Solutions Manual
Tutorial 2: Digital Data Transmission

University of Windsor
Department of Electrical and Computer Engineering
ELEC 4190 - Digital Communications

1.

(a) To check if a set of waveforms are orthonormal, we need to show that:

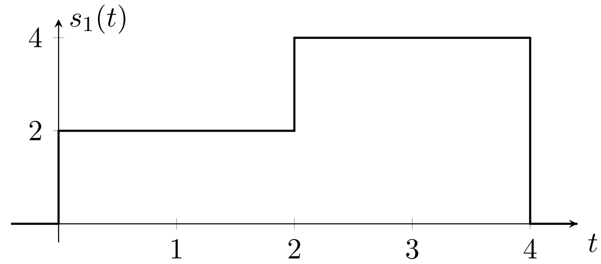
$$\int_0^{T_s} \phi_i(t)\phi_j(t)\mathrm{d}t = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

In this case,

$$\int_0^{T_s} \phi_1^2(t)\mathrm{d}t = 1$$
$$\int_0^{T_s} \phi_2^2(t)\mathrm{d}t = 1$$
$$\int_0^{T_s} \phi_3^2(t)\mathrm{d}t = 1$$
$$\int_0^{T_s} \phi_1(t)\phi_2(t)\mathrm{d}t = 0$$
$$\int_0^{T_s} \phi_1(t)\phi_3(t)\mathrm{d}t = 0$$
$$\int_0^{T_s} \phi_2(t)\phi_3(t)\mathrm{d}t = 0$$

Hence, $\{\phi_i(t)\}$ are orthonormal.

(b) Let's first plot $s_1(t)$:



To express a signal as a linear combination of $\{\phi_i(t)\}$, we need to find the projection of $s_1(t)$ on each basis function. That is,

$$s_{11} = \int_0^{T_s} s_1(t)\phi_1(t)\mathrm{d}t = -2$$
$$s_{12} = \int_0^{T_s} s_2(t)\phi_1(t)\mathrm{d}t = 6$$
$$s_{13} = \int_0^{T_s} s_3(t)\phi_1(t)\mathrm{d}t = 0$$

Then, $s_1(t) = -2\phi_1(t) + 6\phi_2(t)$ or equivalently $\mathbf{s}_1 = (-2, 6, 0)$ in this 3-dimensionale signal space.

2. To check if the of waveforms are orthonormal, we will evaluate the following:

$$\begin{aligned} I_1 &= \int_0^{T_s} \phi_1^2(t)\mathrm{d}t = \frac{2}{T_s}\int_0^{T_s} \cos^2(2\pi f_c t)\mathrm{d}t \\ &= \frac{2}{T_s}\int_0^{T_s} 0.5(1+\cos(4\pi f_c t))\mathrm{d}t = 1 + \frac{\sin(4\pi f_c t)}{4\pi f_c T_s} \approx 1 \end{aligned}$$

Note that the last step follows from the fact that $f_c T_s >> 1$. This is because the numerator in the second term is bounded by 1 and the denominator is very large. Hence, the second term can be neglected.

Similarly,

$$\begin{aligned} I_2 &= \int_0^{T_s} \phi_2^2(t)\mathrm{d}t = \frac{2}{T_s}\int_0^{T_s} \sin^2(2\pi f_c t)\mathrm{d}t \\ &= \frac{2}{T_s}\int_0^{T_s} 0.5(1-\cos(4\pi f_c t))\mathrm{d}t = 1 - \frac{\sin(4\pi f_c t)}{4\pi f_c T_s} \approx 1 \end{aligned}$$

Finally,

$$I_3 = \int_0^{T_s} \phi_1(t)\phi_2(t)\mathrm{d}t = \frac{2}{T_s}\int_0^{T_s} \cos(2\pi f_c t)\sin(2\pi f_c t)\mathrm{d}t$$

$$= \frac{2}{T_s}\int_0^{T_s} 0.5\sin(4\pi f_c t)\mathrm{d}t = -\frac{\cos(4\pi f_c t)}{4\pi f_c T_s} \approx 0$$

Hence, $\phi_1(t)$ and $\phi_2(t)$ are orthonormal.

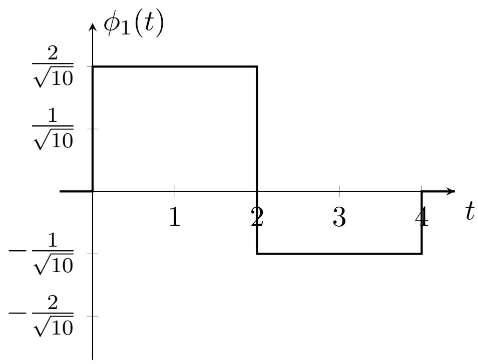
3. Consider the four signal waveforms shown below:

   (a) We will use the Gram-Schmidt orthogonalization procedure to find the basis functions which is as follows:

   (i) We start with $d_1(t) = s_1(t)$ to calculate $\phi_1(t)$ such that:

$$\phi_1(t) = \frac{d_1(t)}{\sqrt{E_{d_1}}} = \frac{s_1(t)}{\sqrt{10}}, \leftarrow E_{d_1} = \int\limits_0^{T_s} d_1^2(t)\mathrm{d}t = 10$$

Hence,



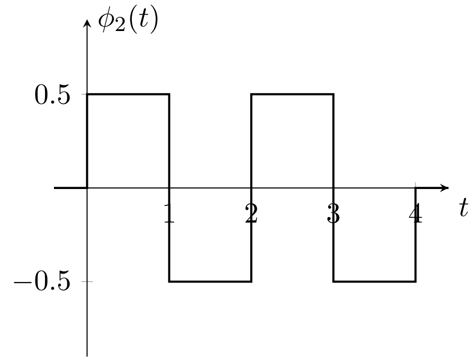
   (ii) Then, we calculate $\phi_2(t)$ using the projection of $s_2(t)$ onto $\phi_1(t)$ by firstly finding $d_2(t)$ as follows:

$$d_2(t) = s_2(t) - c_{21}\phi_1(t) = s_2(t) \leftarrow c_{21} = \int\limits_0^{T_s} s_2(t)\phi_1(t)\mathrm{d}t = 0$$

Then,

$$\phi_2(t) = \frac{d_2(t)}{\sqrt{E_{d_2}}} = \frac{s_2(t)}{2}, \leftarrow E_{d_2} = \int\limits_0^{T_s} d_2^2(t)\mathrm{d}t = 4$$

Hence,



(iii) Then, we calculate $\phi_3(t)$ using the projection of $s_3(t)$ onto $\phi_1(t)$ and $\phi_2(t)$ by firstly finding $d_3(t)$ as follows:

$$\begin{aligned} d_3(t) &= s_3(t) - c_{31}\phi_1(t) - c_{32}\phi_2(t) \\ &= s_2(t) - 4\phi_2(t) \leftarrow c_{31} = 0 \text{ and } c_{32} = -4 \\ &= 0 \end{aligned}$$

Hence, no new basis function is needed to represent $s_3(t)$. You can verify the this result by plotting $d_3(t)$.

(iv) Then, we calculate $\phi_3(t)$ using the projection of $s_4(t)$ onto $\phi_1(t)$ and $\phi_2(t)$ by firstly finding $d_4(t)$ as follows:

$$\begin{aligned} d_4(t) &= s_4(t) - c_{41}\phi_1(t) - c_{42}\phi_2(t) \\ &= s_4(t) - \frac{5}{\sqrt{10}}\phi_1(t) - 2\phi_2(t) \leftarrow c_{41} = \frac{5}{\sqrt{10}} \text{ and } c_{42} = 2 \\ &= 0 \end{aligned}$$

Hence, no new basis function is needed to represent $s_4(t)$. You can verify the this result by plotting $d_4(t)$.

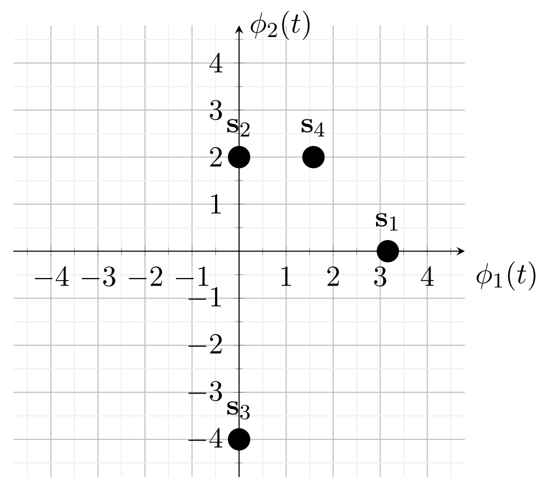
So, the the dimensionality of the waveforms in this case is 2 since we only need two basis functions to represent all signals.

(b) Using the results we obtained, we get

$$\begin{aligned} \mathbf{s}_1 &= \left(\sqrt{10}, 0\right) \leftarrow c_{11} = \int_0^{T_s} s_1(t)\phi_1(t)\mathrm{d}t = \sqrt{10} \\ \mathbf{s}_2 &= (0, 2) \leftarrow c_{22} = \int_0^{T_s} s_2(t)\phi_2(t)\mathrm{d}t = 2 \\ \mathbf{s}_3 &= (0, -4) \end{aligned}$$

$$\mathbf{s}_4 = \left(5/\sqrt{10}, 2\right)$$

The signal space representation is



(c) To find the minimum distance $d_{\min}$, we will find the distance between each pair of vectors as follows:

$$d_{12} = \|\mathbf{s}_1 - \mathbf{s}_2\| = \sqrt{(\sqrt{10} - 0)^2 + (0 - 2)^2} = \sqrt{14}$$
$$d_{13} = \|\mathbf{s}_1 - \mathbf{s}_3\| = \sqrt{(\sqrt{10} - 0)^2 + (0 - (-4))^2} = \sqrt{26}$$
$$d_{14} = \|\mathbf{s}_1 - \mathbf{s}_3\| = \sqrt{(\sqrt{10} - 5/\sqrt{10})^2 + (0 - 2)^2} = \sqrt{6.5}$$
$$d_{23} = \|\mathbf{s}_2 - \mathbf{s}_3\| = \sqrt{(0 - 0)^2 + (2 - (-4))^2} = 6$$
$$d_{24} = \|\mathbf{s}_2 - \mathbf{s}_4\| = \sqrt{(0 - 5/\sqrt{10})^2 + (2 - 2)^2} = \sqrt{2.5}$$
$$d_{34} = \|\mathbf{s}_3 - \mathbf{s}_4\| = \sqrt{(0 - 5/\sqrt{10})^2 + (-4 - 2)^2} = \sqrt{38.5}$$

Then, the minimum distance $d_{\min} = \sqrt{2.5}$.

4. For the first constellation:

(a) Closest symbols are the ones at $(1, 0)$ and $(0, 1)$, hence $d_{\min} = \sqrt{1^2 + 1^2} = \sqrt{2}$. Note that there are other signals that are at the same distance.

(b) $E_s = \frac{1}{8}(4 \times 1^2 + 4 \times 3^2) = 5$.

For the second constellation:

(a) Closest symbols are the ones at $(1, 0.5)$ and $(1, -0.5)$, hence $d_{\min} = \sqrt{0^2 + 1^2} = 1$. Note that there are other signals that are at the same distance.

(b) $E_s = \frac{1}{8}(4 \times (1^2 + 0.5^2) + 4 \times (3^2 + 1.5^2)) = 6.25$.

For the third constellation:

(a) Closest symbols are the ones at $(2, 0)$ and $(2/\sqrt{2}, 2/\sqrt{2})$, hence

$$d_{\min} = \sqrt{(2 - 2/\sqrt{2})^2 + (0 - 2/\sqrt{2})^2} = 1.53$$

Note that there are other signals that are at the same distance.

(b) $E_s = \frac{1}{8}(8 \times 2^2) = 4$. Note that all symbols have the same energy.

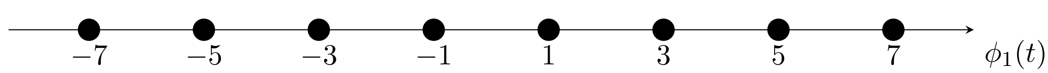
For the fourth constellation:

(a) Closest symbols are the ones at $(1, -1)$ and $(1, 1)$, hence $d_{\min} = \sqrt{0^2 + 2^2} = 2$. Note that there are other signals that are at the same distance.

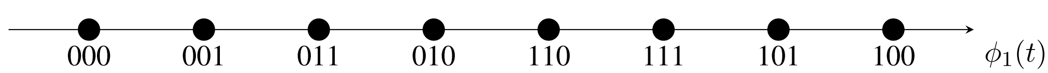
(b) $E_s = \frac{1}{8}(4 \times (1^2 + 1^2) + 4 \times 3^2) = 5.5$.

5. For 8-ASK and unit-energy pulse shaping $g(t)$,

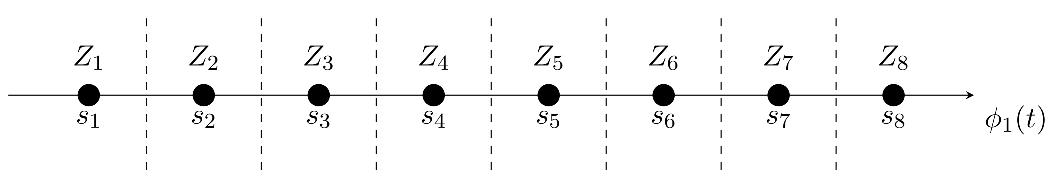
   (a) Given $d_{\min} = 2d = 2$, then $d = 1$ and the signal space representation is



   (b) Gray code is shown below



   (c) To minimize error probability assuming AWGN, decision regions are equivalent to nearest distance as shown below.



   (d) To keep the same bit error probability, we need to keep $d_{\min} = 2$ the same. Hence,

$$E_s[\text{8-ASK}] = \frac{M^2 - 1}{3} d^2 = \frac{8^2 - 1}{3} 1^2 = 21$$

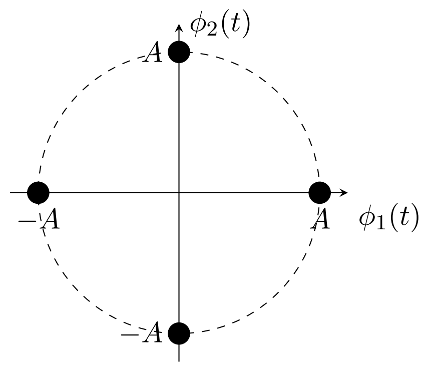
and

$$E_s[\text{64-ASK}] = \frac{M^2-1}{3}d^2 = \frac{64^2-1}{3}1^2 = 1365$$

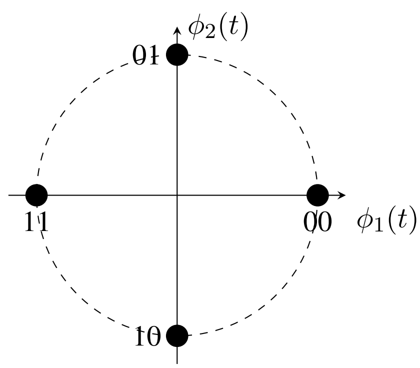
Hence, we need $(1365-21)/21 = 64 = 18$ dB additional energy per symbol to double the number of bits while keeping the same bit error probability.

6. For 4-PSK and unit-energy pulse shaping $g(t)$,

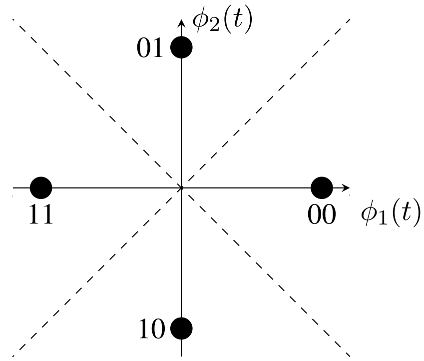
   (a) Given $d_{\text{min}} = 2A\sin(\pi/4) = \sqrt{3}$, then $A = \sqrt{3/2} = 1.2247$ and the signal space representation is

   (b) Gray code is shown below

   (c) To minimize error probability assuming AWGN, decision regions are equivalent to nearest distance as shown below.



(d) To keep the same bit error probability, we need to keep $d_{\min} = \sqrt{3}$ the same. Hence,

$$E_s[\text{4-PSK}] = A^2 = \left(\frac{d_{\min}}{2\sin(\pi/4)}\right)^2 = 3/2$$

and

$$E_s[\text{16-ASK}] = \left(\frac{d_{\min}}{2\sin(\pi/16)}\right)^2 = 19.72$$

Hence, we need $(19.72 - 1.5)/1.5 = 12 = 11$ dB additional energy per symbol to double the number of bits while keeping the same error probability.

7. Consider an M-QAM modulation with a square signal constellation of size $M = L^2$. This system can be seen as two M-ASK systems with a signal constellation of size $L = \sqrt{M}$ (one over the in-phase component and one over the quadrature component). For L-ASK, we know that

$$P_s[\text{L-ASK}] = 2\left(\frac{L-1}{L}\right) Q\left(\sqrt{\frac{3E_s}{(L^2-1)N_0}}\right)$$

where E_s is the average energy per symbol in M-QAM.

The probability that both M-ASK components are received in the correct decision region is $(1 - P_s[\text{L-ASK}])^2$. Hence, the symbol error probability for M-QAM systems is

$$\begin{aligned} P_s[\text{M-QAM}] &= 1 - (1 - P_s[\text{L-ASK}])^2 \\ &= 1 - \left[1 - 2\left(\frac{\sqrt{M}-1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3E_s}{(M-1)N_0}}\right)\right]^2 \end{aligned}$$

8. The condition for orthogonality is

$$\int_0^{T_s} \cos(2\pi f_i t)\cos(2\pi f_j t)\mathrm{d}t = 0$$

which we will use to derive $\Delta f = f_i - f_j$ that satisfies this constraint as follows.

$$\begin{aligned}\int_0^{T_s} \cos(2\pi f_i t)\cos(2\pi f_j t)\mathrm{d}t &= \frac{1}{2}\int_0^{T_s} [\cos(2\pi(f_i+f_j)t) + \cos(2\pi(f_i-f_j)t)]\mathrm{d}t \\ &= \frac{\sin(2\pi(f_i+f_j)t)}{2\pi(f_i+f_j)}\Big|_0^{T_s} + \frac{\sin(2\pi(f_i-f_j)t)}{2\pi(f_i-f_j)}\Big|_0^{T_s} \\ &= \frac{\sin(2\pi(f_i+f_j)T_s)}{2\pi(f_i+f_j)} + \frac{\sin(2\pi(f_i-f_j)T_s)}{2\pi(f_i-f_j)}\end{aligned}$$

The first term goes to zero as $f_i + f_j \gg 1$ in the denominator while $-1 \leq \sin(x) \leq 1$ in the numerator. For the second term to be zero, we have

$$\sin(2\pi(f_i - f_j)T_s) = 0 \quad \rightarrow \quad 2\pi(f_i - f_j)T_s = \pm n\pi$$

For the minimum frequency separation to be orthogonal, $\Delta f = f_i - f_j = \frac{1}{2T_s}$.

9. For an M-PAM communication system with a unit-energy pulse shaping filter, the bit error probability is

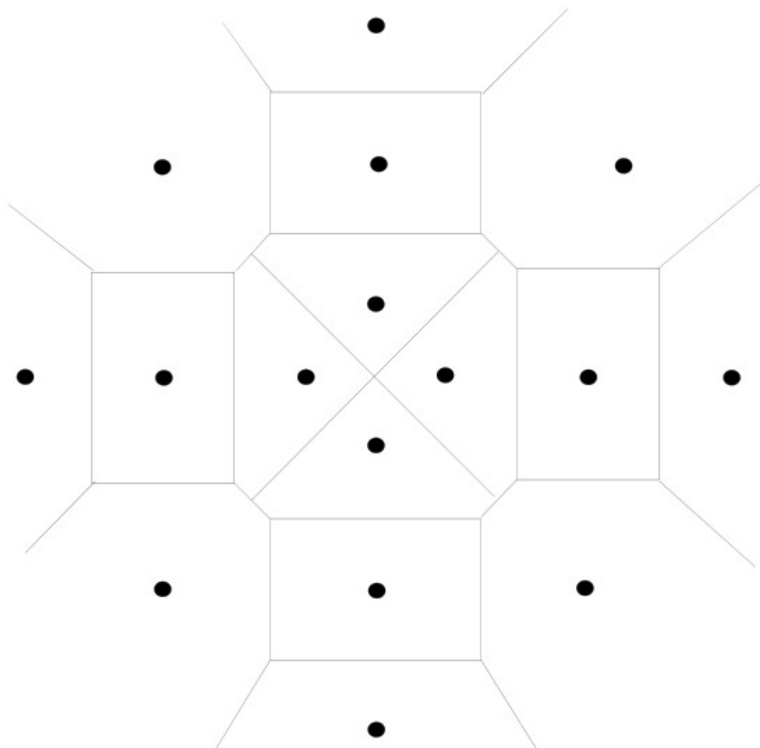
$$P_b \approx \frac{P_s}{\log_2 M} = \frac{2}{\log_2 M}\left(\frac{M-1}{M}\right) Q\left(\sqrt{\frac{6E_s}{(M^2-1)N_0}}\right)$$

Given $M = 8$, we get

$$P_b = \frac{7}{12} Q\left(\sqrt{\frac{6E_s}{63N_0}}\right) = 10^{-6} \rightarrow Q\left(\sqrt{\frac{6E_s}{63N_0}}\right) = 1.714 \times 10^{-6}$$

Using the Q-function table, find that $Q(4.64) = 1.714 \times 10^{-6}$. Hence, $E_s/N_0 = 226$ and $E_b/N_0 = 226/3 = 75.33$.

10. The optimum decision boundary of a point is determined by the perpendicular bisectors of each line segment connecting the point with its neighbors.



11. The bandwidth of the bandpass channel $W = 3000 - 600 = 2400$ Hz. Also, for QPSK system with $R_b = 2400$ bits/sec, the required bandwidth is

$$W = (1+\alpha)\frac{R_b}{\log_2 4} = (1+\alpha)\frac{2400}{2} = 2400 \rightarrow \alpha = 1$$

If the bit rate becomes $R_b = 4800$ bits/sec, the roll-off factor can be calculated as

$$(1+\alpha)\frac{4800}{2} = 2400 \rightarrow \alpha = 0$$

This means the pulse shape is a simple sinc function. For the block diagram, please refer to the lecture notes.

12. The bandwidth of the bandpass channel $W = 3300 - 300 = 3000$ Hz. The number of bits per symbol is $R_b/R_s = 9600/2400 = 4$ bits per symbol. Hence, we need to use a 16-QAM to achieve the desired bit rate.

For a 16-QAM system with $R_b = 9600$ bits/sec, the required bandwidth is

$$W = (1+\alpha)\frac{R_b}{\log_2 16} = (1+\alpha)\frac{9600}{4} = 3000 \rightarrow \alpha = 0.25$$

13. For a QAM system, the error probability is

$$P_s = 1 - \left[1 - 2\left(\frac{\sqrt{M}-1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3E_s}{(M-1)N_0}}\right)\right]^2$$

Consider a digital communication system that transmits information via QAM over a voice-band telephone channel at a rate 2400 symbols per second. The additive noise is assumed to be white and Gaussian.

(a) We know that $k = 4800/2400 = 2$, hence $M = 4$. Given $P_s = 10^{-5}$, we can find $E_b/N_0 = 9.7682 = 9.89$ dB.

(b) We know that $k = 9600/2400 = 4$, hence $M = 16$. Given $P_s = 10^{-5}$, we can find $E_b/N_0 = 25.3688 = 14.04$ dB.

(c) We know that $k = 19200/2400 = 8$, hence $M = 256$. Given $P_s = 10^{-5}$, we can find $E_b/N_0 = 659.8922 = 28.19$ dB.

(d) It is clear that there is an increase in transmitted power of approximately 3 dB per additional bit per symbol.