ELEC 4190 – Digital Communications

Analog-to-Digital Conversion

Outline

- Sampling theorem
- Signal reconstruction and practical considerations
- Quantization and encoding
- Pulse Code Modulation (PCM)
- Delta Modulation (DM)
- Remarks

- Recommended reading: Proakis and Salehi – Chapter 7
- Extra reading: Lathi and Ding – Chapter 5

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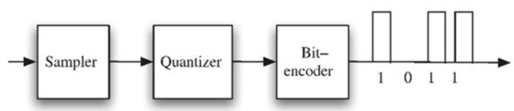
From Analog to Digital

- Thanks to the advances of digital technology and hardware, in many cases, it is more efficient to:
  1. convert the analog signal into a stream of bits
  2. process as a digital signal
  3. convert back to an analog signal if needed

From Analog to Digital (cont.)

- The process of converting a continuous-time signal to a discrete-time signal is called sampling
- The sampled signal is not digital yet
  - Sampled values can take an infinite number of values
- After sampling, quantization is needed to round sampled values to a finite number of values
- Finally, encoding is used to map each sample to a sequence of bits



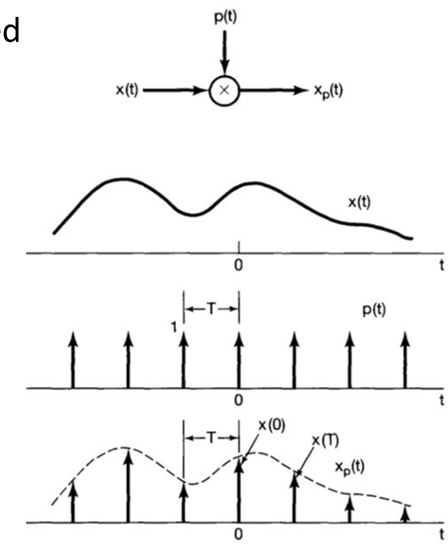
Sampling

- A sampled version of the signal can be obtained by multiplying by a periodic impulse train

$$x_p(t) = x(t) \cdot p(t), \qquad p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{+\infty} x(nT_s)\delta(t - nT_s)$$

- T_s is called the sampling period and $f_s = 1/T_s$ is called the sampling frequency



Sampling (cont.)

- To find the Fourier transform of the sample signal $x_p(t)$, we can use the multiplication property as follows:

$$x(t) \cdot y(t) \xleftrightarrow{\mathcal{F}} X(f) * Y(f)$$

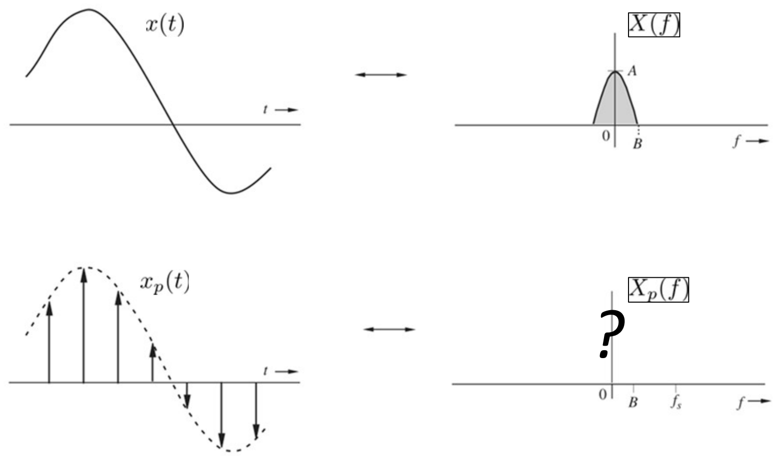
$$P(f) = \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} \delta(f - nf_s)$$

$$x_p(t) = x(t) \cdot p(t) \xleftrightarrow{\mathcal{F}} X_p(f) = \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} X(f - nf_s)$$

The same can be derived using DTFT

Sampling (cont.)

- Let's plot the spectrum using an arbitrary signal $x(t)$ with bandwidth W when (i) $f_s > 2B$, (ii) $f_s = 2B$, (iii) $f_s < 2B$



Sampling Theorem

- It is clear that $x(t)$ can be reconstructed from $x_p(t) = \{x(nT_s)\}$ without distortion when $f_s \geq 2B$

Theorem: *A signal x(t) with a bandwidth B Hz, i.e., $X(f) = 0$ for $f > |B|$, can be uniquely reconstructed from its uniform discrete-time samples using and ideal lowpass filter if the sampling frequency f_s is greater than or equal to 2B.*

- That is, a CT signal can be reconstructed from its samples using a distortionless LPF (also called interpolation filter in time)
- Minimum sampling rate is $f_s = 2B$ and is called Nyquist Rate
- Maximum sampling interval is $1/2B$ and is called Nyquist Interval

Example

Determine the Nyquist sampling rate for the following signal:

$$x(t) = 1 + \cos(6000\pi t) + \sin(3000\pi t)$$

$2B\text{sinc}(2\pi Bt) \leftrightarrow \Pi\left(\frac{f}{2B}\right)$

Example

Determine the Nyquist sampling rate for the following signal:

$$x(t) = 2\ \text{sinc}(600\pi t)$$

$B\,\mathrm{sinc}^2(\pi B t) \leftrightarrow \Lambda\left(\frac{f}{2B}\right)$

Example

Determine the Nyquist sampling rate for the following signal:

$$x(t) = 3\,\mathrm{sinc}^2(200\pi t) + 4\,\mathrm{sinc}(400\pi t)$$

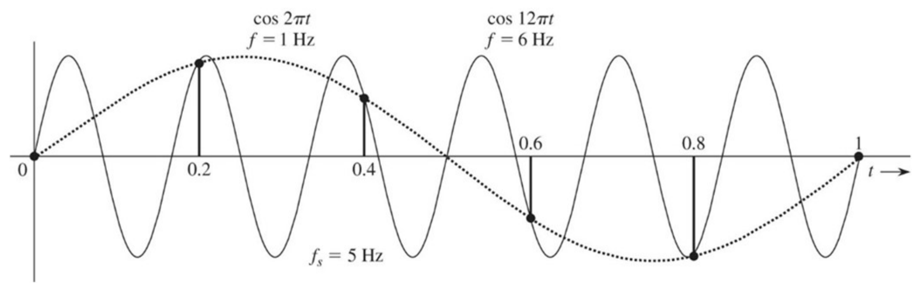
Example

Signal $x(t) = \cos(2\pi f_0 t)$ is sampled every 0.2 sec. Plot the spectrum of the sampled signal when (a) $f_0 = 1$, (b) $f_0 = 6$.

- Note*:*
  - For (a) $x[n] = \cos(0.4\pi n)$ and (b) $x[n] = \cos(2.4\pi n) = ?$
  - The overlap in case (b) is due to undersampling and is called aliasing

Example (cont.)

- To demonstrate the effect of undersampling in time domain:

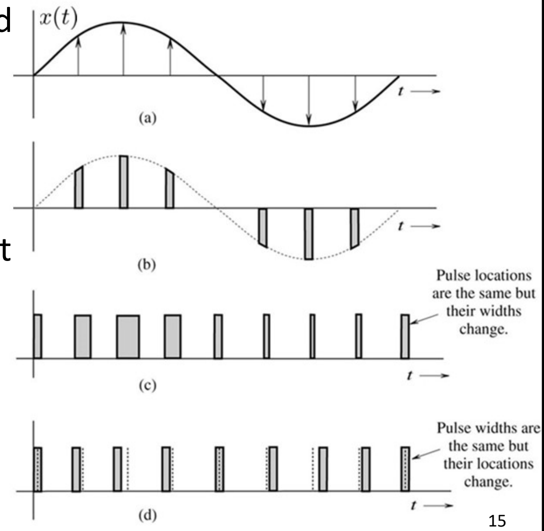


- Samples of two different frequencies (sampled at the same rate) generate identical sets of samples
  - Sampling rate is adequate for the lower-frequency sinusoid but is clearly inadequate for the higher-frequency sinusoid.

Demo

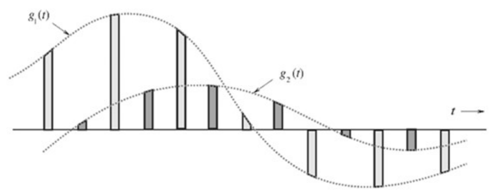
Pulse Modulation

- Once $x(t)$ is sampled, it can be used to modulate characteristics of a periodic pulse train and then transmitted
- Examples
  - Pulse Amplitude Modulation (PAM)
  - Pulse Width Modulation (PWM)
  - Pulse Position Modulation (PPM)
- Pulse-modulated signal contain sufficient information to reconstruct $x(t)$
- This is called baseband pulse modulation in which no carrier modulation is present



Pulse Modulation (cont.)

- Note that pulse-modulated signals occupy only a fraction of the channel time
- Hence, several signals can be multiplexed and transmitted simultaneously → Time Division Multiplexing (TDM)



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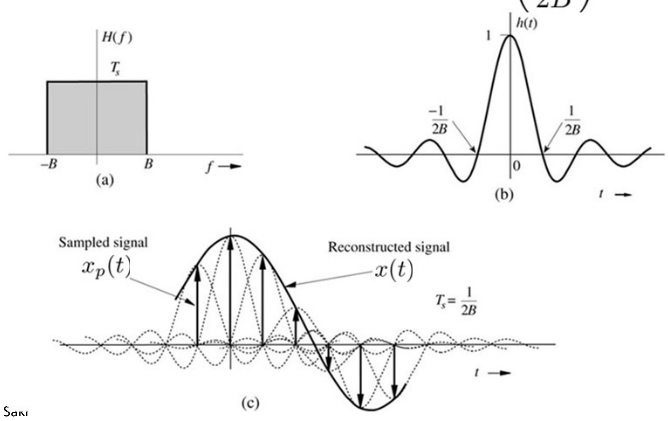
Ideal Reconstruction (Interpolation) Filter

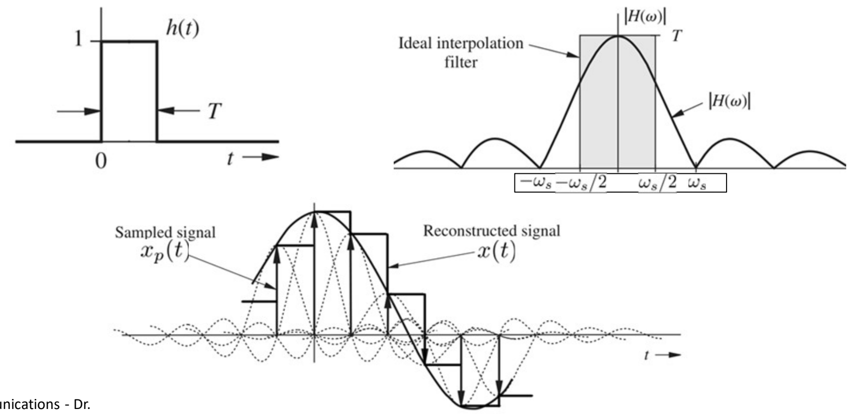
- A continuous time signal can be reconstructed from its samples using a distortionless LPF (also called interpolation filter in time)

$$h(t) = \text{sinc}(2\pi Bt) \leftrightarrow H(f) = T_s \Pi\left(\frac{f}{2B}\right)$$



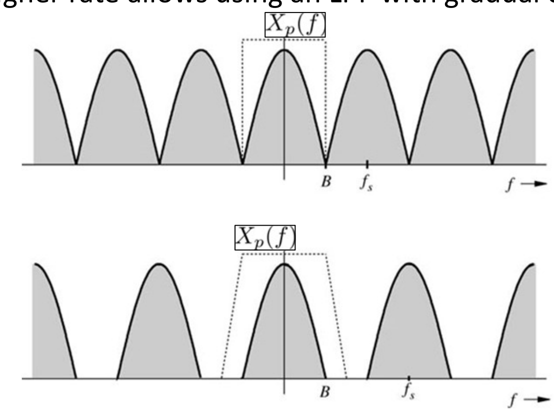
Zero-Order Hold Reconstruction (Interpolation) Filters

- In practice, ideal LPFs are unrealizable
- Zero-order hold:
  - Hold the value of sample until the next instant at which a new sample is given



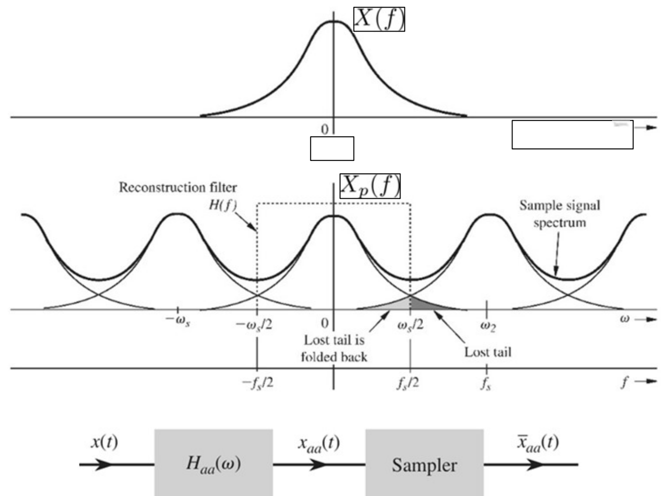
Practical Issues

- Realizability of reconstruction filters
  - Nyquist rate means no gap (guard band) between repetitions which requires an ideal (unrealizable) LPF
  - Sampling at higher rate allows using an LPF with gradual cutoff



Practical Issues (cont.)

- Aliasing
  - Practical signals are non-bandlimited which causes overlaps in $X_p(f)$
  - Sampling at a higher rate reduces aliasing (but does not eliminate it)
  - Use an antialiasing filter before sampling to eliminate frequency components beyond folding frequency $f_s/2$



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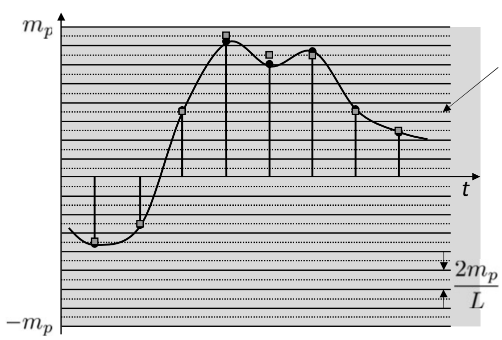
Quantization

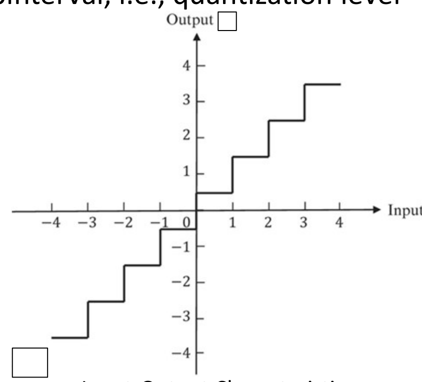
- Sampled signal is not necessarily digital because amplitude is not restricted to a finite number of values
  - i.e., discrete-time continuous-value signal
- Quantization rounds off each sample value to the closest permissible level to produce a digital signal
  - i.e., transformed into a discrete-time discrete-value signal, where the discrete amplitudes belong to a finite set of possible values
- Note that this process is lossy and irreversible
  - i.e., multiple input values can yield the same output value
  - Good news: degradation is in principle unnoticeable if designed carefully such that humans cannot perceive much difference between these two signals

Uniform Quantization

- Not optimum, but simple and commonly used in practice
- Assume signal amplitude ranges between $-m_p$ and m_p:
  1. Divide the range into L equal subintervals of size $\Delta v = 2m_p/L$
  2. Samples are rounded to midpoint of the subinterval, i.e., quantization level



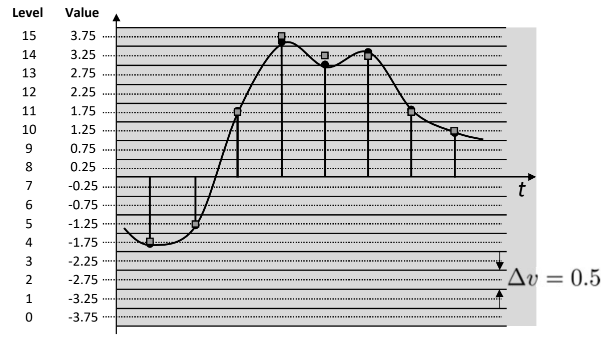
Animation



Input-Output Characteristics

Example

Given the signal below with amplitudes in [-4, 4] find the quantized sequence if quantization is uniform with 16 levels.



Sample	-1.8	-1.3	1.75	3.6	3	3.4	1.77	1.2
Level	4	5	11	15	14	14	11	10
Quantized	-1.75	-1.25	1.75	3.75	3.25	3.25	1.75	1.25

Animation

Uniform Quantization (cont.)

- Difference between two signals is referred to as the quantization error

$$q[k] = m[k] - m_q[k] \text{ where } -\frac{\Delta v}{2} \leq q \leq \frac{\Delta v}{2}$$

- Mean squared quantization error is

$$\mathrm{MSE} = \mathbb{E}[q^2(t)] = \frac{1}{\Delta v}\int_{-\Delta v/2}^{-\Delta v/2} q^2 \mathrm{d}q = \frac{m_p^2}{3L^2}$$

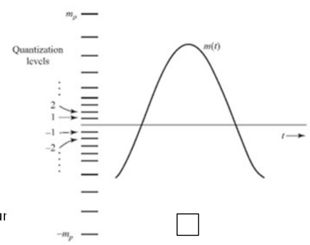
- SNR is

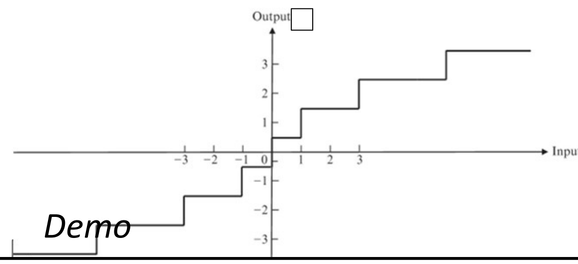
$$\mathrm{SNR} = \frac{3P_m}{m_p^2}L^2$$

- Note how increasing L improves SNR as expected
  - Example: doubling L (or adding one extra bit), quadruples the SNR

Nonuniform Quantization

- In many cases, it is found that smaller/larger amplitudes predominate the signal whereas larger/smaller amplitudes are much less frequent
  - e.g., silent and low voices vs. screaming and loud voices in voice signals
  - Uniform quantization is wasteful for quantizing levels hardly used
- With nonuniform quantization, quantization error can be reduced by dynamically use smaller Δv for predominate amplitudes
  - i.e., improve SNR for lower amplitudes and reduce SNR for higher amplitudes
  - Better overall when lower amplitudes predominate





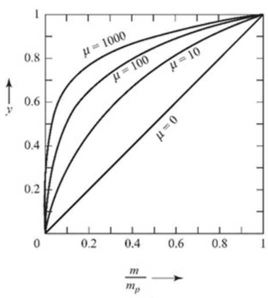
Demo

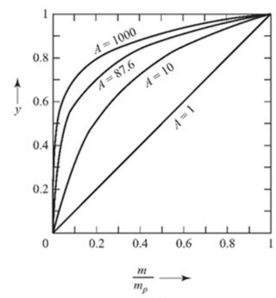
Nonuniform Quantization (cont.)

- Same effect can be achieved by (i) nonlinearly compressing signal and then (ii) pass it through a uniform quantizer
  - e.g., the standardized μ-law and A-law compressors (or compandors)
  - Need an expander at receiver
- For positive amplitudes,

$$y = \frac{1}{\ln(1+\mu)} \ln\left(1 + \frac{\mu m}{m_p}\right), \quad 0 \le \frac{m}{m_p} \le 1$$

$$y = \begin{cases} \dfrac{A}{1+\ln A}\left(\dfrac{m}{m_p}\right), & 0 \le \dfrac{m}{m_p} \le \dfrac{1}{A} \\ \dfrac{1}{1+\ln A}\left(1 + \ln\dfrac{Am}{m_p}\right), & \dfrac{1}{A} \le \dfrac{m}{m_p} \le 1 \end{cases}$$





Demo

Nonuniform Quantization (cont.)

- For the μ-law compandor:

$$\mathrm{SNR} = \frac{3L^2}{(\ln(1+\mu))^2}, \quad \mu^2 \gg \frac{P_m}{m_p^2}$$

Example

Given a signal with amplitudes in [-2, 2] and samples {1.1, -0.1, 0.5, 0.2, 0.7, 1.8},
(a) Find the quantized sequence if quantization is uniform with 4 levels.
(b) Find the quantized sequence if quantization is nonuniform with 4 levels. Assume μ-law quantizer with μ=9. ***Hint****: find nonuniform quantization levels using the inverse formula given y=uniform/m_p*

Uniform	$y=l/m_p$	Nonuniform
-1.5	-0.75	-1.0275
-0.5	-0.25	-0.173
0.5	0.25	0.173
1.5	0.75	1.0275

$$y = \frac{1}{\ln(1+\mu)} \ln\left(1 + \frac{\mu m}{m_p}\right), \quad 0 \le \frac{m}{m_p} \le 1$$

$$\frac{m}{m_p} = \frac{(1+\mu)^y - 1}{\mu}, \quad 0 \le y \le 1$$

1. y=l/m_p
2. Find nonuniform levels m
3. Quantize

Encoding

- The last step is encoding to convert each quantized sample into a codeword of *n* binary bits
  - i.e., map each quantization level to a unique codeword

$$n = \log_2 L \text{ bits/symbol}$$

- Example: Natural Binary Code (NBC) for 8 levels:
  - 000, 001, 010, 011, 100, 101, 110, 111
- The bit rate in bps = sampling rate x *n*
- The required channel bandwidth = bit rate/2

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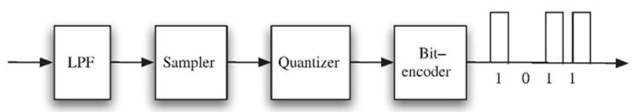
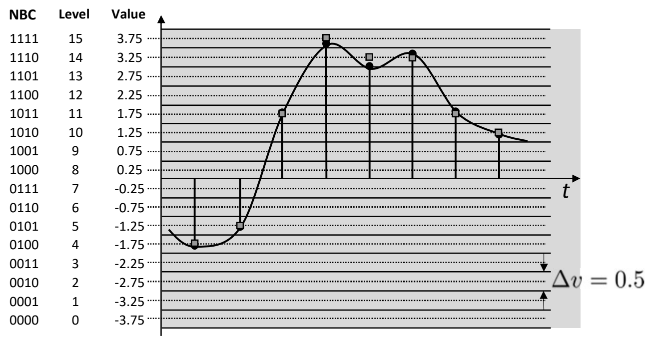
Pulse Code Modulation (PCM)

- A digital transmission system that provides analog-to-digital conversion in the transmitter and digital-to-analog conversion in the receiver
  - Filtering: anti-aliasing LPF to remove high-frequency components of the analog signal that are not essential
  - Sampling: conversion to discrete-time continuous-value signals
  - Quantization: conversion to discrete-time discrete-value signals
  - Encoding: Map quantized sample values into a unique binary codeword



Example

Given the signal below with amplitudes in [-4, 4] find the binary sequence if quantization is uniform with 16 levels.

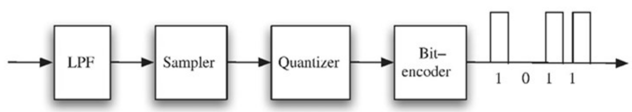


Sample	-1.8	-1.3	1.75	3.6	3	3.4	1.77	1.2
Level	4	5	11	15	14	14	11	10
Quantized	-1.75	-1.25	1.75	3.75	3.25	3.25	1.75	1.25
Codeword	0100	0101	1011	1111	1110	1110	1011	1010

Animation

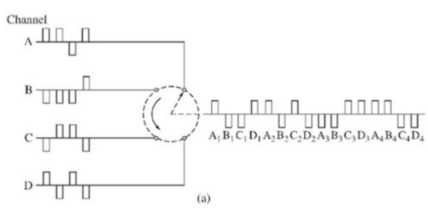
PCM in Digital Telephony

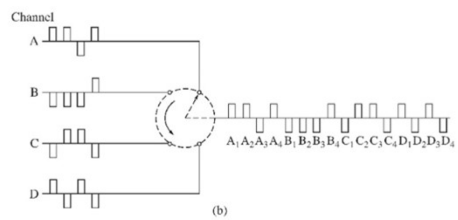
- A/D conversion of an audio signal in digital landline telephone:
  - Bandwidth about 15 kHz
  - Suppress frequency components above 3.4 kHz (but still intelligible) (also prevents aliasing)
  - Sample at 8 kHz (above Nyquist rate for realizability of reconstruction filter)
  - Quantize into 256 levels (L=256 or 8 bits per pulse)
  - Bit rate 64 kbps (for dial-up internet, 8 kbps are reserved for signaling and data rate is 56 kbps)



Time-Division Multiplexing (TDM)

- Several low-bit-rate signals are multiplexed, or combined, to form one high-bit-rate signal to be transmitted over a high-frequency medium
- The channel in TDM is time-shared by several signals
  - e.g., digitized PCM signals
- Multiplexing can be done by digit interleaving (i.e., bit-by-bit) or word interleaving (i.e., word-by-word)





Max. acceptable quantization error = $\Delta v/2$
Channel bandwidth = bit rate/2
Bit rate = sampling rate × n

Example

A signal $m(t)$ bandlimited to 3 kHz is sampled at a rate 1/3 higher than its Nyquist rate. The maximum acceptable quantization error is 0.5% of the peak signal amplitude m_p. The quantized samples are binary coded. Find the minimum bandwidth of the channel required to transmit the encoded binary signal. If 24 such signals are time-division-multiplexed, determine the minimum transmission bandwidth required to transmit the multiplexed signal.

Differential Pulse Code Modulation (DPCM)

- PCM is not very efficient because it generates many bits that require too much bandwidth to transmit
- We can exploit the characteristics of the source signal to reduce the number of bits (and bandwidth) required for transmission
- For example, samples values are usually correlated (i.e., previous samples give some information about current and next samples)
- Simplest way is to quantize the difference between successive samples
    - Smaller values
    - Fewer levels
    - Fewer bits needed

DPCM (cont.)

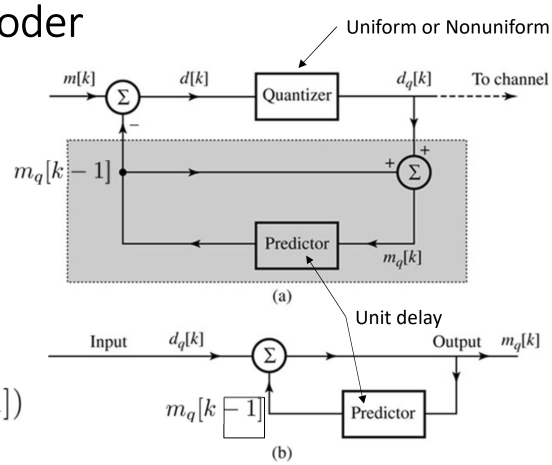
$$\text{MSE} = \frac{m_p^2}{3L^2}$$

- In DPCM, instead of sending sample *m[k]*, we send the difference *d*[*k*]

$$d[k] = m[k] - m[k-1]$$

- At the receiver, knowing *d*[*k*] and *m*[*k-1*], we can reconstruct *m*[*k*]
- Trivial example:
  - Signal: *m = 1, 2, 3, 1, 3, 4, 5*
  - Transmitted: *d = 1, 1, 1, -2, 2, 1, 1*
  - Reconstructed: m_q *= 1, 2, 3, 1, 3, 4, 5*
  - Notice the peak amplitude of the values in *m* and *d*
- Since peak amplitude d_p is much smaller than m_p:
  - For the same number of levels (and bits), we can reduce quantization error by a factor of $(m_p/d_p)^2$
  - For the same SNR, we can reduce the number of levels (and bits)

DPCM System: Encoder and Decoder



- The input and output of the quantizer are

$$d[k] = m[k] - m_q[k-1]$$

$$d_q[k] = m_q[k] - m_q[k-1]$$

- The input to the predictor is

$$m_q[k] = d_q[k] + m_q[k-1]$$

- At the receiver, the output is

$$m_q[k] = d_q[k] + m_q[k-1]$$

- The quantization error is

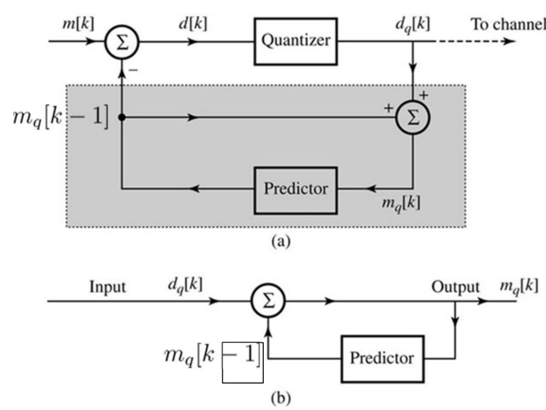
$$\begin{aligned} d_q[k] - d[k] &= d_q[k] - (m[k] - m_q[k-1]) \\ &= m_q[k] - m[k] \end{aligned}$$

- Notes:
  - Initial value $m_q[-1]$ is assumed to be the same at transmitter and receiver
  - Quantization error is the same for $m[k]$ and $d[k]$, smaller than error in PCM
  - Using $m_q[k\text{-}1]$ rather that $m[k\text{-}1]$ ensures error does not accumulate (i.e., $|q| \leq \frac{\Delta v}{2}$)

Example

Given a signal with samples {1, 2, 3, 1, 3, 4, 5}, a very small quantization step size, and zero initial condition. Show how this DPCM system works.

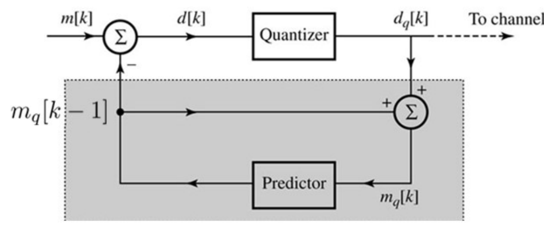
Encoder							
k	0	1	2	3	4	5	6
$m[k]$	1	2	3	1	3	4	5
$m_q[k-1]$	0	1	2	3	1	3	4
$d[k]$	1	1	1	-2	2	1	1
$d_q[k]$	1	1	1	-2	2	1	1
$m_q[k]$	1	2	3	1	3	4	5
Decoder							
$d_q[k]$	1	1	1	-2	2	1	1
$m_q[k-1]$	0	1	2	3	1	3	4
$m_q[k]$	1	2	3	1	3	4	5



Example

Given a signal with samples {1, 2, 3, 1, 3, 4, 5}, a quantization step size of 1.5, and zero initial condition. Show how this DPCM encoder works and track the quantization error. Hint: quantization levels are at ±0.75, ±2.25, ±3.75, …

Encoder							
k	0	1	2	3	4	5	6
m[k]	1	2	3	1	3	4	5
$m_q[k-1]$	0	0.75	1.5	3.75	1.5	3.75	4.5
d[k]	1	1.25	1.5	-2.75	1.5	0.25	0.5
$d_q[k]$	0.75	0.75	2.25	-2.25	2.25	0.75	0.75
$m_q[k]$	0.75	1.5	3.75	1.5	3.75	4.5	5.25
$d[k]-d_q[k]$	0.25	0.5	-0.75	-0.5	-0.75	-0.5	-0.25

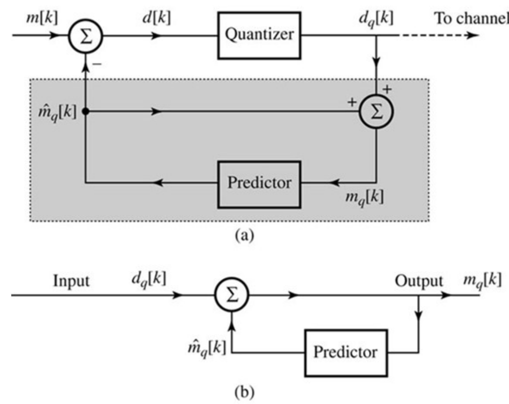


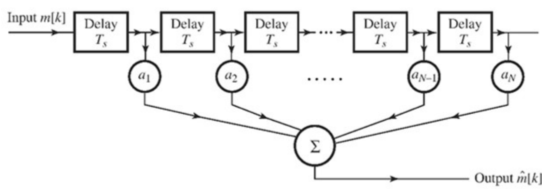
DPCM System: Encoder and Decoder (cont.)

- This can be further improved using previous *N* samples to predict the value of the k-th sample $m[k]$
- That is, the predictor becomes

$$\hat{m}_q[k] = \sum_{n=1}^{N} a_n m[k-n]$$

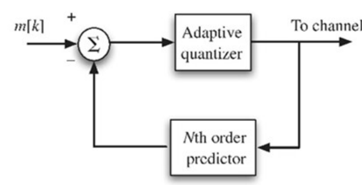
- This can be achieved by a transversal filter (tapped delay line)





Adaptive DPCM

- In DPCM, assume the number of quantization level *L* is fixed:
  - If step size it too big, quantization error is too large
  - If step size it too small, quantizer cannot cover the necessary signal range
- ADPCM improve efficiency of encoding by using an adaptive quantizer
- That is, quantization step size Δv is adaptive depending on the prediction error for quantizing
  - Increase step size when prediction error $d[k]$ is very large
  - Decrease step size when prediction error $d[k]$ is near zero



Adaptive DPCM (cont.)

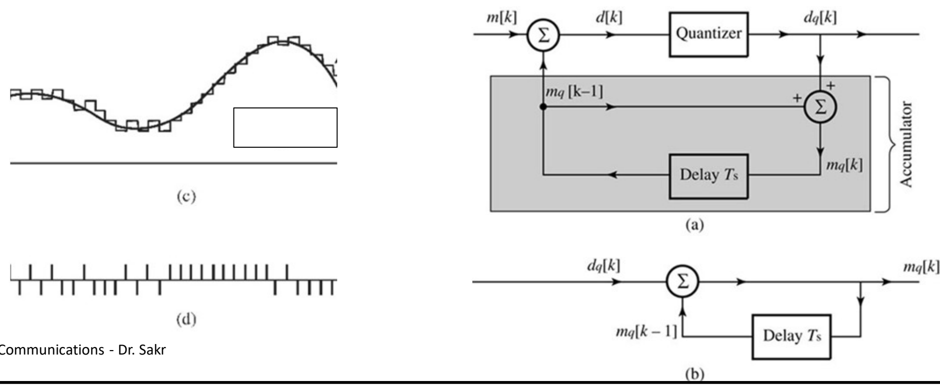
- Both transmitter and receiver need to apply the same algorithm to adjust the step size identically
- Example: ITU-T standard G.726 codec for speech signals
  - Used in applications such as speech transmission over digital networks, video conferencing, and multimedia
  - 8 kHz sampling rate and four different ADPCM bit rates at 16, 24, 32, and 40 kbps
  - That is, four different quantization levels of 4, 8, 16, and 32 (or equivalently bit sizes of 2 bits, 3 bits, 4 bits, and 5 bits per sample)

Outline

- Sampling theorem
- Signal reconstruction and practical considerations
- Quantization and encoding
- Pulse Code Modulation (PCM)
- Delta Modulation (DM)
- Remarks

Delta Modulation (DM)

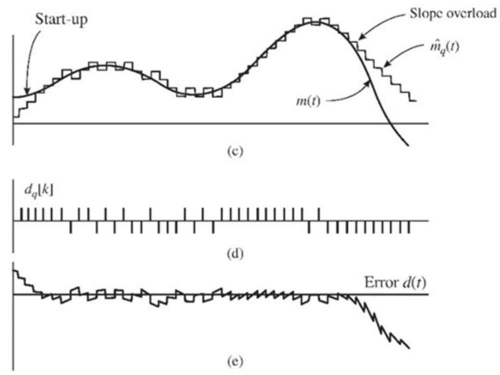
- Simplified version of DPCM system in two steps:
  - Oversample the baseband signal to increase correlation (i.e., much higher than the Nyquist rate)
  - Then, use 1-bit DPCM (i.e., only signal one bit per sample)
- It is a staircase approximation of the signal



DM (cont.)

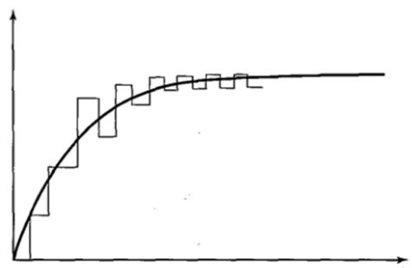
- Step size is very important
  - Large: follows rapid changes but increases quantization noise for slow changes
    - This called granular noise
  - Small: cannot follow rapid changes (i.e., high slopes) and takes a long time for the output to follow the input
    - This called slope overload distortion
    - Condition for no overload:

$$|\dot{m}(t)|_{\max} < \Delta v f_s$$



Adaptive DM (ADM)

- Change the step size according to changes in the input
  - If the input changes rapidly (i.e., high slop), increase step size to follow the input quickly
  - If the input changes slowly (i.e., mostly flat), decrease step size to prevent granular noise
- So, the slope of the input signal can be used as indicator to adaptively change the step size



Outline

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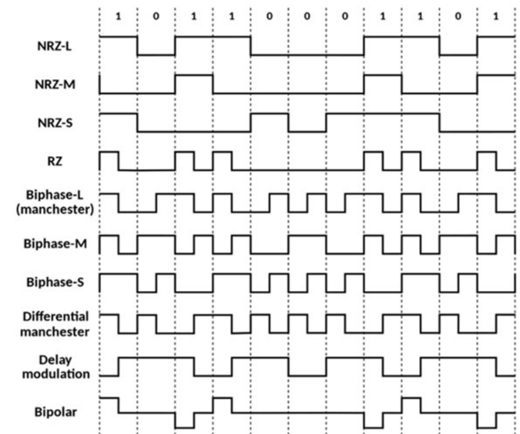
Notes of Line Codes

- Line codes are used to convert binary digital data to digital signals for effective transmission or storage
- There are numerous line codes for an array of applications
- Selection criteria of a particular code depends on many factors:
  - Spectral properties, e.g., bandwidth
  - Power efficiency
  - Ease of synchronization
  - Complexity
  - Error detection and correction
  - DC component elimination



Wikipedia

Notes on the Discrete Fourier Transform

- Recall: Discrete and periodic in time-domain → periodic and discrete in frequency-domain
- DFT is used for the numerical computation of the Fourier transform
  - Used for signals known only at N samples separated by sampling time T_s
  - DFT treats the data as if it were periodic with period $T_0 = NT_s$
  - Evaluate FT for N points in the frequency domain: 0, $2\pi/T_0$, etc.

Notes on the Discrete Fourier Transform (cont.)

- The DFT can be derived as

$$X[n] = \sum_{k=0}^{N-1} x[k] e^{-j\frac{2\pi}{N}nk}, \quad n = 0, 1, 2, \cdots, N-1$$

- Let W = exp(-j2π/N), this can be written in the matrix form as

$$\begin{pmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W & W^2 & \cdots & W^{N-1} \\ 1 & W^2 & W^3 & \cdots & W^{N-2} \\ \vdots & & & & \\ 1 & W^{N-1} & W^{N-2} & \cdots & W \end{pmatrix} \begin{pmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{pmatrix}$$

- The inverse DFT is straightforward
  - Inverse matrix is 1/N times the complex conjugate of the original (symmetric) matrix

Notes on the Discrete Fourier Transform (cont.)

- The number of computations required to compute DFT was dramatically reduced by the fast Fourier transform (FFT) algorithm
- DFT is only an approximation since it provides only for a finite set of frequencies, so there are two sources of errors:
  - Aliasing: time-limited means non-bandlimited, increase sampling rate or pre-filter the signal to reduce effect
  - Truncation error: if not time-limited, increase truncation interval T_0

Summary

- By now you should know:
  - How to sample an analog signal such that it can reconstructed from its samples
  - How to reduce/eliminate effects of aliasing and ideal interpolation filters
  - Analog pulse modulation: PAM, PWM, PPM
  - Digital pulse modulation: PCM, DPCM, ADPCM, DM, ADM
  - The connection between CT and DT signals and systems and their frequency-domain representations