ELEC 4190 – Digital Communications

Digital Data Transmission

Notes:

sBJksDRPrf96kfrB1XsIog

Outline

- Introduction
- Geometric representation of signals
- Baseband and passband PAM
- PSK, QAM, and FSK
- Optimal receivers and performance over AWGN channels
- Pulse shaping
- Eye diagram

- Recommended reading: Proakis and Salehi – Chapter 8, 9, and 10
- Extra reading: Goldsmith – Chapters 5 and 6

Notes:

Outline

- Introduction
- Geometric representation of signals
- Baseband and passband PAM
- PSK, QAM, and FSK
- Optimal receivers and performance over AWGN channels
- Pulse shaping
- Eye diagram

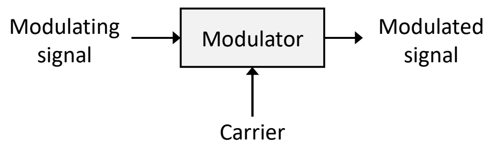
Notes:

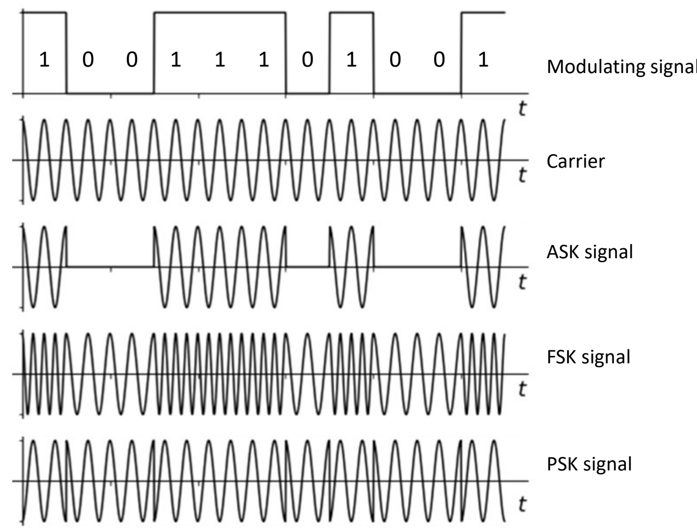
Digital Modulation

- Linear Modulation:
  - Amplitude of modulated signal varies linearly with the modulating signal
  - Information encoded in the amplitude and/or phase of the carrier signal
  - Example: PAM, ASK, PSK, and QAM
  - Better spectral efficiency, but power inefficient
- Nonlinear Modulation:
  - Amplitude of modulated signal is typically constant and does not change with the modulating signal
  - Information encoded in the frequency of the carrier signal
  - Example: FSK, MSK, and CPFSK
  - Better power efficiency, but bandwidth inefficient

Notes:

Example: Binary Modulation





Notes:

Binary vs. M-ary Digital Modulation

- In binary modulation, the digital modulator maps each information bit into one of two possible signal waveforms
  - e.g., 0: $s_1(t)$ and 1: $s_2(t)$
- Alternatively, in M-ary modulation, the modulator may map k bits (i.e., symbol) at a time into one of $M = 2^k$ distinct waveforms
  - e.g., 00: $s_1(t)$, 01: $s_2(t)$, 10: $s_3(t)$, and 11: $s_4(t)$ when $k = 2$ (or $M=4$)
- Given symbol duration T_s and bit duration T_b the symbol rate and bit rate is

$$R_s = \frac{1}{T_s} \qquad T_b = \frac{T_s}{k} = \frac{T_s}{\log_2 M}$$

$$R_b = \frac{1}{T_b} = \frac{k}{T_s} = \frac{\log_2 M}{T_s}$$

Notes:

Outline

- Introduction
- Geometric representation of signals
- Baseband and passband PAM
- PSK, QAM, and FSK
- Optimal receivers and performance over AWGN channels
- Pulse shaping
- Eye diagram

Notes:

Geometric Representation of Signals

- Any set of M finite-energy signal waveforms $S = \{s_i(t), I = 1, 2, \ldots, M\}$ defined over $[0, T_s)$ can be represented as a linear combination of $N \leq M$ real orthonormal basis functions $\{\phi_m(t), m = 1, 2, \ldots, M\}$

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t), \quad 0 \leq t < T_s \quad \text{where} \quad s_{ij} = \int_0^{T_s} s_i(t)\phi_j(t)dt$$

- That is, a signal $s_i(t)$ can be represented as a vector $\mathbf{s}_i$

$$\mathbf{s}_i = (s_{i1}, \cdots, s_{iN}) \in \mathbb{R}^N \longleftarrow$$

signal constellation point in N dimensional signal space

- A set of signals are said to be orthonormal when

$$\int_0^{T_s} \phi_i(t)\phi_j(t)dt = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Notes:

Example

Check if the following can be a basis function assuming $f_c T_s >> 1$

$$\phi_1(t) = \sqrt{\frac{2}{T_s}} \cos(2\pi f_c t), \quad 0 \le t < T_s$$

If yes, use it to represent the set of signals $S = \{s_1(t), s_2(t)\}$

$$s_1(t) = \alpha \cos(2\pi f_c t), \quad 0 \le t < T_s$$

$$s_2(t) = -\alpha \cos(2\pi f_c t), \quad 0 \le t < T_s$$

Notes:

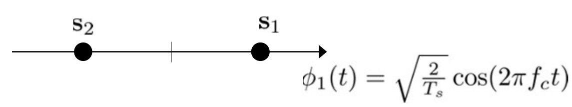
Signal Space Representation

- It is the representation of $s_i(t)$ in terms of its constellation point $\mathbf{s}_i$

$$s_i(t) \Leftrightarrow \mathbf{s}_i = (s_{i1}, \cdots, s_{iN}) \in \mathbb{R}^N \quad \text{where} \quad s_{ij} = \int_0^{T_s} s_i(t)\phi_j(t)dt$$

- The vector space containing the constellation is called the signal space

$$\begin{aligned} s_1(t) &= \alpha\cos(2\pi f_c t), \quad 0 \le t < T_s \\ s_2(t) &= -\alpha\cos(2\pi f_c t), \quad 0 \le t < T_s \end{aligned} \longleftrightarrow \begin{aligned} \mathbf{s}_1 &= \left(\alpha\sqrt{\frac{T_s}{2}}\right) \\ \mathbf{s}_2 &= \left(-\alpha\sqrt{\frac{T_s}{2}}\right) \end{aligned}$$



Notes:

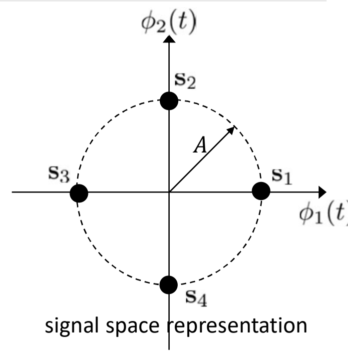
Example

Given the following two basis functions

$$\phi_1(t) = \sqrt{\frac{2}{T_s}}\cos(2\pi f_c t), \quad 0 \le t < T_s$$

$$\phi_2(t) = \sqrt{\frac{2}{T_s}}\sin(2\pi f_c t), \quad 0 \le t < T_s$$

Find signals $s_1(t), s_2(t), s_3(t), s_4(t)$.



Notes:

sBJksDRPrf96kfrB1XsIog

Vector Space Properties

- For a vector $\mathbf{s}_i$, we can define the following:
  - Vector length (or norm of signal)

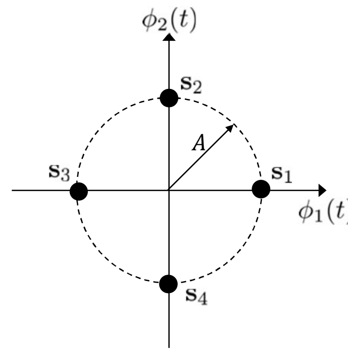
$$\|\mathbf{s}_i\| = \sqrt{\sum_{j=1}^{N} |s_{ij}|^2}$$

  - Distance between two signal constellation points

$$\|\mathbf{s}_i - \mathbf{s}_k\| = \sqrt{\sum_{j=1}^{N} (s_{ij} - s_{kj})^2}$$

  - Inner product between two vectors

$$\langle \mathbf{s}_i, \mathbf{s}_k \rangle = \mathbf{s}_i \mathbf{s}_k^T = \int_0^T s_i(t) s_k(t) dt$$



Notes:

sBJksDRPrf96kfrB1XsIog

Vector Space Properties (cont.)

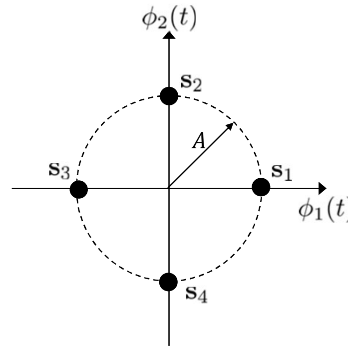
- For a vector $\mathbf{s}_i$, we can define the following:
  - Signal energy

$$E_{s_i} = \int_{-\infty}^{\infty} s_i^2(t)dt = \sum_{j=1}^{N} s_{ij}^2$$

  - Average signal energy

$$E_s = \frac{1}{M}\sum_{i=1}^{M} E_{s_i}$$

  - A set of vectors are orthonormal if their inner product is 0 and each vector has a unit norm

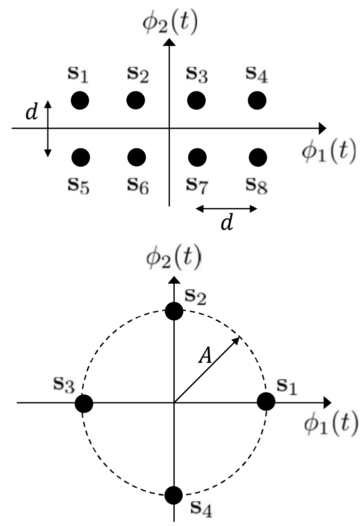


Notes:

sBJksDRPrf96kfrB1XsIog

Example

Calculate the average energy and $d_{\min}$ of the two constellations below.



Notes:

sBJksDRPrf96kfrB1XsIog

Gram-Schmidt Orthogonalization Procedure

- Given $\{s_1(t), \ldots, s_M(t)\}$, we need to construct a set of $N \leq M$ orthonormal basis functions:
  1. Begin with $\mathrm{d}_1(\mathrm{t}) = s_1(t)$ to calculate $\phi_1(t)$ such that
  $$\phi_1(t) = \frac{d_1(t)}{\sqrt{E_{d_1}}}, \quad E_{d_i} = \int_{-\infty}^{+\infty} d_i^2(t)dt$$
  2. Calculate $\phi_2(t)$ using the projection of $s_2(t)$ onto $\phi_1(t)$ as follows
  $$\phi_2(t) = \frac{d_2(t)}{\sqrt{E_{d_2}}} = \frac{s_2(t) - c_{21}\phi_1(t)}{\sqrt{E_{d_2}}}, \quad c_{21} = \int_{-\infty}^{+\infty} s_2(t)\phi_1(t)dt$$
  3. Repeat until all M waveforms are exhausted
     - If at any step $\phi_k(t) = 0$, then there is no new basis function
  $$d_k(t) = s_k(t) - \sum_{i=1}^{k-1} c_{ki}\phi_i(t), \quad c_{ki} = \int_{-\infty}^{+\infty} s_k(t)\phi_i(t)dt$$

Notes:

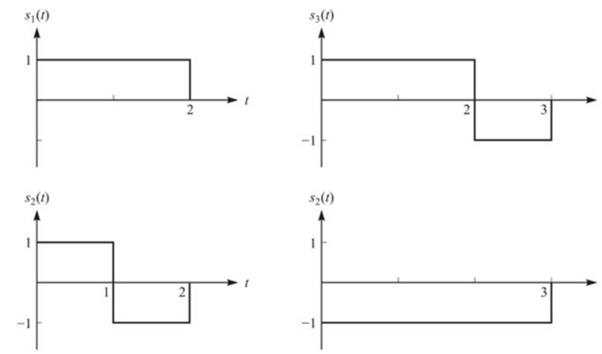
sBJksDRPrf96kfrB1XsIog

Example

Apply the Gram-Schmidt procedure to the set of four waveforms below.

- Hint: to determine dimensionality, you can check how many columns are linearly independent

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & -1 & -1 \end{pmatrix}$$



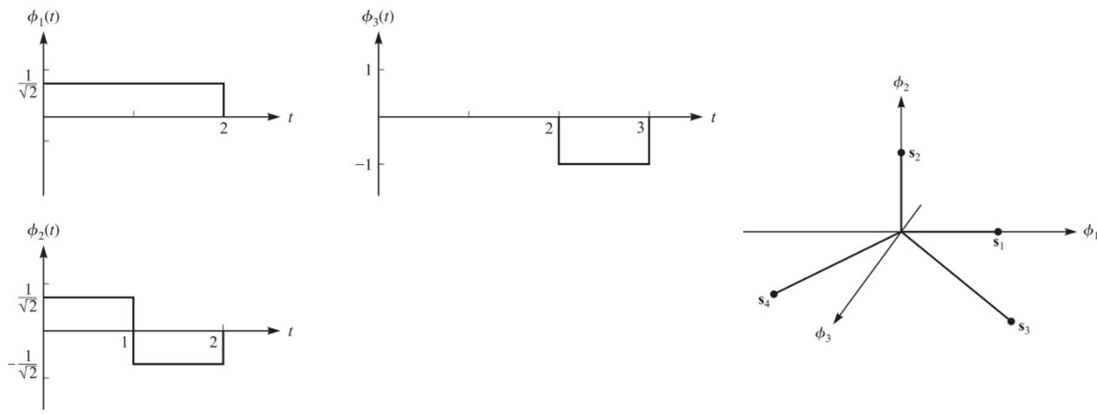
Notes:

sBJksDRPrf96kfrB1XsIog

Example (cont.)

Apply the Gram-Schmidt procedure to the set of four waveforms below.

- Solution:



Notes:

Outline

- Introduction
- Geometric representation of signals
- Baseband and passband PAM
- PSK, QAM, and FSK
- Optimal receivers and performance over AWGN channels
- Pulse shaping
- Eye diagram

Notes:

Baseband Pulse Amplitude Modulation (PAM)

- In M-ary baseband PAM, only one basis function is used (i.e., 1-D signal space)

$$\phi_1(t) = g(t)$$

- $g(t)$ is a unit-energy baseband pulse shaping filter to improve the spectral characteristics of the transmitted signal
  - Example 1: Non-Return-to-Zero (NRZ) pulse

$$g(t) = \sqrt{\frac{1}{T_s}}, \quad 0 \le t < T_s$$

  - Example 2: Half-sine (HS) pulse

$$g(t) = \sqrt{\frac{2}{T_s}} \sin\left(\frac{\pi t}{T_s}\right), \quad 0 \le t < T_s$$

Notes:

$\phi_1(t) = g(t)$

M-PAM

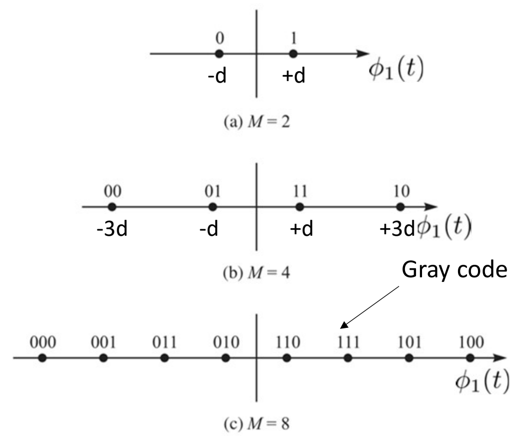
- In general, the transmitted signal is

$$s_i(t) = \boxed{A_i} g(t), \quad 0 \le t < T_s$$
$$A_i = (2i - 1 - M)d, \quad i = 1, 2, \ldots, M$$

  for some distance *d*

- The signal can be represented in the vector space as

$$s_i(t) = s_{i1}\phi_1(t) \quad \Leftrightarrow \quad \mathbf{s}_i = (A_i)$$

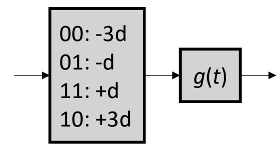


signal space representation

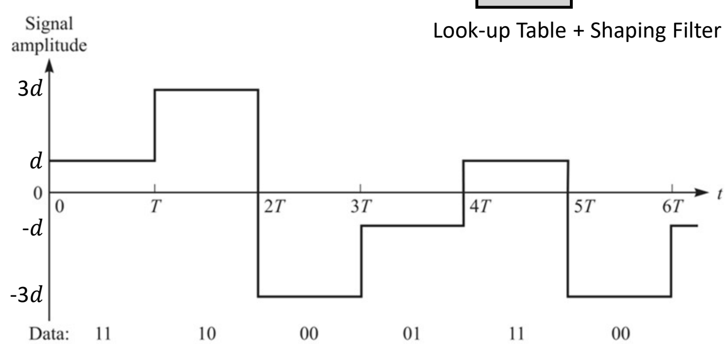
Notes:

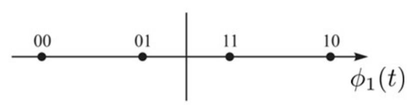
Example

- 4-PAM for data 111000011100



Look-up Table + Shaping Filter



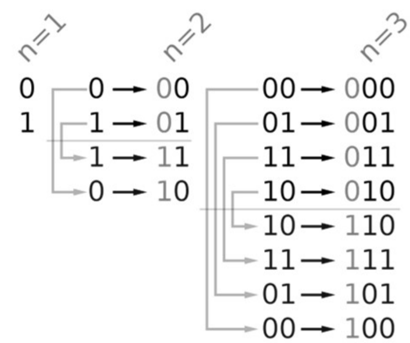


Notes:

sBJksDRPrf96kfrB1XsIog

Notes on Gray Coding

- Mapping is done such that adjacent symbols differ in only one bit
- Steps to generate n-bit gray code
  1. Start with 1-bit code
  2. Reflect
  3. Prefix old entries with 0
  4. Prefix new entries with 1
  5. Concatenate
  6. Repeat



Wikipedia

Notes:

sBJksDRPrf96kfrB1XsIog

Bandpass Pulse Amplitude Modulation (PAM)

- In M-ary bandpass PAM, only one basis function is used (i.e., 1-D signal space)

$$\phi_1(t) = \sqrt{2}g(t)\cos(2\pi f_c t), \quad 0 \leq t < T_s$$

- $g(t)$ is a unit-energy baseband pulse shaping filter
- In this case, the transmitted signal is

$$s_i(t) = \boxed{A_i}\underline{\sqrt{2}g(t)\cos(2\pi f_c t)}, \quad 0 \leq t < T_s \gg 1/f_c$$
$$A_i = (2i - 1 - M)d, \quad i = 1, 2, \ldots, M$$

for some distance *d*

- Bandpass PAM is also called amplitude shift keying (ASK)

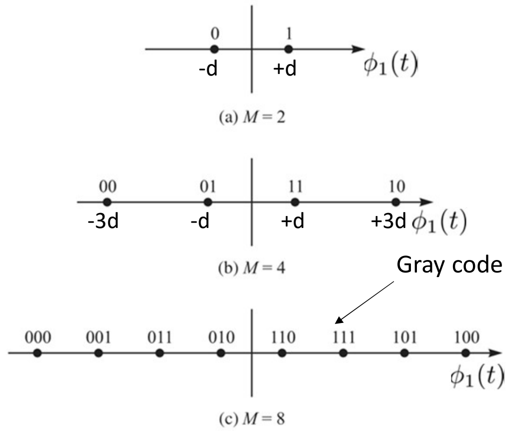
Notes:

$$\phi_1(t) = \sqrt{2}g(t)\cos(2\pi f_c t), \quad 0 \le t < T_s$$

M-ASK

- The signal can be represented in the vector space as

$$s_i(t) = s_{i1}\phi_1(t) \quad \Leftrightarrow \quad \mathbf{s}_i = (A_i)$$



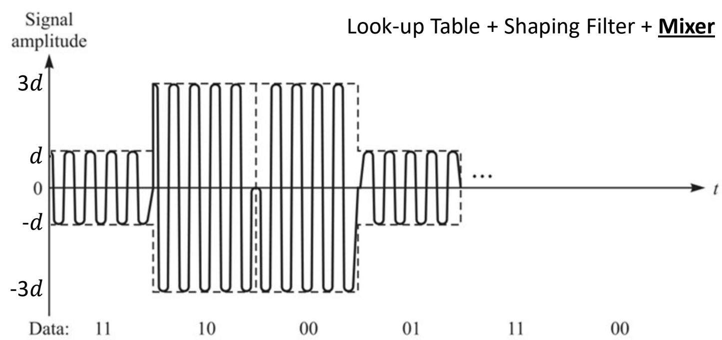
signal space representation

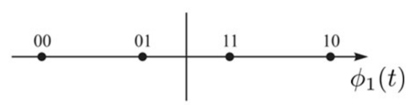
Notes:

sBJksDRPrf96kfrB1XsIog

Example

- 4-ASK for data 111000011100



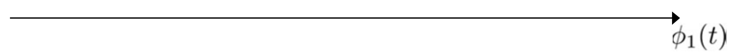


Notes:

sBJksDRPrf96kfrB1XsIog

Example

For a unit-energy g(t), sketch the signal space representation for M-ASK when the symbol length is 3 bits. What is the minimum distance between symbols?

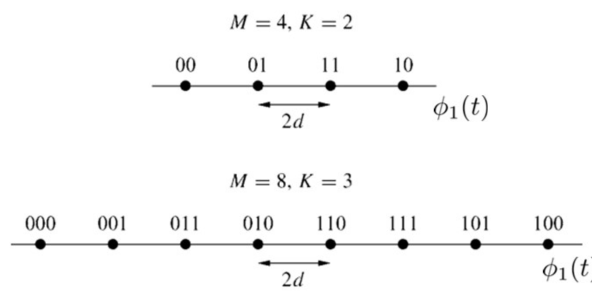


$$d_{\min} = 2d$$

Notes:

Example

For a unit-energy g(t), find the average symbol energy of 4-ASK and 8-ASK modulations.



Notes:

sBJksDRPrf96kfrB1XsIog

Recap

- The average energy of M-ASK (and M-PAM) modulation is given by

$$E_{s_i} = A_i^2 \rightarrow E_s = \frac{1}{M}\sum_{i=1}^{M} A_i^2 = \frac{(M^2-1)d^2}{3} \rightarrow E_b = \frac{E_s}{\log_2 M}$$

- The minimum distance is

$$d_{\min} = 2d = \sqrt{\frac{12\log_2 M}{M^2-1}E_b}$$

- The special case when $M = 2$ is sometimes called binary antipodal signaling since $s_1(t) = -s_2(t)$

Notes:

Outline

- Introduction
- Geometric representation of signals
- Baseband and passband PAM
- PSK, QAM, and FSK
- Optimal receivers and performance over AWGN channels
- Pulse shaping
- Eye diagram

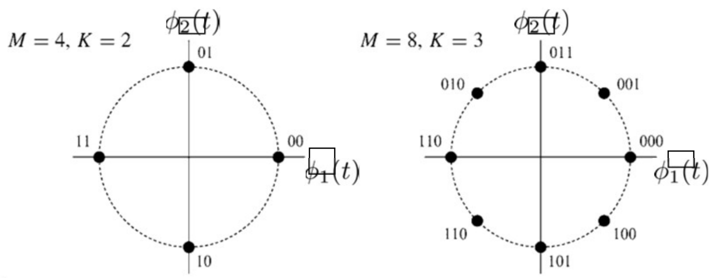
Notes:

Phase Shift Keying (PSK)

- What if we change the phase of the cosine instead of the amplitude?
- The transmitted signal is

$$s_i(t) = A\sqrt{2}g(t)\cos\left(2\pi f_c t + \frac{2\pi(i-1)}{M}\right), \quad 0 \le t < T_s, \quad i = 1, \ldots M$$

$$= \boxed{A\cos\left(\frac{2\pi(i-1)}{M}\right)}\underline{\sqrt{2}g(t)\cos(2\pi f_c t)} - \boxed{A\sin\left(\frac{2\pi(i-1)}{M}\right)}\underline{\sqrt{2}g(t)\sin(2\pi f_c t)}$$



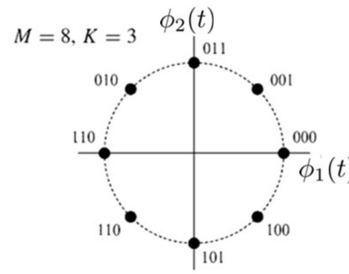
Notes:

M-PSK

- Note that we need two basis function (i.e., 2-D signal space)

$$s_i(t) = s_{i1}\phi_1(t) + s_{i2}\phi_2(t) \Leftrightarrow \mathbf{s}_i = \left(A\cos\left(\frac{2\pi(i-1)}{M}\right), A\sin\left(\frac{2\pi(i-1)}{M}\right)\right)$$

$$\phi_1(t) = \sqrt{2}g(t)\cos(2\pi f_c t), \quad 0 \le t < T_s$$
$$\phi_2(t) = -\sqrt{2}g(t)\sin(2\pi f_c t), \quad 0 \le t < T_s$$

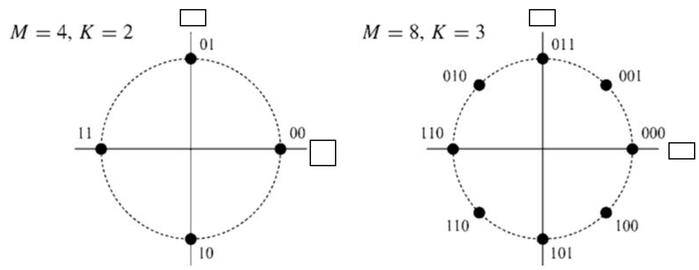


Notes:

sBJksDRPrf96kfrB1XsIog

Example

For a unit-energy g(t), find the average energy and $d_{\min}$ of Quaternary PSK (QPSK) and 8-PSK modulations.



$$d_{\min} = 2A \sin\left(\frac{\pi}{M}\right)$$

Notes:

Differential PSK

- Information in M-PSK signals is carried in the signal phase, thus requires coherent demodulation
  - That is, the phase of the transmitted signal carrier must be matched to the phase of the receiver carrier
- In general, techniques for phase recovery typically require more complexity and cost at the receiver
- In differential encoding, the information is conveyed by phase shifts between any two successive signal intervals
  - That is, the symbol transmitted depends on:
    - The bits associated with the current symbol to be transmitted and
    - The bits transmitted over prior symbol time

Notes:

Differential PSK (cont.)

- So, phase of previous symbol is used as a reference instead
  - Hence, called modulation with memory (vs. memoryless)
- For example, in differential BPSK,
  - Information bit 0 is transmitted by a zero-phase shift relative to the phase in the preceding signaling interval
  - Information bit 1 may be transmitted by shifting the phase of the carrier by π
- Similarly, in DQPSK, the phase transitions are
  - 00: 0
  - 01: $\pi/2$
  - 11: π
  - 10: $-\pi/2$

Notes:

Example

Find the sequence of symbols transmitted using DPSK for the bit sequence 101110 starting at the k-th symbol time, assuming the transmitted symbol at the (k-1)-th symbol time was $\mathbf{s}(k\text{-}1) = Ae^{j\pi}$.

Notes:

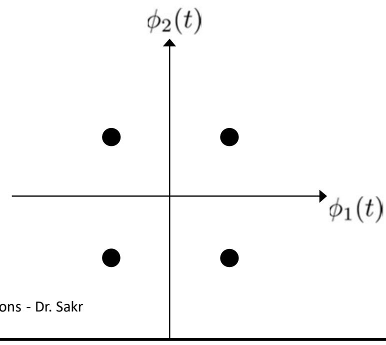
Quadrature Amplitude Modulation (QAM)

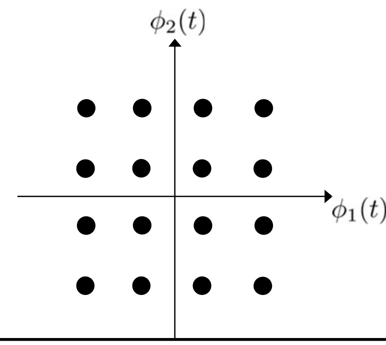
- What if we change both the phase and amplitude of the cosine?
- Typically consists of two $\sqrt{M}$-ASK signals in-phase and quadrature

$$s_i(t) = \boxed{A_i}\underline{\sqrt{2}g(t)\cos(2\pi f_c t)} - \boxed{B_i}\underline{\sqrt{2}g(t)\sin(2\pi f_c t)}, \quad 0 \le t < T_s$$

$$A_i = (2i - 1 - \sqrt{M})d, \quad i = 1, 2, \ldots, \sqrt{M}$$

$$B_i = (2i - 1 - \sqrt{M})d, \quad i = 1, 2, \ldots, \sqrt{M}$$





Notes:

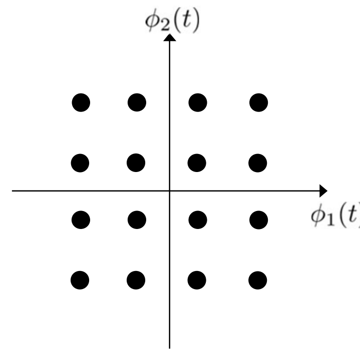
sBJksDRPrf96kfrB1XsIog

M-QAM Modulation (cont.)

- Note that we need two basis function (i.e., 2-D signal space)

$$s_i(t) = s_{i1}\phi_1(t) + s_{i2}\phi_2(t) \Leftrightarrow \mathbf{s}_i = (A_i, B_i)$$

$$\phi_1(t) = \sqrt{2}g(t)\cos(2\pi f_c t), \quad 0 \le t < T_s$$
$$\phi_2(t) = -\sqrt{2}g(t)\sin(2\pi f_c t), \quad 0 \le t < T_s$$

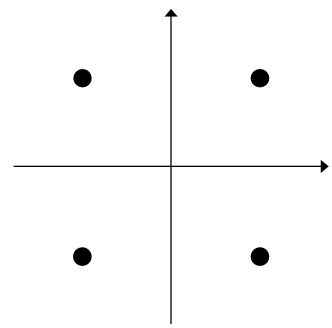


Notes:

$$A_i = (2i - 1 - \sqrt{M})d, \quad i = 1, 2, \ldots, \sqrt{M}$$
$$B_i = (2i - 1 - \sqrt{M})d, \quad i = 1, 2, \ldots, \sqrt{M}$$

Example

For a unit-energy g(t), find the average energy and $d_{\min}$ of 4-QAM modulation.



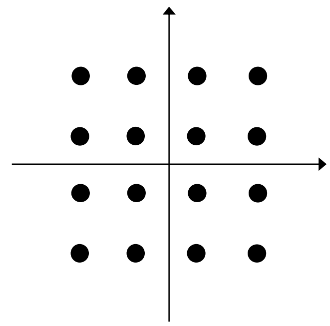
Notes:

sBJksDRPrf96kfrB1XsIog

$$A_i = (2i - 1 - \sqrt{M})d, \quad i = 1, 2, \ldots, \sqrt{M}$$
$$B_i = (2i - 1 - \sqrt{M})d, \quad i = 1, 2, \ldots, \sqrt{M}$$

Example

For a unit-energy g(t), find the average energy and $d_{\min}$ of 16-QAM modulation.



- For square constellations:

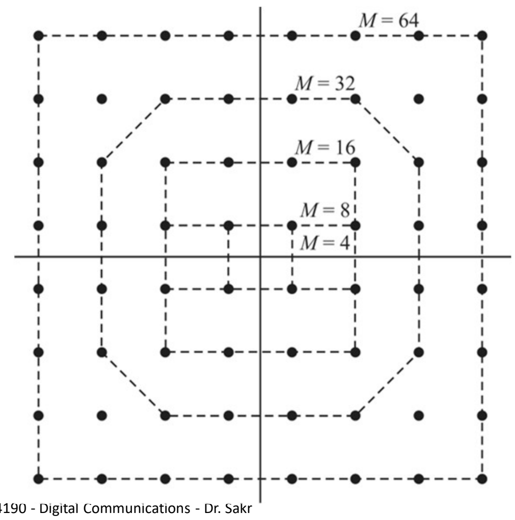
$$d_{\min} = 2d \qquad\qquad E_s = \frac{2}{3}(M-1)d^2$$

Notes:

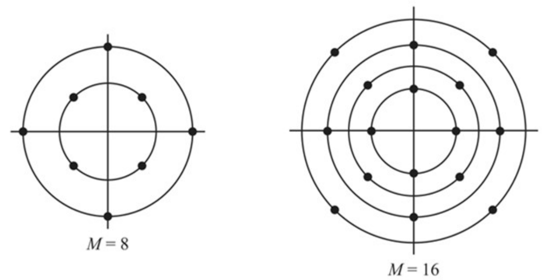
sBJksDRPrf96kfrB1XsIog

Signal-space Constellations for QAM

Rectangular:



PAM-PSK:



Notes:

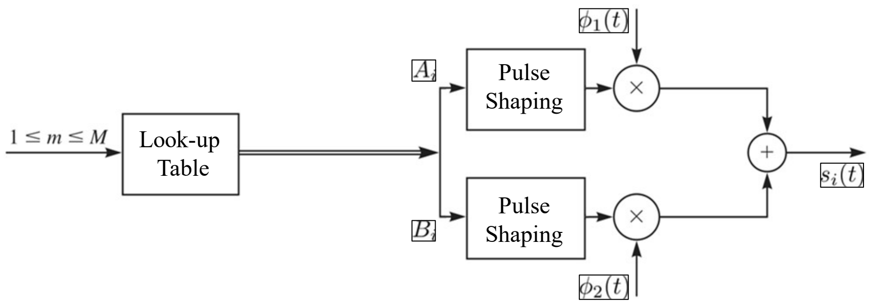
sBJksDRPrf96kfrB1XsIog

Recap

- It is clear that ASK, PSK, and QAM belong to the same family
- In general, the transmitted signal is of the general form

$$s_i(t) = A_i\sqrt{2}g(t)\cos(2\pi f_c t) - B_i\sqrt{2}g(t)\sin(2\pi f_c t), \quad 0 \le t < T_s, \quad i = 1, \dots M$$

- A general structure of the modulator looks like this:



Notes:

sBJksDRPrf96kfrB1XsIog

Frequency Shift Keying (FSK)

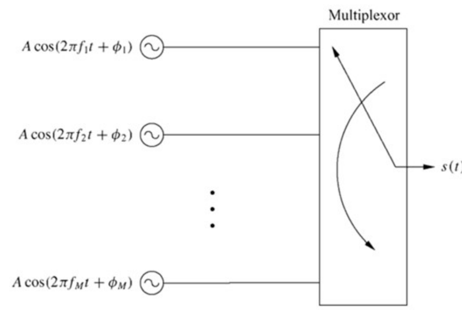
- What if we change the frequency of the cosine?
- The transmitted signal is

$$s_i(t) = \boxed{A}\sqrt{2}g(t)\underline{\cos\left(2\pi(f_c + \alpha_i \Delta f_c)t\right.} + \left.\phi_i\right), \quad 0 \le t < T_s, \quad i = 1, \ldots M$$

$$\alpha_i = (2i - 1 - M), \quad i = 1, 2, \ldots, M$$

- To ensure orthogonality, minimum frequency separation should be

$$\Delta f = 2\Delta f_c = \begin{cases} \dfrac{1}{2T_s}, & \phi_i = \phi_j \\ \dfrac{1}{T_s}, & \phi_i \neq \phi_j \end{cases}$$



Notes:

sBJksDRPrf96kfrB1XsIog

M-FSK

- Note that we need *M* basis functions (i.e., multidimensional signaling)

$$\phi_j(t) = \sqrt{2}g(t)\cos(2\pi(f_c + \alpha_j \Delta f_c)t + \phi_j), \quad 0 \leq t < T_s$$

- M-FSK is a special case called orthogonal signaling, i.e.,

$$s_i(t) = s_{i1}\phi_i(t)$$

- Discontinuous phase transitions may happen due to phase offsets between oscillators
- Minimum-shift Keying (MSK) is a special case of binary FSK where $\phi_1 = \phi_2$
- Continuous-Phase FSK (CPFSK) is a version of FSK that eliminates phase discontinuity using signaling with memory

Notes:

Example

For a unit-energy $g(t)$, express the basis functions and find the average energy of 4-FSK modulation. Express the four signals as vectors.

Notes:

s1 = (A, 0, 0, 0)

sBJksDRPrf96kfrB1XsIog

Outline

- Introduction
- Geometric representation of signals
- Baseband and passband PAM
- PSK, QAM, and FSK
- Optimal receivers and performance over AWGN channels
- Pulse shaping
- Eye diagram

Notes:

$\mathbf{s}_i = (s_{i1}, \cdots, s_{iN})$
$\mathbf{n} = (n_1, \cdots, n_N)$
$\mathbf{r} = (r_1, \cdots, r_N)$
$r_j = s_{ij} + n_j$

Additive White Gaussian Noise (AWGN) Channel

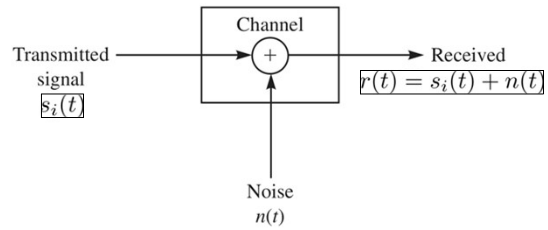
- Channel corrupts the transmitted signal by the addition of white Gaussian noise
- The received signal is

$$r(t) = s_i(t) + n(t), \quad 0 \le t < T_s$$

AWGN with PSD=$N_0/2$

$$\Updownarrow$$

$$\mathbf{r} = \mathbf{s}_i + \mathbf{n}$$



- Noise components are i.i.d. zero-mean Gaussian random variables with distribution $N(0, N_0/2)$

Number of basis functions

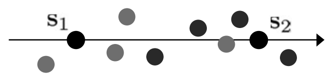
$$p(\mathbf{n}) = \frac{1}{(\pi N_0)^{N/2}} \exp\left[-\frac{1}{N_0}\sum_{j=1}^{N} n_j^2\right]$$

Noise vector $\mathbf{r}$-$\mathbf{s}_i$

Notes:

Optimal Receiver under AWGN

- Let's consider the Binary ASK case



- The receiver should make this decision over each time interval T_s:
  - For a received signal $\mathbf{r}$, which constellation point $\mathbf{s}_i$ (i.e., message m_i) was sent?

$$\hat{\mathbf{s}} = \arg\max_{\mathbf{s}_i} \; p(\mathbf{s}_i \text{ sent} \mid \mathbf{r}) \longleftarrow \text{Conditional probability}$$

Demo

Notes:

sBJksDRPrf96kfrB1XsIog

$r_j = s_{ij} + n_j$

Optimal Receiver under AWGN (cont.)

- Conditioned on $\mathbf{s}_i$, we know that r_j is an independent Gaussian random variable with mean s_{ij} and variance $N_0/2$, hence,

Likelihood function

$$L(\mathbf{s}_i) = p(\mathbf{r} \mid \mathbf{s}_i \text{ sent}) = \frac{1}{(\pi N_0)^{N/2}} \exp\left[\underbrace{-\frac{1}{N_0}\sum_{j=1}^{N}(r_j - s_{ij})^2}_{\text{Log likelihood function}}\right]$$

- From Bayes' rule

$$p(\mathbf{s}_i \mid \mathbf{r}) = \frac{p(\mathbf{r} \mid \mathbf{s}_i)p(\mathbf{s}_i)}{p(\mathbf{r})} \rightarrow \hat{\mathbf{s}} = \arg\max_{\mathbf{s}_i} \quad p(\mathbf{r} \mid \mathbf{s}_i)$$

- For maximum likelihood receiver, the optimal receiver is

$$\hat{\mathbf{s}} = \arg\min_{\mathbf{s}_i} \|\mathbf{r} - \mathbf{s}_i\|^2$$

Notes:

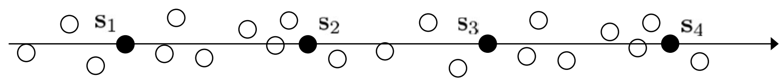
p(si)'s are not necessarily equal

sBJksDRPrf96kfrB1XsIog

$$\hat{\mathbf{s}} = \arg\min_{\mathbf{s}_i} \|\mathbf{r} - \mathbf{s}_i\|^2$$

Decision Regions

- Closest distance translates to a decision region Z_i for each symbol



$$Z_i = \{\mathbf{r} : \|\mathbf{r} - \mathbf{s}_i\| < \|\mathbf{r} - \mathbf{s}_j\| \forall j = 1, \ldots, M, j \neq i\}, \quad i = 1, \ldots M$$

- In other words, each received symbol is estimated to be the closest constellation point
- Main assumption:
  - All symbols are equiprobable with probability of 1/M

Notes:

$$s_i(t) = A_i g(t)\cos(2\pi f_c t), \quad 0 \le t < T_s$$
$$A_i = (2i-1-M)d, \quad i = 1, 2, \ldots, M$$

Example

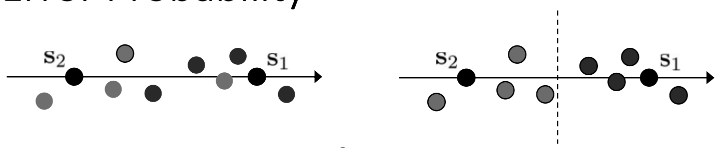
Find the decision regions corresponding to the constellations of the Binary ASK and 8-ASK.

Notes:

Z1 = {r1<0}

$\gamma_s = E_s/N_0$: SNR per symbol
$\gamma_b = E_b/N_0$: SNR per bit

Symbol Error Probability



- Decision regions are not error-free
- The error probability quantifies the error in detecting $\mathbf{s}_i$ given $\mathbf{r}$

$$P_s = \sum_{i=1}^{M} p(\mathbf{r} \notin Z_i \mid \mathbf{s}_i \text{ sent}) p(\mathbf{s}_i \text{ sent}) = \frac{1}{M} \sum_{i=1}^{M} p(\mathbf{r} \notin Z_i \mid \mathbf{s}_i \text{ sent})$$

- BER is

$$P_b \approx \frac{P_s}{\log_2 M}$$

Notes:

sBJksDRPrf96kfrB1XsIog

$\gamma_s = E_s/N_0$: SNR per symbol
$\gamma_b = E_b/N_0$: SNR per bit

Example

For Binary ASK with gray coding under AWGN, prove that:

$$P_s = P_b = Q\left(\frac{d}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

***Hint**: If X is a Gaussian random variable with mean μ and variance σ, then*
P(X>x) = Q((x- μ)/σ)

Notes:

sBJksDRPrf96kfrB1XsIog

$\gamma_s = E_s/N_0$: SNR per symbol
$\gamma_b = E_b/N_0$: SNR per bit

Example

For M-ASK with gray coding under AWGN, prove that:

$$P_s = 2\left(\frac{M-1}{M}\right) Q\left(\frac{d}{\sqrt{N_0/2}}\right) = 2\left(\frac{M-1}{M}\right) Q\left(\sqrt{\frac{6E_s}{(M^2-1)N_0}}\right)$$

Hint*: If X is a Gaussian random variable with mean μ and variance σ, then*

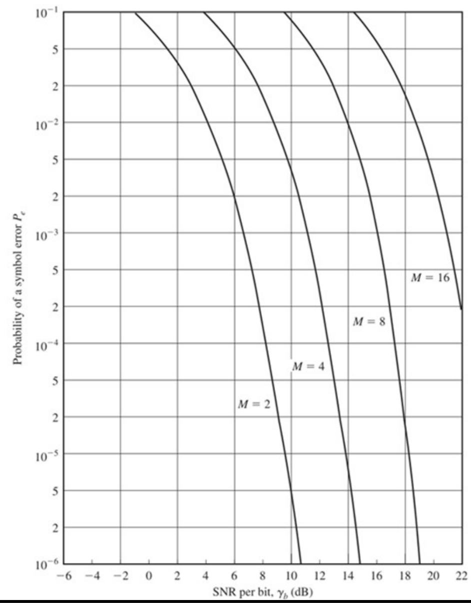
P(X>x) = Q((x- μ)/σ)

Notes:

sBJksDRPrf96kfrB1XsIog

ASK Symbol Error Probability vs. SNR per Bit

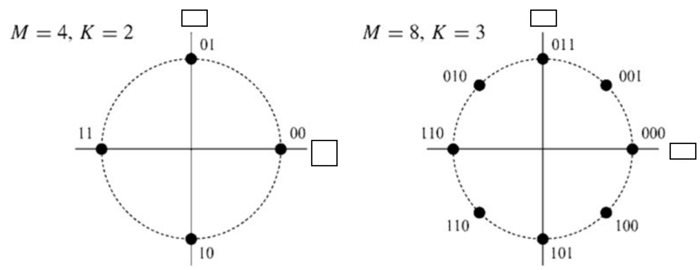
- It is clear that
  - Increasing *M* deteriorates the performance for the same SNR per bit
  - Increasing SNR improves the performance for the same M



Notes:

Example

Find the decision regions corresponding to the PSK constellations shown below.



$$Z_i = \left\{ \mathbf{r} = re^{j\theta} : \frac{2\pi(i-1.5)}{M} \le \theta < \frac{2\pi(i+1.5)}{M} \right\}, \quad i = 1, \dots M$$

Notes:

sBJksDRPrf96kfrB1XsIog

$\gamma_s = E_s/N_0$: SNR per symbol
$\gamma_b = E_b/N_0$: SNR per bit

PSK Symbol Error Probability vs. SNR per Bit

- BPSK:

$$P_s = Q\left(\sqrt{\frac{E_s}{N_0/2}}\right)$$

- QPSK:

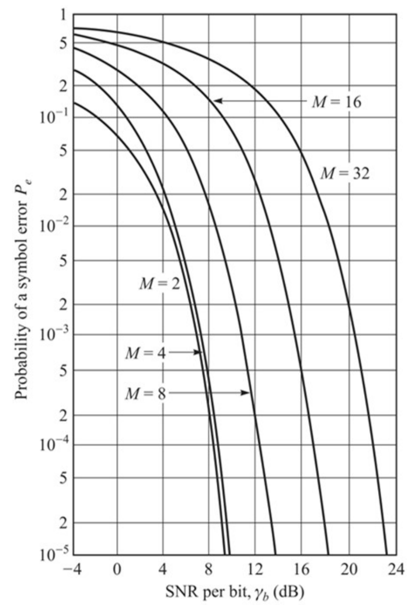
$$P_s = 1 - \left[1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right]^2 \leq 2Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

- M-PSK:

$$P_s \leq 2Q\left(\sqrt{\frac{E_s}{N_0/2}}\sin\left(\frac{\pi}{M}\right)\right)$$

- BER is

$$P_b \approx \frac{P_s}{\log_2 M}$$

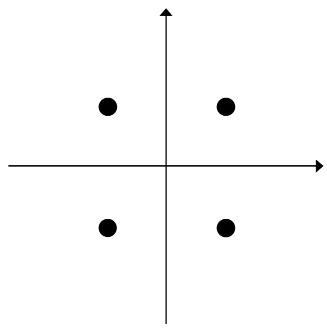


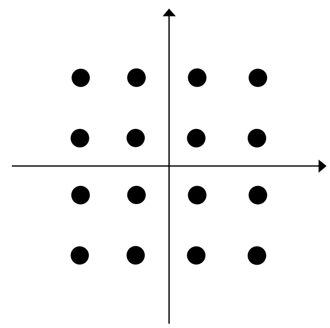
Notes:

sBJksDRPrf96kfrB1XsIog

Example

Find the decision regions corresponding to the QAM constellations shown below.



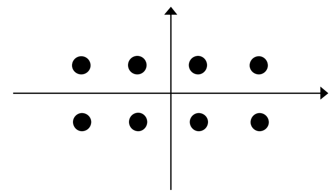


Notes:

sBJksDRPrf96kfrB1XsIog

Example

For a unit-energy g(t), find the average energy, $d_{\min}$, and decision regions of 8-QAM constellation below.



Notes:

sBJksDRPrf96kfrB1XsIog

$\gamma_s = E_s/N_0$: SNR per symbol
$\gamma_b = E_b/N_0$: SNR per bit

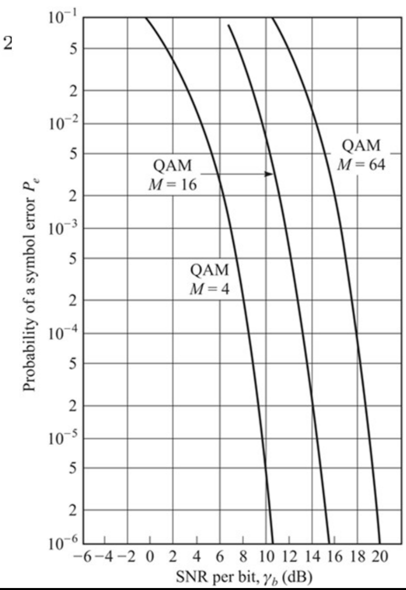
QAM Symbol Error Probability vs. SNR per Bit

- M-QAM:

$$P_s = 1 - \left[1 - 2\left(\frac{\sqrt{M}-1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3E_s}{(M-1)N_0}}\right)\right]^2$$
$$\leq 4\left(\frac{\sqrt{M}-1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3E_s}{(M-1)N_0}}\right)$$

- BER is

$$P_b \approx \frac{P_s}{\log_2 M}$$



Notes:

sBJksDRPrf96kfrB1XsIog

Implementation of the Optimal Receiver in AWGN

- We will present two different but equivalent implementations
  - The correlator receiver
  - The matched filter

Notes:

Correlator Receiver

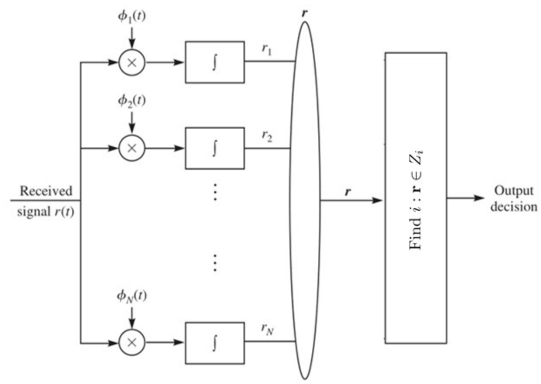
- We derived the optimal decision criteria

$$\hat{\mathbf{s}} = \arg\min_{\mathbf{s}_i} \|\mathbf{r} - \mathbf{s}_i\|^2$$

- The receiver knows all possible $\mathbf{s}_i$ points and needs to compute the vector $\mathbf{r}$ from the received signal is $r(t)$

$$\mathbf{r} = (r_1, \cdots, r_N)$$

$$r_j = \int_0^{T_s} r(t)\phi_j(t)dt = s_{ij} + n_j$$



Notes:

sBJksDRPrf96kfrB1XsIog

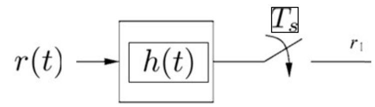
$$r(t) = s_{i1}\phi_1(t) + s_{i2}\phi_2(t) + \cdots + n(t), \quad 0 \le t < T_s$$

Matched Filter Receiver

- An LTI filter is called a matched filter to the signal $\phi_1(t)$ if its impulse response is

$$h(t) = \phi_1(T_s - t), \quad 0 \le t < T_s$$

- The filter output is



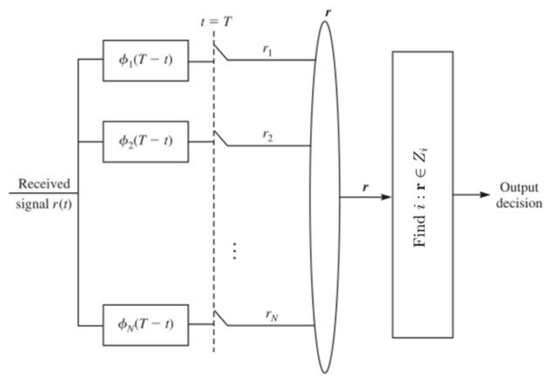
$$y(t) = \int_0^t r(\tau)h(t-\tau)d\tau$$
$$= \int_0^t r(\tau)\phi_1(T_s - t + \tau)d\tau$$
$$y(T_s) = \int_0^{T_s} r(\tau)\phi_1(\tau)d\tau = \int_0^{T_s} (s_{i1}\phi_1(\tau) + s_{i2}\phi_2(\tau) + \cdots + n(\tau))\phi_1(\tau)d\tau$$
$$= s_{i1}\int_0^{T_s} \phi_1^2(\tau)d\tau + \int_0^{T_s} n(\tau)\phi_1(\tau)d\tau = s_{i1} + n_i = r_i$$

Notes:

sBJksDRPrf96kfrB1XsIog

Matched Filter Receiver (cont.)

- So, the sampled output of the j-th filter is r_j and the output SNR is maximum at T_s when the input signal is $\phi_j(t)$
- Alternative receiver that uses a bank of filters matched to each basis function



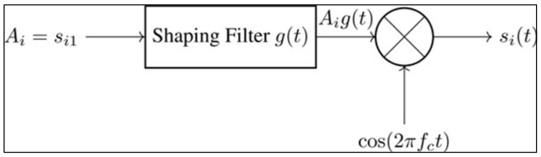
Notes:

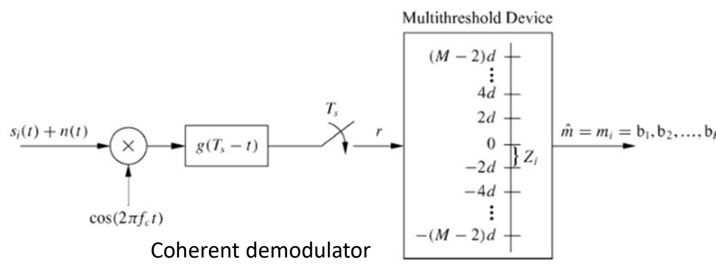
sBJksDRPrf96kfrB1XsIog

M-ASK Optimal Receiver

$$s_i(t) = A_i g(t)\cos(2\pi f_c t), \quad 0 \le t < T_s$$
$$A_i = (2i - 1 - M)d, \quad i = 1, 2, \ldots, M$$

Design a transmitter and a receiver for the M-ASK modulation scheme.



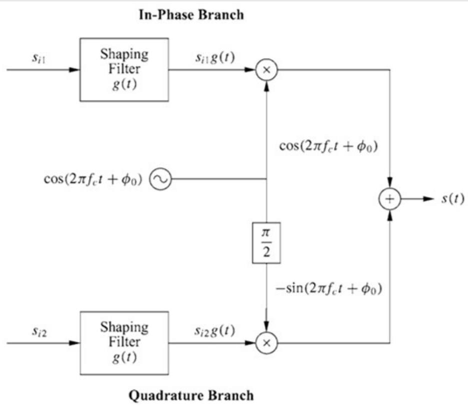


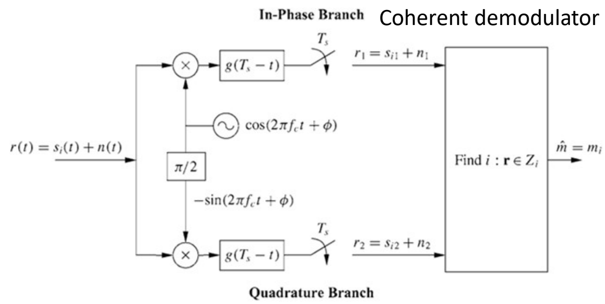
Notes:

$$s_i(t) = A\cos\left(\frac{2\pi(i-1)}{M}\right)\sqrt{2}g(t)\cos(2\pi f_c t) - A\sin\left(\frac{2\pi(i-1)}{M}\right)\sqrt{2}g(t)\sin(2\pi f_c t)$$

Example

Design a transmitter and a receiver for the M-PSK modulation scheme.





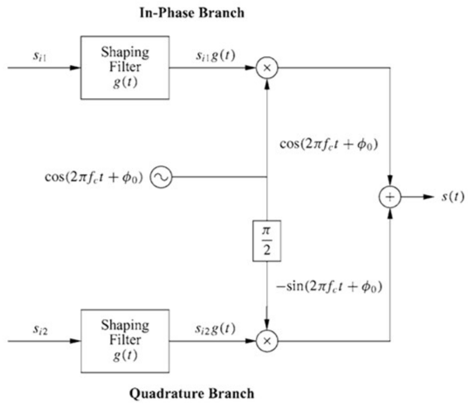
Notes:

sBJksDRPrf96kfrB1XsIog

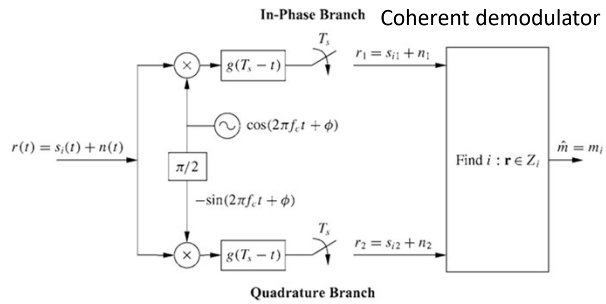
$$s_i(t) = A_i\sqrt{2}g(t)\cos(2\pi f_c t) - B_i\sqrt{2}g(t)\sin(2\pi f_c t)$$

Example

Design a transmitter and a receiver for the M-QAM modulation scheme.



Coherent demodulator



Notes:

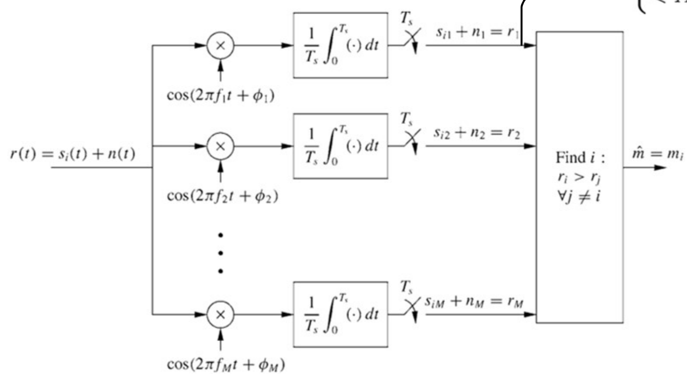
sBJksDRPrf96kfrB1XsIog

$$s_i(t) = A\sqrt{2/T_s}\cos(2\pi f_i t + \phi_i)$$
$$= A\cos(\phi_i)\sqrt{2/T_s}\cos(2\pi f_i t) + A\sin(\phi_i)\sqrt{2/T_s}\sin(2\pi f_i t)$$

M-FSK Demodulation

- Coherent detection

$$= \begin{cases} A/2, & f_i = f_j, \phi_i = \phi_j \\ < A/2, & \text{otherwise} \end{cases}$$



- Main issue: j-th carrier signal at the receiver needs to be matched in phase of the j-th carrier signal at the transmitter (very complex)

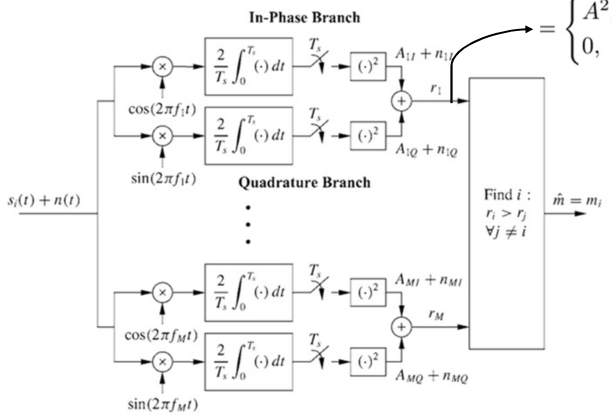
Notes:

$$s_i(t) = A\sqrt{2/T_s}\cos(2\pi f_i t + \phi_i)$$

M-FSK Demodulation (cont.)

$$= A\cos(\phi_i)\sqrt{2/T_s}\cos(2\pi f_i t) + A\sin(\phi_i)\sqrt{2/T_s}\sin(2\pi f_i t)$$

- Noncoherent detection eliminates the need for a coherent phase reference:
  - First, detect the energy at each frequency
  - Second, select branch with highest energy



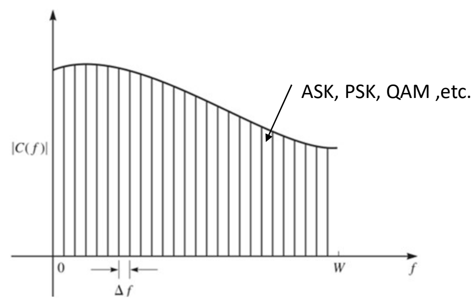
$$= \begin{cases} A^2, & f_i = f_j \\ 0, & \text{otherwise} \end{cases}$$

Notes:

sBJksDRPrf96kfrB1XsIog

Notes on Multicarrier Modulation and OFDM

- Sometimes, sending data serially over the whole bandwidth results in ISI caused by the nonlinearity of the channel → single-carrier
- OFDM subdivides the available channel bandwidth W into a number of subchannels such that each subchannel $\Delta f = W/N$ is nearly ideal (i.e., constant) → multicarrier



Notes:

sBJksDRPrf96kfrB1XsIog

Notes on Single-carrier vs. Multicarrier Modulation

- For each subchannel, a carrier signal is used

$$c_i(t) = \cos(2\pi f_i t)$$

  such that $\Delta f = 1/T_s$ to ensure orthogonality
- Each subchannel can use ASK, PSK, QAM, etc.
- OFDM is very popular in modern communication systems
  - Examples: DSL, Wi-Fi, LTE, and more
- OFDM can be implemented the FFT algorithm which is very fast and efficient
- Recommended reading: Proakis and Salehi – Chapter 11

Notes:

Outline

- Introduction
- Geometric representation of signals
- Baseband and passband PAM
- PSK, QAM, and FSK
- Optimal receivers and performance over AWGN channels
- Pulse shaping
- Eye diagram

Notes:

Pulse Shaping

- Pulse shape *g*(*t*) is chosen to improve the spectral characteristics of the transmitted signal
  - For example, rectangular pulse has high spectral sidelobes → band-limited channel → spread in time → intersymbol interference (ISI)
- Nyquist criterion: To avoid ISI, a necessary and sufficient condition is

$$p(t) = g(t) * g^*(-t) = \begin{cases} 1, & t = 0 \\ 0, & t \neq nT_s \end{cases}$$

- It must also maintain the orthonormal properties of basis functions

$$\int_0^{T_s} 2g^2(t)\cos^2(2\pi f_c t)dt = 1$$
$$\int_0^{T_s} 2g^2(t)\cos(2\pi f_c t)\sin(2\pi f_c t)dt = 0$$

Notes:

Example: Full-response Pulse Shapes

- Pulse shapes whose time support is equal to the symbol time (i.e., BW is infinity)
- Note the trade-off between the ease of synchronization and bandwidth

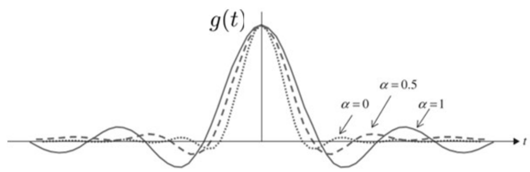
Pulse	*g*(*t*)	*p*(*t*)	90% BW
Non-Return-to-Zero (NRZ)	$= \sqrt{\frac{1}{T_s}}, \quad 0 \leq t \leq T_s$	$= 1 - \frac{\lvert t \rvert}{T_s}, \quad -T_s \leq t < T_s$	$0.85/T_s$
Return-to-Zero (RZ)	$= \sqrt{\frac{2}{T_s}}, \quad 0 \leq t \leq \frac{T_s}{2}$	$= 1 - 2\frac{\lvert t \rvert}{T_s}, \quad -\frac{T_s}{2} \leq t < \frac{T_s}{2}$	$1.70/T_s$
Manchester (MAN)	$= \begin{cases} \sqrt{\frac{1}{T_s}}, & 0 \leq t \leq \frac{T_s}{2} \\ -\sqrt{\frac{1}{T_s}}, & \frac{T_s}{2} \leq t < T_s \end{cases}$		$3.05/T_s$
Half-Sine (HS)	$= \sqrt{\frac{2}{T_s}} \sin\left(\frac{\pi t}{T_s}\right), \quad 0 \leq t \leq T_s$		$0.78/T_s$

Notes:

Example: Partial-response Pulse Shapes

- Follows the Nyquist non-ISI criterion, it turns out there are pulses that satisfy the criteria while having infinite support (not limited in time between 0 and T_s)
- Have much better spectral properties (spectrum is limited, not infinite)
- Square-root raised-cosine (SRRC) pulse:
  - Most popular pulse shaping filter used in mobile communication
  - α is a roll off factor (or excess bandwidth as a percentage)

$$g(t) = \frac{1}{\sqrt{T_s}} \frac{\sin\left(\pi(1-\alpha)\frac{t}{T_s}\right) + \frac{4\alpha t}{T_s}\cos\left(\pi(1+\alpha)\frac{t}{T_s}\right)}{\frac{\pi t}{T_s}\left[1-\left(\frac{4\alpha t}{T_s}\right)^2\right]}$$



Notes:

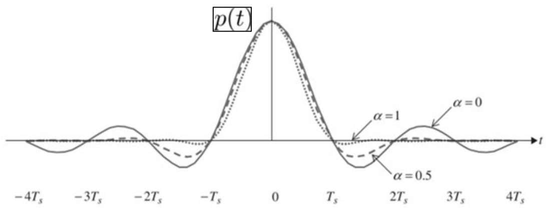
sBJksDRPrf96kfrB1XsIog

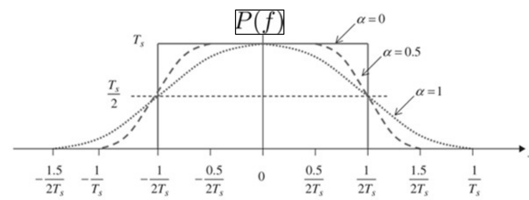
Example: Partial-response Pulse Shapes (cont.)

- Square-root raised-cosine (SRRC) pulse:
  - Practical implementations truncate the pulse shape to span *L* symbols on both sides (2*L*+1)
  - Then, shifted to the right to be causal (i.e., realizable)
  - Baseband bandwidth is limited to

$$W = \frac{1+\alpha}{2T_s}$$

$$p(t) = \frac{\sin\left(\pi \frac{t}{T_s}\right)}{\frac{\pi t}{T_s}} \times \frac{\cos\left(\pi \alpha \frac{t}{T_s}\right)}{1 - \left(\frac{2\alpha t}{T_s}\right)^2}$$





Notes:

Matlab: rcosdesign(alpha, span, sps)

sBJksDRPrf96kfrB1XsIog

Spectral Efficiency

- Dimensionality Theorem: an approximation for an *N*-dimensional band-limited space of signals

Number of basis functions

$$W = (1+\alpha)\frac{N}{2T_s} = (1+\alpha)\frac{NR_s}{2}\ \text{Hz}$$

- For ASK and BPSK, given that *N*=1,

$$W = (1+\alpha)\frac{R_b}{2\log_2 M}\ \text{Hz}$$

- Bandwidth of bandpass ASK signal can be reduced by a factor of 2 using single-sideband (SSB) transmission

Notes:

Spectral Efficiency (cont.)

- For PSK and QAM, given that *N*=2,

$$W = (1+\alpha)\frac{R_b}{\log_2 M} \text{ Hz}$$

- Note that this formula is valid for M-PSK where *M*>2

- For FSK, given that *N*=*M*,

$$W = (1+\alpha)\frac{M R_b}{2\log_2 M} \text{ Hz}$$

Notes:

Example

Assume a SRRC pulse, what is the spectral efficiency (i.e., R_b/W) of M-ASK, M-PSK, M-QAM, and M-FSK passband signals when M = 8?

Notes:

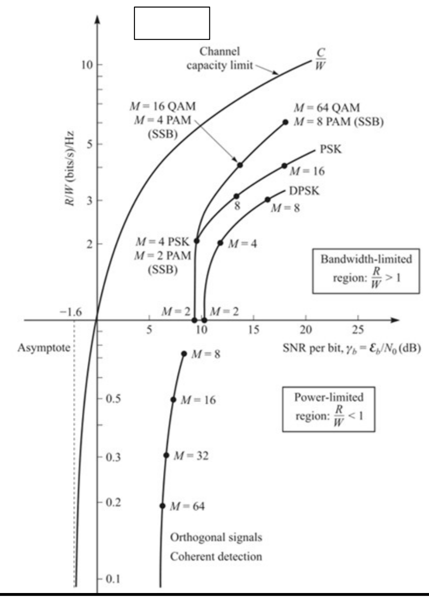
Spectral Efficiency vs. Power Efficiency trade-off

- The size of constellation M determines the tradeoff
- For ASK, PSK, and QAM systems, as M increases, the spectral efficiency R_b/W increases but power efficiency (or minimum SNR γ_b) decreases
  - Appropriate for band-limited channels and desired bit rate-to-bandwidth ratio > 1 and where there is sufficiently high SNR to support increases in M
  - Example: telephone and mobile communications
- For FSK systems, as M increases, the spectral efficiency R_b/W decreases but power efficiency (or minimum SNR γ_b) improves
  - Appropriate for power-limited channels that have sufficiently large bandwidth to accommodate a large number of signals
  - Example: deep space communications

Notes:

Example: R_b/W vs. γ_b for P_b=10^{-5}

$$\frac{C}{W} = \log_2\left(1 + \frac{E_b}{N_0}\right)$$



Notes:

How to Choose a Modulation Scheme

- Digital modulation methods discussed so far can be compared in a number of ways
  - Data rate [bps]: number of transmitted bits per second
  - Spectral (bandwidth) efficiency [bps/Hz]: minimum bandwidth W required for a desired data rate R_b, i.e., R_b/W
  - Power efficiency: minimum SNR per bit γ_b required for a desired BER P_b
  - Robustness to channel impairments
  - Ease of implementation

Notes:

Outline

- Introduction
- Geometric representation of signals
- Baseband and passband PAM
- PSK, QAM, and FSK
- Optimal receivers and performance over AWGN channels
- Pulse shaping
- Eye diagram

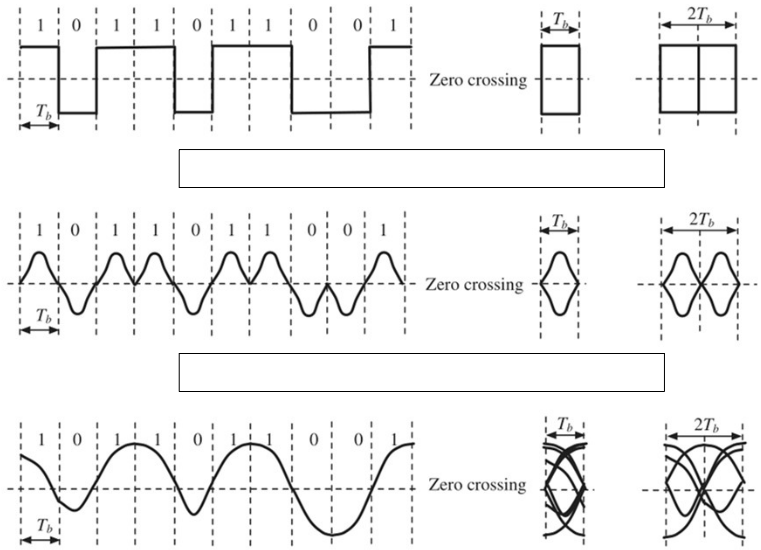
Notes:

The Eye Pattern

- A practical diagnostic tool, easy to generate for visual inspection of the received signals to determine the severity of ISI, the accuracy of timing extraction, the noise immunity, and other important factors
- The eye pattern is produced by:
  - Apply the received signal to the vertical input of the oscilloscope
  - The time base of the scope is triggered at the same (or a fraction of) the bit rate $1/T_b$
  - The oscilloscope shows the superposition of many traces of length one (or multiple) T_b
- In other words, what appears on the scope is simply the input that is cut up into individual pieces of T_b in duration and then superimposed on top of one another the results in a pattern that looks like a human eye

Notes:

Examples: Time Base T_b and $2T_b$

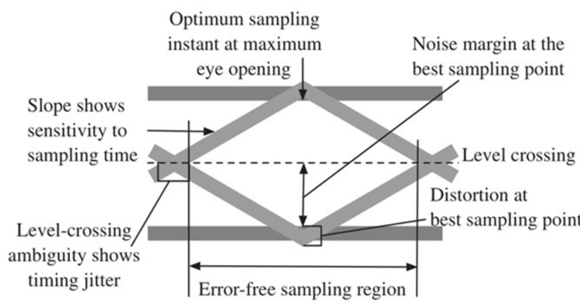


Notes:

sBJksDRPrf96kfrB1XsIog

Key Measures for Signal Quality

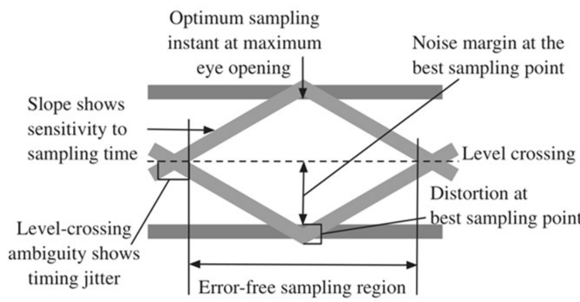
- Maximum opening point:
  - Eye opening amount at the sampling instant indicates what amount of noise the detector can tolerate without making an error
  - This quantity is known as the noise margin
  - The instant of maximum eye opening is the optimum sampling instant



Notes:

Key Measures for Signal Quality

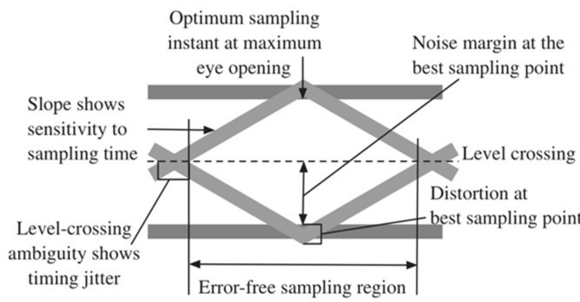
- Sensitivity to timing jitter:
  - The width of the eye is the time interval over which a correct decision can still be made
    - If the sampling deviates from the instant with the maximum vertical opening, the margin of noise tolerance is reduced
  - It is desirable to have an eye with the maximum horizontal opening
  - The slope of the eye shows how fast the noise tolerance is reduced and the sensitivity to timing jitter



Notes:

Key Measures for Signal Quality

- Level-crossing (timing) jitter:
  - Practical receivers extract timing information from the (zero) level crossing of the received signal waveform
  - The variation of level crossing can be seen from the width of the eye corners
  - This measure provides information about the timing jitter such a receiver is expected to experience from its timing extractor

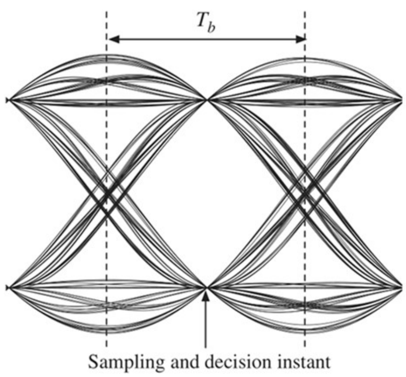


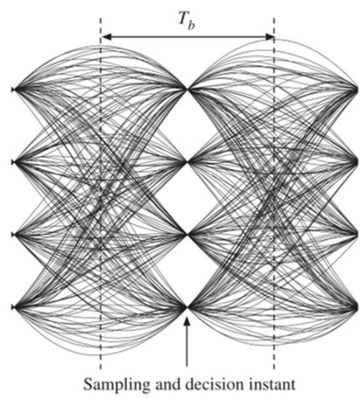
Notes:

sBJksDRPrf96kfrB1XsIog

Practical Example

- SRRC pulse with a roll-of factor of 0.5 in binary and M-ary signals
- We can clearly see:
  - The zero ISI at the optimum sampling instant (Nyquist criteria)
  - There is still timing jitter if we do not select sampling instant carefully





Notes:

sBJksDRPrf96kfrB1XsIog

Summary

- By now you should know:
  - Why digital transmission is superior to analog transmission
  - Geometric representation of signals
  - Passband modulation: amplitude, phase, and frequency
  - How to design an optimal receiver under AWGN
  - How to calculate the error probability of ASK, PSK, QAM, FSK
  - Effect of pulse shaping on spectral efficiency
  - How to choose a modulation scheme based on bandwidth and energy constraints

Notes: