Practice Problems: Error-Control Coding

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ELEC 4190 - Digital Communications

> **Instructions**
>
> Solutions are posted on Brightspace. You should review the problems before the tutorial session. Your solutions will not be collected or graded.

> **Note**
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1. In a single parity-check code where a single parity-check bit is appended to a block of seven bits,

(a) Compute the probability of an undetected bit error when the bit error probability is 10^{-3}.

(b) Compare the throughput of the Stop-and-Wait, Go-Back-N, and Selective Repeat ARQ systems when the delay-bandwidth product (i.e., $D = 2RT$) is 40.

Solution:

(a) Given $k = 7$, $n = 8$, and $P_b = 10^{-3}$,

$$P_u = \frac{k(k+1)}{2} P_b^2 = \frac{7 \times 8}{2} 10^{-6} = 2.8 \times 10^{-5} \tag{1}$$

(b) Given $2RT = 1$, we get that $W = \frac{2RT}{n} = 5$. Given $P_b = 10^{-3}$, we get $P_c = (1 - P_b)^n = 0.992$ and $P_d = 1 - P_c - P_u = 7.9441 \times 10^{-3}$. Then,

$$\eta_{\text{SW}} = \frac{\left(\frac{k}{n}\right)(1 - P_d)}{1 + W} = 0.1447$$

$$\eta_{\text{GB}} = \frac{\left(\frac{k}{n}\right)(1 - P_d)}{1 + WP_d} = 0.8349$$

$$\eta_{\text{SR}} = \left(\frac{k}{n}\right)(1 - P_d) = 0.8680$$

2. Check if the code consisting of 000, 111, and 101 codewords is linear or not. Then, find the minimum distance of the code.

Solution:

A code is said to be linear if and only if the sum of any two codewords, in modulo-2 arithmetic, is a codeword in the code. So,

$$\begin{aligned} 000 + 111 &= 111 \checkmark \\ 000 + 101 &= 101 \checkmark \\ 111 + 101 &= 010 \times \end{aligned}$$

Then the code is not linear.

For the minimum distance,

$$\begin{aligned} d(000, 111) &= 3 \\ d(000, 101) &= 2 \\ d(111, 101) &= 2 \end{aligned}$$

Then, the minimum distance of the code is 2.

3. Consider $g(x) = x^3 + x + 1$, with 1001 as the information sequence. Determine the transmitted CRC codeword. With the rightmost received bit in error, find the syndrome at the receiver.

Solution:

Given $k = 4$, $n - k = 3$, $n = 7$, $g(x) = x^3 + x + 1$, and $m(x) = x^3 + 1$, following the steps to generate the CRC bits:

- $p(x) = x^{n-k} m(x) = x^6 + x^3$.
- Polynomial $p(x)$ is divided by $g(x)$ (using modulo-2 division) to obtain the remainder $r(x)$, which can have maximum degree $n - k - 1 = 2$ or lower as follows:

$$x^3 + x + 1 \left| \begin{array}{l} x^3 + x \\ \hline x^6 + x^3 \\ \underline{x^6 + x^4 + x^3} \\ x^4 \\ \underline{x^4 + x^2 + x} \\ x^2 + x \end{array} \right. \tag{2}$$

- Then, the redundancy bits are $r = 110$ and the CRC code word is $c = 1001110$.

If the received codeword is $b = 1001111$ or $b(x) = x^6 + x^3 + x^2 + x + 1$, the syndrome is calculated by diving $b(x)$ by $g(x)$ as follows:

$$
\begin{array}{r|l}
 & x^3 + x \\ \cline{2-2}
x^3 + x + 1 & x^6 + x^3 + x^2 + x + 1 \\
 & \underline{x^6 + x^4 + x^3} \\
 & x^4 + x^2 + 1 \\
 & \underline{x^4 + x^2 + x} \\
 & x + 1
\end{array} \tag{3}
$$

It is clear that the syndrome (i.e., the remainder) is not zero due to the error in the received codeword.

4. Assuming the information bits are 111000101 and the generator polynomial is $g(x) = x^8 + x^2 + x + 1$, determine the transmitted bit sequence consisting of the information bits and the CRC bits.

   Solution:

   Following the same steps, the transmitted codeword is $c = 11100010101000000$.

5. The generator polynomial for a (15, 7) cyclic code is $g(x) = 1 + x^4 + x^6 + x^7 + x^8$. Find the codeword (in the systematic form) for the message polynomial $m(x) = x^2 + x^3 + x^4$. Assuming the first and the last bits of the codeword are in error, determine the corresponding syndrome.

   Solution:

   Following the same steps, the transmitted codeword is $c = 1110011000001$. When the first and last bits are flipped with $b = 0110011000000$, the syndrome is $s = 00111011$.

6. Consider the input bit sequence is as follows:

   $$010010011011001110111101100110010110011001101101$$

   Assuming the bit sequence is divided into 16-bit segments, determine the checksum sent along with data, and show how the checksum checker operates on the received bit stream.

   Solution:

   Given $L = 16$ bits/section, $k = 48$ bits, and $q = 3$ sections, following the steps to generate the checksum bits:

$$
\begin{array}{r|c|cccccccccccccccc}
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\
 & & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
 & & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
 & & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
\hline
 & & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
\text{Carry:} & & & & & & & & & & & & & & & & & 1 \\
\hline
\text{Sum:} & & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
\text{Checksum:} & & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1
\end{array}
\tag{4}
$$

Note that the sum is complemented (i.e., all bits are inverted) in the last step to form the checksum. So, the checksum is 1011001001000101.

At the receiver, following the steps to check the received data:

$$
\begin{array}{r|c|cccccccccccccccc}
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 10 & \\
 & & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
 & & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
 & & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
 & & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
\hline
 & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
\text{Carry:} & & & & & & & & & & & & & & & & & 1 \\
\hline
\text{Sum:} & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\text{Syndrome:} & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\tag{5}
$$

It is clear that the syndrome is zero due to receiving the same transmitted data without errors.

7. The parity-check bits of a (8, 4) linear systematic code are generated by

- $c_5 = c_1 + c_2 + c_4$,
- $c_6 = c_1 + c_2 + c_3$,
- $c_7 = c_1 + c_3 + c_4$, and
- $c_8 = c_2 + c_3 + c_4$.

Determine the following:

(a) The code rate.

(b) The generator matrix and the parity-check matrix for this code.

(c) All the codewords.

(d) Minimum weight and error detecting and correcting capabilities.

(e) The syndrome **s** when the received codeword is 11000011.

(f) The syndrome $\mathbf{s}$ when the received codeword is 11001111.

Solution:

(a) Given $n = 8$, $k = 4$, the code rate is $\frac{4}{8} = 0.5$.

(b) The 8×4 generator matrix is given by:

$$\mathbf{G} = [\mathbf{I}_k|\mathbf{P}] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{array}\right] \tag{6}$$

Note that the fifth column is the result of $c_5 = c_1 + c_2 + c_4$, the sixth column is the result of $c_6 = c_1 + c_2 + c_3$, etc.

The parity-check matrix is given by:

$$\mathbf{H} = [\mathbf{P}^T|\mathbf{I}_{n-k}] = \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \tag{7}$$

(c) The codeword is generated by multiplying all possible messages from $\mathbf{m}_1 = [0000]$ to $\mathbf{m}_{16} = [1111]$ by the generator matrix as follows:

Message	Codeword
0000	00000000
0001	00011011
0010	00100111
0011	00111100
0100	01001101
0101	01010110
0110	01101010
0111	01110001
1000	10001110
1001	10010101
1010	10101001
1011	10110010
1100	11000011
1101	11011000
1110	11100100
1111	11111111

(d) Since the code os linear, the minimum distance of the code is equal to the minimum weight such that $d_{\min} = 4$. This means that the code can detect up to 3 errors and correct up to 1 error.

(e) The syndrome is generated by multiplying the received codeword by the transpose of the parity-check matrix $\mathbf{H}$. Therefore,

$$\mathbf{s} = \mathbf{b}\mathbf{H}^T = [11000011] \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [0000] \rightarrow \text{valid codeword}$$

(f) The syndrome is

$$\mathbf{s} = \mathbf{b}\mathbf{H}^T = [11001111] \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [1100] \rightarrow \text{invalid codeword}$$