Using the DFT method, obtain the circular convolution of the following:

$$x_1(n) = [1 \; 2 \; 1 \; -2]$$

$$x_2(n) = [3 \; -2 \; 1 \; -3]$$

From the DFT property we know,

$$x_1(n) \circledast x_2(n) \xleftrightarrow{DFT} X_1(k) \cdot X_2(k)$$

we begin with computing the DFT of $x_1(n)$ and $x_2(n)$

$$x_1(n) = [1, 2, 1, -2]$$

$$X_1(k) = [W_4] x_1(n)$$

$$\therefore X_1(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+1-2 \\ 1-2j-1-2j \\ 1-2+1+2 \\ 1+2j-1+2j \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -4j \\ 2 \\ 4j \end{bmatrix}$$

Similarly,

$$X_2(k) = [W_4] \cdot x_{2N}$$

$$\therefore X_2(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3-2+1-3 \\ 3+2j-1-3j \\ 3+2+1+3 \\ 3-2j-1+3j \end{bmatrix} = \begin{bmatrix} -1 \\ 2-j \\ 9 \\ 2+j \end{bmatrix}$$

Let $Y(k) = X_1(k) \cdot X_2(k)$

$$= \{2, -4j, 2, 4j\} \cdot \{-1, 2-j, 9, 2+j\}$$

$$\therefore Y(k) = \{-2, -4-8j, 18, -4+8j\}$$

The result of circular convolution is obtained by performing IDFT of $Y(k)$ as follows,

$$y(n) = \frac{1}{N} [W_4^*] \, Y(k)$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} -2 \\ -4-8j \\ 18 \\ -4+8j \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -2-4-8j+18-4+8j \\ -2-4j+8-18+4j+8 \\ -2+4+8j+18+4-8j \\ -2+4j-8-18-8-4j \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 8 \\ -4 \\ 24 \\ -36 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \\ -9 \end{bmatrix}$$

$\therefore y(n) = \{2, -1, 6, -9\}$

This can be verified by performing direct circular convolution. We generate circular matrix of $x_1(n)$:

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 & 2 \\ 2 & 1 & -2 & 1 \\ 1 & 2 & 1 & -2 \\ -2 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \\ -9 \end{bmatrix}$$

$\therefore y(n) = \{2, -1, 6, -9\}$

$y(n) = [2 \ -1 \ 6 \ -9]$

which is same as obtained using DFT