\text{Mean } \mu_1' = m$$

$$\therefore \text{Variance} = \mu_2 - \mu_1'^2 = (m+m^2) - m^2 = m$$

Q. 3 Average mark scored by 32 boys is 72 with standard deviation of 8 while that for 36 girls is 70 with standard deviation of 6. Test at 1% LoS whether the boys perform better than the girls. May 2014

Ans.: $n_1 = 32$ and $n_2 = 36$ (> 30, it is large sample)

$\bar{x}_1 = 72$; $\bar{x}_2 = 70$; $s_1 = 8$; $s_2 = 6$

Step 1 :

Null Hypothesis $(H_0) : \mu_1 = \mu_2$ (i.e. performance of boys and girls is equal).

Alternative Hypothesis $(H_a) : \mu_1 > \mu_2$ (i.e. boys perform better than the girls) (One tailed test)

Step 2 :

$$\text{LOS} = 1\% \text{ (Two tailed test)}$$

$$\therefore \text{LOS} = 2\% \text{ (One tailed test)}$$

$$\therefore \text{Critical value } (z_\alpha) = 2.33$$

Step 3 : Since samples are large,

$$\text{S.E.} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = \sqrt{\frac{8^2}{32} + \frac{6^2}{36}} = 1.732$$

Step 4 : Test statistic

$$z_{cal} = \frac{\bar{x}_1 - \bar{x}_2}{\text{S.E.}} = \frac{72-70}{1.732} = 1.1547$$

Step 5 : Decision

Since $|z_{cal}| < z_\alpha$, H_0 is accepted.

$\therefore$ Boys do not perform better than the girls.

Q. 4 In a distribution exactly normal 7 % of items are under 35 and 89 % of the items are under 63. Find the probability that an item selected at random lies between 45 and 56. May 2015

Ans. :

Let the mean and standard deviation be 'm' and 'σ'.

Let SNV corresponding to x = 35 be z_1.

$P(x < 35) = 7\%$

$\therefore P(z < z_1) = 0.07$

$\therefore$ 0.5 – Area between 'z = 0' to 'z = – z_1' is 0.43

Form z-table – z_1 = 1.4758

$$\therefore z_1 = -1.4758$$

$$\text{But } z = \frac{x-m}{\sigma}$$

$$\therefore z_1 = \frac{35-m}{\sigma}$$

$$\therefore -1.4758 = \frac{35-m}{\sigma}$$

$$\therefore m - 1.4758\,\sigma = 35 \qquad \ldots(1)$$

Let SNV corresponding to x = 63 be z_2

$$P(x < 63) = 89\%$$

$$\therefore p(z < z_2) = 0.89$$

$\therefore$ 0.5 + Area between '$z = 0$' to '$z = z_2$' is 0.39

$\therefore$ Area between '$z = 0$' to '$z = z_2$' is 0.39

From z-table, $z_2 = 1.2265$

$$\text{But } z_2 = \frac{63 - m}{\sigma}$$

$$\therefore 1.2265 = \frac{63 - m}{\sigma}$$

$$\therefore m + 1.2265\,\sigma = 63 \qquad \ldots(2)$$

Solving Equation (1) and (2) simultaneously,

$m = 50.2916$ and $\sigma = 10.3615$

Now, probability that an item selected at random lies between 45 and 56 = $P(45 < x < 56)$

$$= P\left(\frac{45 - 50.2916}{10.3615} < \frac{x - m}{\sigma} < \frac{56 - 50.2916}{10.3615}\right)$$

$$= P(-0.5106 < z < 0.5509)$$

= Area between '$z = 0$' to '$z = -0.5106$' + Area between '$z = 0$' to '$z = 0.5509$'

$$= 0.1952 + 0.2092 = 0.4044$$

Mean = 50.2916, standard deviation = 10.3615

Probability that an item selected at random lies between 45 and 56 = 0.4044

Q. 5 A continuous random variable has probability density function $f(x) = 6(x - x)^2$, $0 \le x \le l$. Find (i) mean (ii) variance. May 2015

Ans. :

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = \int_0^1 x \cdot 6(x - x^2)\, dx$$

$$= 6\int_0^1 (x^2 - x^3)\, dx$$

$$= 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 6\left\{\left[\frac{1^3}{3} - \frac{1^4}{4}\right] - [0 - 0]\right\} = 0.5$$

$$\text{Consider,} \quad E(x^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx = \int_0^1 x^2 \cdot 6(x - x^2)\, dx$$

$$= 6\int_0^1 (x^3 - x^4)\, dx = 6\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1$$

$$= 6\left\{\left[\frac{1^4}{4} - \frac{1^5}{5}\right] - [0 - 0]\right\} = 0.3$$

$$\therefore \text{Variance} = E(x^2) - [E(x)]^2 = 0.3 - 0.5^2 = 0.05$$

Hence mean = 0.5, Variance = 0.05

Q. 6 Find the moment generating function of Binomial distribution and hence find mean and variance. May 2015

Ans. :

For binomial distribution, $P(X = x) = {}^nC_x\, p^x q^{n-x}$

By definition moment generating function about origin $M_0(t) = E(e^{tx})$

$$= \sum_{x=0}^{n} p_t e^{tx} = \sum_{x=0}^{n} {}^nC_x\, p^x q^{n-x} e^{tx} = \sum_{x=0}^{n} {}^nC_x (pe^t)^x q^{n-x}$$

$$= (q + pe^t)^n \qquad \ldots\left[\because \sum_{x=0}^{n} {}^nC_x\, a^x b^{n-x} = (a+b)^n\right]$$

$$\text{Now } \mu_r' = \left[\frac{d^r}{dt^r} M_0(t)\right]_{t=0}$$

$$\therefore \mu_1' = \left[\frac{d}{dt} M_0(t)\right]_{t=0} = \left[\frac{d}{dt}(q + pe^t)^n\right]_{t=0}$$

$$= \left[n(q + pe^t)^{n-1} \cdot pe^t\right]_{t=0}$$

$$= \left[n(q + pe^0)^{n-1} pe^0\right] = \left[n(q+p)^{n-1} p\right]$$

$$= \left[n \cdot 1^{n-1} \cdot p\right] \qquad (\because q + p = 1)$$

$$= np$$

$$\mu_2' = \left[\frac{d^2}{dt^2} M_0(t)\right]_{t=0} = \left[\frac{d^2}{dt^2}(q + pe^t)^n\right]_{t=0}$$

$$= \left[\frac{d}{dt} np(q + pe^t)^{n-1} e^t\right]_{t=0}$$

$$\left[np\left\{q + pe^t\right\}^{n-1} \cdot e^t + e^t \cdot (n-1)(q + e^t)^{n-2} pe^t\right]_{t=0}$$

$$= np\left\{(q+p)^{n-1} \cdot 1 + 1 \cdot (n-1)(q+p)^{n-2} p \cdot 1\right\}$$

$$= np\left\{(1)^{n-1} + (n-1)(1)^{n-2} p\right\} \qquad \ldots(\because q + p = 1)$$

$$= np\{1 + np - p\} = np\{(q + np)\} \qquad \ldots(\because 1 - p = q)$$

$$= npq + n^2p^2$$

$$\therefore \text{Mean} = \mu_1' = np$$

$$\therefore \text{Variance } \mu_2 = \mu_2' - \mu_1'^2 = (npq + n^2p^2) - (np)^2 = npq$$

Q. 7 A die was thrown 132 times and the following frequencies were observed. May 2015

No. obtained :	1	2	3	4	5	6	Total
Frequency :	15	20	25	15	29	28	132

Ans. :

Binomial distribution is applied only in those experiments which have exactly two out-comes viz success and failure. A dice has six equally likely customers. Assuming 'x' to be a binomial variate then its first value should be zero and not one. This problem is solved just for the sake of solving.

x	f	fx	Theoretical frequency
0	0	0	$0.31 \approx 0$
1	15	15	$3.25 \approx 3$
2	20	40	$14.16 \approx 14$
3	25	75	$32.86 \approx 33$
4	15	60	$42.89 \approx 43$
5	29	145	$29.86 \approx 30$
6	28	168	$8.66 \approx 9$
Total	132	503	132

$$\text{Mean} = \frac{\sum f_i x_i}{\sum f_i} = \frac{503}{132} = 3.8106$$

But for binomial distribution, mean = np

$$\therefore\ 38106 = 6p$$

$$\therefore\ p = 0.6351$$

$$\therefore\ q = 1 - p = 1 - 0.6351 = 0.3649$$

$$n = 6 \text{ and } N = 132$$

$$\therefore\ P(X = x) = {}^nC_x\, p^x q^{n-x} = {}^6C_x \times (0.6351)^x \times (0.3649)^{6-x}$$

$$\text{Theoretical frequency} = N \times P(X = x)$$

$$= 132 \times {}^6C_x \times (0.6351)^x \times (0.3649)^{6-x}$$

Q. 8 A random sample of 50 items gives the mean 6.2 and standard deviation 10.24, can it be regarded as drawn from a normal population with mean 5.4 at 5 % level of significance ? May 2015

Ans. :

n = 50 (> 30, so it is large sample)

$\bar{x} = 6.2$; $\sigma = 10.24$

Step 1 :

Null hypothesis (H_0) : $\mu = 5.4$ (i.e. sample belongs to the population with mean 5.4). Alternative hypothesis (H_a) : $\mu \neq 5.4$ (i.e. sample does not belong to population with mean 5.4) (two tailed test)

Step 2 :

LOS 5 % (Two tailed test)

$\therefore$ Critical value $(Z_\alpha) = 1.96$

Step 3 :

Since sample is large,

$$\text{S. E.} = \frac{\sigma}{\sqrt{n}} = \frac{10.24}{\sqrt{50}} = 1.4482$$

Step 4 : Test Statistic

$$Z_{cal} = \frac{\bar{x} - \mu}{\text{S. E.}} = \frac{6.2 - 5.4}{1.4482} = 0.5524$$

Step 5 : Decision

Since $Z_{cal} < z_\alpha$, H_0 is accepted

$\therefore$ Sample can be regarded as drawn from a normal population with mean 5.4 at 5 % LOS

Q. 9 Find the M. G. F. of the following distribution.

X :	– 2	3	1
P (X = x)	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

Hence find first four central moments. May 2015

Ans. :

By definition, Mean $\bar{x} = E(X)$

$$= \sum x_i p_i = -2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{2} + 1 \cdot \frac{1}{6} = 1$$

By definition, moment generating function (M. G. F.) about mean $M_{\bar{x}}(t) = E = \left[e^{t(x - \bar{x})} \right]$

$$= \sum e^{t(x-\bar{x})} p_i = \sum e^{t(x-1)} p_i = e^{-3t} \cdot \frac{1}{3} + e^{2t} \cdot \frac{1}{2} + e^{0t} \cdot \frac{1}{6}$$

$$= \frac{1}{3} e^{-3t} + \frac{1}{2} e^{2t} + \frac{1}{6}$$

Now, central moments are given by

$$\mu_r = \left[\frac{d^r}{dt^r} M_{\bar{x}}(t) \right]_{t=0}$$

$\therefore$ First central moment $\mu_1 = \left[\frac{d}{dt} M_{\bar{x}}\right](t))_{t=0}$

$$= \left[\frac{d}{dt}\left(\frac{1}{3}e^{-3t} + \frac{1}{2}e^{2t}\right) + \frac{1}{6}\right]_{t=0}$$

$$= \left[\frac{1}{3}e^{-3t} \cdot -3 + \frac{1}{2}e^{2t} - 2 + 0\right]_{t=0} \quad \text{...(1)}$$

$$= -1 + 1 + 0 = 0$$

$\therefore$ Second central moment $\mu_2' = \left[\frac{d^2}{dt^2} M_{\bar{x}}(t)\right]_{t=0}$

$$= \left[\frac{d^2}{dt^2}\left(\frac{1}{3}e^{-3t} + \frac{1}{2}e^{2t}\right) + \frac{1}{6}\right]_{t=0}$$

$$= \left[\frac{d}{dt}(e^{-3t} + e^{2t})\right]_{t=0} \quad \text{...(From Equation (1))}$$

$$= [e^{-3t} \cdot -3 + e^{2t} \cdot 2]_{t=0}) \quad \text{...(2)}$$

$$= 3 + 2 = 5$$

$\therefore$ Third central moment

$$\mu_3' = \left[\frac{d^3}{dt^3} M_{\bar{x}}(t)\right]_{t=0} = \left[\frac{d^3}{dt^3}\left(\frac{1}{3}e^{-3t} + \frac{1}{2}e^{2t}\right) + \frac{1}{6}\right]_{t=0}$$

$$= \left[\frac{d}{dt}(3e^{-3t} + 2e^{2t})\right]_{t=0} \quad \text{...(From Equation (2))}$$

$$= [3e^{-3t} \cdot -3 + 2e^{2t} \cdot 2]_{t=0} \quad \text{...(3)}$$

$$= -9 + 4 = -5$$

$\therefore$ Fourth central moment

$$\mu_4' = \left[\frac{d^4}{dt^4} M_{\bar{x}}(t)\right]_{t=0} = \left[\frac{d^4}{dt^4}\left(\frac{1}{3}e^{-3t} + \frac{1}{2}e^{2t}\right) + \frac{1}{6}\right]_{t=0}$$

$$= \left[\frac{d}{dt}(-9e^{-3t} + 4e^{2t})\right]_{t=0} \quad \text{...(From Equation (3))}$$

$$= [-9e^{-3t} \cdot -3 + 4e^{2t} \cdot 2]_{t=0} = 27 + 8 = 35$$

Q. 10 The probability density function of a random variable x is.

x	−2	−1	0	1	2	3
p(x)	0.1	k	0.2	2k	0.3	k

Find (i) k (ii) mean (iii) variance

Dec. 2015

Ans. : Consider the probability density function of a random variable 'x' as

x	−2	−1	0	1	2	3
P(x)	0.1	k	0.2	2k	0.3	k

Consider, $\sum p_i S = 1$

$$0.1 + k + 0.2 + 2k + 0.3 + k = 10.6 + 4k = 1$$

$$\therefore 4k = 1 - 0.6 = 0.4 \qquad 4k = 0.4$$

$$\therefore k = 0.1$$

Now,

Mean $= E(X) = \sum x_i p_i = (-2)(0.1) + (-1)(0.1) + 0 + 1(0.2) + 2(0.3) + 3(0.1)$

$$= -0.2 - 0.1 + 0.2 + 0.6 + 0.3 = 0.8$$

$\therefore$ **Mean = E(X) = 0.8**

Consider $E(X^2) = \sum x_i^2 p_i = 0.4 + 0.1 + 0.2 + 1.2 + 0.9$

$$E(X^2) = 2.8$$

$\therefore$ Variance $= E(X^2) - [E(X)]^2 = 2.8 - [0.8]^2$

$$= 2.8 - 0.64 = 2.16$$

$\therefore$ **Variance = 2.16**

Q. 11 If the height of 500 students is normally distributed with mean 68 inches and standard deviation of 4 inches, estimate the number of students having heights (i) less than 62 inches (ii) between 65 and 71 inches.

Dec. 2015

Ans. :

Let x be a normal variate with $\mu = 68$ inches, $\sigma = 4$ inches, N = 500

$$Z = \frac{x - \mu}{\sigma} = \frac{x - 68}{4}$$

(i) Consider

$$P(x < 62) = P\left(Z < \frac{62 - 68}{4}\right) = P(Z < -1.5)$$

$$= \text{Area between } Z = -\infty \text{ to } Z = -1.5$$

$$= 0.5 - \text{Area between } Z = 0 \text{ and } Z = 1.5 = 0.5 - 0.4332$$

$$P(x < 62) = 0.0668$$

$\therefore$ No.of students with height less than 62 inches.

$$= N\,P(X < 62) = 500\,(0.0668) = 33.4 \cong 33$$

(ii) Now consider

$$P(65 < x < 71) = P(-0.75 < Z < 0.75) = 2\,(\text{Area between } Z = 0 \text{ to } Z = 0.75) = 2(0.2734)$$

$$P(65 < x < 71) = 0.5468$$

No. of students with height between 65 and 71 inches.

$$= N\,P(65 < x < 71) = 500\,(0.5468) = 273.4 \cong 273$$

$\therefore$ No. of students with height between 65 inches and 71 inches = 273

Q.12 Fit a Poisson distribution to the following data

x	0	1	2	3	4	5	6	7	8
f	56	156	132	92	37	22	4	0	1

Dec. 2015

Ans. ;

Let fm be the mean of Poisson distribution. The probability function is given by

$$P(X = x) = \frac{e^{-m} \cdot m^x}{x!}$$

Consider the following table.

x	f	$x_i f_i$	$P(X = x_i) = P_i$	Expected frequency $= N \cdot Pi$
0	56	0	0.1392	$69.6 \cong 70$
1	156	156	0.2745	$137.2 \cong 137$
2	132	264	0.2707	$135.3 \cong 135$
3.	92	276	0.1779	$88.9 \cong 87$
4.	37	148	0.0877	$43.8 \cong 44$
5.	22	110	0.0346	$17.2 \cong 17$
6.	4	24	0.0114	$5.6 \cong 6$
7	0	0	0.0032	$1.6 \cong 2$
8.	1	8	0.0008	$0.3 \cong 0$

$$m = \frac{\sum x_i f_i}{\sum f_i} = \frac{986}{500} = 1.972$$

$$P(X = x) = \frac{e^{-1.972} \cdot (1.972)^x}{x!}$$

Q. 13 If x is a continuous random variable with the probability density function given by

$$f(x) = \begin{cases} k(x - x^3) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find (i) k (ii) the mean of distribution.

May 2016

Ans. :

Let x be a continuous random variable with probability density function.

$$f(x) = \begin{cases} k(x - x^3) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\int_0^1 k(x - x^3)\, dx = 1$$

$$k\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = 1$$

$$\therefore k\left[\frac{1}{2} - \frac{1}{4}\right] = 1 \qquad \therefore \frac{K}{4} = 1 \qquad \therefore K = 4$$

$$\text{Mean} = \int_0^1 x\, f(x)\, dx = \int_0^1 4 \cdot x\, (x - x^3)\, dx$$

$$= 4 \int_0^1 (x^2 - x^4)\, dx = 4\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$

$$= 4\left[\frac{1}{3} - \frac{1}{5}\right] = 4\left[\frac{2}{15}\right] = \frac{8}{15}$$

$$\textbf{Mean} = \frac{8}{15}$$

Q. 14 The marks of 1000 students in an examination are found to be normally distributed with mean 70 and standard deviation 5, estimate the number of students. Whose marks will be (i) between 60 and 75 (ii) more than 75.

May 2016

Ans. :

$N = 1000,\ \mu = 70,\ \sigma = 5$

standard normal variable as

$$Z = \frac{x - \mu}{\sigma} = \frac{x - 70}{5}$$

(i) Now consider the probability for 60 and 70

$\therefore$ For $x = 60 \qquad z = \frac{60 - 70}{5} = -2$

$x = 75 \qquad z = \frac{75 - 70}{5} = 1$

$$P(60 < x < 75) = P(-2 \le z \le 1)$$

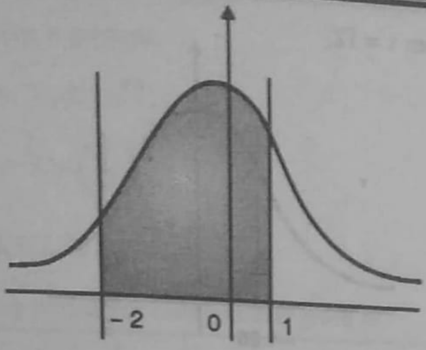

Fig. 4.1

= Area from $z = -2$ to $z = 0$ + Area from $z = 0$ to $z = 1$

$= 0.4772 + 0.3413 = 0.8185$

$\therefore$ Numbers of students between 60 and 75

$= 1000\ P(60 < x < 75)$

$= 1000\ (0.8185) = 818.5 \cong 819$

$= 819$

(ii) Numbers of students more than 75

For $x = 75$, $\therefore$ $z = 1$

$P(x > 75) = P(z > 1)$

$= 0.5 -$ Area from $z = 0$ to $z = 1$

$= 0.5 - 0.3413$

$= 0.1587$

$P(X > 75) = 0.1587$

Numbers of students more than 75 = NP (X > 75)

$= 0.1587 \times 1000 = 158.7 \cong 159$

$\therefore$ Numbers of students more than 75 marks = 159

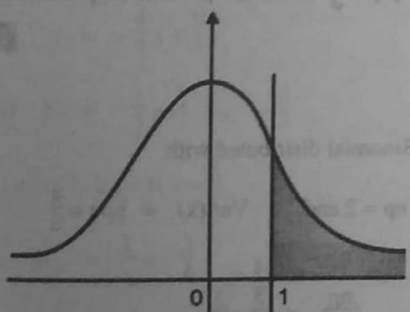

Fig. 4.2

Q. 15 **Fit a Binomial distribution to the following data.**

x	0	1	2	3	4	5	6
f	5	18	28	12	7	6	4

May 2016

Ans. : Let x be a binomial variate. n = 6.

The probability distribution function is given as and N = 80

$$P(X = x) = {}^6C_x\ P^x\ q^{6-x}$$

Consider the following table

x	f	$x_i f_i$	$P(x = x_i) = P_i$	Expected frequency $= N P_i$
0	5	0	0.0467	$3.7325 \cong 4$
1	18	18	0.1866	$14.9299 \cong 15$
2	28	56	0.3110	$24.8832 \cong 25$
3	12	36	0.2765	$22.1184 \cong 22$
4	7	28	0.1382	$11.0592 \cong 11$
5	6	30	0.0369	$2.9491 \cong 3$
6	4	24	0.0041	$0.3277 \cong 0$

$$\bar{x} = \frac{\sum xi\ fi}{N} = \frac{192}{80} = 2.4$$

and $\bar{x} = np$ $\therefore np = 2.4$ $\therefore P = \frac{2.4}{6} = 0.4$

$P = 0.4$; $q = 1 - P = 1 - 0.4$

$\therefore q = 0.6$

Consider $P(X = x) = {}^6C_x\ (0.4)^x\ (0.6)^{6-x}$

$P(X = 0) = 0.0467$ Expected frequency $= NP_i$

$P(X = 1) = 0.1866$ $P(X = 2) = 0.3110$

$P(X = 3) = 0.2765$ $P(X = 4) = 0.1382$

$P(X = 5) = 0.0369$ $P(X = 6) = 0.0041$

Q. 16 **If a random variable X follows the poission distribution such that,**

P (X = 1) = 2 P (X = 2)

Find mean, the variance and the distribution and P (X = 3)

May 2016

Ans. :

Let m be the parameter of poission distribution and x be a poisons variable

$\therefore$

$$P(X = x) = \frac{e^{-m} \cdot m^x}{x!}$$

We consider, $P(X = 1) = 2P(X = 2)$

$$\frac{e^{-m} \cdot m^1}{1!} = \frac{2e^{-m} \cdot m^2}{2!}$$

$\therefore \quad m = 1$

$\therefore$ The mean = variance = 1

Now

$$P(X = 3) = \frac{e^{-m} \cdot m^{x}}{x\,!} = \frac{e^{-1} \cdot 1^{3}}{3\,!} = 0.0613$$

$$\therefore \quad P(X = 3) = 0.0613$$

Q. 17 If x is a normal variate with mean 10 and standard deviation 4 find (i) p(|x – 14| < 1) (ii) p(5 ≤ x ≤ 18) (iii) p (x ≤ 12). Dec. 2016

Ans. :

Let, $\mu = 10, \sigma = 4$, we define standard normal vitiate z as

$$z = \frac{x-\mu}{\sigma} = \frac{x-10}{4}$$

(i) when $x = 14, z = \frac{14-10}{4} = 1$

$\therefore$

$p(|x - 14| \le 1) = p(|z| \le 1)$ = Area between $z = -1$ and $z = 1$

= 2 area between $z = 0$ and $z = 1$

$= 2\,(0.3413) = 0.6826$

$\therefore$ **$p(|x - 14| \le 1) = 0.6826$**

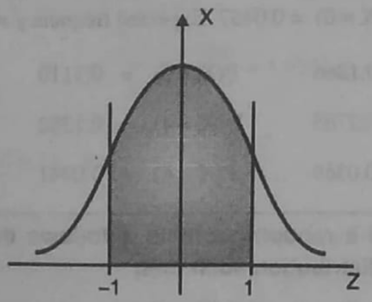

Fig. 4.3

(ii) When $x = 5, z = \frac{5-10}{4} = -1.25$

$x = 18, z = \frac{18-10}{4} = 2$

$\therefore p(5 \le x \le 18) = p(-1.25 \le z \le 2)$

= Area between $z = -1.25$ and $z = 2$

= Area between $z = 0$ and

$z = 1.25$ + Area between $z = 0$ and $z = 2$.

$P(5 \le x \le 18) = 0.3944 + 0.4772 = 0.8716$

(iii) When $x = 12$,

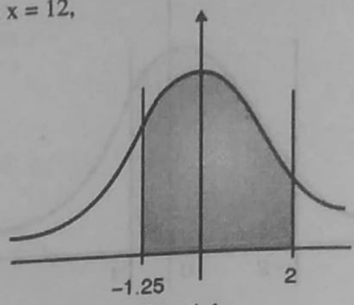

Fig. 4.4

$$z = \frac{12-10}{4} = 0.5$$

$p(x \le 12) = p(z < 0.5)$

= Area between $-\infty$ to 0.5

= (Area between $z = -\infty$ to $z = 0$)

+ (Area between $z = 0$ to $z = 0.5$)

$= 0.5 + 0.1915 = 0.6915$

$p(x \le 12) = 0.6915$

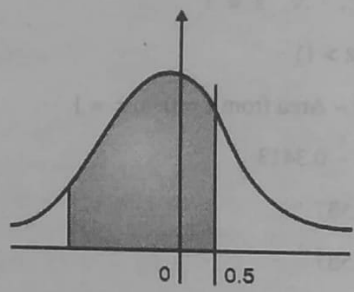

Fig. 4.5

Q. 18 If x is Binomial distributed with E(x) = 2 and $V(x) = \frac{4}{3}$. Find the probability distribution of x. Dec. 2016

Ans. :

Let x is Binomial distributed with

$E(x) = np = 2$ and $\quad Var(x) = npq = \frac{4}{3}$

$$\frac{np}{npq} = \frac{\frac{4}{3}}{2} = \frac{3}{2}$$

$\therefore \frac{1}{q} = \frac{3}{2} \qquad \therefore q = \frac{2}{3}$

$\therefore p = 1 - q = 1 - \frac{2}{3} = \frac{1}{3} \quad \therefore p = \frac{1}{3}$

Since $np = 2$

$\therefore n \cdot \frac{1}{3} = 2 \qquad \therefore n = 6$

The distribution is given as

$$p(X=x) = {}^nC_x\, p^x \cdot q^{n-x}$$

$$p(X=x) = {}^6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}$$

where x = 0, 1, 2, 3, …6,

the following probability distribution is given as

x	0	1	2	3	4	5	6
p(X = x)	$\frac{64}{729}$	$\frac{192}{729}$	$\frac{240}{729}$	$\frac{160}{729}$	$\frac{60}{729}$	$\frac{12}{729}$	$\frac{1}{729}$

Q. 19 If a random variable x follows Poisson distribution such that p(x = 1) = 2 p(x = 2) Find the mean and variance of the distribution. Also find p(x = 3). Dec. 2016

Ans. : Let x follows Poisson Distribution. The probability function is given as

$$P(X=x) = \frac{e^{-m}\,m^x}{x!} \quad \text{where m = mean}$$

$$p(X=1) = 2P(X=2)$$

$$\frac{e^{-m}\,m^1}{1!} = 2\frac{e^{-m}\,m^2}{2!} \qquad \therefore\ m = 1$$

$$\therefore \text{ Mean } = m = 1$$

Now

$$p(X=3) = \frac{e^{-m}\,m^3}{3!} = \frac{e^{-1}\cdot m^1}{3!} = 0.0613$$

$$\therefore p(X=3) = 0.0613$$

Chapter 5 : Correlation

Q. 1 State true or false with justification: If the two lines of regression are x + 3y – 5 = 0 and 4x + 3y – 8 = 0 then the correlation coefficient is + 0.5. May 2014

Ans. :

$$x + 3y - 5 = 0$$

$$\therefore 3y = 5 - x$$

$$\therefore y = -\frac{1}{3}x + \frac{5}{3} \qquad \text{...(1)}$$

And, $4x + 3y - 8 = 0$

$$\therefore 3y = -4x + 8$$

$$\therefore y = -\frac{4}{3}x + \frac{8}{3} \qquad \text{...(2)}$$

Let $b_1 = -\frac{1}{3}$ and $b_2 = -\frac{4}{3}$

Since, $|b_1| < |b_2|$, $b_{yx} = b_1 = -\frac{1}{3}$

$$b_{xy} = \frac{1}{b_2} = -\frac{3}{4}$$

Hence Equation (1) is regression equation of Y on X and Equation (2) is regression equation of X on Y.

$$\therefore r = \pm\sqrt{b_{yx}\,b_{xy}} = \pm\sqrt{\frac{-1}{3}\times\frac{-3}{4}}$$

$$= \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$$

$$= \pm 0.5$$

Q. 2 Obtain the equation of the line of regression of cost on age from the following table giving the age of a car of certain make and the annual maintenance cost. Also find maintenance cost if age of the car is 9 years.

Age of car (in years) : x	**2**	**4**	**6**	**8**	
Maintenance cost : y (in thousands)	**5**	**7**	**8**	**5**	**11**

May 2014

Ans. :

Let a = 5, b = 8, C = 1

Here n = – 4

X	Y	u = X – 5	V = y – 8	u^2	uV
2	5	– 3	– 3	9	9
4	7	– 1	– 1	1	1
6	8.5	1	0.5	1	0.5
8	11	3	3	9	9
	Σ	0	– 0.5	20	19.5

$$\bar{x} = a + c\,\bar{u} = ac\frac{\Sigma u}{n} = 5 + 1\times\frac{0}{4} = 4$$

$$\bar{y} = b + c\,\bar{v} = b + c\frac{\Sigma v}{n} = 8 + 1\times\frac{-0.5}{4} = 7.875$$

$$b_{yx} = b_{vu} = \frac{n\sum uv - \sum u \sum v}{n\sum u^2 - (\sum u)^2}$$

$$= \frac{4\,(19.5) - (0)\,(-0.5)}{4\,(20) - (0)^2} = 0.975$$

$\therefore$ Regression Equation of Y on X is

$$Y - \overline{Y} = b_{yx}(X - \overline{X})$$

$$\therefore\ y - 7.875 = 0.975\,(X - 5)$$

$$y = 0.975\,x + 3$$

when $X = 9$

$$\therefore\ y = 0975\,(9) + 3 = 11.775$$

$\therefore$ Maintenance cost for a car 9 years old is $11.775 \times 1000 = 11775$ units.

Q. 3 **If the first four moments of a distribution about the value 4 of the random variable are – 1.5, 17, -30 and 108 then find first four raw moments.** May 2014

Ans. :

Given a = 4

By definition, Row moments about 'a' :

$$\mu_r' = E[(x-a)^r]$$

$$\therefore\ \mu_r' = E[(x-4)^r] \quad \ldots(1)$$

and

Raw moments about origin : $\mu_r' = E[x^r]$

Put $r = 1$ in Equation (1), $\mu_1' = E[(x-4)^1]$

$$\therefore\ 1.5 = E(x) - 4$$

$$\therefore\ E(x) = 2.5 \quad \ldots(2)$$

Put $r = 2$ in Equation (1), $\mu_2' = E[(x-4)^2]$

$$\therefore\ 17 = E(x^2 - 8x + 16)$$

$$\therefore\ 17 = E(x^2) - 8E(x) + 16$$

$$\therefore\ 17 = E(x^2) - 8 \times 2.5 + 16 \quad \ldots\text{(From Equation (2))}$$

$$\therefore\ E(x^2) = 17 + 20 - 16$$

$$\therefore\ E(x^2) = 21 \quad \ldots(3)$$

Put $r = 3$ in Equation (1), $\mu_3' = E[(x-4)^3]$

$$\therefore\ -30 = E(x^3 - 3 \times x^2 \times 4 + 3 \times x \times 4^2 - 4^3)$$

$$\therefore\ -30 = E(x^3) - 12E(x^2) + 48\,E(x) - 64$$

$$\therefore\ -30 = E(x^3) - 12 \times 21 + 48 \times 2.5 - 64$$

...(From Equations (2) and (3))

$$\therefore\ E(x^3) = -30 + 12 \times 21 - 48 \times 2.5 + 64$$

$$\therefore\ E(x^3) = 166 \quad \ldots(4)$$

Put $r = 4$ in Equation (1), $\mu_4' = E[(x-4)^4]$

$$\therefore\ 108 = E({}^4C_0 x^4 - {}^4C_1 \times x^3 \times 4 + {}^4C_2 \times x^2 \times 4^2 - {}^4C_3 \times x \times 4^3 + {}^4C_4 \times 4^4)$$

$$\therefore\ 108 = E(x^4) - 16E(x^3) + 96E(x^2) - 256\,E(x) + 256$$

$$\therefore\ 108 = E(x^4) - 16 \times 166 + 96 \times 21 - 256 \times 2.5 - 256$$

...(From Equations (2), (3) and (4))

$$\therefore\ E(x^4) = 108 + 16 \times 166 - 96 \times 21 + 256 \times 2.5 - 256$$

$$\therefore\ E(x^4) = 1132$$

Hence, four moments about origin are 2.5, 21, 166, 1132.

Q. 4 **It is given that the means of x and y are 5 and 10. If the line of regression of y on x is parallel to the line 20y = 9 x + 40, estimate the value of y for x = 30.** May 2015

Ans. :

Given means of x and y are 5 and 10

$$\therefore\ \overline{x} = 5\ ;\ \overline{y} = 10$$

Given line is $20y = 9x + 40$

$$\therefore\ y = \frac{9}{20}x + \frac{40}{20}$$

Slope of the above line $(m_1) = \frac{9}{20}$

Slope of the regression of y on x $(m_2) = b_{yx}$

Since two lines are parallel $m_1 = m_2$

$$\therefore\ b_{yx} = \frac{9}{20}$$

$\therefore$ Regression equation of Y on X is $y - \overline{y} = b_{yx}\,(x - \overline{x})$

$$\therefore\ y - 10 = \frac{9}{20}(x - 5)$$

$$\therefore\ 20y - 200 = 9x - 45$$

$$\therefore\ 20y = 9x + 155$$

When $x = 30$

$$\therefore\ 20y = 9\,(30) + 155$$

$$\therefore\ y = 21.25$$

Estimate value of y for x = 30 is 21.55

Q. 5 **Calculate the coefficients of correlation from the following data.**

x	30	33	25	10	33	75	40	85	90	95	65	55
y	68	65	80	85	70	30	55	18	15	10	35	45

Dec. 2015

Ans. : Consider the coefficients of correlation

$$r = \frac{\Sigma (x_i - \bar{x}) \cdot (y_i - \bar{y})}{\sqrt{\Sigma (x_i - \bar{x})^2 \cdot \Sigma (y_i - \bar{y})^2}}$$

here N = 12, consider the following table.

x	y	$x - \bar{x}$	$y - \bar{y}$	$(x - \bar{x})^2$	$(y - \bar{y})^2$	$(x - \bar{x})(y - \bar{y})$
30	68	– 23	20	529	400	– 460
33	65	– 20	17	400	289	– 340
25	80	– 28	32	784	1024	– 896
10	85	– 43	37	1849	1369	– 1591
33	70	– 20	22	400	484	– 440
75	30	22	– 18	484	324	– 396
40	55	– 13	7	169	49	– 91
85	18	32	– 30	1024	900	–960
90	15	37	– 33	1369	1089	– 1221
95	10	42	– 38	1764	1444	– 1596
65	35	12	– 13	144	169	– 156
55	45	2	– 3	4	9	– 6
636	579	-	-	8920	7550	– 8153

$$\bar{x} = \frac{\Sigma xi}{N} = \frac{636}{12} = 53 \qquad \bar{y} = \frac{\Sigma y_i}{N} = \frac{576}{12} = 48$$

$$r = \frac{-8153}{\sqrt{7550 \times 8920}} = -0.9934, \ \therefore r = -0.9934$$

Q. 6 **Compute spearman's rank corretation coefficient from the following data.** **May 2016**

x	18	20	34	52	12
y	39	23	35	18	46

Ans. : Let R be the spearman's rank correlation coefficient of x and y.

$$R = 1 - \frac{6 \Sigma d_i^2}{N^3 - N}$$

Where, N = Numbers of observations

Here, N = 5

Consider the following table

x	y	R_x	R_y	$d_i = R_x - R_y$	d_i^2
18	39	4	2	2	4
20	23	3	4	– 1	1
34	35	2	3	– 1	1
52	18	1	5	– 4	16
15	46	5	1	4	16
					38

$$R = 1 - \frac{6\,(38)}{5^3 - 5} = 1 - \frac{6\,(38)}{125 - 5} = 1 - \frac{228}{120}$$

$$R = -0.9$$

Q. 7 **Find the equation of lines of regression for the following data.** **May 2016**

x	5	6	7	8	9	10	11
y	11	14	14	15	12	17	16

Ans. :

The line of regression of y on x is given as

$$(y - \bar{y}) = byx \ (x - \bar{x}) \qquad ...(1)$$

Where
$$b_{yx} = \frac{\sum dxdy - \frac{\sum dx \cdot \sum dy}{N}}{\sum dx^2 - \frac{(\sum dx)^2}{N}}$$

The line of regression of x on y is given as

$$(x - \bar{x}) = bxy \ (y - \bar{y}) \qquad ...(2)$$

Where,
$$bxy = \frac{\sum dxdy - \frac{\sum dx \cdot \sum dy}{N}}{\sum dy^2 - \frac{(\sum dy)^2}{N}}$$

Hence N = 7, consider the following table

x	dx = x − 8	dx^2	y	dy = y − 14	dy^2	dx dy
5	−3	9	11	−3	9	9
8	−2	4	14	0	0	0
7	−1	1	14	0	0	0
8	0	0	15	1	1	0
9	1	1	12	−2	4	−2
10	2	4	17	3	9	6
11	3	9	16	2	4	6
56	0	28	99	1	27	19

$\sum dx = 0,\ \sum dx^2 = 28,\ \sum dy = 1, \sum dy^2 = 27,\ \sum dxdy = 19,$

$$\therefore \bar{x} = A + \frac{\sum dx}{N} = 8 + 0 = 8$$

$$\bar{y} = B + \frac{\sum dy}{N} = 14 + \frac{1}{7} = 14.14$$

$$\text{byx} = \frac{\sum dxdy - \frac{\sum dx \cdot \sum dy}{N}}{\sum dx^2 + \frac{\left(\sum dx\right)^2}{N}} = \frac{19-0}{28} = 6.6786$$

$$\text{bxy} = \frac{\sum dxdy - \frac{\sum dx \sum dy}{N}}{\sum dy^2 - \frac{\left(\sum dy\right)^2}{N}} = \frac{19-0}{27-1/7} = 0.7074$$

The line of regression of y on x is

$$(y - \bar{y}) = \text{byx}\,(x - \bar{x})$$

$$y - 14.14 = 0.6786\,(x - 8)$$

$$\therefore y = 0.678\,x + 8.71$$

Now consider the tone of regression of X on sY

$$(x - \bar{x}) = \text{bxy}\,(y - \bar{y})$$

$$(x - 8) = 0.707y - 1.99$$

$$\therefore x = 0.707\,y - 1.99$$

Q. 8 **Calculate karl Pearson's coefficient of correlation from the following data :**

x	100	200	300	400	500
y	30	40	50	60	70

Dec. 2016

Ans. :

Let 'r' be the karl pearson's coefficient of correlation of x and y.

the following table, hence N = 5.

X	$x = \frac{x_i - \bar{x}}{100}$	x^2	y	$x = \frac{x_i - \bar{x}}{100}$	y^2	xy
100	−2	4	30	−2	4	4
200	−1	1	40	−1	1	1
300	0	0	50	0	0	0
400	1	1	60	1	1	1
500	2	4	70	2	4	4
1500	–	10	250	–	10	10

From above table

$$\bar{x} = \frac{\sum x_i}{N} = \frac{1500}{5} = 300 \therefore\ \bar{y} = \frac{\sum y_i}{N} = \frac{250}{5} = 50$$

$$\text{Consider } r = \frac{\sum xy}{\sqrt{\sum x^2 \cdot \sum y^2}} = \frac{10}{\sqrt{10 \times 10}} = \frac{10}{10} = 1$$

$$\therefore r = 1$$

Chapter 6 : Complex Integration

Q. 1 **Evaluate $\oint_c \frac{e^{kz}}{z} dz$ over the circle |z| = 1 and k is real. Hence prove that $\int_0^{\pi} e^{k\cos\theta} \cos(k \sin\theta)\, d\theta = 2\pi$.**

May 2014

Ans. :

Circle |z| = 1 has centre (0, 0) and radius 1

$$\text{Let } I = \int_c \frac{e^{kz}}{z} dz$$

Here, $z_0 = 0$ lies inside the circle.

$z_0 = 0$ is a simple pole.

$$R = \text{Residue of f(z) at "z = 0"} = \lim_{z \to z_0} (z - z_0) \times f(z)$$

$$= \lim_{z \to z_0} (Z - 0) \times \frac{e^{kz}}{z} = \lim_{z \to z_0} e^{kz} = e^0 = 1$$

$$= 1+\frac{3}{z}\left(1-\frac{2}{z}+\frac{2^2}{z^2}-\frac{2^3}{z^3}\ldots\ldots\right)$$
$$-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{3^2}-\frac{z^3}{3^3}\ldots\ldots\right)$$
$$= 1+3\left(\frac{1}{z}-\frac{2}{z^2}+\frac{2^2}{z^3}-\frac{2^3}{z^4}\ldots\ldots\right)$$
$$-8\left(\frac{1}{3}-\frac{z}{3^2}+\frac{z^2}{3^3}-\frac{z^3}{3^4}\ldots\ldots\right)$$

Region of convergence

Above series is convergent for $|2/z| < 1$ and $|z/3| < 1$ i.e. $2 < |z| < 3$, which is the annular region between the concentric circles with centre (0, 0) and radii 2 and 3

Case 3 :

For $|z| > 3$ Obviously $|z| > 2$

$\therefore\ 2 < |z|$ and $3 < |z|$

$\therefore \left|\frac{2}{z}\right| < 1$ and $\left|\frac{3}{z}\right| < 1$

$$\therefore\ f(z) = 1+\frac{3}{z(1+2/z)}-\frac{8}{z(1+3/z)}$$
$$= 1+\frac{3}{2}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$
$$= 1+\frac{3}{z}\left(1-\frac{z}{2}+\frac{2^2}{z^2}-\frac{2^3}{z^3}\ldots\ldots\right)$$
$$-\frac{8}{z}\left(1-\frac{3}{z}+\frac{3^2}{z^2}-\frac{3^3}{z^3}\ldots\ldots\right)$$
$$= 1+3\left(\frac{1}{z}-\frac{2}{z^2}+\frac{2^2}{z^3}-\frac{2^3}{z^4}\ldots\ldots\right)$$
$$-8\left(\frac{1}{z}-\frac{3}{z^2}+\frac{3^2}{z^3}-\frac{3^3}{z^4}\ldots\ldots\right)$$

Q. 3 Find f(3), f′(1 + i), f″(1 − i), if $f(a)=\oint_c \frac{3z^2+11z+7}{z-a}dz$, c : |z| = 2.

May 2014

Ans. : The circle $|z| = 2$ has centre (0, 0) and radius 2.

$z_0 = i, 1+i, 1-i$ lies inside while $z_0 = 3$ lies outside the circle $|z| = 2$

$$\text{Let } \phi(z) = 3z^2+11z+7$$

We assume z_0 = a lies inside the circle $|z| = 2$

"z_0 = 0" is a simple pole.

$$\therefore\ f(a) = \int_c \frac{3z^2+11z+7}{z-a}dz$$
$$= 2\pi i\,\phi(a) \text{ (Cauchy's integral formula)}$$
$$\therefore\ f(a) = 2\pi i\,\phi(a)$$
$$\therefore\ f(a) = 2\pi i\,(3a^2+11a+7) \quad \ldots(1)$$
$$\therefore\ f'(a) = 2\pi i\,(6a+11) \quad \ldots(2)$$
$$\therefore\ f''(a) = 2\pi i\,(6)$$
$$\therefore\ f''(a) = 12\pi i \quad \ldots(3)$$

From Equation (2),

$$\therefore\ f'(1+i) = 2\pi i\,[6(1+i)+11]$$
$$= 2\pi i\,(17+6i)$$

From Equation (3),

$$\therefore\ f''(1-i) = 12\pi i$$

since, z_0 = − 3 lies outside the circle

$$\int_c f(z)\,dz = 0 \text{ (Cauchy's integral theorem)}$$
$$\therefore\ f(3) = 0$$

Hence $\therefore\ f'(1+i) = 2\pi i\,(17+6i)$; $f''(1-i) = 12\pi i$; $f(3) = 0$

Q. 4 Evaluate $\int_0^{\infty} \frac{x^3 \sin x}{(x^2+a^2)^2}$ using contour integration.

May 2014

Ans. :

$$\text{Let } I = \int_{-\infty}^{\infty} \frac{x^2 \cos mx}{(x^2+a^2)^2}dx$$

Step 1 : Consider the contour of a large semicircle with diameter on real axis, centre at origin and above the real axis.

Step 2 : Let $f(z) = \frac{z^2}{(z^2+a^2)^2}$

$$\text{Let } F(z) = \frac{z^2 e^{imz}}{(z^2+a^2)^2} = e^{imz} f(z)$$

As $z \to \infty$, $z f(z) \to 0$

Step 3 : For singularity,

$$\therefore\ (z^2+a^2)^2 = 0$$
$$\therefore\ z^2+a^2 = 0$$
$$\therefore\ z^2 = -a^2 = i^2a^2$$
$$\therefore\ z = \pm ai,$$

Here, z_0 = − ai lies outside while z_0 = ai lies inside the contour.

z_0 = "a i" is a pole order 2.

Step 4 : R_1 = Residue of f(z) at "z = a i"

$$= \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n \times f(z)$$

$$= \frac{1}{(2-1)!} \lim_{z \to z_{ai}} \frac{d}{dz} (z - ai)^2 \times \frac{z^2 e^{imz}}{(z - ai)^2 (z + ai)^2}$$

$$= \frac{1}{1!} \lim_{z \to z_{ai}} \frac{d}{dz} z^2 e^{imz} (z + ai)^{-2}$$

$$= \lim_{z \to z_{ai}} [z^2 e^{imz} - 2(z + ai)^{-3} \cdot 1 + (z + ai)^{-2} e^{imz} \cdot 2z$$
$$+ z^2 (z + ai)^{-2} e^{imz} \cdot im]$$

$$= \lim_{z \to z_{ai}} e^{imz} \left[\frac{-2z^2}{(z + ai)^3} + \frac{2z}{(z + ai)^2} + \frac{imz^2}{(z + ai)^2} \right]$$

$$= e^{i^2 ma} \left[\frac{-2a^2 i^2}{(2ai)^3} + \frac{2ai}{(2ai)^2} + \frac{im \times a^2 i^2}{(2ai)^2} \right]$$

$$= e^{-ma} \left[\frac{-1}{4ai} + \frac{1}{2ai} + \frac{im}{4} \right]$$

$$= e^{-ma} \left[\frac{-1 + 2 + ai^2 m}{4ai} \right] = e^{-ma} \left[\frac{1 - am}{4ai} \right]$$

By Cauchy's Residue theorem

$$\int_c F(z)\, dz = 2\pi i\, (R_1 + R_2 + \ldots)$$

$$\therefore \int_c \frac{z^2 e^{imz}}{(z^2 + a^2)^2}\, dz = 2\pi i\, e^{-ma} \left[\frac{1 - am}{4ai} \right]$$

$$\int_c \frac{z^2 (\cos mz\, i + \sin mz)}{(z^2 + a^2)}\, dz = \frac{\pi}{2a} e^{-ma} (1 - am)$$

$$\therefore \int_{-\infty}^{\infty} \frac{z^2 (\cos mz\, i \sin mz)}{(z^2 + a^2)^2}\, dz = \frac{\pi}{2a} e^{-m} (1 - am)$$

$$\int_{-\infty}^{\infty} \left[\frac{x^2 \cos mx}{(x^2 + a^2)^2} + i \frac{x^2 \sin mx}{(x^2 + a^2)^2} \right] dx = \frac{\pi}{2a} e^{-ma} (1 - am) + 0i$$

Equating Real parts,

$$\therefore \int_{-\infty}^{\infty} \frac{x^2 \cos mx}{(x^2 + a^2)^2}\, dx = \frac{\pi}{2a} e^{-ma} (1 - am)$$

$$\therefore 2 \int_0^{\infty} \frac{x^2 \cos mx}{(x^2 + a^2)^2}\, dx = \frac{\pi}{2a} e^{-ma} (1 - am) \quad (\because \text{Even function})$$

$$\therefore \int_0^{\infty} \frac{x^2 \cos mx}{(x^2 + a^2)^2}\, dx = \frac{\pi}{4a} e^{-ma} (1 - am)$$

Differentiating w.r.t. 'm' and applying DUIS

$$\therefore \int_0^{\infty} \frac{x^2}{(x^2 + a^2)^2} (-\sin mx \cdot x)\, dx = \frac{\pi}{4a} [e^{-am} \cdot -a + (1 - am)$$
$$e^{-am} \cdot -a]$$

$$\therefore \int_0^{\infty} \frac{-x^3 \sin mx}{(x^2 + a^2)^2}\, dx = \frac{\pi}{4a} \times -e^{-am} (1 + 1 - am)$$

put m = 1

$$\therefore \int_0^{\infty} \frac{-x^3 \sin x}{(x^2 + a^2)^2}\, dx = \frac{-\pi}{4} e^{-a} (2 - a)$$

$$\therefore \int_0^{\infty} \frac{x^3 \sin x}{(x^2 + a^2)^2}\, dx = \frac{\pi}{4} e^{-a} (2 - a)$$

Q. 5 Evaluate $\int_0^{1+i} (x^2 + iy)\, dz$ along $y = x$ and $y = x^2$ **Dec. 2016**

Ans. :

(a) Along the straight line $y = x$

$\therefore z = x + iy = x + ix$

$dz = dx + idx \;\therefore\; x$ varies from $x = 0$ to $x = 1$

$$\therefore \int_0^{1+i} (x^2 + iy)\, dz = \int_0^1 (x^2 + ix)(dx + idx)$$

$$= (1 + i) \int_0^1 (x^2 + ix)\, dx$$

$$= (1 + i) \left[\frac{x^3}{3} + i \frac{x^2}{2} \right]_0^1$$

$$= (1 + i) \left[\frac{1}{3} + \frac{1}{2} i \right]$$

$$= \frac{1}{3} + i \frac{1}{3} + \frac{1}{2} i - \frac{1}{2} = -\frac{1}{6} + \frac{5}{6} i$$

(b) Along the parabola $y = x^2$ (OAB)

$\therefore dy = 2x\, dx \;;\; z = x + iy = x + ix^2$

$\Rightarrow dz = dx + i\, 2x dx = (1 + i\, 2x)\, dx$

and x varies from 0 to 1.

$$\therefore \int_0^{1+i} (x^2 + iy)\, dz = \int_0^1 (x^2 + ix^2)(1 + i2x)\, dx$$

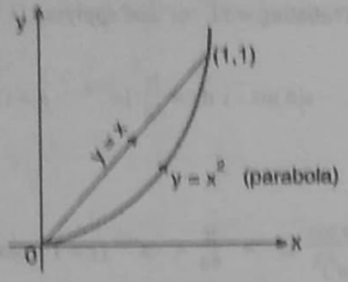

Fig. 6.1

$$= \int_0^1 (x^2 + i2x^3 - ix^2 + 2x^3) \cdot dx$$

$$= \left[\frac{x^3}{3} + i\frac{2x^4}{4} + i\frac{x^3}{3} - \frac{2x^4}{4}\right]_0^1 = \left[\frac{1}{3} + i\frac{1}{2} + \frac{1}{3}i - \frac{1}{2}\right]$$

$$= \frac{1}{6} + i\frac{5}{6}$$

Q. 6 Obtain Taylor's and two distinct Laurent's series expansion of $f(z) = \frac{z-1}{z^2 - 2z - 3}$ about $z = 0$ indicating the region of convergence.

Dec. 2014

Ans. :

$$\text{Given} = \frac{z-1}{z^2 - 2z - 3} = \frac{z-1}{(z+1)(z-3)}$$

We shall now resolve F(z) into partial fractions

$$\text{Let } \frac{z-1}{(z+1)(z-3)} = \frac{A}{z+1} + \frac{B}{z-3}$$

This gives, $z - 1 = A(z-3) + B(z+1)$

when $z = -1, -2 = -4A \therefore A = \frac{1}{2}$

when $z = 3, 2 = 4B \therefore B = \frac{1}{2}$

$$\therefore f(z) = \frac{1/2}{z+1} + \frac{1/2}{z-3} \qquad \ldots(1)$$

f(z) is analytic everywhere except at $z = -1$ and $z = 3$. Thus f(z) is analytic for

(i) $|z| < 1$

$$f(z) = \frac{1}{2} \cdot \frac{1}{1+z} - \frac{1}{6}\,\frac{1}{\left(1 - \frac{z}{3}\right)}$$

$$= \frac{1}{2}(1+z)^{-1} - \frac{1}{6}\left(1 - \frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2}\{1 - z + z^2 - z^3 + \ldots\} - \frac{1}{6}\left\{1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \ldots\right\}$$

$$= \frac{1}{3} - \frac{1}{9}z + \frac{13}{27}z^2$$

(ii) **When $1 < |z| < 3$**

In this case $1 < |z|$

$$\therefore \left|\frac{1}{z}\right| < 1 \text{ and } \left|\frac{z}{3}\right| < 1$$

$$f(z) = \frac{1}{2z\left(1 + \frac{1}{z}\right)} - \frac{1}{6\left(1 - \frac{z}{3}\right)}$$

$$= \frac{1}{2z}\left(1 + \frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1 - \frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2z}\left\{1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \ldots\right\} - \frac{1}{6}\left\{1 + \frac{z}{3} + \frac{z^2}{9} + \ldots\right\}$$

(iii) **when $|z| < 3$ then $|z| > 1$**

$$\left|\frac{1}{z}\right| < 1 \text{ and } \left|\frac{3}{z}\right| < 1$$

$$f(z) = \frac{1}{2z\left(1 + \frac{1}{z}\right)} + \frac{1}{2z\left(1 - \frac{3}{z}\right)}$$

$$= \frac{1}{2z}\left(1 + \frac{1}{z}\right)^{-1} + \frac{1}{2z}\left(1 - \frac{3}{z}\right)^{-1}$$

$$= \frac{1}{2z}\left\{1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \ldots\right\} + \frac{1}{2z}\left\{1 + \frac{3}{z} + \frac{9}{z^2} + \ldots\right\}$$

$$= \frac{1}{2z}\left\{1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \ldots + 1 + \frac{3}{z} + \frac{9}{z^2} + \frac{27}{z^3} + \ldots\right\}$$

$$= \frac{1}{z} + \frac{1}{z^2} + \frac{5}{z^3} + \frac{13}{z^4} + \ldots$$

Q. 7 Evaluate by using Residue theorem.

$$\int_0^{2\pi} \frac{d\theta}{(2 + \cos\theta)^2}$$

Dec. 2014

Ans. :

$$I = \int_0^{2\pi} \frac{d\theta}{(2 + \cos\theta)^2} \qquad \ldots(1)$$

Taking $z = e^{i\theta}$ on unit circle $|z| = 1$

$$\therefore dz = i e^{i\theta} \cdot d\theta = iz\, d\theta$$

$$\therefore d\theta = \frac{1}{iz}\, dz$$

and
$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2} = \frac{z^2 + 1}{2z}$$

$$2 + \cos\theta = 2 + \frac{z^2 + 1}{2z} = \frac{4z + z^2 + 1}{2z}$$

$$(2+\cos\theta)^2 = \left(\frac{z^2+4z+1}{2z}\right)^2$$

$$\frac{1}{(2+\cos\theta)^2} = \frac{4z^2}{(z^2+4z+1)^2}$$

The Equation (1) becomes

$$I = \int_c \frac{4z^2}{iz\,(z^2+4z+1)^2}\,dz$$

$$I = \frac{4}{i}\int_c \frac{z}{(z^2+4z+1)^2}\,dz \quad \ldots(2)$$

$$\text{here } f(z) = \frac{z}{(z^2+4z+1)^2}$$

Poles are obtained from $(z^2+4z+1)^2 = 0$

$$\therefore z^2+4z+1 = 0$$

$$z = \frac{-4\pm\sqrt{16-4}}{2(1)} = -2\pm\sqrt{3}$$

$$z = -2+\sqrt{3},\ -2-\sqrt{3}$$

Both poles are of order of $m = 2$

Let $z_1 = -2+\sqrt{3}$ and $-2-\sqrt{3}$

Only $z_1 = -2+\sqrt{3}$ inside the circle $|z| = 1$

Then

$$R = \frac{1}{(2-1)!}\lim_{z\to z_1}\frac{d}{dz}\{(z-z_1)\cdot f(z)\}$$

$$= \lim_{z\to z_1}\frac{d}{dz}\left[(z-z)^2\frac{z}{(z-z_1)^2(z-z_2)^2}\right]$$

$$= \lim_{z\to z_1}\frac{d}{dz}\left[\frac{z}{(z-z_2)^2}\right]$$

$$= \lim_{z\to z_1}\left[\frac{(z-z_2)^2\cdot(1)-z\cdot 2(z-z_2)}{(z-z_2)^4}\right]$$

$$= \lim_{z\to z_1}\frac{z-z_2-2z}{(z-z_2)^3} = \lim_{z\to z_1}\frac{-(z+z_2)}{(z-z_2)^3}$$

$$= \frac{-(z_1+z_2)}{(z-z_2)^3} = \frac{-(4)}{(2\sqrt{3})^3} = \frac{4}{8\cdot 3\sqrt{3}} = \frac{1}{6\sqrt{3}}$$

Then by Cauchy's residue theorem

$$\int_c f(z)\,dz = 2\pi i\,(R) = 2\pi i\,\frac{1}{6\sqrt{3}} = \frac{\pi i}{3\sqrt{3}} \quad \ldots(3)$$

Using Equations (3) in (2),

$$I = \frac{1}{i}\times\frac{\pi i}{3\sqrt{3}} = \frac{2\pi}{3\sqrt{3}}$$

$$\text{Thus } \int_c^{\pi/2}\frac{d\theta}{(2+\cos\theta)^2} = \frac{2\pi}{3\sqrt{3}}$$

Q. 8 **Evaluate** $\int_{-\infty}^{\infty}\frac{\cos 3x}{(x^2+1)(x^2+4)}\,dx$ **using Cauchy Residue Theorem.** **Dec. 2014**

Ans. :

$$\text{Let } F(z) = \frac{\cos 3x}{(z^2+1)(z^2+4)}$$

The poles of F (z) are $\pm i$, $\pm 2i$

The poles i and 2i lie within C, By residue theorem

$$\int_C F(z)\,dz = 2\pi i\,[\text{Sum of residues of } F(z)] \quad \ldots(1)$$

Let R_1 = Residue of F (z) at z = i

$$= \lim_{z\to i}(z-i)\cdot f(z)$$

$$= \lim_{z\to i}\cancel{(z-i)}\cdot\frac{\cos 3x}{\cancel{(z-i)}(z+i)(z^2+4)}$$

$$= \frac{\cos 3x}{(i+i)(-1+4)} = \frac{\cos 3x}{2i\times 3} = \frac{\cos 3x}{6i} \quad \ldots(2)$$

R_2 = Residue of F (z) at z = 2i

$$= \lim_{z\to 2i}(z-2i)\cdot f(z)$$

$$= \lim_{z\to 2i}\cancel{(z-2i)}\frac{\cos 3x}{(z^2+1)(z+2i)\cancel{(z-2i)}}$$

$$= \frac{\cos 3x}{(-4+1)(2i+2i)} = \frac{\cos 3x\ i^2}{-3\times 4i} = \frac{-\cos 3x}{12\,i} \quad \ldots(3)$$

Using Equations (2) and (3) in Equation (1)

$$\int_C F(z)\,dz = 2\pi i\,[R_1+R_2]$$

$$= 2\pi i\left[\frac{\cos 3x}{6i}-\frac{\cos 3x}{12i}\right] = \frac{2\pi i\cos 3x}{i}\left[\frac{1}{6}-\frac{1}{12}\right]$$

$$= 2\pi\cos 3x\,\frac{5}{12} = \frac{10\,\pi\cos 3x}{12}$$

$$\text{Thus } \int_{-\infty}^{\infty}\frac{x^2\,dx}{x^2+1\,(x^2+4)} = \frac{10\pi\cos 3x}{12}$$

Q. 9 Evaluate $\int_c |z|\,dz$, where c is the left half of unit circle $|z| = 1$ from $z = -i$ to $z = i$. May 2015

Ans. : $|z| = 1$ is unit circle with centre (0,0) and radius = 1

$$\text{Put } z = re^{i\theta}$$

$$\therefore dz = e^{i\theta} \cdot i\,\theta \text{ and}$$

Limits for left half of circle: $\theta = -\pi/2$ to $\theta = \pi/2$

$$\therefore \int_c |z|\,dz = \int_c 1\,dz = \int_{-\pi/2}^{\pi/2} 1 \times e^{i\theta}\, i\, d\theta$$

$$= i\left[\frac{e^{i\theta}}{i}\right]_{-\pi/2}^{\pi/2} = e^{i\pi/2} - e^{-i\pi/2} = 2i\sin\frac{\pi}{2} = 2i(1)$$

$$\therefore \int_c |z|\,dz = 2i$$

Q. 10 Obtain two distinct Laurent's series expansions of $f(z) = \dfrac{2z-3}{z^2-4z+3}$ in powers of (z – 4) indicating the region of convergence in each case. May 2015

Ans. : By partial fractions

$$f(z) = \frac{2z-3}{z^2-4z+3} = \frac{2z-3}{(z-1)(z-3)} = \frac{1/2}{z-1} + \frac{3/2}{z-3}$$

$$\text{put } u = z-4$$

$$\therefore z = u+4$$

$$\therefore f(z) = \frac{1/2}{(u+4)-1} + \frac{3/2}{(u+4)-3} = \frac{1}{2(u+3)} + \frac{3}{2(u+1)}$$

Case 1 : For $|u| < 1$

Obviously, $|u| < 3$

$$\therefore |u| < 1 \text{ and } \left|\frac{u}{3}\right| < 1$$

$$\therefore f(z) = \frac{1}{2\cdot 3(u/3+1)} + \frac{3}{2(1+u)}$$

$$= \frac{1}{2\times 3}\left(1+\frac{u}{3}\right)^{-1} + \frac{3}{2}(1+u)^{-1}$$

$$= \frac{1}{2\times 3}\left(1-\frac{u}{3}+\frac{u^2}{3^2}-\ldots\right) + \frac{3}{2}(1-u+u^2-\ldots)$$

$$= \frac{1}{2}\left(\frac{1}{3}-\frac{u}{3^2}+\frac{u^2}{3^3}-\ldots\right) + \frac{3}{2}(1-u+u^2-\ldots)$$

$$= \frac{1}{2}\left[\frac{1}{3}-\frac{(z-4)}{3^2}+\frac{(z-4)^2}{3^3}-\ldots\right] + \frac{3}{2}\left[1-(z-4)+(z-4)^2-\ldots\right]$$

Region of Convergence :

Above series is convergent

For $|u| < 1$ and $|u| < 3$

i.e. $|u| < 1$,

i.e. $|z-4| < 1$, which is the interior of the circle with centre (4, 0) and radius 1.

Case 2 : $1 < |u| < 3$

$\therefore 1 < |u|$ and $|u| < 3$

$$\therefore \left|\frac{1}{u}\right| < 1 \text{ and } \left|\frac{3}{u}\right| < 1$$

$$\therefore f(z) = \frac{1}{2\cdot 3(u/3+1)} + \frac{3}{2\cdot u(1/u+1)}$$

$$= \frac{1}{2\times 3}\left(1+\frac{1}{3}\right)^{-1} + \frac{3}{2u}\left(1+\frac{1}{u}\right)^{-1}$$

$$= \frac{1}{2\times 3}\left(1-\frac{u}{3}+\frac{u^2}{3^2}-\ldots\right) + \frac{3}{2u}\left(1-\frac{1}{u}+\frac{1}{u^2}-\ldots\right)$$

$$= \frac{1}{2}\left(\frac{1}{3}-\frac{u}{3^2}+\frac{u^2}{3^3}-\ldots\right) + \frac{3}{2}\left(\frac{1}{u}-\frac{1}{u^2}+\frac{1}{u^3}-\ldots\right)$$

$$= \frac{1}{2}\left[\frac{1}{3}-\frac{(z-4)}{3^2}+\frac{(z-4)^2}{3^3}-\ldots\right] + \frac{3}{2}\left[\frac{1}{z-4}-\frac{1}{(z-4)^2}+\frac{1}{(z-4)^3}-\ldots\right]$$

Region of Convergence :

Above series is convergent for $|1/u| < 1$ and $|u/3| < 1$;

i.e. $1 < |u| < 3$,

i.e. $1 < |z-4| < 3$, which is the annular region between the concentric circles with centre (4, 0) and radii 1 and 3.

Case 3 : for $|u| > 3$

Obviously, $|u| > 1$

$\therefore 1 < |u|$ and $3 < |u|$

$$\therefore \left|\frac{1}{u}\right| < 1 \text{ and } \left|\frac{3}{u}\right| < 1$$

$$\therefore f(z) = \frac{1}{2\cdot u(1+3/u)} + \frac{3}{2\cdot u(1/u+1)}$$

$$= \frac{1}{2u}\left(1+\frac{3}{u}\right)^{-1} + \frac{3}{2u}\left(1+\frac{1}{u}\right)^{-1}$$

$$= \frac{1}{2u}\left(1-\frac{3}{u}+\frac{3^2}{u^2}-\ldots\right) + \frac{3}{2u}\left(1-\frac{1}{u}+\frac{1}{u^2}-\ldots\right)$$

$$= \frac{1}{2}\left(\frac{1}{u}-\frac{3}{u^2}+\frac{3^2}{u^3}-\ldots\right) + \frac{3}{2}\left(\frac{1}{u}-\frac{1}{u^2}+\frac{1}{u^3}-\ldots\right)$$

$$= \frac{1}{2}\left[\frac{1}{z-4}-\frac{3}{(z-4)^2}+\frac{3^2}{(z-4)^3}-\ldots\right] + \frac{3}{2}\left[\frac{1}{z-4}-\frac{1}{(z-4)^2}+\frac{1}{(z-4)^3}-\ldots\right]$$

Region of Convergence :

Above series is convergent for $|1/u| < 1$ and $|3/u| < 1$

i.e. $|u| > 3$.

i.e. $|z - 4| > 3$.

Which is the exterior region of the circle with centre (4, 0) and radius 3.

Q. 11 **Evaluate $\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$, where c is the circle $|z| = 3$.**

May 2015

Ans. : $I = \int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$

For singularity, $(z-1)(z-2) = 0$

$\therefore z = 1$ or $z = 2$

$|z| = 3$ represents a circle with centre (0, 0) and radius is 3.

Here, $z_0 = 1$ and $z_0 = 2$ lies inside $|z| = 3$.

$z_0 = 1$ and $z_0 = 2$ are simple poles.

$$R_1 = \text{Residue of } f(z) \text{ at "} z = 2 \text{"} = \lim_{z \to z_0} (z - z_0) \times f(z)$$

$$= \lim_{z \to z_0} (z-2) \times \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$$

$$= \frac{\sin \pi(2)^2 + \cos \pi(2)^2}{2-1} = \frac{0+1}{1} = 1$$

$$R_2 = \text{Residue of } f(z) \text{ at "} z = 1 \text{"}$$

$$= \lim_{z \to z_0} (z-2) \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$$

$$= \frac{\sin \pi(1)^2 + \cos \pi(1)^2}{1-2} = \frac{0-1}{-1} = 1$$

Q. 12 **Evaluate $\int_0^{2\pi} \frac{d\theta}{13 + 5 \sin \theta}$.**

May 2015

Ans. :

$$\text{Let, } I = \int_0^{2\pi} \frac{d\theta}{13 + 5 \sin \theta}$$

Consider a circle $|z| = 1$ which has centre (0, 0) and radius 1.

$$\text{Put } z = re^{i\theta} = 1e^{i\theta} = e^{i\theta}$$

$$\therefore dz = e^{i\theta} \cdot i\, d\theta = iz\, d\theta$$

$$\therefore d\theta = \frac{dz}{iz}$$

$$\text{Also, } \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i} = \frac{z^2 - 1}{2iz}$$

On substituting, $I = \int_c \frac{1}{13 + 5 \times \frac{z^2-1}{2iz}} \cdot \frac{dz}{iz}$

$$= \int_c \frac{2\cancel{iz}}{26iz + 5z^2 - 5} \cdot \frac{dz}{\cancel{iz}}$$

$$= \int_c \frac{2}{5z^2 + 26iz - 5} dz$$

For singularity $5z^2 + 26iz - 5 = 0$

$$\therefore z = \frac{-(26i) \pm \sqrt{(26i)^2 - 4(5)(-5)}}{2(5)}$$

$$= \frac{-26i \pm \sqrt{-576}}{10} = \frac{-26i \pm 24i}{10}$$

$$\therefore z = -5i \text{ or } z = \frac{-i}{5}$$

Here, $z_0 = -5i$ lies outside while '$z_0 = \frac{-i}{5}$' lies inside the circle $|z| = 1$

$z_0 = \frac{-i}{5}$ is a simple pole.

$$R_1 = \text{Residue of } f(z) \text{ at "} z = \frac{-i}{5} \text{"}$$

$$= \lim_{z \to z_0} (z - z_0) \times f(z)$$

$$= \lim_{z \to -i/5} (z + i/5) \times \frac{2}{5(z + i/5)(z + 5i)}$$

$$= \frac{2}{5(-i/5 + 5i)} = \frac{1}{12i}$$

By Cauchy's Residue theorem

$$\int_c f(z)\, dz = 2\pi i (R_1 + R_2 + \ldots)$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{13 + 5 \sin \theta} = 2\pi i \times \frac{1}{12i}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{13 + 5 \sin \theta} = \frac{\pi}{6}$$

Q. 13 **Verify Cayley Hamilton theorem for the matrix A and hence find A^{-1} and A^4.**

$$\textbf{Where } A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

May 2015

Ans. :

Let λ be eigen value of matrix A

Characteristic equation is $|A - \lambda I| = 0$

$$\therefore \begin{bmatrix} 1-\lambda & 2 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & -2 & 1-\lambda \end{bmatrix} = 0$$

On solving ,

λ^3 – (sum of diagonal elements) λ^2 + (sum of the minors of diagonal elements) $\lambda - |A| = 0$

$$\therefore \lambda^3 - (1+3+1)\lambda^2 + (3+1+5)\lambda - 1 = 0$$

$$\therefore \lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$$

Cayley Hamiton Theorem states that the characteristic equation is satisfied by matrix A.

$$\therefore A^3 - 5A^2 + 9A - I = 0 \quad \ldots(1)$$

$$\text{Now, } A^2 = A \times A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$$

$$\therefore \text{LHS} = A^3 - 5A^2 + 9A - I$$

$$= \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 5 \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + 9 \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{RHS}$$

$\therefore$ Cayley Hamilton Theorem is verified.

Part II : Pre multiply Equation (1) by A –1

$$A^2 - 5A + 9I - A^{-1} = 0$$

$$\therefore A^{-1} = A^2 - 5A + 9I$$

$$= \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} - 5 \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} + 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

Part III :

Pre multiply Equation (1) by A

$$A^4 - 5A^3 + 9A^2 - A = 0$$

$$\therefore A^4 = 5A^3 - 9A^2 + A$$

$$= 5 \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 9 \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\therefore A^4 = \begin{bmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -40 & -23 \end{bmatrix}$$

Q. 14 **Evaluate $\int \frac{dx}{z^3\,(z+4)}$ where $|z| = 4$**

Dec. 2015

Ans. : $\int_C \frac{dz}{z^3\,(z+4)}$ $\quad C : |z| = 4$

It is not analytic at $z = 0$ and $z = -4$

$z = 0, \quad |0| = 0 < 2 \quad$ It is inside

$z = -4, \quad |-4| = 4 > 2 \quad$ It is outside

$$= \int_C \frac{\frac{1}{z+4}}{z^3}\,dz = \frac{2\pi i}{2!}\left[\frac{d^2}{dz^2}\,\frac{1}{(z+4)}\right]_{z=0}$$

$$= \pi i\left[-\frac{d}{dz}\,\frac{1}{(z+4)^2}\right]_{z=0}$$

$$= \pi i\left[\frac{2}{(z+4)^3}\right]_{z=0} = \frac{2\pi i}{(4)^3} = \frac{2\pi i}{64} = \frac{\pi i}{32}$$

Q. 15 **Find the complete solution of $\int_{x_0}^{x_1} \left(2xy - y'^2\right) dx$**

Dec. 2015

Ans. :

$$\int_{x_0}^{x_1} \left(2\,xy - y''^2 \right) dx$$

$$F(x, y, y', y'') = 2xy - y''^2$$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0$$

$$2x - \frac{d}{dx}(0) + \frac{d^2}{dx^2}(-2y'') = 0$$

$$2x - 2y''^{v} = 0$$

$$2y'^{v} = x$$

$$y''' = \frac{x^2}{x} + C_1$$

$$y'' = \frac{x^3}{6} + C_1 x + C_2$$

$$y' = \frac{x^4}{24} + C_1\frac{x^2}{2} + C_2 x + C_3$$

$$y = \frac{x^5}{120} + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3 x + C_4$$

Q. 16 Find expansion of $f(z) = \dfrac{1}{(1+z^2)(z+2)}$ indicating region of convergence. Dec. 2015

Ans. :

$$f(z) = \frac{1}{(1+z^2)(z+2)} = \frac{1}{(z+i)(z-i)(z+2)}$$

Its singularity are $z = -i, i, -2$

$$\frac{1}{(z+i)(z-i)(z+2)} = \frac{A}{(z+i)} + \frac{B}{(z-i)} + \frac{C}{(z+2)}$$

$$1 = A(z-i)(z+2) + B(z+i)(z+2) + C(z+i)(z-i)$$

Put $z - i = 0 \Rightarrow z = i$

$$1 = B(2i)(i+2) \Rightarrow B = \frac{1}{2i(i+2)}$$

Put $z + i = 0 \Rightarrow z = -i$

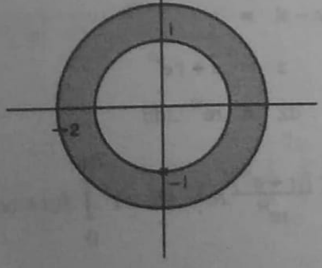

Fig. 6.2

$$1 = A(-2i)(-i+2) \Rightarrow A = -\frac{1}{2i(-i+2)}$$

Put $z + 2 = 0 \Rightarrow z = -2$

$$1 = C(-2+i)(-2-i) \Rightarrow 1 = C((-2)^2 - i^2) \Rightarrow 1 = C(4+1)$$

$$C = 1/5$$

$$f(z) = -\frac{1}{2i(2-i)}\frac{1}{(z+i)} + \frac{1}{2i(i+2)}\frac{1}{(z-i)} + \frac{1}{5}\frac{1}{(z+2)}$$

(i) $|z| < 1$ (ii) $1 < |z| < 2$ (iii) $|z| > 2$

(i) $|z| < 1 \Rightarrow |z| < |i| \Rightarrow \dfrac{|z|}{|i|} < 1 \Rightarrow |z| < 2 \Rightarrow \dfrac{|z|}{2} < 1$

$$f(z) = -\frac{1}{2i(2-i)}\frac{1}{i\left(1+\frac{z}{i}\right)} + \frac{1}{2i(i+2)}\frac{1}{(-i)\left(1-\frac{z}{i}\right)} + \frac{1}{5}\frac{1}{(2)\left(1+\frac{z}{2}\right)}$$

$$= \frac{1}{2(2-i)}\left(1+\frac{z}{i}\right)^{-1} + \frac{1}{2(i+2)}\left(1-\frac{z}{i}\right)^{-1} + \frac{1}{10}\left(1+\frac{z}{2}\right)^{-1}$$

$$= \frac{1}{2(2-i)}\left(1 - \frac{z}{i} + \frac{z^2}{i^2} - \frac{z^3}{i^3}\ldots\right) + \frac{1}{2(i+2)}\left(1 + \frac{z}{i} + \frac{z^2}{i^2} + \frac{z^3}{i^3}\ldots\right) + \frac{1}{10}\left(1 - \frac{z}{2} - \frac{z^2}{2^2}\ldots\right)$$

(ii) $1 < |z| < 2$

$$\frac{1}{|z|} < 1 \text{ and } \frac{|z|}{2} < 1 \Rightarrow |i| < |z| \Rightarrow \frac{|i|}{|z|} < 1$$

$$f(z) = -\frac{1}{2i(2-i)}\frac{1}{z\left(1+\frac{i}{z}\right)} + \frac{1}{2i(i+2)}\frac{1}{z\left(1-\frac{i}{z}\right)} + \frac{1}{10}\frac{1}{\left(1+\frac{z}{2}\right)}$$

$$= -\frac{1}{2i(2-i)z}\left(1+\frac{i}{z}\right)^{-1} + \frac{1}{2i(i+2)z}\left(1-\frac{i}{z}\right)^{-1} + \frac{1}{10}\left(1+\frac{z}{2}\right)^{-1}$$

$$= -\frac{1}{2i(2-i)z}\left(1 - \frac{i}{z} + \frac{i^2}{z^2} - \frac{i^3}{z^3}\ldots\right) + \frac{1}{2i(i+2)z}\left(1 + \frac{i}{z} + \frac{i^2}{z^2} + \frac{i^3}{z^3}\ldots\right) + \frac{1}{10}\left(1 - \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3}\ldots\right)$$

(iii) $|z| > 2 \Rightarrow |z| > |i| \Rightarrow \dfrac{|i|}{|z|} < 1 \Rightarrow \dfrac{2}{|z|} < 1$

$$f(z) = -\frac{1}{2i(2-i)}\frac{z}{z\left(1+\frac{i}{z}\right)} + \frac{1}{2i(i+2)}\frac{1}{z}\frac{1}{\left(1-\frac{i}{z}\right)} + \frac{1}{5z}\frac{1}{\left(1+\frac{2}{z}\right)}$$

$$= -\frac{1}{2i(2-i)z}\left(1+\frac{i}{z}\right)^{-1} + \frac{1}{2i(i+2)z}\left(1-\frac{i}{z}\right)^{-1} + \frac{1}{5z}\left(1+\frac{2}{z}\right)^{-1}$$

$$= -\frac{1}{2i(2-i)z}\left(1-\frac{i}{z}+\frac{i^2}{z^2}-\frac{i^3}{z^3}\ldots\ldots\right) + \frac{1}{2i(i+2)z}\left(1+\frac{i}{z}+\frac{i^2}{z^2}+\frac{i^3}{z^3}\ldots\right) + \frac{1}{5z}\left(1-\frac{2}{z}+\frac{2^2}{z^2}-\frac{2^3}{z^3}\ldots\ldots\right)$$

Q. 17 **Using Cauchy Residue Theorem evaluate** $\int_{-\infty}^{\infty}\frac{x^2}{x^6+1}dx$

Dec. 2015

Ans. : $\int_{-\infty}^{\infty}\frac{x^2}{x^6+1}dx = \int_C \frac{z^2}{z^6+1}dz$

Its poles are $z^6+1=0$

$$z^6 = -1$$

$$z^6 = \cos\pi + i\sin\pi$$

$$z^6 = \cos(2k\pi+\pi) + i\sin(2k\pi+\pi)$$

$$z = [\cos(2k\pi+\pi)+i\sin(2k\pi+\pi)]^{1/6} = \cos\left(\frac{2k\pi+\pi}{6}\right) + i\sin\left(\frac{2k\pi+\pi}{6}\right)$$

$$k = 0, 1, 2, 3, 4, 5, 6$$

$$z_0 = \cos\frac{\pi}{6}+i\sin\frac{\pi}{6} = e^{i\pi/6}$$

$$z_1 = \cos\frac{3\pi}{6}+i\sin\frac{3\pi}{6} = e^{i3\pi/6}$$

$$z_2 = \cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6} = e^{i5\pi/6}$$

These poles lies in upper half plane

Residue at $z = e^{i\pi/6} = \frac{z^2}{6z^5} = \frac{1}{6z^3} = \frac{1}{6\left(e^{i\pi/6}\right)^3} = \frac{1}{6e^{i\pi/2}} = \frac{1}{6i}$

Residue at $z = e^{i3\pi/6} = \frac{z^2}{6z^5} = \frac{1}{6z^3} = \frac{1}{6\left(e^{i3\pi/6}\right)^3}$

$$= \frac{1}{6e^{i3\pi/2}} = \frac{1}{-6i}$$

Residue at $z = e^{i5\pi/6} = \frac{1}{6z^3} = \frac{1}{6\left(e^{i5\pi/6}\right)^3} = \frac{1}{6e^{i5\pi/2}} = \frac{1}{6i}$

$$\therefore \int_C \frac{z^2}{z^6+1}dz = 2\pi i\left(\frac{1}{6i}-\frac{1}{6i}+\frac{1}{6i}\right) = \frac{2\pi i}{6i} = \frac{\pi}{3}$$

$$\int_{-\infty}^{\infty}\frac{x^2}{x^6+1}dx = \frac{\pi}{3}$$

Q. 18 **State and prove Cauchy's Integral Formula for the simply connected region and hence evaluate** $\int \frac{z+6}{z^2-4}$, $|z-2|=5$

Dec. 2015

Ans. : **Cauchy Integral formula :** Let f(z) in an analytic function inside

a region R as well as on the boundary C and 'a' is a point inside the region R then

$$f(z)\big|_{z=a} = f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}dz$$

Proof :

Let f(z) is an analytic function inside a region R with the boundary C then $\frac{f(z)}{(z-a)}$ is also analytic inside the region R except the point z = a.

Consider z = a as a centre and construct a circle C_1 with centre a and radius r which lies completely inside C

$$C_1 : |z-a| = r$$

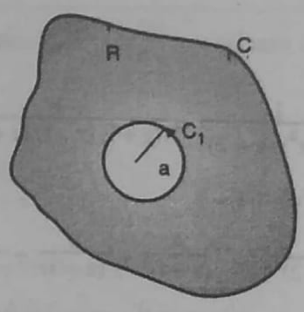

Fig. 6.3

By Cauchy integral theorem for multiply connected region

$$\int_C \frac{f(z)}{(z-a)}dz = \int_{C_1}\frac{f(z)}{(z-a)}dz \quad \ldots(1)$$

$$C_1 : |z-a| = r$$

$$z = a + re^{i\theta}$$

$$dz = re^{i\theta}\cdot id\theta$$

$$\int_C \frac{f(z)}{(z-a)}dz = \int_0^{2\pi}\frac{f(r+e^{i\theta})}{re^{i\theta}}re^{i\theta}\cdot id\theta = i\int_0^{2\pi}f(a+re^{i\theta})\,d\theta$$

$$= i \lim_{r \to 0} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta \left(\begin{array}{c} \text{to reduce the} \\ \text{cure } C_1 \text{ to a only} \end{array} \right)$$

$$= i \int_0^{2\pi} f(a)\, d\theta = i\, f(a)\, (\theta)_0^{2\pi} = 2\pi i\, f(a)$$

$$\int_C \frac{f(z)}{(z-a)} dz = 2\pi i\, f(a)$$

$$\Rightarrow \quad f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)} dz$$ **hence proved.**

$$\int_C \frac{z+6}{z^2-4} dz, \quad C : |z-2| = 5$$

$$= \int_C \frac{(z+6)}{(z+2)(z-2)} dz$$ it is not analytic at $z = -2, 2$

$z = 2$, $|2 - 2| = 0 < 5$ it is inside.

$z = -2$, $|-2 - 2| = 4 < 5$ it is inside.

$$\frac{z+6}{(z+2)(z-2)} = \frac{A}{(z+2)} + \frac{B}{(z-2)} \Rightarrow z + 6 = A(z-2) + B(z+2)$$

Put $z - 2 = 0 \Rightarrow z = 2$ Put $z + 2 = 0 \Rightarrow = -2$

$8 = B(4) \Rightarrow B = 2$ $-2 + 6 = A(-2-2)$

$4 = -4A \Rightarrow A = -1$

$$= \int_C \left[\frac{-1}{(z+2)} + \frac{2}{(z-2)} \right] dz$$

$$= 2 \int_C \frac{1}{(z-2)} dz - \int_C \frac{1}{(z+2)} dz$$

$$= 2(2\pi i)(1)|_{z=2} - 2\pi i\, (1)_{z=-2} = 4\pi i - 2\pi i = 2\pi i$$

Q. 19 **Show that** $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta}\, d\theta = \frac{2\pi}{b^2} \left(a - \sqrt{a^2 - b^2}\right), 0 < b < a$

Dec. 2015

Ans. :

$$\int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta} d\theta = \int_0^{2\pi} \frac{(1 - \cos 2\theta)}{2(a + b\cos\theta)} d\theta = \frac{1}{2} \int_0^{2\pi} \frac{\cos 2\theta}{(a + b\cos\theta)} d\theta \quad \ldots(1)$$

$C : |z| = 1$; $\quad z = e^{i\theta}, \ \frac{1}{z} = e^{-i\theta}$;

$$dz = e^{i\theta} \cdot i\, d\theta \quad \Rightarrow \quad d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{z + \frac{1}{z}}{2} = \frac{z^2 + 1}{2z}$$

From Equation (1)

$$= \frac{1}{2} \int_0^{2\pi} \frac{\text{Re}\,(1 - e^{i2\theta})}{(a + b\cos\theta)} d\theta = \frac{\text{Re}}{2} \int_0^{2\pi} \frac{(1 - e^{i2\theta})}{(a + b\cos\theta)} d\theta$$

$$= \frac{\text{Re}}{2} \int_C \frac{(1 - z^2)}{\left(a + b\frac{(z^2+1)}{2z}\right)} \frac{dz}{iz} = \frac{\text{Re}}{2} \int_C \frac{(1 - z^2)\, 2\cancel{z}}{(2az + bz^2 + b)} \frac{dz}{i\cancel{z}}$$

$$= \frac{\text{Re}}{2} \cdot \frac{2}{i} \int_C \frac{(1 - z^2)}{(bz^2 + 2az + b)} dz \quad \ldots(2)$$

It is not analytic at

$$bz^2 + 2az + b = 0 \Rightarrow z = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b}$$

$$z = \frac{-a \pm \sqrt{a^2 - b^2}}{b} = \frac{-a}{b} \pm \sqrt{\frac{a^2}{b^2} - 1} \ \text{Let, } \alpha$$

$$= \frac{-a + \sqrt{a^2 - b^2}}{b}, \ \beta = \frac{-a - \sqrt{a^2 - b^2}}{b}$$

$$z = -\frac{a}{b} + \sqrt{\frac{a^2}{b^2} - 1} = \alpha \text{ lies inside c and } \frac{1}{\alpha} = \beta$$

$\therefore$ Residue at $z = \alpha$

$$= \lim_{z \to \alpha} \cancel{(z - \alpha)} \frac{(1 - z^2)}{b\cancel{(z - \alpha)}(z - \beta)} = \frac{(1 - \alpha^2)}{b(\alpha - \beta)}$$

$$= \frac{\alpha\left(\frac{1}{\alpha} - \alpha\right)}{b(\alpha - \beta)} = \frac{\alpha(\beta - \alpha)}{b(\alpha - \beta)} = -\frac{\alpha}{b} = \frac{a - \sqrt{a^2 - b^2}}{b^2}$$

$\therefore$ From (2)

$$= \frac{\text{Re}}{\cancel{2}} \frac{\cancel{2}}{\cancel{i}} 2\pi \cancel{i} \left(\frac{a - \sqrt{a^2 - b^2}}{b^2} \right)$$

$$= \text{Re}\, \frac{2\pi}{b^2} \left(a - \sqrt{a^2 - b^2}\right) = \frac{2\pi}{b^2} \left(a - \sqrt{a^2 - b^2}\right)$$

Q. 20 **Evaluate** $\displaystyle\int_C \frac{e^{2z}}{(z+1)^4} dz$ **where C : |z| = 2**

May 2016

Ans. :

$$\int \frac{e^{2z}}{(z+1)^4} dz, \ C : |z| = 2$$

It has pole at $z = -1$ which lies inside $C : |z| = 2$

$$\oint_C \frac{e^{2z}}{(z+1)^4}\, dz = \frac{2\pi i}{3!}\left[\frac{d^3}{dz^3} e^{2z}\right]_{z=-1} = \frac{2\pi i}{6}\,[2.2.2e^{2z}]_{z=-1} = \frac{4}{3}\pi i\, e^{-2}$$

Q. 21 **Obtain Taylor's and two distinct Laurent's expansion of $f(z) = \dfrac{z-1}{z^2-2z-3}$ indicating the region of convergence.** May 2016

Ans. :

$$f(z) = \frac{z-1}{z^2-2z-3} = \frac{z-1}{z^2-3z+z-3}$$

$$= \frac{z-1}{z(z-3)+1(z-3)} = \frac{z-1}{(z+1)(z-3)} = \frac{A}{(z+1)} + \frac{B}{(z-3)}$$

$$z-1 = A(z-3) + B(z+1)$$

Put $z - 3 = 0 \Rightarrow z = 3$

$$3-1 = B(3+1) \Rightarrow B = \frac{2}{4} = \frac{1}{2}$$

Put $z + 1 = 0 \Rightarrow z = -1$

$$-1-1 = A(-1-3) + B(0)$$

$$-2 = A(-4) \Rightarrow A = \frac{1}{2}$$

$$\therefore\ f(z) = \frac{1}{2(z+1)} + \frac{1}{2(z-3)} \qquad \ldots(1)$$

It has singularity at $z = -1$, $z = 3$

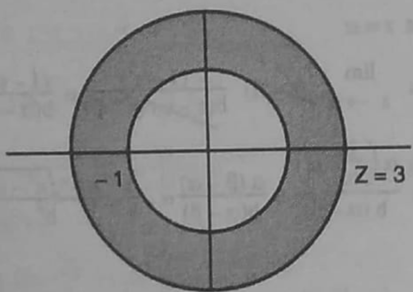

Fig. 6.4

Region of convergence are

(i) $|z| < 1$ (ii) $1 < |z| < 3$

(iii) $|z| > 3$

In first region of convergence $|z| < 1$ and $|z| < 3 \Rightarrow \dfrac{|z|}{3} < 1$

$$\therefore\ f(z) = \frac{1}{2(z+1)} + \frac{1}{2(z-3)}$$

$$= \frac{1}{2(1+z)} - \frac{1}{6\left(1-\frac{z}{3}\right)} = \frac{1}{2}(1+z)^{-1} - \frac{1}{6}\left(1-\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2}[1 - z + z^2 - z^3 ----] - \frac{1}{6}\left[1 + \frac{z}{3} + \frac{z^2}{3^2} + \frac{z^3}{3^3} ----\right]$$

In second region of convergence $1 < |z| < 3$

$$\Rightarrow \frac{1}{|z|} < 1 \quad \text{and} \quad \frac{|z|}{3} < 1$$

$$f(z) = \frac{1}{2z\left(1+\frac{1}{z}\right)} - \frac{1}{6\left(1-\frac{z}{3}\right)}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1-\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} ----\right) - \frac{1}{6}\left(1 + \frac{z}{3} + \frac{z^2}{3^2} + \frac{z^3}{3^3} ----\right)$$

In third region of convergence $|z| > 3 \Rightarrow |z| > 1$

$$\Rightarrow \frac{3}{|z|} < 1 \text{ and } \frac{1}{|z|} < 1$$

$$f(z) = \frac{1}{2z\left(1+\frac{1}{z}\right)} + \frac{1}{2z\left(1-\frac{3}{z}\right)}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} + \frac{1}{2z}\left(1-\frac{3}{z}\right)^{-1}$$

$$= \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} ----\right) + \frac{1}{2z}\left(1 + \frac{3}{z} + \frac{3^2}{z^2} + \frac{3^3}{z^3} ---\right)$$

Q. 22 **Using Cauchy Residue Theorem. evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\, dx$** May 2016

Ans. :

$$\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\, dx = \int_C \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}\, dz$$

It is not analytic at $z^4 + 10z^2 + 9 = 0$

$$z^4 + 9z^2 + z^2 + 9 = 0; \qquad z^2(z^2+9) + 1(z^2+9) = 0$$

$$(z^2+1)(z^2+9) = 0 \Rightarrow z = \pm i, \pm 3i$$

$z = i, 3i$ lies in upper half plane

Residue at $z = i$

$$= \lim_{z \to i} \cancel{(z-i)} \frac{z^2 - z + 2}{(z+i)\cancel{(z-i)}(z^2+9)}$$

$$= \frac{i^2 - i + 2}{2i(i^2+9)} = \frac{-1-i+2}{2i(-1+9)} = \frac{1-i}{2i(8)} = \frac{1-i}{16i}$$

Residue at $z = 3i$

$$= \lim_{z \to 3i} \cancel{(z-3i)} \frac{z^2 - z + 2}{(z^2+1)\cancel{(z-3i)}(z+3i)}$$

$$= \frac{-9-3i+2}{(-9+1)(3i+3i)} = \frac{-7-3i}{(-8)(6i)} = \frac{-7-3i}{-48i}$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,dx \approx \int_C \frac{z^2-z+2}{z^4+10z^2+9}\,dz = 2\pi \cancel{i}\left(\frac{1-i}{16\cancel{i}}+\frac{-7-3i}{-48\cancel{i}}\right)$$

$$=2\pi\left(\frac{3-3i+7+3i}{48}\right)=2\pi\left(\frac{10}{48}\right)=\frac{5\pi}{12}$$

Q. 23 **By using Cauchy Residue Theorem evaluate** $\displaystyle\int_0^{2\pi} \frac{\cos^2\theta}{5+4\cos\theta}\,d\theta$

Ans. : May 2016

$$\int_0^{2\pi} \frac{\cos^2\theta}{5+4\cos\theta}\,d\theta = \int_0^{2\pi} \frac{(1+\cos 2\theta)}{2(5+4\cos\theta)}\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{\text{Re}\,(1+e^{i2\theta})}{(5+4\cos\theta)}\,d\theta$$

$$= \frac{\text{Re}}{2}\int_0^{2\pi} \frac{(1+e^{i2\theta})}{(5+4\cos\theta)}\,d\theta$$

Let $C : |z| = 1$

$$z=e^{i\theta} \Rightarrow \cos\theta = \frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z},\ d\theta=\frac{dz}{iz}$$

$$= \frac{\text{Re}}{2}\int_C \frac{(1+z^2)}{\left(5+4\frac{z^2+1}{2z}\right)}\frac{dz}{iz}$$

$$= \frac{\text{Re}}{2}\int_C \frac{1+z^2}{\frac{(5z+2z^2+2)}{z}}\frac{dz}{iz}$$

$$= \text{Re}\,\frac{1}{2i}\int_C \frac{z^2+1}{2z^2+5z+2}\,dz \qquad \ldots(1)$$

It is not analytic at

$$2z^2+5z+2 = 0$$

$$2z^2+4z+z+2 = 0 \Rightarrow 2z(z+2)+1(z+2)=0$$

$$\Rightarrow (2z+1)(z+2) = 0 \Rightarrow z=\frac{-1}{2},\ z=-2$$

$z = \frac{-1}{2}$ lies inside the residue at z

$$=\frac{-1}{2} = \lim_{z\to\frac{-1}{2}} \left(z+\frac{1}{2}\right)\frac{z^2+1}{(2z+1)(z+2)}$$

$$= \lim_{z\to\frac{-1}{2}} \frac{\cancel{(2z+1)}}{2}\,\frac{z^2+1}{\cancel{(2z+1)}(z+2)}$$

$$= \frac{\left(\frac{-1}{2}\right)^2+1}{2\left(2-\frac{1}{2}\right)} = \frac{1+\frac{1}{4}}{2\left(\frac{3}{2}\right)} = \frac{\frac{5}{4}}{3} = \frac{5}{12}$$

$\therefore$ From Equation (1)

$$= \text{Re}\,\frac{1}{\cancel{2}\cancel{i}}\,\cancel{2}\pi\cancel{i}\left(\frac{5}{12}\right)=\frac{5\pi}{12}$$

Q. 24 **Evaluate** $\displaystyle\int_C \frac{z+4}{z^2+2z+5}\,dz$ **where C : |z + 1+ i| = 2.** May 2016

Ans. :

$$\int_C \frac{z+4}{z^2+2z+5}\,dz \qquad C : |z+1+i|=2$$

It is not analytic at $z^2+2z+5=0$

$$z = \frac{-2\pm\sqrt{4-20}}{2} = \frac{-2\pm\sqrt{-16}}{2} = \frac{-2\pm 4i}{2} = -1\pm 2i$$

$z=-1+2i \Rightarrow |-1+2i+1+i| = |3i| = 3 > 2$ It is outside

$z=-1-2i \Rightarrow |-1-2i+1+i| = |-i| = 1 < 2$ It is inside

Residue at $z=-1-2i = \lim_{z\to -1-2i}$

$$\cancel{(z+1+2i)}\frac{(z+4)}{(z+\cancel{1}+2i)(z+1-2i)}$$

$$= \frac{-1-2i+4}{(-1-2i+1-2i)} = \frac{3-2i}{(-4i)} = \left(\frac{-3+2i}{4i}\right)$$

$$\therefore \int_C \frac{z+4}{z^2+2z+5}\,dz = \cancel{2}\pi\cancel{i}\left(\frac{-3+2i}{\cancel{4i}}\right)=\frac{\pi}{2}(-3+2i)$$

Q. 25 **Evaluate** $\displaystyle\int_C \frac{z}{(z-1)^2(z-2)}\,dz$ **, where C is the circle | z – 2 | = 2.5.** Dec. 2016

Ans. : $\displaystyle\int_C \frac{z}{(z-1)^2(z-2)}\,dz \quad : C\,|z-2|=2.5$

It in not analytical at z = 1, z = 2

$z=1 \Rightarrow |1-2| = 1 < 2.5$ It is inside and pole of order 2

$z=2 \Rightarrow |2-2| = 0 < 2.5$ It is inside and pole of order 1

residue at z = 2 $\Rightarrow \displaystyle\lim_{z\to 2} \cancel{(z-2)}\frac{z}{(z-1)^2\cancel{(z-2)}}$

$$= \frac{2}{(2-1)^2} = 2$$

Residue at z = 1 $\Rightarrow \dfrac{1}{(2-1)!} \lim\limits_{z \to 1} \left[\dfrac{d}{dz} (z \cancel{-} 1) \dfrac{z}{(z \cancel{-} 1)^2 (z-2)} \right]$

$$= \lim_{z \to 1} \left[\frac{(z-2)-z}{(z-2)^2} \right]$$

$$= \lim_{z \to 1} \left[\frac{-2}{(z-2)^2} \right] = \frac{-2}{(1-2)^2} = -2$$

∴ By Cauchy's residue theorem

$$\int_C \frac{z}{(z-1)^2 (z-2)} dz = 2\pi i (2-2) = 0$$

Q. 26 **Evaluate** $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5 + 4\cos\theta} d\theta$

Dec. 2016

Ans. :

$$\int_0^{2\pi} \frac{\cos 3\theta}{5+4\cos\theta} d\theta = \int_0^{2\pi} \frac{\text{Re } e^{i3\theta}}{5+4\cos\theta} d\theta$$

Let C : | z | = 1 $\quad \therefore \; z = e^{i\theta}, \; \frac{1}{z} = e^{-i\theta}$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2} = \frac{z^2+1}{2z}$$

$$dz = e^{i\theta} \cdot i d\theta \Rightarrow d\theta = \frac{dz}{iz} = \text{Re} \int_C \frac{z^3}{5 + 4\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$$

$$= \text{Re} \int_C \frac{z^3}{5 + 2\frac{(z^2+1)}{z}} \frac{dz}{iz}$$

$$= \text{Re} \int_C \frac{z^3}{\frac{5z + 2(z^2+1)}{z}} \frac{dz}{iz} = R_e \frac{1}{i} \int_C \frac{z^3}{2z^2 + 5z + 2} dz$$

$$= \text{Re} \frac{1}{i} \int_C \frac{z^3}{2z^2 + 4z + z + 2} dz$$

$$= \text{Re} \frac{1}{i} \int_C \frac{z^3}{2z(z+2) + 1(z+2)} dz$$

$$= \text{Re} \frac{1}{i} \int_C \frac{z^3}{(2z+1)(z+2)} dz$$

It is not analytic at $z = \frac{-1}{2}, -2$

$z = \frac{-1}{2}$ lies inside C : | z | = 1

Residue at z $= \frac{-1}{2}$

$$= \lim_{z \to -1/2} \left(z + \frac{1}{2}\right) \frac{z^3}{(2z+1)(z+2)}$$

$$= \lim_{z \to -1/2} \frac{(2z \cancel{+} 1)}{2} \frac{z^3}{(2z \cancel{+} 1)(z+2)}$$

$$= \frac{\left(-\frac{1}{2}\right)^3}{2\left(-\frac{1}{2}+2\right)} = \frac{-\frac{1}{8}}{\cancel{2}\left(\frac{3}{\cancel{2}}\right)} = -\frac{1}{24}$$

$$\therefore \text{ Re} \frac{1}{\cancel{i}} 2\pi \cancel{i} \left(-\frac{1}{24}\right)$$

$$\Rightarrow \text{Re} \left(-\frac{\pi}{12}\right) \Rightarrow -\frac{\pi}{12}$$

Q. 27 **Find all possible Laurent's expansions of** $\dfrac{z}{(z-1)(z-2)}$ **about z = – 2 indicating the region of convergence.**

Dec. 2016

Ans. : Given function is

$$f(z) = \frac{z}{(z-1)(z-2)} = \frac{2(z-1)-(z-2)}{(z-1)(z-2)}$$

$$f(z) = \frac{2}{(z-2)} - \frac{1}{(z-1)}$$

(i) $|z+2| < 3$ (ii) $3 < |z+2| < 4$ (iii) $|z+2| > 4$

(i) $|z+2| < 3 \Rightarrow \frac{|z+2|}{3} < 1, \quad |z+2| < 4 \Rightarrow \frac{(z+2)}{4} < 1$

$$f(z) = \frac{2}{(z+2)-4} - \frac{1}{(z+2)-3}$$

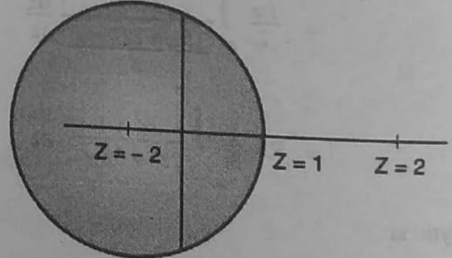

Fig. 6.5

$$f(z) = \frac{2}{-4\left[1 - \frac{(z+2)}{4}\right]} - \frac{1}{-3\left[1 - \frac{(z+2)}{3}\right]}$$

$$= -\frac{1}{2}\left[1 - \frac{(z+2)}{4}\right]^{-1} + \frac{1}{3}\left[1 - \frac{(z+2)}{3}\right]^{-1}$$

$$= -\frac{1}{2}\left[1 + \frac{(z+2)}{4} + \frac{(z+2)^2}{4^2} + \frac{(z+2)^3}{4^3} \ldots\ldots\right] + \frac{1}{3}\left[1 + \frac{(z+2)}{3} + \frac{(z+2)^2}{3^2} \ldots\right]$$

(ii) $3 < |z+2| < 4$

$$\frac{3}{|z+2|} < 1 \quad \text{and} \quad \frac{|z+2|}{4} < 1$$

$$f(z) = \frac{2}{-4\left[1-\frac{(z+2)}{4}\right]} - \frac{1}{(z+2)\left[1-\frac{3}{(z+2)}\right]}$$

$$= -\frac{2}{4}\left[1-\frac{(z+2)}{4}\right]^{-1} - \frac{1}{(z+2)}\left[1-\frac{3}{(z+2)}\right]^{-1}$$

$$= -\frac{1}{2}\left[1-\frac{(z+2)}{4}\right]^{-1} - \frac{1}{(z+2)}\left[1-\frac{3}{(z+2)}\right]^{-1}$$

$$= -\frac{1}{2}\left[1+\frac{(z+2)}{4}+\frac{(z+2)^2}{4^2}+\ldots\right] - \frac{1}{(z+2)}\left[1+\frac{3}{(z+2)}+\frac{3^2}{(z+2)^2}\ldots\right]$$

(iii) $f(z) = \frac{2}{(z+2)-4} - \frac{1}{(z+2)-3}$,

$$|z+2| > 4 \Rightarrow \frac{4}{|z+2|} < 1$$

$$\Rightarrow |z+2| > 3 \Rightarrow \frac{3}{|z+2|} < 1$$

$$= \frac{2}{(z+2)\left[1-\frac{4}{(z+2)}\right]} - \frac{1}{(z+2)\left[1-\frac{3}{(z+2)}\right]}$$

$$= \frac{2}{(z+2)}\left[1-\frac{4}{(z+2)}\right]^{-1} - \frac{1}{(z+2)}\left[1-\frac{3}{(z+2)}\right]^{-1}$$

$$= \frac{2}{(z+2)}\left[1+\frac{4}{(z+2)}+\frac{4^2}{(z+2)^2}+\frac{4^3}{(z+2)^3}\ldots\right] - \frac{1}{(z+2)}\left[1+\frac{3}{(z+2)}+\frac{3^2}{(z+2)^2}\ldots\right]$$

Q. 28 **If $\phi(\alpha) = \int_C \frac{4z^2+z+5}{z-\alpha}\,dz$, where C is the contour of the ellipse $\frac{x^2}{4}+\frac{y^2}{9} = 1$, find the values of $\phi(3.5)$, $\phi(i)$, $\phi'(-1)$, $\phi''(-i)$** **Dec. 2016**

Ans. :

$$\phi(\alpha) = \int_C \frac{4z^2+z+5}{(z-\alpha)}\,dz \text{ where C is the contour}$$

$$\frac{x^2}{4}+\frac{y^2}{9} = 1$$

$$\phi(3.5) = \int_C \frac{4z^2+z+5}{(z-3.5)}\,dz \text{ It is not analytic at}$$

$z = 3.5$ and it lies inside the contour $\frac{x^2}{4}+\frac{y^2}{9} = 1$.

…ormula

$$\phi(3.5) = \int_C \frac{4z^2+z+5}{(z-3.5)} = 2\pi i\,(4z^2+z+5)_{z=3.5}$$

$$= 2\pi i\left(4\times\frac{7^2}{2}+\frac{7}{2}+5\right) = 2\pi i\left(49+5+\frac{7}{2}\right)$$

$$= 2\pi i\left(54+\frac{7}{2}\right) = 2\pi i\left(\frac{115}{2}\right) = 115\pi i$$

$$\phi(i) = \int_C \frac{4z^2+z+5}{(z-i)}\,dz$$

$z = i$ also lies inside the contour therefore by Cauchy integral formula.

$$= \int_C \frac{4z^2+z+5}{(z-i)}\,dz = 2\pi i\,(4z^2+z+5)_{z=i}$$

$$= 2\pi i\,(4\cdot i^2+i+5) = 2\pi i\,(i+1)$$

$$= 2\pi\,(i^2+i) = 2\pi\,(i-1)$$

$$\phi'(\alpha) = \int_C \frac{4z^2+z+5}{(z-\alpha)^2}\,dz$$

$$\therefore\ \phi'(-1) = \int_C \frac{4z^2+z+5}{(z+1)^2}\,Dz$$

It is not analytic at $z = -1$ which lies inside the counter and it is pole of order 2.

Residue at $z = -1$

$$= \frac{1}{(2-1)!}\lim_{z\to -1}\left[\frac{d}{dz}\cancel{(z+1)^2}\frac{(4z^2+z+5)}{\cancel{(z+1)^2}}\right]$$

$$= \lim_{z\to -1}[8z+1] = -8+1 = -7$$

$$\phi'(-1) = \int_C \frac{4z^2+z+5}{(z+1)^2} = 2\pi i\,(-7) = -14\pi i$$

$$\phi''(\alpha) = 2\int_C \frac{4z^2+z+5}{(z-\alpha)^3}\,Dz$$

$$\phi''(-i) = 2\int_C \frac{4z^2+z+5}{(z+i)^3}\,dz$$ It is not analytic at $z = -i$ which lies inside the contour and it is a pole of order 3

Residue at $z = -i$

$$= \frac{1}{(3-1)!}\lim_{z\to -i}\left[\frac{d^2}{dz^2}\cancel{(z+i)^3}\frac{(4z^2+z+5)}{\cancel{(z+i)^3}}\right]$$

$$= \frac{1}{2!}\lim_{z\to -i}\left[\frac{d}{dz}(8z+1)\right] = \frac{1}{2}\lim_{z\to -i}(8) = 4$$

$\therefore \quad \phi''(-1) = 2 \int_C \frac{4z^2+z+5}{(z-1)^3} dz = 2 \times 2\pi i (4) = 16\pi i$

Q. 29 **Using Cauchy's Residue theorem, evaluate $\oint_C \frac{z^2+3}{z^2-1} dz$ where C is the circle**

(i) $|z-1|=1$ (ii) $|z+1|=1$ **Dec. 2016**

Ans. :

$$\oint_c \frac{z^2+3}{z^2-1} dz = \oint_c \frac{z^2+3}{(z+1)(z-1)} dz$$

It is not analytic at z = 1, −1

(i) $|z-1|=1$, $z=1 \Rightarrow |1-1| = 0 < 1$ it is inside and pole of order 1

$z=-1 \Rightarrow |-1-1| = 2 > 1$ It is outside

Residue at z = 1

$$= \lim_{z\to 2} \cancel{(z-1)} \frac{(z^2+3)}{\cancel{(z-1)}(z+1)} = \frac{1^2+3}{(1+1)} = \frac{4}{2} = 2$$

$$\therefore \oint_c \frac{z^2+3}{(z-1)(z-1)} dz = 2\pi i (2) = 4\pi i$$

(ii) $C : |z+1| = 1$

$z=-1$, $|-1+1| = |0| = 0 < 1$. It is inside and pole of order 1

Residue at z = −1

$$= \lim_{z\to -1} \cancel{(z+1)} \frac{z^2+3}{(z-1)\cancel{(z+1)}}$$

$$= \frac{(-1)^2+3}{(-1-1)} = \frac{4}{-2} = -2$$

$\therefore$ By Cauchy's residue theorem

$$\oint_c \frac{z^2+3}{(z+1)(z-1)} dz = 2\pi i(-2) = -4\pi i$$

Q. 30 **Find the sum of the residues at singular points of $f(z) = \frac{z}{(z-1)^2(z^2-1)}$** **Dec. 2016**

Ans. :

$$f(z) = \frac{z}{(z-1)^2(z^2-1)} = \frac{z}{(z-1)^2(z+1)(z-1)}$$

$$= \frac{z}{(z-1)^3(z+1)}$$

It has singular point at $z = 1$, $z = -1$

$z = 1$ is a pole of order 3 $\therefore$ $z = -1$ is a simple pole

$$\text{Residue at } z = -1 = \lim_{z\to -1} \cancel{(z+1)} \frac{z}{(z-1)^3\cancel{(z+1)}}$$

$$= \frac{-1}{(-1-1)^3} = \frac{-1}{-8} = \frac{1}{8}$$

$$\text{Residue at } z = 1 = \frac{1}{(3-1)!} \lim_{z\to 1}\left[\frac{d^2}{dz^2}\cancel{(z-1)^3}\frac{z}{\cancel{(z-1)^3}(z-1)}\right]$$

$$= \frac{1}{2!} \lim_{z\to 1}\left[\frac{d}{dz}\left\{\frac{(z+1)-z}{(z+1)^2}\right\}\right]$$

$$= \frac{1}{2} \lim_{z\to 1}\left[\frac{d}{dz}\frac{1}{(z+1)^2}\right] = \frac{1}{2} \lim_{z\to 1}\left[\frac{-2}{(z+1)^3}\right]$$

$$= \frac{-1}{(1+1)^3} = \frac{-1}{8}$$

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