Digital Communications Homework 4
Amey Mahendra Thakur 110107589

Q1

a) The entropy is

$$H(x) = -\sum_i p_i \log_2 p_i$$

$$= -(0.12 \log_2 (0.12) + 0.08 \log_2 (0.08) + 0.2 \log_2 (0.2) + 0.35 \log_2 (0.35) + 0.2 \log_2 (0.2) + 0.05 \log_2 (0.05))$$

$$= 2.33 \text{ bits}$$

b) The average codeword length is lower bounded by the entropy of the source for error free reconstruction. So, the minimum possible average codeword length is $H(x) = 2.33$ bits.

c) For a uniformly distributed source with the same alphabet, each symbol would have equal probability i.e $\frac{1}{6}$

So, the entropy of the uniformly distributed source is :

$$H(x) = -\sum_{i=1}^{6} \frac{1}{6} \log_2 \left(\frac{1}{6}\right)$$

$$= -6 \times \frac{1}{6} \log_2 \left(\frac{1}{6}\right)$$

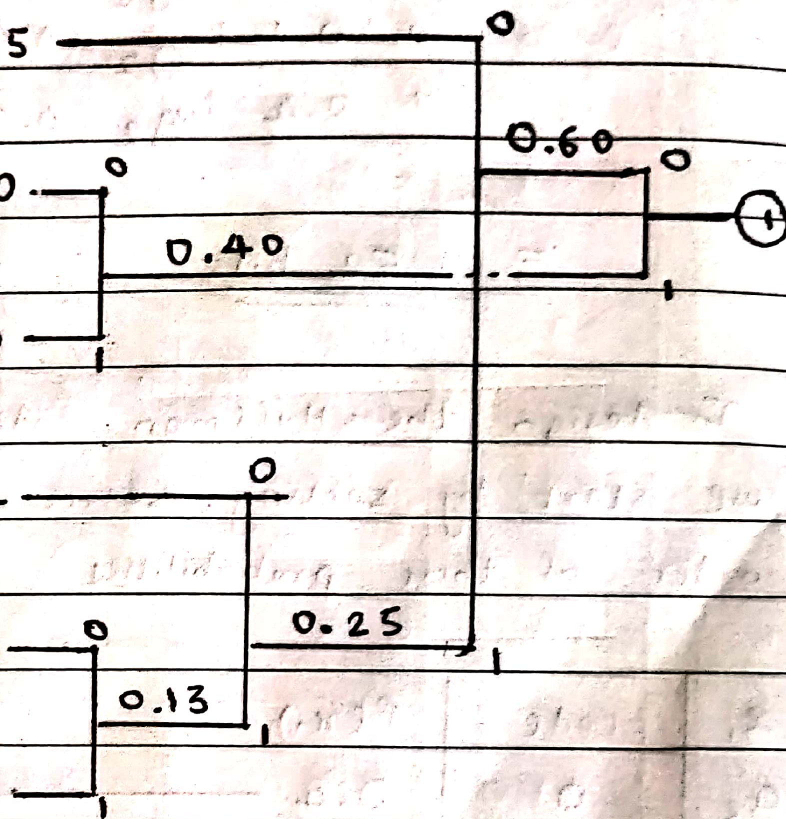
$$= \log_2 (6)$$

$$= \underline{\underline{2.585 \text{ bits}}}$$

Comparing this with the entropy of the original source, we can see that the original source has lower entropy, indicating less uncertainty.

d) To design the Huffman code, we start by sorting source outputs in decreasing order of their probabilities.

x_i	code	$P(x_i)$
x_1	00	0.35
x_2	10	0.20
x_3	11	0.20
x_4	010	0.12
x_5	0110	0.08
x_6	0111	0.05



The average codeword length is

$$L_{avg} = \sum_{i=1}^{6} P(x_i) \times \text{length}(x_i)$$

$$= 0.35 \times 2 + 0.20 \times 2 + 0.20 \times 2 + 0.12 \times 3 + 0.08 \times 4 + 0.05 \times 4$$

$= \underline{2.38}$ bits/source output

It is clear that the average codeword length is very close to the source entropy

Q2.

a) The entropy is:

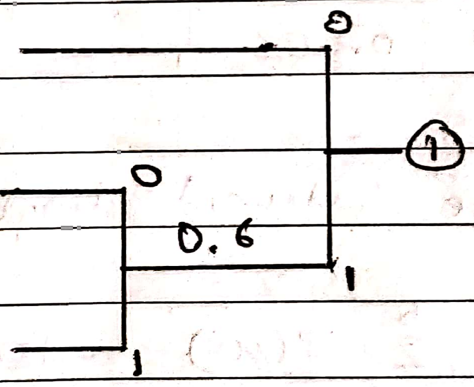
$$H(x) = -\sum_i p_i \log_2 p_i$$

$$= -\left(0.2 \log_2 (0.2) + 0.4 \log_2 (0.4) + 0.4 \log_2 (0.4) \right)$$

$$= \underline{\underline{1.52 \text{ bits}}}$$

b) To design the Huffman code, we start by sorting source outputs in decreasing order of their probabilities.

x_i	code	$P(x_i)$
a_3	0	0.4
a_2	10	0.4
a_1	11	0.2



The average codeword length is

$$L_{avg} = \sum_{i=1}^{3} P(x_i) \times \text{length}(x_i)$$

$$= 0.4 \times 1 + 0.4 \times 2 + 0.2 \times 2$$

$$= \underline{\underline{1.6 \text{ bits / source output}}}$$

c)

For the second extension of the source (taking 2 letters at a time), we first calculate the probabilities of each pair.

$p(a_1 a_1) = 0.2 \times 0.2 = 0.04$
$p(a_1 a_2) = 0.2 \times 0.4 = 0.08$
$p(a_1 a_3) = 0.2 \times 0.4 = 0.08$

$p(a_2 a_1) = 0.4 \times 0.2 = 0.08$
$p(a_2 a_2) = 0.4 \times 0.4 = 0.16$
$p(a_2 a_3) = 0.4 \times 0.4 = 0.16$

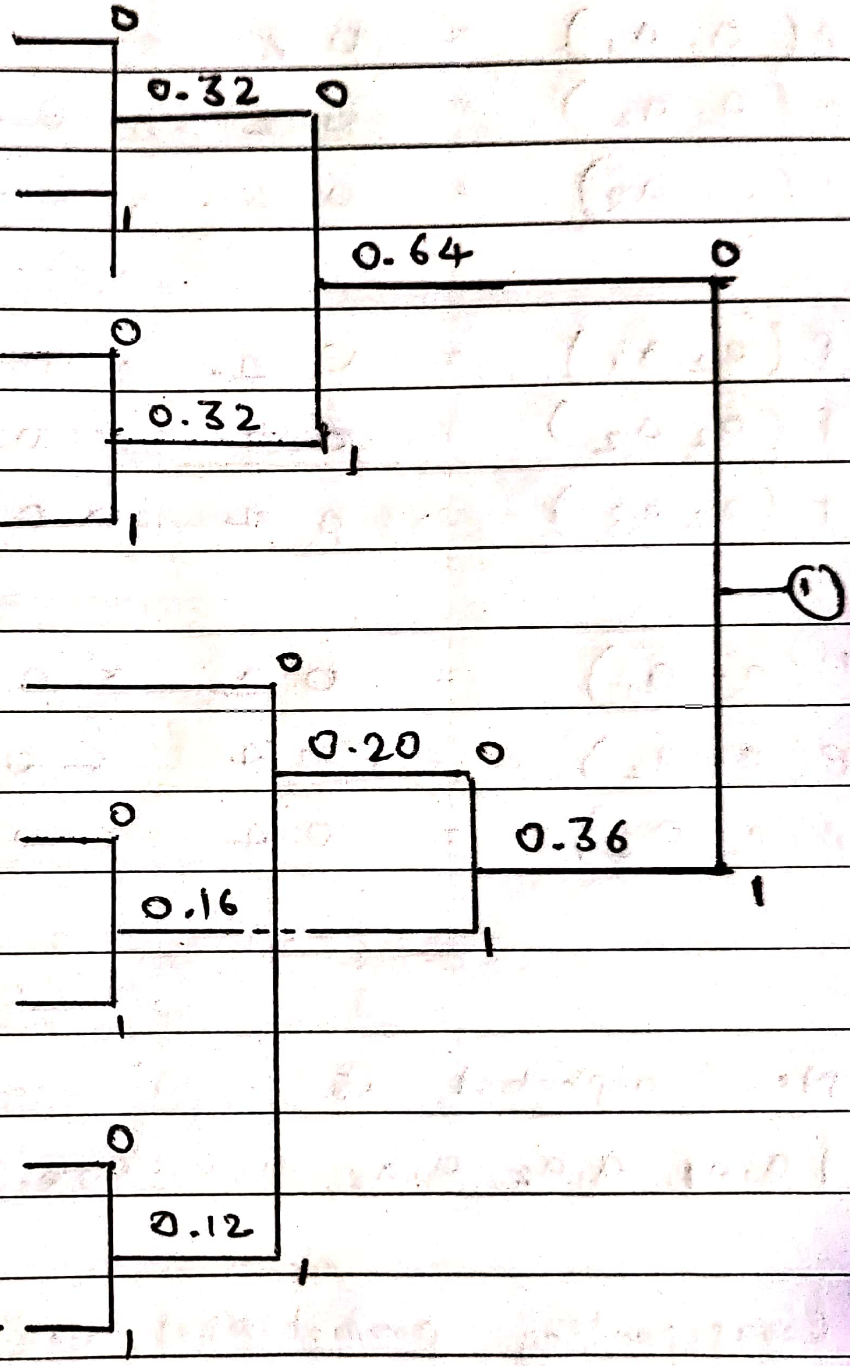
$p(a_3 a_1) = 0.4 \times 0.2 = 0.08$
$p(a_3 a_2) = 0.4 \times 0.4 = 0.16$
$p(a_3 a_3) = 0.4 \times 0.4 = 0.16$

$\therefore$ New alphabet is
$\{a_1a_1, a_1a_2, a_1a_3, a_2a_1, a_2a_2, a_2a_3, a_3a_1, a_3a_2, a_3a_3\}$

$\therefore$ Corresponding probabilities are
$\{0.04, 0.08, 0.08, 0.08, 0.16, 0.16, 0.08, 0.16, 0.16\}$

To design the Huffman code, we start by sorting source outputs in decreasing order of their probabilities.

x_i	Code	$P(x_i)$
a_3a_3	000	0.16
a_3a_2	001	0.16
a_2a_3	010	0.16
a_2a_2	011	0.16
a_3a_1	100	0.08
a_2a_1	110	0.08
a_1a_3	111	0.08
a_1a_2	1010	0.08
a_1a_1	1011	0.04



The average codeword length is

$$L_{avg} = \sum_{i=1}^{9} P(x_i) \times \text{length}(x_i)$$

$$= 0.16 \times 3 + 0.16 \times 3 + 0.16 \times 3$$
$$+ 0.16 \times 3 + 0.08 \times 3 + 0.08 \times 3$$
$$+ 0.08 \times 3 + 0.08 \times 4 + 0.04 \times 4$$

$$= \underline{\underline{3.12 \text{ bits}}} \text{ / source output pair}$$

The average number of required binary letters per each source output letter is $\frac{L_{avg}}{2}$

$$= \frac{3.12}{2} = \underline{\underline{1.56 \text{ bits}}} \text{ / source output.}$$

d)

Huffman coding of the original source requires 1.6 bits / source output letter.

$\because$ Efficiency (in %) $= \frac{\text{Entropy } (H(x))}{L_{avg}} = \frac{1.52}{1.6} \times 100 = 95\%.$

Huffman coding of second extention source requires 1.56 bits / source output letter.

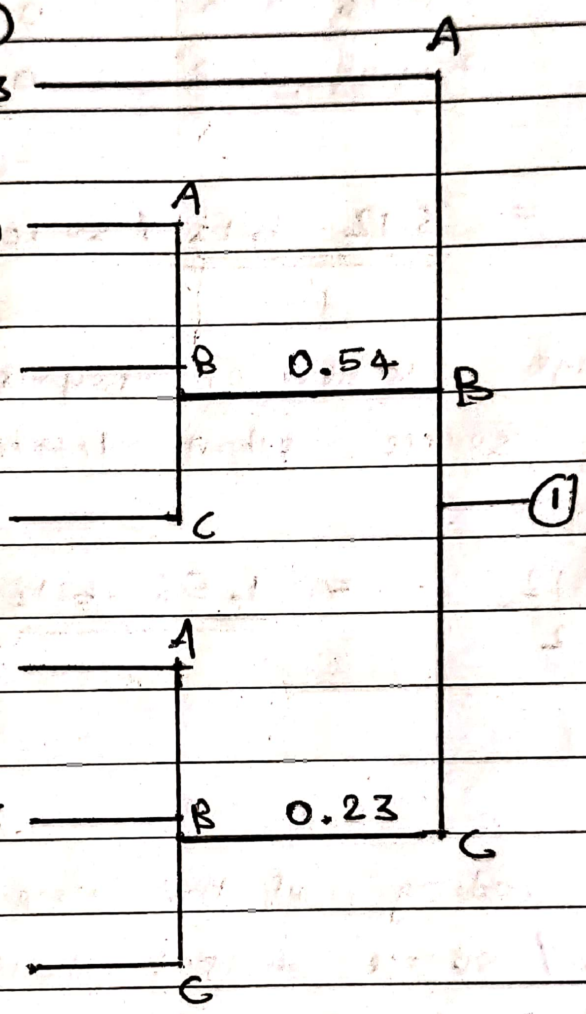
$\because$ Efficiency $= \frac{1.52}{1.56} \times 100 = 97.4\%$

$\therefore$ Huffman coding of the second extension of the source is more efficient coding scheme.

Q3.

To design the Huffman code,
we start by sorting source outputs in decreasing order of their probabilities.

x_i	code	$P(x_i)$
x_1	A	0.23
x_2	BA	0.20
x_3	BB	0.18
x_4	BC	0.16
x_5	CA	0.13
x_6	CB	0.05
x_7	CC	0.05



The average codeword length is

$$L_{avg} = \sum_{i=1}^{7} P(x_i) \times \text{length}(x_i)$$

$= 0.23 \times 1 + 0.20 \times 2 + 0.18 \times 2 + 0.16 \times 2 + 0.13 \times 2 + 0.05 \times 2 + 0.05 \times 2 = 1.77$

$= 1.77$ ternary symbol / source output

For a fair comparison of the average codeword length with the entropy of the source, we should compute the entropy of ternary code with logarithms in base 3. So,

$$H(x) = -\sum_i P_i \log_3 P_i$$

$$= -(0.05 \log_3 (0.05) + 0.2 \log_3 (0.2) + 0.05 \log_3 (0.05) + 0.16 \log_3 (0.16) + 0.18 \log_3 (0.18) + 0.23 \log_3 (0.23) + 0.13 \log_3 (0.13))$$

$= \underline{\underline{1.66}}$ ternary symbols / source output.

Average code word (L_{avg}) $= 1.77$

Entropy ($H(x)$) $= 1.66$

As expected, $L_{avg} \geq H(x)$

Q4.

Encoding (Lempel - Ziv compression)

Location	Content	Codeword
1	B	0 B
2	A	0 A
3	AA	2 A
4	AB	2 B
5	AAA	3 A
6	ABA	4 A
7	BA	1 A
8	BB	1 B
9	BBB	8 B
10	BBA	8 A
11	ABAB	6 B
12	AAB	3 B
13	ABABB	11 B
14	ABABBA	13 A
15	ABB	4 B
16	AABB	12 B
17	ABB	15

Decoding (Recovering the original sequence)

Location	Codeword	Content
1	0 B	B
2	0 A	A
3	2 A	A A
4	2 B	A B
5	3 A	A A A
6	4 A	A B A
7	1 A	B A
8	1 B	B B
9	8 B	B B B
10	8 A	B B A
11	6 B	A B A B
12	3 B	A A B
13	11 B	A B A B B
14	13 A	A B A B B A
15	4 B	A B B
16	12 B	A A B B
17	15	A B B

Q5

Given:

$P(X=0, Y=0)$
$= P(X=0, Y=1)$
$= P(X=1, Y=1)$
$= \frac{1}{5}$

and $\sum_i \sum_j P(x_i, y_j) = 1$,
we get that

$P(X=1; Y=0) = 1 - \frac{3}{5} = \frac{2}{5}$

The marginal probabilities are

$P(X=0) = P(X=0, Y=0) + P(X=0, Y=1) = \frac{2}{5}$

$P(X=1) = P(X=1, Y=0) + P(X=1, Y=1) = \frac{3}{5}$

$P(Y=0) = P(X=0, Y=0) + P(X=1, Y=0) = \frac{3}{5}$

$P(Y=1) = P(X=0, Y=1) + P(X=1, Y=1) = \frac{2}{5}$

a) $H(X) = -\sum_{i=1} P_i \log_2 P_i$

$= -\left(\frac{2}{5}\log_2\left(\frac{2}{5}\right) + \frac{3}{5}\log_2\left(\frac{3}{5}\right)\right)$

$= \underline{0.971 \text{ bits / source output}}$

b) $H(Y) = -\sum_{i=1} P_i \log_2 P_i$

$$= -\left(\frac{3}{5}\log_2\left(\frac{3}{5}\right) + \frac{2}{5}\log_2\left(\frac{2}{5}\right)\right)$$

$$= \underline{\underline{0.971}} \text{ bits / source output.}$$

c) $H(X,Y)$

The joint entropy

$$H(X,Y) = -\sum_i \sum_j P(x_i, y_j) \log_2 P(x_i, y_j)$$

$$= -\left(\frac{1}{5}\log_2\left(\frac{1}{5}\right) + \frac{1}{5}\log_2\left(\frac{1}{5}\right) + \frac{1}{5}\log_2\left(\frac{1}{5}\right) + \frac{2}{5}\log_2\left(\frac{2}{5}\right)\right)$$

$$= \underline{\underline{1.922}} \text{ bits / source output.}$$

d) The conditional entropy is

$$H(X|Y) = H(X,Y) - H(Y)$$
$$= 1.922 - 0.971$$
$$= \underline{\underline{0.951}} \text{ bits / source output}$$

$$H(Y|X) = H(X,Y) - H(X)$$
$$= 1.922 - 0.971$$
$$= \underline{\underline{0.951}} \text{ bits / source output}$$

e) The mutual information is

$$I(X;Y) = H(X) - H(X|Y)$$
$$= 0.971 - 0.951$$
$$= 0.02 \text{ bits/source output}$$