Q1

a) To check if a set of waveforms are Orthonormal, we need to show that:

$$\int_0^{T_s} \phi_i(t)\,\phi_j(t)\,dt = \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases}$$

In this case,

$$\int_0^{T_s} \phi_1^2(t)\,dt = 1 \qquad \int_0^{T_s} \phi_1(t)\,\phi_2(t)\,dt = 0$$

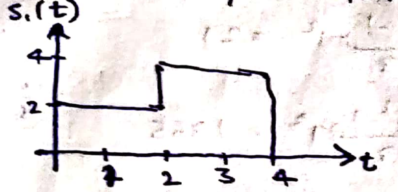
$$\int_0^{T_s} \phi_2^2(t)\,dt = 1 \qquad \int_0^{T_s} \phi_1(t)\,\phi_3(t)\,dt = 0$$

$$\int_0^{T_s} \phi_3^2(t)\,dt = 1 \qquad \int_0^{T_s} \phi_2(t)\,\phi_3(t)\,dt = 0$$

Hence $\{\phi_i(t)\}$ are orthonormal

b) $s_1(t) = \begin{cases} 2, & 0 \le t < 2 \\ 4, & 2 \le t < 4 \end{cases}$

Let's first plot $s_1(t)$:



To express a signal as a linear combination of $\{\phi_i(t)\}$, we need to find the projection of $s_1(t)$ on each basis function.

$$s_{11} = \int_0^{T_s} s_1(t)\,\phi_1(t)\,dt$$

$$= \int_0^2 (2)(0.5)\,dt + \int_2^4 (4)(-0.5)\,dt$$

$$= -2$$

$$s_{12} = \int_0^{T_s} s_1(t)\,\phi_2(t)\,dt$$

$$= \int_0^2 (2)(0.5)\,dt + \int_2^4 (4)(0.5)\,dt$$

$$= 6$$

$$s_{13} = \int_0^{T_s} s_1(t)\,\phi_3(t)\,dt$$

$$= \int_0^1 (2)(0.5)\,dt + \int_1^2 (2)(-0.5)\,dt + \int_2^3 (4)(0.5)\,dt + \int_3^4 (4)(-0.5)\,dt$$

$$= 0$$

$\therefore\ s_1(t) = -2\,\phi_1(t) + 6\,\phi_2(t) + 0\,\phi_3(t)$

or $s_1 = (-2, 6, 0)$

Q2

$$\phi_1(t) = \sqrt{\frac{2}{T_s}}\cos(2\pi f_c t), \quad 0 \le t < T_s$$

$$\phi_2(t) = \sqrt{\frac{2}{T_s}}\sin(2\pi f_c t), \quad 0 \le t < T_s$$

To check if the waveforms are orthonormal.

$$I_1 = \int_0^{T_s} \phi_1^2(t)\,dt$$

$$= \int_0^{T_s} \left(\sqrt{\frac{2}{T_s}}\cos(2\pi f_c t)\right)^2 dt$$

$$= \frac{2}{T_s}\int_0^{T_s} \cos^2(2\pi f_c t)\,dt$$

$$= \frac{2}{T_s}\int_0^{T_s} \frac{1}{2}\left(1 + \cos 2(2\pi f_c t)\right) dt$$

$$= \frac{2}{T_s} \times \frac{1}{2}\left[\int_0^{T_s} 1\,dt + \int_0^{T_s} \cos 2(2\pi f_c t)\,dt\right]$$

$$= \frac{1}{T_s}\left[[t]_0^{T_s} + \left[\frac{\sin(4\pi f_c t)}{4\pi f_c}\right]_0^{T_s}\right]$$

$$= \frac{1}{T_s}\left[(T_s - 0) + \left[\frac{\sin(4\pi f_c T_s)}{4\pi f_c}\right]\right]$$

$$= \frac{T_s}{T_s} + \frac{\sin 4\pi f_c T_s}{4\pi f_c T_s}$$

$$= 1 + \frac{\sin 4\pi f_c T_s}{4\pi f_c T_s}$$

$$\approx 1$$

Note: $f_c T_s >> 1$

$$I_2 = \int_0^{T_s} \phi_2^2(t)\,dt$$

$$= \frac{2}{T_s}\int_0^{T_s} \sin^2(2\pi f_c t)\,dt$$

$$= \frac{2}{T_s}\int_0^{T_s} \frac{1}{2}\left(1 - \cos 2(2\pi f_c t)\right) dt$$

$$= \frac{2}{T_s} \times \frac{1}{2}\left[\int_0^{T_s} 1\,dt - \int_0^{T_s} \cos(4\pi f_c t)\,dt\right]$$

$$= \frac{1}{T_s}\left[[t]_0^{T_s} - \left[\frac{\sin(4\pi f_c t)}{4\pi f_c}\right]_0^{T_s}\right]$$

$$= \frac{1}{T_s}\left([T_s - 0] - \frac{\sin 4\pi f_c T_s}{4\pi f_c}\right)$$

$$= \frac{T_s}{T_s} - \frac{\sin 4\pi f_c T_s}{4\pi f_c T_s} = 1 - \frac{\sin 4\pi f_c T_s}{4\pi f_c T_s}$$

≈ 1 Note: $f_c T_s >> 1$

$I_3 = \int_0^{T_s} \phi_1(t)\,\phi_2(t)\,dt$

$= \frac{2}{T_s}\int_0^{T_s}\cos(2\pi f_c t)\sin(2\pi f_c t)\,dt$

$= \frac{2}{T_s}\int_0^{T_s}\frac{1}{2}\sin(2\pi f_c t + 2\pi f_c t) + \sin(2\pi f_c t - 2\pi f_c t)\,dt$

$= \frac{2}{T_s}\times\frac{1}{2}\int_0^{T_s}\sin(4\pi f_c t) + \sin(0)\,dt$

$= \frac{1}{T_s}\int_0^{T_s}\sin(4\pi f_c t)\,dt$

$= \frac{1}{T_s}\left[-\frac{\cos(4\pi f_c t)}{4\pi f_c}\right]_0^{T_s}$

$= -\frac{\cos(4\pi f_c T_s)}{4\pi f_c T_s}$

≈ 0 Note $f_c T_s >> 1$

Hence $\phi_1(t)$ and $\phi_2(t)$ are orthonormal

Q4

For 1st constellation:

closest symbols: (1, 0) & (0, 1)

Minimum distance: $d_{min} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$d_{min} = \sqrt{(0-1)^2+(1-0)^2} = \sqrt{1^2+1^2} = \sqrt{2}$

Average signal energy

$E_s = \frac{1}{8}(4\times1^2 + 4\times3^2) = 5$

For 2nd constellation:

closest symbols: (1, 0.5) & (1, -0.5)

$d_{min} = \sqrt{(1-1)^2+(-0.5-0.5)^2} = \sqrt{0^2+1^2} = 1$

$E_s = \frac{1}{8}(4\times(1^2+0.5^2) + 4\times(3^2+1.5^2))$

$= 6.25$

For 3rd constellation

Closest symbol: (2, 0) ($\sqrt{2}$, $\sqrt{2}$)

$d_{min} = \sqrt{(\sqrt{2}-2)^2+(\sqrt{2}-0)^2} = 1.53$

$E_s = \frac{1}{8}(8\times2^2) = 4$

For 4th constellation

closest symbol: (1, 1) & (1, -1)

$d_{min} = \sqrt{(1-1)^2+(-1-1)^2} = \sqrt{0^2+2^2} = 2$

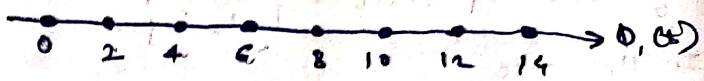
$E_s = \frac{1}{8}(4\times(1^2+1^2) + 4\times3^2) = 5.5$

Q5

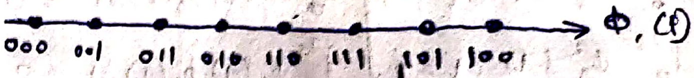
a) 8-ASK $d_{min} = 2d$ $\therefore 2d = 2$

$d_{min} = 2$ $d = 1$

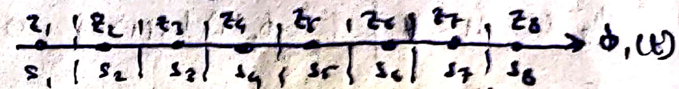
Signal space representation



b) Gray code



c) Decision region



d) To keep the same bit error probability

$d_{min} = 2$ $\therefore d = 1$

$E_s[M\text{-ASK}] = \frac{M^2-1}{3}d^2$

$E_s(8\text{-ASK}) = \frac{8^2-1}{3}\times1^2 = 21$

$E_s(64\text{-ASK}) = \frac{64^2-1}{3}\times1^2 = 1365$

$\frac{(1365-21)}{21} = 64 = 10\log(64) = 18\text{ dB}$

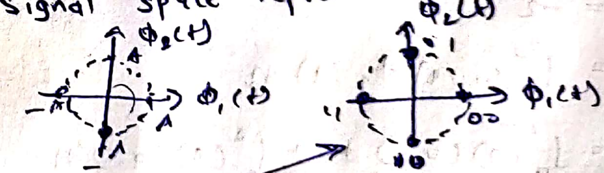
Q6

a) 4-PSK $d_{min} = 2A\sin\left(\frac{\pi}{4}\right)$

$d_{min} = \sqrt{3}$

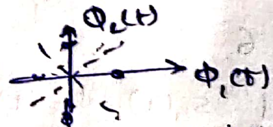
$d_{min} = 2A\sin\left(\frac{\pi}{4}\right) = \sqrt{3}$ $\therefore A = 1.2247$

Signal space representation



b) Gray code

c) Decision region



d) To keep the same bit error probability

$d_{min} = \sqrt{3}$ $\therefore$

$E_s[M\text{-PSK}] = A^2 = \left(\frac{d_{min}}{2\sin(\pi/M)}\right)^2$

$E_s[4\text{-PSK}] = \left(\frac{d_{min}}{2\sin(\pi/4)}\right)^2 = \left(\frac{\sqrt{3}}{2\sin(\frac{\pi}{4})}\right)^2 = 1.5$

$E_s[16\text{-PSK}] = \left(\frac{d_{min}}{2\sin(\pi/16)}\right)^2 = \left(\frac{\sqrt{3}}{2\sin(\frac{\pi}{16})}\right)^2 = 19.71$

$\left(\frac{19.71-1.5}{1.5}\right) = 12.14 = 10\log(12.14) = 10.84\text{ dB}$

Q9

For m-PAM communication system, the bit error probability is

$$P_b \approx \frac{P_s}{\log_2 M} = \frac{2}{\log_2 M}\left(\frac{M-1}{M}\right) Q\left(\sqrt{\frac{6E_s}{(M^2-1)N_0}}\right)$$

Given: $M=8$ & $P_b = 10^{-6}$ SNR/bit $\frac{E_b}{N_0}$?

$$\frac{2}{\log_2 8}\left(\frac{8-1}{8}\right) Q\left(\sqrt{\frac{6E_s}{(8^2-1)N_0}}\right) = 10^{-6}$$

$$\Rightarrow \frac{7}{12} Q\left(\sqrt{\frac{6E_s}{(63)N_0}}\right) = 10^{-6}$$

$$\Rightarrow Q\left(\sqrt{\frac{6E_s}{63N_0}}\right) = \frac{12}{7}\times 10^{-6}$$

$$= 1.7142 \times 10^{-6}$$

Q function: $Q(x) = 0.5\exp\left(\frac{x^2}{2}\right)$

$Q(0.1) = 0.5\exp\left(\frac{0.1^2}{2}\right)$

$Q(x) = 0.2 \quad x = ?$

$0.2 = 0.5\exp\left(\frac{x^2}{2}\right)$

$$1.7142\times10^{-6} = 0.5\exp\left(\frac{x^2}{2}\right)$$

$$\frac{1.7142\times10^{-6}}{0.5} = \exp\left(\frac{x^2}{2}\right)$$

$$3.4284\times10^{-6} = \exp\left(\frac{x^2}{2}\right)$$

$$\ln(3.4284\times10^{-6}) = \frac{x^2}{2}$$

$$-12.58 = \frac{x^2}{2}$$

$$2(-12.58) = x^2$$

$$-25.16 = x^2$$

$$\sqrt{25.16} = x$$

$$x = 5.02$$

Now

$$Q(x) = Q(5.02) = 1.7142\times10^{-6}$$

$$Q\left(\sqrt{\frac{6E_s}{63N_0}}\right) = Q(5.02)$$

$$\frac{6E_s}{63N_0} = 5.02^2 \Rightarrow$$

$$\frac{E_s}{N_0} = 5.02^2 \times \frac{63}{6}$$

$$= 264.6$$

$$\frac{E_s}{N_0} = 264.6$$

$$\frac{E_b}{N_0} = \frac{264.6}{3 \leftarrow \log_2(M)}$$

$= 88.2$ per bit

dB $= 10\log(88.2) = 19.45$ dB

Q.10

Bandwidth of bandpass channel

$W = 3000 - 600 = 2400$ Hz

QPSK system $\rightarrow$ Quad $:4$ $M=4$

$R_b = 2400$ bits/sec

$$W = (1+\alpha)\frac{R_b}{\log_2 M} = (1+\alpha)\frac{2400}{2}$$

$$2400 = (1+\alpha)\frac{2400}{2}$$

$$\frac{2400\times2}{2400} - 1 = \alpha$$

$$\therefore \alpha = 1$$

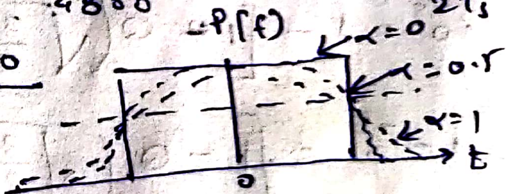
If R_b become 4800 bits/sec

$$W = (1+\alpha)\frac{4800}{2} = 2400$$

$$\alpha = \frac{2400\times2}{4800} - 1$$

$$\therefore \alpha = 0$$

$W = \frac{1+\alpha}{2T_s}$



Q12

Bandwidth of bandpass channel

$W = 3300 - 300 = 3000$ HZ

$R_b = 9600$ bits/sec

$R_s = 2400$ bits/sec

$\frac{R_b}{R_s} = \frac{9600}{2400} = 4$ bits/sec

$\therefore 2^4 = 16$ QAM

For 16 QAM system $R_b = 9600$ bits/sec

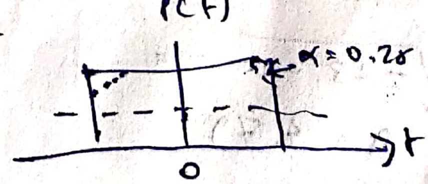
$$W = (1+\alpha)\frac{R_b}{\log_2 M}$$

$$= (1+\alpha)\frac{9600}{\log_2 16}$$

$$= (1+\alpha)\frac{9600}{4}$$

$$\frac{3000\times4}{9600} - 1 = \alpha$$

$$\alpha = 0.25$$



Q13

$P_s[M\text{-}QAM] = 1 - (1 - P_s[L\text{-}ASK])^2$

$$= 1 - \left[1 - 2\left(\frac{\sqrt{M}-1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3E_s}{(M-1)N_o}}\right)\right]^2$$

To find: $\frac{E_b}{N_o}$

① $K = \frac{4800}{2400} = 2$

$M = 2^2 = 4$

$P_s = 10^{-5}$

$$10^{-5} = 1 - \left[1 - 2\left(\frac{\sqrt{4}-1}{\sqrt{4}}\right) Q\left(\sqrt{\frac{3E_s}{(4-1)N_o}}\right)\right]^2$$

$$10^{-5} = 1 - \left[1 - 2\left(\frac{1}{2}\right) Q\left(\sqrt{\frac{3E_s}{3N_o}}\right)\right]^2$$

$$10^{-5} = 1 - \left[1 - Q\left(\sqrt{\frac{3E_s}{3N_o}}\right)\right]^2$$

$$1 - 10^{-5} = \left[1 - Q\left(\sqrt{\frac{3E_s}{3N_o}}\right)\right]^2$$

$$\sqrt{1-10^{-5}} = 1 - Q\left(\sqrt{\frac{3E_s}{3N_o}}\right)$$

$$Q\left(\sqrt{\frac{3E_s}{3N_o}}\right) = 1 - \sqrt{1-10^{-5}}$$

$$= 5.00 \times 10^{-6}$$

$$5.00\times10^{-6} = 0.5 \exp\left(\frac{x^2}{2}\right)$$

$$\frac{5.00\times10^{-6}}{0.5} = \exp\left(\frac{x^2}{2}\right)$$

$$\ln\left(1\times10^{-5}\right) = \frac{x^2}{2}$$

$$2(-11.51) = x^2$$

$$\sqrt{-23.02} = x$$

$$x = 4.79$$

Now

$Q(4.79) = 5.00\times10^{-6}$

$Q(4.79) = Q\left(\sqrt{\frac{3E_s}{3E_o}}\right)$

$4.79 = \sqrt{\frac{3E_s}{3E_o}}$

$4.79^2 = \frac{3E_s}{3E_o}$

$22.94 = \frac{3E_s}{3E_o}$

$\frac{E_s}{E_o} = \frac{22.94\times3}{3}$

22.94

$\frac{E_b}{N_o} = \frac{22.94}{\log_2 M} = \frac{22.94}{2} = 11.47$

$= 10\log(11.47)$

$= 10.60\,dB$

Q2 HOMEWORK

a)

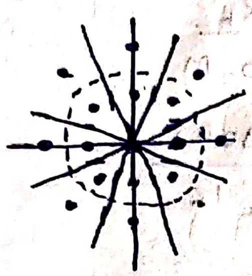
Closest symbols: (2, 0) & (3, 0)

$d_{min} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{(3-2)^2+(0-0)^2} = 1$

b) Average symbol energy

$E_s = \frac{1}{16}(8\times2^2+8\times3^2) = 3.125$

c)



Q3

QPSK 8-PSK

$d_{min} = 2A_1\sin\left(\frac{\pi}{4}\right)$ $d_{min} = 2A_2\sin\left(\frac{\pi}{8}\right)$

d_{min} is the radii

so comparing both of them

$2A_1\sin\left(\frac{\pi}{4}\right) = 2A_2\sin\left(\frac{\pi}{8}\right)$

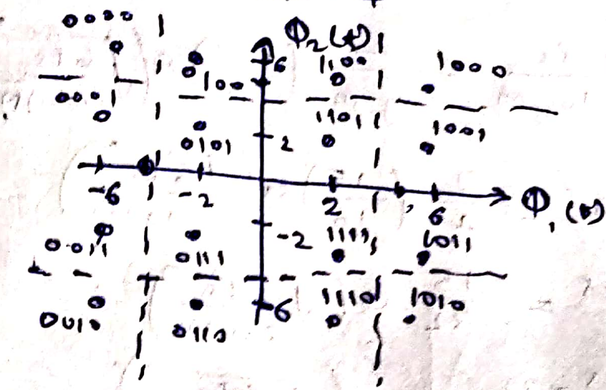
$A_2 = A_1\frac{\sin\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{8}\right)}$

$A_2 = A_1\frac{\frac{\sqrt{2}}{2}}{0.3826}$

$A_2 = A_1(1.848162)$

Q4

16 QAM $d_{min} = 4$



To keep the same bit error probability

$d_{min} = 4$ $2d = 4$ $d = 2$

$d_{min} = 2d$

$E_s = \frac{2}{3}(M-1)\,d^2$

16 QAM $= \frac{2}{3}(16-1)\,2^2$

$= \frac{2}{3}\times15\times4$

$= 40$

To increase bits per symbol by 1

$k = 5$ $\therefore 2^5 = 32 = M$

32 QAM $= \frac{2}{3}(32-1)\,2^2$

$= \frac{2}{3}\times31\times4$

$= 82.67$

$\therefore$ Additional energy per symbol

$\Rightarrow \frac{82.67-40}{40} = 1.07$

$= 10\log(1.07)$

$= 0.29\,dB$

Q5

16 PAM $M=16$ $P_b=1.1\times10^{-5}$

$$P_b \approx \frac{P_s}{\log_2 M} = \frac{2}{\log_2 M}\left(\frac{M-1}{M}\right) Q\left(\sqrt{\frac{6E_s}{(M^2-1)N_0}}\right)$$

$$1.1\times10^{-5} = \frac{2}{4}\left(\frac{15}{16}\right) Q\left(\sqrt{\frac{6E_s}{255N_0}}\right)$$

$$1.1\times10^{-5} = \frac{15}{32} Q\left(\sqrt{\frac{6E_s}{255N_0}}\right)$$

$$\frac{1.1\times10^{-5}\times32}{15} = Q\left(\sqrt{\frac{6E_s}{255N_0}}\right)$$

$$2.2\times10^{-5} = Q\left(\sqrt{\frac{6E_s}{255N_0}}\right)$$

$$2.2\times10^{-5} = 0.5\exp\left(\frac{x^2}{2}\right)$$

$$\ln\left(\frac{2.2\times10^{-5}}{0.5}\right) = \frac{x^2}{2}$$

$$2\times-10.03 = x^2$$

$$\sqrt{20.06} = x$$

$$4.47 = x$$

$$Q(4.47) = Q\left(\sqrt{\frac{6E_s}{255N_0}}\right)$$

$$4.47 = \sqrt{\frac{6E_s}{255N_0}}$$

$$\frac{4.47^2\times255}{6} = \frac{E_s}{N_0}$$

$$\frac{E_s}{N_0} = 849.19$$

$$\frac{E_b}{N_0} = \frac{849.19}{4} = 212.30 \text{ per bit}$$

$$dB = 10.\log(212.30) = \underline{23.27\,dB}$$

Q6

Bandwidth of bandpass channel

$B = 25000\,Hz$

Symbol rate $= 4800$ symbols/sec $= R_s$

$R_b = 19200$ bits/sec

$$\frac{R_b}{R_s} = \frac{19200}{4800} = 4$$

$2^4 = 16$

$M = 16$

M6 FSK

$$W = (1+\alpha)\frac{MR_b}{2\log_2 M}$$

$$25000 = (1+\alpha)\frac{16\times19200}{2\log_2 16}$$

$$\frac{25000\times8}{16\times19200} - 1 = \alpha$$

$$\underline{\alpha = -0.35}$$