Computing DFT by Matrix Method

The DFT is given by the equation,

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-\frac{j2\pi kn}{N}} \rightarrow (1)$$

$; k = 0, 1, 2, \cdots, N-1$

"We define a new term W_N which is called the twiddle factor"

$$W_N = e^{-\frac{j2\pi}{N}} \rightarrow (2)$$

Substituting (2) in the DFT equation (1) we get,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$; k = 0, 1, 2, \cdots, N-1$

$\rightarrow (3)$

Similarly the IDFT which is given by the equation,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{+\frac{j2\pi kn}{N}} \rightarrow (4)$$

$; n = 0, 1, \cdots, N-1$

Can be written in terms of W_N as,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}$$

$\rightarrow (5)$

Eqn (3) can be represented in the matrix form as

$$DFT \rightarrow X(k) = [W_N] x(n) \rightarrow (6)$$

where,

$$x(n) = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}_{N\times 1}$$

$$X(k) = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix}_{N\times 1}$$

&

W_N^{nk} is given as,

$$W_N^{nk} = \begin{array}{c|c} & n \rightarrow \\ \downarrow k & \begin{array}{ccccc} 0 & 1 & 2 & \cdots & N-1 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \end{array} & \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}_{N\times N} \end{array}$$

$W_N \rightarrow N\times N$ DFT matrix

$$
\begin{array}{c|cccc}
 & n \rightarrow & & & \\
k \downarrow & 0 & 1 & 2 & 3
\end{array}
$$

$$
W_4^{nk} = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array}
\begin{bmatrix}
W_4^{0\cdot 0} & W_4^{0\cdot 1} & W_4^{0\cdot 2} & W_4^{0\cdot 3} \\
W_4^{1\cdot 0} & W_4^{1\cdot 1} & W_4^{1\cdot 2} & W_4^{1\cdot 3} \\
W_4^{2\cdot 0} & W_4^{2\cdot 1} & W_4^{2\cdot 2} & W_4^{2\cdot 3} \\
W_4^{3\cdot 0} & W_4^{3\cdot 1} & W_4^{3\cdot 2} & W_4^{3\cdot 3}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
W_4^0 & W_4^0 & W_4^0 & W_4^0 \\
W_4^0 & W_4^1 & W_4^2 & W_4^3 \\
W_4^0 & W_4^2 & W_4^4 & W_4^6 \\
W_4^0 & W_4^3 & W_4^6 & W_4^9
\end{bmatrix}
$$

where we know; $W_N = e^{-j\frac{2\pi}{N}}$

We calculate each value of the matrix

$W_4^0 = e^{-j\frac{2\pi 0}{4}} = 1$

$W_4^1 = e^{-j\frac{2\pi 1}{4}} = -j$

$W_4^2 = e^{-j\frac{2\pi \cdot 2}{4}} = -1$

$W_4^3 = e^{-j\frac{2\pi}{4}\cdot 3} = +j$

$W_4^4 = e^{-j\frac{2\pi}{4}\cdot 4} = +1$

$W_4^6 = e^{-j\frac{2\pi 6}{4}} = -1$

$W_4^9 = e^{-j\frac{2\pi 9}{4}} = -j$

$\therefore$

$$
W_4^{nk} = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & -j & -1 & j \\
1 & -1 & +1 & -1 \\
1 & j & -1 & -j
\end{bmatrix}
$$