CHAPTER 1

Introduction to Digital Signal Processing

Syllabus :

Introduction to Digital Signal Processing, Sampling and Reconstruction, Standard DT Signals, Concept of Digital Frequency, Representation of DT signal using Standard DT Signals, Signal Manipulations(shifting, reversal, scaling, addition, multiplication).

1.1 Introduction

– A signal is defined as any physical quantity that varies with time, space or any other independent variable. Anything that carries some information can be called a signal.

– Examples of signals that we encounter frequently in our daily life are speech, music, image, video etc. Some other examples of signals are Electrocardiograms (ECG) which provides us information about the health of a person's heart, Seismic signals which are used to measure the intensity of an earthquake, AC power supply signals that we have at our homes and offices etc.

– Mathematically, we define a signal as a function of one or more independent variables. Consider a voltage signal and a pressure signal,

$$v(t) = 15t$$
$$p(t) = 10e^{-2t} \qquad ...(1.1.1)$$

MATLAB program for displaying 1-dimensional signals

```
clc
clear all
for t=1:1:10
    v(t)=5*t;
    p(t)=10*exp(-0.2*t);
end
subplot (2, 1, 1)
plot (v)
xlabel ('Time')
ylabel ('voltage v(t)')
grid on
subplot (2, 1, 2)
plot (p)
xlabel ('Time')
ylabel ('Pressure p(t) ')
grid on
```

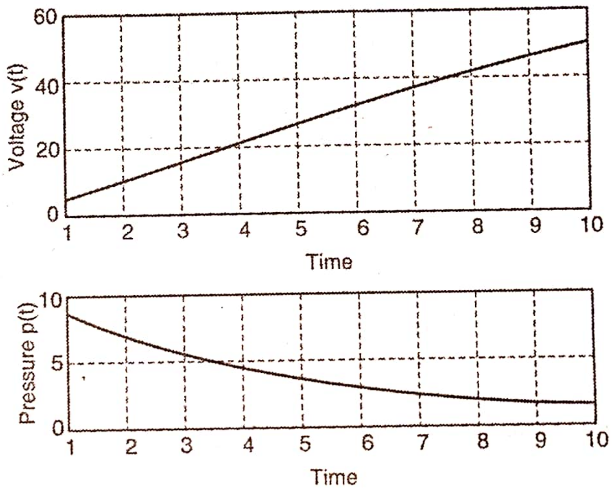


Fig. 1.1.1

– Both these signals in the given example depend on an independent variable t i.e. both voltage and pressure change with respect to time. Such signals are known as 1-Dimensional signals. In the Fig. 1.1.1, the x-axis is the time axis (independent variable) while the y-axis is the voltage (dependent variable).

– Another example of a signal is an image signal that we see on the computer monitor. A pixel on the computer could be represented as,

$$f(x, y) = 10x + 12y \qquad ...(1.1.2)$$

Here, f depends on two independent variables x and y and hence are known as 2-Dimensional signals. This subject deals with only 1-Dimensional signals.

– The Equation (1.1.2) represent a class of signals that are precisely defined. However there are many such signals that cannot be directly represented by a mathematical formula. For example a speech signal shown in Fig. 1.1.1 cannot be represented by a mathematical equation of the kind shown in Equations (1.1.1) and (1.1.2).

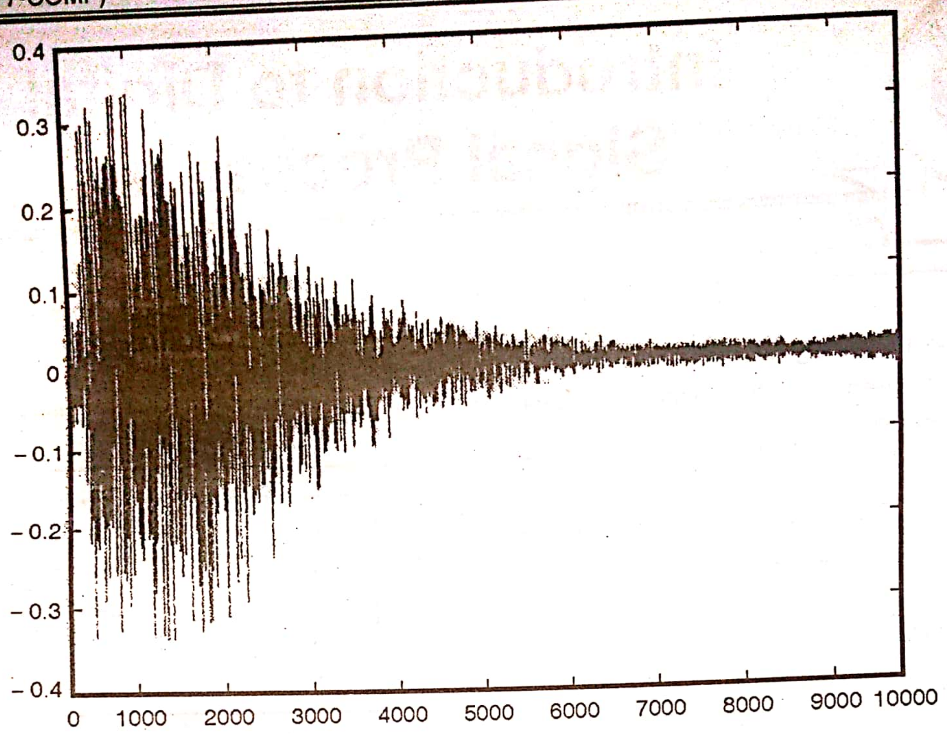


Fig. 1.1.2 : Speech signal

- Associated with all the above mentioned signals are the means by which these signals are generated. For example speech signals are generated by the vocal chords; the ECG signal is generated by the polarization and depolarization of the ventricles of the heat.
- Hence signals can be said to be generated by a system.
- A system is a device that produces a signal and also performs an operation on the signal to give the desired output.
- Suppose we want to amplify a signal. In this case the amplifier is a system. Similarly, if we want to filter out some noise, we pass it through a filter. In this case, the filter is also a system. Hence a system could be an interconnection of components that performs an operation on an input signal and produces an output signal.
- This production of a desired output signal from an input signal is known as **signal processing.**

  Let us define the three important terms discussed.

  a) **Signal :** A signal is defined as any physical quantity that varies with time, space or any other independent variable. Anything that carries some information can be called a signal.

  b) **System :** A system is defined as a device or a combination of devices which produces signals and also performs operation on signals to get the desired output.

  c) **Signal processing :** Production of a desired output signal from an input signal is known as signal processing.

  This chapter provides an overview of signals, systems and signal processing methods.

1.1.1 Digital Signal Processing System

- Most of the signals that occur in nature are continuous (analog) i.e., they are a function of a continuous variable such as time (e.g. speech and ECG)
- In many cases, this continuous input signal can be directly processed in the continuous form itself.

  Let us take an example,
- Suppose we want to amplify our voice. The set up would be as shown in Fig. 1.1.3.

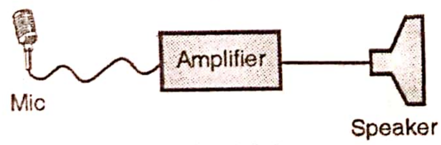


Fig. 1.1.3

- The amplifier which is made up resistors, capacitors, transistors etc., takes the input from the microphone which is a continuous signal and amplifies it to gives us a continuous output signal.
- Such an operation is called Analog Signal Processing. A general analog processing system is shown in Fig. 1.1.4. In this, the signal is directly processed in its continuous (analog) form.

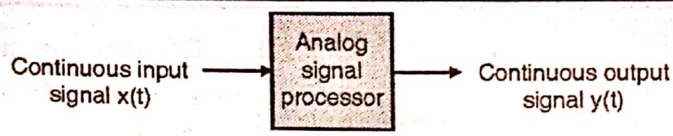


Fig. 1.1.4

- Digital signal processing, which this chapter is all about, provides an alternate method of processing the same signal. In this method we convert the continuous signal to a discrete time signal, also known as a digital signal, using an Analog to Digital Converter (ADC).
- The output of the ADC is fed to a digital processor (could also be a computer) which processes the discrete time signal. The output of the digital processor is again in digital form. This needs to be converted back to the continuous signal for any output device to understand it. This is done by using a Digital to Analog Converter (DAC). Let us take the same example of trying to amplify a voice signal. If we do it using a digital signal processor, we would have the following block diagram as shown in Fig. 1.1.5.

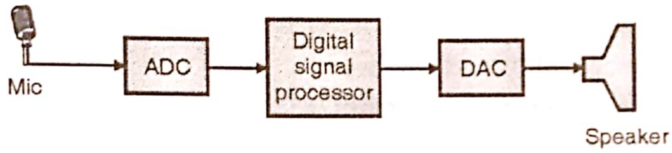


Fig. 1.1.5

In general we have,

Fig. 1.1.6

Here, $x(t) \rightarrow$ Continuous time input signal. $x(n) \rightarrow$ Discrete time input signal.

$y(n) \rightarrow$ Discrete time output signal. $y(t) \rightarrow$ Continuous time output signal.

1.1.2 Advantages of Digital Signal Processing over Analog Signal Processing

Though a DSP system has more blocks than an ASP system, there are many advantages of working in the digital domain. We will list down the advantages here.

1. Digital Signal Processing (DSP) systems are flexible i.e., they can be reconfigured simply by changing the program. Hence these systems are extremely versatile.
2. DSP systems can be easily replicated (duplicated) as they do not depend on component tolerances.
3. Accuracy also plays a pivotal role in determining the form of the signal processor. To design an analog system, we need components like resistors, capacitors and inductors. The tolerances of these components reduce accuracy of the analog system. On the other hand, DSP systems provide much better control of the accuracy.
4. Digital signal can be stored without deterioration. This is not true with analog signals.
5. Complex mathematical algorithms can be easily implemented using DSP systems while it is extremely difficult to build these algorithms using circuits.
6. Because of the use of software, DSP systems can be easily upgraded compared to Analog Signal Processing systems.
7. In most cases, DSP systems are cheaper compared to ASP systems.

Based on the above discussion we shall now form a table for a quick reference.

1.1.3 Difference between DSP and ASP

Table 1.1.1

Sr. No.	DSP	ASP
1.	More flexible	Less flexible
2.	Better accuracy	Less accuracy
3.	Digital signals can be easily stored.	It is difficult to store analog signals.

Sr. No.	DSP	ASP
4.	Digital systems can be easily replicated.	Tedious to replicate analog systems.
5.	Complex mathematical algorithms can be easily implemented.	It is extremely difficult to implement complex mathematical algorithms.
6.	Easily upgradable	Difficult to upgrade

1.1.4 Limitations of Digital Signal Processing

In spite of the obvious advantages of DSP systems over the ASP systems, it would be unfair if we do not highlight the disadvantages of the DSP systems.

1. A DSP system makes use of an ADC's and DAC's. These devices slow down the process. Hence a DSP system is not as fast as an ASP system.
2. A typical DSP chip contains thousands of transistors and hence the power consumption is more compared to ASP systems.
3. The most important limitation of DSP systems is that they are not suitable for signals which have a huge bandwidth. The ADC needs to sample the input signals at twice the bandwidth of the input signal to avoid aliasing. If the frequency content in the input signal is very larger, the ADC will need to sample it at least twice that frequency. There is a practical limitation on the speed of the ADC. Hence it is advisable to use ASP systems when the bandwidth of the input signal is very large.

With this background, we now move to discuss signals and systems.

We shall first study the various types of signals and then study the different systems.

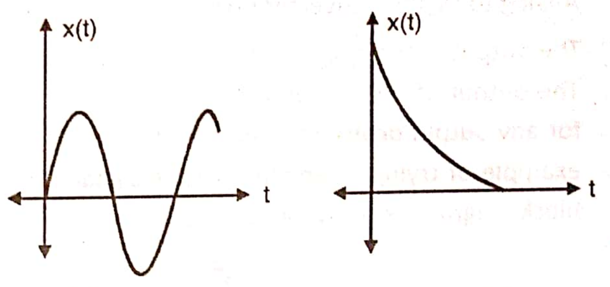
1.2 Signals

Signals can be broadly classified as,

1. Continuous and discrete time signals.
2. Continuous valued and discrete valued signals.
3. Deterministic and random signals.

1.2.1 Continuous and Discrete Time Signals

- A signal that exists for all time in a given interval is said to be a continuous signal (also called analog signal). Most of the naturally occurring signals are continuous in nature. Some of the continuous signals are shown in Fig. 1.2.1.



(a) (b)

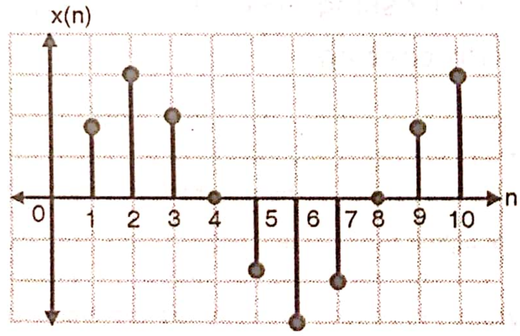
Fig. 1.2.1

These signals can be represented as,

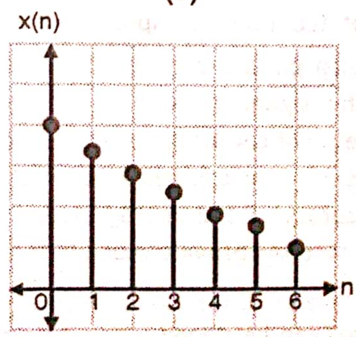
a) $x(t) = A \sin (\Omega t + \theta)$

b) $x(t) = e^{-t/RC}$

- As can be seen in Fig.1.2.1, continuous time signals are defined for every instance of time.
- Some of the examples of continuous time signals are speech signals, vibrations, ECG, seismic waves etc.
- A discrete time signal is one that is defined only at specific values of time. Two discrete time signals are shown in Fig. 1.2.2.



(a)



(b)

Fig. 1.2.2

- We do not know what is the value between 0 and 1 or between 1 and 2. Hence a discrete time signal exists only at specific values of time.
- A discrete time signal is obtained from a continuous time signal using the sampling operation.

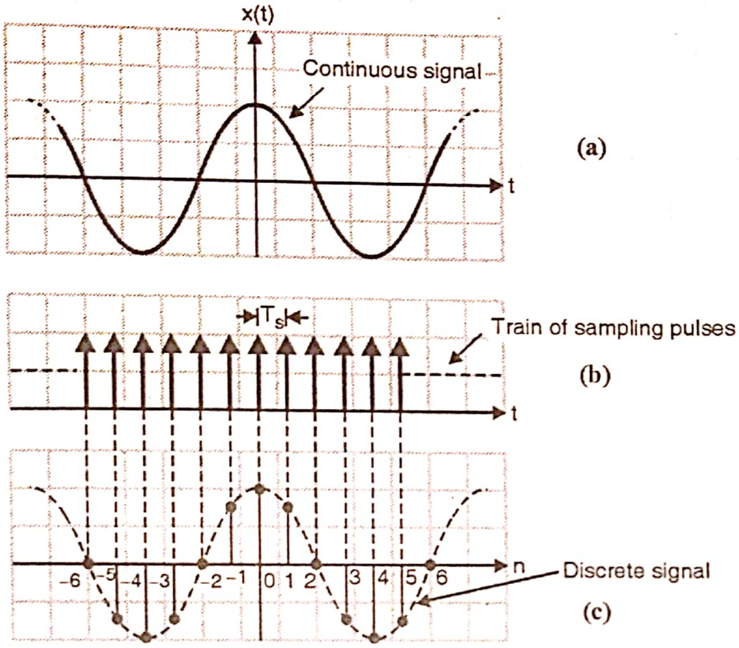


Fig. 1.2.3 : Sampling of a Continuous time signal to obtain a Discrete time signal

- Hence a discrete time signal is an approximation of the continuous time signal. To get a good approximation we need to take more samples in an interval i.e., reduce the sampling period (T_s) (increase sampling frequency). A typical discrete time signal can be represented by the equation.

$$x(n) = A \cos \omega n$$

We will discuss more about sampling in the next section.

- There are some signals that are inherently discrete in time and do not require the sampling of a continuous time signal. One such example is the stock market data. In this the value of a stock is studied at every hour or every minute. Nobody studies the stock value at every millisecond. Hence the data obtained would be discrete in time. Fig.1.2.4 represents a stock at different instances of time.

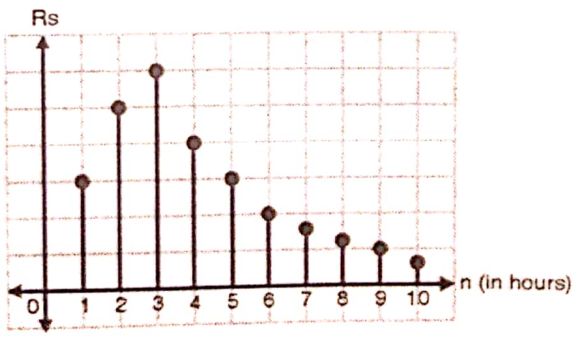


Fig. 1.2.4

1.2.2 Continuous Valued and Discrete Valued Signals

- The values of a continuous time or discrete time signal can be continuous or discrete. If the signal (whether continuous or discrete) can take all possible values, it is said to be continuous valued.
- On the other hand if the signal can take only a finite set of values, it is called a discrete valued signal. To put it simply, if the y-axis can be divided into infinite number of levels, it is called a continuous valued signal. If the y-axis can be divided into only a finite number of levels, it is called as a discrete valued signal. One such discrete valued signal is shown in Fig. 1.2.5. A discrete time signal having a set of discrete values is called a digital signal.

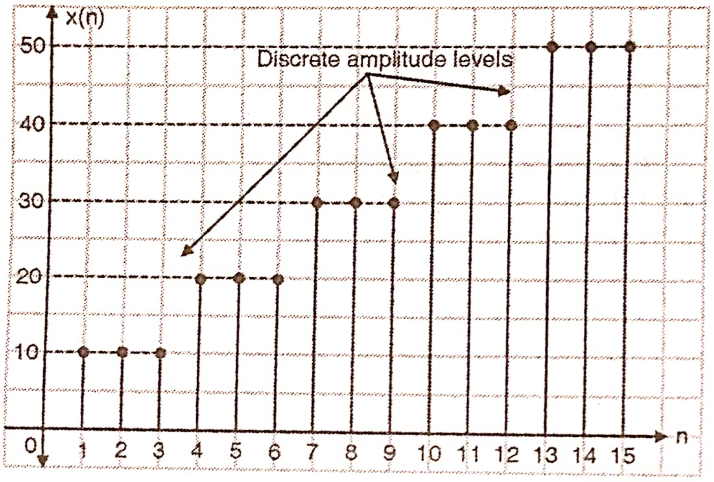


Fig. 1.2.5 : Discrete valued signal which is also discrete in nature

1.2.3 Deterministic and Random Signals

A signal which can be uniquely represented by a mathematical formula is called a deterministic signal. Since there is a mathematical formula, all past, present and future values are known and hence the name deterministic.

A signal which cannot be represented by a mathematical formula is called a Random signal. Such signals evolve over time and are unpredictable. Seismic waves, voice signals etc. are examples of random signals.

1.3 Concept of Frequency in the Continuous and Discrete Time Signals

1.3.1 Concept of Frequency in Continuous Time Signals

A continuous time domain signal is represented as,

$$x(t) = A \cos (2\pi Ft + \theta) \qquad ...(1.3.1)$$

Where F is the frequency in Hz and has a range $-\infty \leq F \leq +\infty$.

One such continuous time signal is shown in Fig.1.3.1.

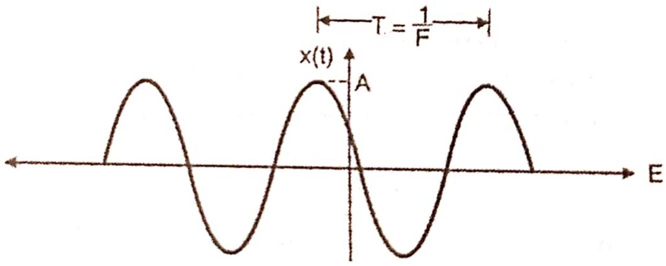


Fig. 1.3.1

This analog signal has three very nice properties associated with it.

1) For every fixed value of F, x(t) is periodic i.e., $x(t + T) = x(t)$ (10 Hz, 20 Hz, 100 Hz ... etc. are all periodic signals.
2) All continuous time signals with distinct frequencies are distinct i.e., 10 Hz looks different from 11 Hz which looks different from 100 Hz. Every frequency looks different.
3) Range of frequency, F is $-\infty \leq F \leq +\infty$.

Though frequency is a positive quantity, we consider negative frequency as well from Euler's formula.

Let us understand the concept of Negative frequencies

- A continuous time signal is represented as,

$$x(t) = A \cos (2\pi Ft + \theta)$$

We know $\cos\theta = \dfrac{e^{j\theta}+e^{-j\theta}}{2}$

$$\therefore \quad x(t) = A\left[\frac{e^{j(2\pi Ft+\theta)}+e^{-j(2\pi Ft+\theta)}}{2}\right]$$

$$\therefore \quad x(t) = \frac{A}{2}e^{-j(2\pi Ft+\theta)}+\frac{A}{2}e^{-j(2\pi Ft+\theta)}$$

– Hence x(t) can be considered to be made up of two phases moving in opposite directions.

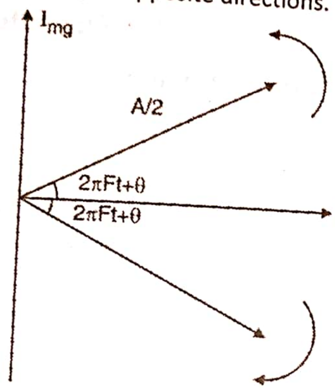


Fig. 1.3.2

– As time progresses one phasor moves in the counter clockwise direction while the other moves in the clockwise direction.

– A positive frequency corresponds to a counter clockwise uniform angular motion and negative frequency corresponds to negative clockwise uniform angular motion. Hence the range of Analog frequency is $-\infty \le F \le +\infty$.

– Remember, like negative numbers, negative frequencies do not exist and are used for mathematical convenience.

– When we pass this x(t), having the three properties just discussed, through an Analog to Digital Converter (ADC) we obtain x(n).

– We will now see if these 4 properties of x(t) are retained in x(n).

1.3.2 Concept of Frequency in Discrete Time Signals

A discrete time domain signal in represented as,

$$x(n) = A\cos(2\pi fn+\theta) \quad -\infty \le n \le +\infty \quad \ldots(1.3.2)$$

One such discrete time signal is shown in Fig. 1.3.3.

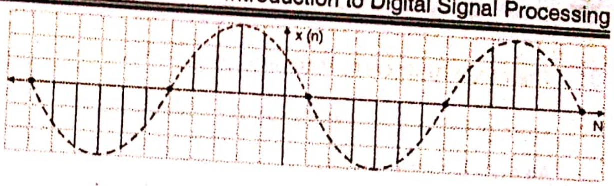


Fig. 1.3.3

1. **A discrete time signal is periodic only if its frequency is a rational number.**

Let us prove it

For periodicity $x(n+N) = x(n)$

Hence, for our x(n) to be periodic, we should have

$A\cos(2\pi f(n+N)+\theta) = A\cos(2\pi fn+\theta)$

$\therefore \quad A\cos(2\pi fn + 2\pi fN)+\theta) = A\cos(2\pi fn+\theta)$

This relation will be true only if

$$2\pi fN = 2k\pi$$

$$\therefore \quad f = \frac{k}{N}$$

Hence only if $f = \dfrac{k}{N}$ (Rational number)

$$A\cos(2\pi f(n+N)+\theta) = A\cos(2\pi fn+\theta)$$

Therefore x(t), which was always periodic, when passed through a ADC remains periodic only if ' f ' is a rational number.

2. **Discrete time signals whose frequencies are separated by $2\pi k$ are identical.**

Consider $x_1(n) = A\cos(2\pi f_0 n+\theta)$

We know $\omega_0 = 2\pi f_0$.

$\therefore \quad x_1(n) = A\cos(\omega_0 n+\theta)$

This signal has a frequency ω_0

Consider another discrete time signal $x_2(n)$ which as a frequency $(\omega_0+2\pi)$

$$x_2(n) = A\cos((\omega_0 n+2\pi)n+\theta)$$

$$\therefore \quad x_2(n) = A\cos(\omega_0 n+2\pi n+\theta)$$

$$= A\cos(\omega_0 n+\theta)$$

$$= x_1(n)$$

Hence $x_2(n)$ looks identical to $x_1(n)$ even though $x_1(n)$ has a frequency ω_0 and $x_2(n)$ has a frequency $(\omega_0+2\pi)$.

This can be generalized to state that all time signals

$x_k(n) = A\cos(\omega_k n+\theta) \qquad k = 0, 1, 2, 3$

where, $\omega_k = \omega_0 + 2\pi k \qquad -\pi \le \omega_0 \le +\pi$

are identical

Therefore while all x(t)'s were distinct, their discrete time domain counterparts, x(n)'s are not

3. The range of discrete frequency is $-\frac{1}{2} \leq f \leq +\frac{1}{2}$

From point number (2) we write,

All discrete time signals

$$x_k(n) = A\cos(\omega_k n + \theta); \qquad k = 0, 1, 2, 3\ldots$$

where, $\omega_k = \omega_0 + 2K\pi\ ; \quad -\pi \leq \omega_0 \leq \pi$

are identical.

This means only those frequencies which are in the range $-\pi \leq \omega_0 \leq +\pi$ are distinct.

$$-\pi \leq \omega \leq +\pi$$

$\because \omega = 2\pi f$, we can write

$$-\pi \leq 2\pi f \leq +\pi$$

$$\therefore \quad -\frac{1}{2} \leq f \leq +\frac{1}{2}$$

Hence the distinct range of discrete time signals is only $-\frac{1}{2} \leq f \leq +\frac{1}{2}$

Therefore while x(t)'s had a range of $-\infty \leq F \leq +\infty$, their discrete time domain counterparts, x(n)'s, have a range of $\frac{1}{2} \leq f \leq +\frac{1}{2}$

Hence when x(t), which has a range $-\infty \leq F \leq +\infty$, is passed through a ADC, it becomes x(n) and has a frequency range of $-\frac{1}{2} \leq f \leq +\frac{1}{2}$

> **Note :** The range of digital frequency is,
>
> $-\pi \leq \omega \leq +\pi$ OR $-\frac{1}{2} \leq f \leq +\frac{1}{2}$

How did a large range $-\infty \leq F \leq +\infty$ get reduced to $-\frac{1}{2} \leq f \leq +\frac{1}{2}$? What does this mean ? How can discrete time signals have such low frequencies ?

These questions will be answered if we know what 'f' is.

This is what the next section is all about.

1.3.3 Converting a Analog Signal to a Discrete Time Signal

- The process of converting an analog signal (continuous signal) to a discrete time signal is known as sampling.
- Sampling converts x(t) to x(n) and is performed by an Analog to Digital Converter (ADC).

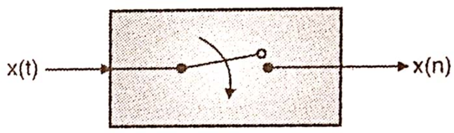


Fig. 1.3.4

- The sampling process can be visualized as a simple switch which opens and closes at fixed intervals of time, T_s.

Consider a switch which closes every T_s sec.

Let, x(t) be a pure sinusoidal function given by the formula.

$$x(t) = A\cos(2\pi F t + \theta) \qquad \ldots (1.3.3)$$

Here, F is the frequency in Hz.

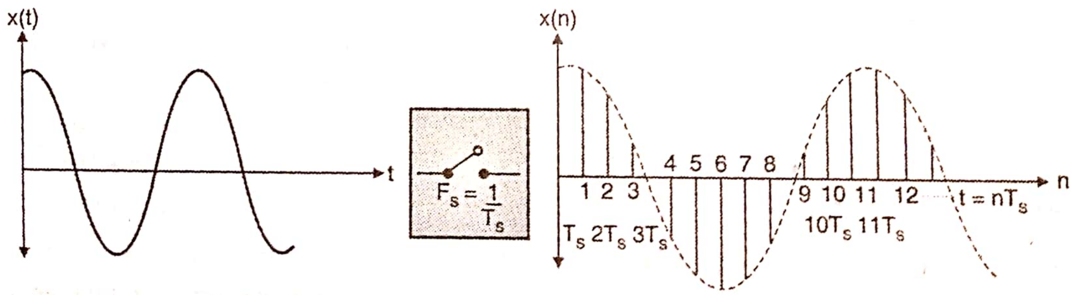


Fig. 1.3.5

- Since the switch closes every T seconds, we get the output only at fixed intervals of time.

  The sampling rate $F_s = \frac{1}{T_s}$

- From Fig. 1.3.5, it is clear that the x-axis which represents time gets replaced by nT_s in the discrete domain.

  $\therefore$ $t = nT_s$ in the discrete domain

- We can have thus convert an analog signal to a discrete time signal by replacing t=nT, where T is the Sampling period.

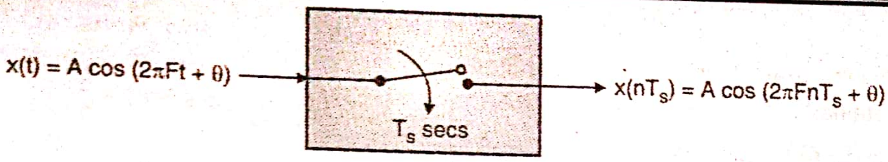


Fig.1.3.6

- Hence we obtain a digital signal that is represented as,

$$x(nT) = x(n) = A\cos(2\pi F n T_s + \theta) \quad \ldots(1.3.4)$$

- We know that T_s is the sampling period i.e, $T_s = \frac{1}{F_s}$, where, F_s is the sampling frequency.
- Substituting $T_s = \frac{1}{F_s}$ in Equation (1.3.4) we get,

$$\therefore \quad x(n) = A\cos\left(2\pi \frac{F}{F_s} n + \theta\right) \quad \ldots(1.3.5)$$

- Hence, an Analog signal $x(t) = A\cos(2\pi F t + \theta)$ can be converted to a digital signal by replacing $t = nT = \frac{n}{F_s}$.

Let $f = \frac{F}{F_s}$

- Substituting this value of f in Equation (1.3.5), we get the standard representation of a discrete time signal.

$$x(n) = A\cos(2\pi f n + \theta) \quad \ldots(1.3.6)$$

where, $f = \frac{F}{F_s}$

We know that the range of f is $-\frac{1}{2} \le f \le \frac{1}{2}$

Since $f = \frac{F}{F_s}$

$$\therefore \quad -\frac{1}{2} \le \frac{F}{F_s} \le \frac{1}{2}$$

$$\therefore \quad \frac{-F_s}{2} \le F \le +\frac{F_s}{2}$$

Hence $-\frac{1}{2} \le f \le +\frac{1}{2}$ basically means $\frac{-F_s}{2} \le F \le +\frac{F_s}{2}$

- To understand this range we consider the positive frequency.

$$F \le \frac{F_s}{2}$$

$$\text{i.e.,} \quad F_s \ge 2F \quad \ldots(1.3.7)$$

Here F is the frequency of input signal (in Hz) while F_s is the sampling frequency (in Hz).

- From Equation (1.3.7) we realize that sampling frequency F_s should be greater than or equal to twice the maximum frequency of the signal being sampled.

Only if $F_s \ge 2F$, will f remain in the distinct range of $-\frac{1}{2} \le f \le +\frac{1}{2}$

This is known as the Nyquist criteria.

If $F_s \le 2F$, f will no longer remain in the distinct range and will give rise to aliasing.

Therefore a range of $-\frac{1}{2} \le f \le +\frac{1}{2}$ implies that $F_s \ge 2F$

1.4 Aliasing and Nyquist Rate

- Aliasing is a phenomenon where high frequencies look identical to low frequencies because of low sampling frequency.
- In order to represent a continuous signal faithfully, we need to take as many samples as possible. This is done by an Analog to Digital Converter having a high sampling.

What is the ideal sampling required ?

- Sampling theorem was introduced in 1949 by Shannon and is also called as Shannon sampling's theorem.
- It is stated as '*A continuous time signal x(t) can be completely represented in its sampled form and recovered back from the sampled form if the sampling frequency $F_s \ge 2F$ where F is the maximum frequency in the continuous signal*'.
- A minimum sampling rate of 2F is called the Nyquist rate. Low sampling frequency gives rise to aliasing.
- We will see the effect of aliasing by solving a couple of examples.

Solved Example

Ex. 1.4.1 : Consider the give continuous time signals having frequencies 5 Hz, 10 Hz, 15 Hz, 45 Hz, 50 Hz and 130 Hz.

$x_1(t) = 5\cos 2\pi(5)t$

$x_2(t) = 5\cos 2\pi(10)t$

$x_3(t) = 5\cos 2\pi(15)t$

$x_4(t) = 5\cos 2\pi(45)t$

$x_5(t) = 5\cos 2\pi(50)t$

$x_6(t) = 5\cos 2\pi(130)t$

Obtain the discrete time signals by using a sampling frequency of 40 Hz.

Soln. :

We will first solve this example mechanically and then analyze the results obtained.

The given signals have the following frequencies.

$x_1(t) = 5$ Hz, $x_2(t) = 10$ Hz, $x_3(t) = 15$ Hz, $x_4(t) = 45$ Hz, $x_5(t) = 50$ Hz and $x_6(t) = 130$ Hz,

A continuous time signal is given by the equation,

$$x(t) = A\cos(2\pi Ft + \theta)$$

We convert this to a discrete time signal by replacing $t = nT_s = \frac{n}{F_s}$

$$\therefore \quad x(n) = A\cos\left(2\pi\frac{F}{F_s}n + \theta\right) \quad \ldots(1)$$

$$\therefore \quad x(n) = A(\cos 2\pi fn + \theta) \quad \ldots(2)$$

Here $\frac{F}{F_s} = f$

In our case $F_s = 40$ Hz

We will convert each of the given continuous signals into discrete time signals using Equation (1).

$\mathbf{x_1(t)} \rightarrow x_1(t) = 5\cos 2\pi\, 5t$

$$\therefore \quad x_1(n) = 5\cos 2\pi\left(\frac{5}{40}\right)n$$

$$\therefore \quad x_1(n) = 5\cos 2\pi\left(\frac{1}{8}\right)n$$

Comparing this with Equation (1) we have

$$f_1 = \frac{1}{8} = 0.125 \quad \ldots(3)$$

$\mathbf{x_2(t)} \rightarrow x_2(t) = 5\cos 2\pi 10t$

$$\therefore \quad x_2(n) = 5\cos 2\pi\left(\frac{10}{40}\right)n$$

$$\therefore \quad x_2(n) = 5\cos 2\pi\left(\frac{1}{4}\right)n$$

$$\therefore \quad f_2 = \frac{1}{4} = 0.25 \quad \ldots(4)$$

$\mathbf{x_3(t)} \rightarrow x_3(t) = 5\cos 2\pi 15t$

$$\therefore \quad x_3(n) = 5\cos 2\pi\left(\frac{15}{40}\right)n$$

$$\therefore \quad x_3(n) = 5\cos 2\pi\left(\frac{3}{8}\right)n$$

$$\therefore \quad f_3 = \frac{3}{8} = 0.325 \quad \ldots(5)$$

$\mathbf{x_4(t)} \rightarrow x_4(t) = 5\cos 2\pi 45t$

$$\therefore \quad x_4(n) = 5\cos 2\pi\left(\frac{45}{40}\right)n$$

$$\therefore \quad x_4(n) = 5\cos 2\pi\left(\frac{9}{8}\right)n$$

Since $\frac{9}{8} > 1$, it can be written as,

$$x_4(n) = 5\cos 2\pi\left(1 + \frac{1}{8}\right)n \qquad \left(\because \frac{9}{8} = 1 + \frac{1}{8}\right)$$

$$\therefore \quad x_4(n) = 5\cos\left(2\pi + 2\pi\cdot\frac{1}{8}\right)n$$

Since $\cos(2\pi + \theta) = \cos\theta$ we write,

$$x_4(n) = 5\cos 2\pi\left(\frac{1}{8}\right)n$$

$$\therefore \quad f_4 = \frac{1}{8} = 0.125 \quad \ldots(6)$$

$\mathbf{x_5(t)} \rightarrow x_5(t) = 5\cos 2\pi\, 50t$

$$\therefore \quad x_5(n) = 5\cos 2\pi\left(\frac{50}{40}\right)n$$

$$\therefore \quad x_5(n) = 5\cos 2\pi\left(\frac{5}{4}\right)n$$

Since $\frac{5}{4} > 1$, it can be written as,

$$x_5(n) = 5\cos 2\pi\left(1 + \frac{1}{4}\right)n \qquad \left(\because \frac{5}{4} = 1 + \frac{1}{4}\right)$$

$$\therefore \quad x_5(n) = 5\cos\left(2\pi + 2\pi\cdot\frac{1}{4}\right)n$$

Since $\cos(2\pi + \theta) = \cos\theta$ we write,

$$x_5(n) = 5\cos 2\pi\left(\frac{1}{4}\right)n$$

$$\therefore \quad f_5 = \frac{1}{4} = 0.25 \quad \ldots(7)$$

$\mathbf{x_6(t)} \rightarrow x_6(t) = 5\cos 2\pi 130t$

$$\therefore \quad x_6(n) = 5\cos 2\pi\left(\frac{130}{40}\right)n = 5\cos 2\pi\left(\frac{13}{4}\right)n$$

Since $\frac{13}{4} > 1$, it can be written as,

$$= 5\cos 2\pi\left(3 + \frac{1}{4}\right)n \qquad \left(\because \frac{5}{4} = 3 + \frac{1}{4}\right)$$

$$\therefore \quad x_6(n) = 5\cos\left(6\pi + 2\pi\cdot\frac{1}{4}\right)n$$

Since $\cos(6\pi + \theta) = \cos\theta$ we write,

$$x_6(n) = 5\cos 2\pi\left(\frac{1}{4}\right)n$$

$$\therefore \quad f_6 = \frac{1}{4} = 0.25 \quad \ldots(8)$$

We have thus been successful in converting the 6 continuous time signals into discrete time signals.

Let us now analyze what we have got. We form a table of the results obtained.

Sr. No.	Continuous time signal	Value of analog frequency	Is the Nyquist condition satisfied ? $F_s \geq 2F$ (Fs = 40 Hz)	Discrete time signal	Value of Discrete frequency(f)	Is f distinct
1.	$x_1(t) = 5\cos 2\pi 5t$	$F_1 = 5$ Hz	Yes	$x_1(n) = 5\cos 2\pi\left(\frac{1}{8}\right)n$	$f_1 = \frac{1}{8} = 0.125$	Yes
2.	$x_2(t) = 5\cos 2\pi 10t$	$F_2 = 10$ Hz	Yes	$x_2(n) = 5\cos 2\pi\left(\frac{1}{4}\right)n$	$f_2 = \frac{1}{4} = 0.25$	Yes
3.	$x_3(t) = 5\cos 2\pi 15t$	$F_3 = 15$ Hz	Yes	$x_3(n) = 5\cos 2\pi\left(\frac{3}{8}\right)n$	$f_3 = \frac{3}{8} = 0.375$	Yes
4.	$x_4(t) = 5\cos 2\pi 45t$	$F_4 = 45$ Hz	No	$x_4(n) = 5\cos 2\pi\left(\frac{1}{8}\right)n$	$f_4 = \frac{1}{8} = 0.125$	No (f_4 same as f_1)
5.	$x_5(t) = 5\cos 2\pi 50t$	$F_5 = 50$ Hz	No	$x_5(n) = 5\cos 2\pi\left(\frac{1}{4}\right)n$	$f_5 = \frac{1}{4} = 0.25$	No (f_5 same as f_2)
6.	$x_6(t) = 5\cos 2\pi 130t$	$F_6 = 130$ Hz	No	$x_6(n) = 5\cos 2\pi\left(\frac{1}{4}\right)n$	$f_6 = \frac{1}{4} = 0.25$	No (f_6 same as f_2)

We observe that, $x_4(n)$ looks identical to $x_1(n)$. Similarly $x_5(n)$ and $x_6(n)$ are identical to $x_2(n)$. This is known as Aliasing. It is important to note that the continuous time signals $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$, $x_5(t)$ and $x_6(t)$ were all distinct having different frequencies. However on sampling, $x_4(n)$ becomes identical to $x_1(n)$. Similarly $x_5(n)$ and $x_6(n)$ resemble $x_2(n)$.

On performing digital to analog conversion again, we would not be able to distinguish between $x_4(t)$ and $x_1(t)$ similarly we would not be able to distinguish between $x_5(t)$, $x_6(t)$ and $x_2(t)$.

In other words, while converting the discrete time signal back to a continuous time signal, 45 Hz signal ($x_4(t)$) will look the same as a 5 Hz signal ($x_1(t)$). Similarly 50 Hz signal($x_5(t)$) and 130 Hz signal ($x_6(t)$) will look the same as 10 Hz signal ($x_2(t)$). This distortion has taken place because the Nyquist condition was not satisfied while sampling $x_4(t)$, $x_5(n)$ and $x_6(n)$. (Look at the 4th column on the table).

This phenomena of higher frequencies resembling low frequencies is known as aliasing. A simple way to avoid this is to ensure that the sampling frequency F_s is always greater than or equal to twice the maximum frequency that needs to be sampled.

$$F_s \geq 2F_{max}$$

In the above example, $x_6(t)$ had the maximum frequency of 130 Hz. To avoid aliasing, we should have sampled all the signals at a sampling frequency of $F_s \geq 2 \times 130$ Hz

$\therefore F_s \geq 260$ Hz

Try solving the earlier example with $F_s = 300$ Hz and you will observe that $x_1(n)$, $x_2(n)$, $x_3(n)$, $x_4(n)$, $x_5(n)$ and $x_6(n)$ are all distinct.

Given below is the MATLAB code for the above example.

Aliasing

```
clc
clear all

Fs=40; %sampling Frequency
F1=5;F2=10;F3=15;F4=45;F5=50;F6=130 ;
% various frequencies present
For n=1:1:50;
x1(n)=5*cos(2*pi*(F1/Fs)*n);
x2(n)=5*cos(2*pi*(F2/Fs)*n);
x3(n)=5*cos(2*pi*(F3/Fs)*n);
x4(n)=5*cos(2*pi*(F4/Fs)*n);
x5(n)=5*cos(2*pi*(F5/Fs)*n);
x6(n)=5*cos(2*pi*(F6/Fs)*n);
```

```
end
subplot(6,1,1)
stem(x1)
ylabel('x1(n)')
subplot (6,1,2)
stem(x2)
ylabel ('x2(n)')
subplot(6,1,3)
stem(x3)
ylabel('x3(n)')
subplot(6,1,4)
stem(x4)
ylabel('x4(n)')
subplot(6,1,5)
stem(x5)
ylabel('x5(n)')
subplot(6,1,6)
stem(x6)
ylael('x6(n)')
```

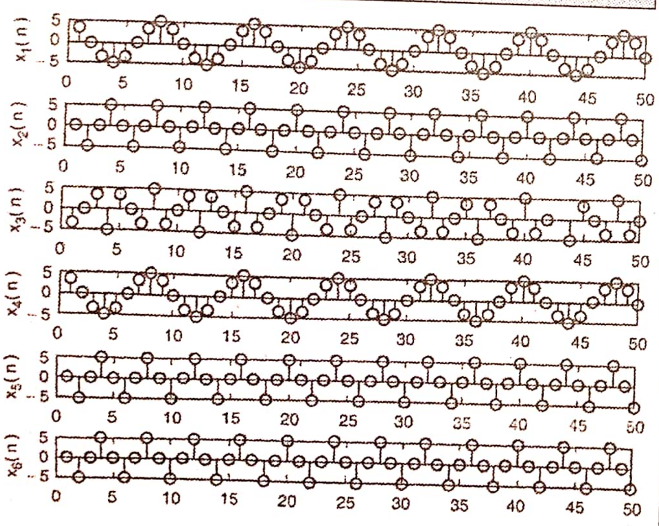


Fig. P. 1.4.1

We observe that, $x_4(n)$ is identical to $x_1(n)$ where as $x_5(n)$ and $x_6(n)$ are identical to $x_2(n)$.

If we simply change the sampling frequency (F_s) to 300 Hz in the program, each discrete time signal will be distinct. This is shown in Fig. P.1.4.1(a).

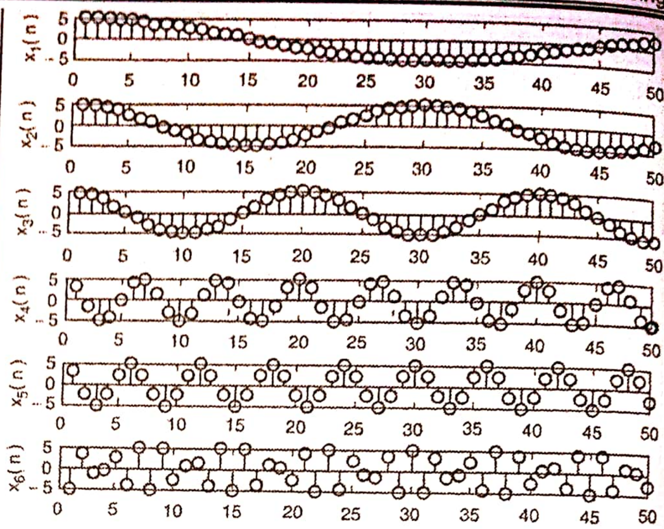


Fig. P.1.4.1(a)

Ex.1.4.2 : Consider the analog sinusoidal signal

x (t) = 5 sin (500 πt)

If the signal is sampled at F_s = 1500 Hz,

(i) What is the discrete time signal obtained after sampling.

(ii) Find the frequency of discrete time signal.

(iii) If sampling frequency F_s = 300 Hz then obtain discrete time signal.

Soln. :

Given : x (t) = 5 sin (500 πt)

(i) F_s = 1500 Hz

Now discrete time signal is obtained by replacing

$$t = nT_s = \frac{n}{F_s}$$

$$\therefore \quad x(n) = 5 \sin\left(\frac{500\,\pi n}{1500}\right)$$

$$\therefore \quad x(n) = 5 \sin\left(\frac{\pi}{3}\right)n \qquad \ldots(1)$$

This is the discrete time signal obtained after sampling.

(ii) The standard equation of the Discrete time signal is,

$$x(n) = A \sin \omega_n \qquad \ldots(2)$$

Comparing Equations (1) and (2) we get,

$$\omega = \frac{\pi}{3}$$

$$\because \quad \omega = 2\pi f$$

$$\therefore \quad f = \frac{1}{6}$$

Hence the frequency of discrete time signal is $\omega = \frac{\pi}{3}$ or $f = \frac{1}{6}$

(iii) Here F_s = 300 Hz

Discrete time signal is obtained by replacing

$$t = nT_s = \frac{n}{F_s}$$

$$x(n) = 5 \sin\left(\frac{500\pi \cdot n}{300}\right)$$

$$\therefore \quad x(n) = 5 \sin\left(\frac{5\pi n}{3}\right)$$

Now $$\frac{5\pi}{3} = \left(2\pi - \frac{\pi}{3}\right)$$

$$\therefore \quad x(n) = 5 \sin\left(2\pi - \frac{\pi}{3}\right) n$$

$$\therefore \quad x(n) = 5 \sin\left(-\frac{\pi}{3} n\right)$$

$$x(n) = -5\sin\left(\frac{\pi}{3}\right) n$$

This is the discrete time signal.

1.4.1 Anti Aliasing Filter

The minimum sampling frequency which is essential is 2F. where F is the maximum frequency. This $F_s = 2F_{max}$ is called the Nyquist rate.

Note : Most of the naturally occurring signals are not band limited. This means that they have infinite frequencies present in them (The extremely high frequencies have very small amplitudes, but they exist).

Nyquist criteria of $F_s \geq 2 F_{max}$ will never be satisfied in such cases and we will always encounter aliasing. These Band unlimited signals are converted to Band limited signals by passing them through a low pass filter. The sole purpose of this low pass filter is to remove very high frequencies thereby making the signal band limited and hence removing the affects of aliasing. These low pass filters are called anti-aliasing filters.

1.5 Basic Elements of a DSP System

We have discussed the basics DSP systems in this chapter. We had drawn a basic block diagram of a DSP system in Fig. 1.5.1. We will now draw a modified block diagram of a DSP system and explain each block.

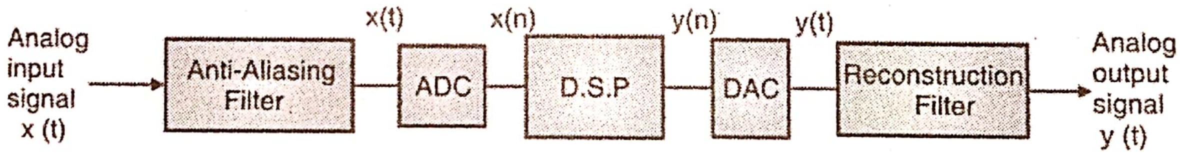


Fig.1.5.1

1. **Input signal :** A signal is defined as any physical quantity that varies with time, space or any other independent variable. Anything that carries some information can be called a signal.
2. **Anti-Aliasing Filter :** Most of the naturally occurring signals are not band limited. This means that they have infinite frequencies present in them. Anti-Aliasing filters are basically low pass filters which are used to remove the high frequencies from the input signal thereby making them band limited.
3. **Analog to Digital Converter (ADC) :** As the name suggests, ADC converts a continuous time signal x(t) to a discrete time signal x(n).
4. **Digital Signal Processor (DSP) :** This is the heart of the DSP system. It processes the discrete time input signal x(n) and gives us the required discrete time output signal y(n). One could either use a computer or dedicated DSP's manufactured by companies such as Motorola and Texas Instruments.
5. **Digital to Analog Converter (DAC) :** The output of the DSP is in digital form but most of the output devices only work in continuous time signals. DAC converts a discrete time signal y(n) to a continuous time signal y(t) which can be then fed to the out device.
6. **Reconstruction filter:** The output of the DAC might contain high frequencies due to the interpolation of sinc functions. These spurious high frequencies are eliminated using the reconstruction filter which is basically a low pass filter.

We will now solve a few examples to understand the sampling concepts.

1.5.1 Solved Examples

Ex. 1.5.1 : An analog signal is given as :

$x(t) = \sin(10\pi t) + 2\sin(20\pi t) + 2\cos(30\pi t)$.

(i) What is the Nyquist rate of this signal ?

(ii) If the signal is sampled with sampling frequency of 20 Hz, what is the discrete time signal obtained after sampling ?

Soln. :

Given :

$x(t) = \sin(10\pi t) + 2\sin(20\pi t) + 2\cos(30\pi t)$

(i) This signal consists of following frequencies,

$$2\pi F_1 = 10\pi,\ 2\pi F_2 = 20\pi \text{ and } 2\pi F_3 = 30\pi$$

$$\therefore F_1 = 5\text{ Hz},\ F_2 = 10\text{ Hz, and } F_3 = 15\text{ Hz}$$

Thus the maximum frequency present in the signal is,

$$F_{max} = W = 15\text{ Hz}$$

$\therefore$ Nyquist rate $= 2F_{max} = 30$ Hz

(ii) The discrete time signal is obtained by substituting

$t = \frac{n}{F_s} = \frac{n}{20}$ in the given equation.

$$\therefore x(n) = \sin\left(10\pi\frac{n}{20}\right) + 2\sin\left(20\pi\frac{n}{20}\right) + 2\cos\left(30\pi\frac{n}{20}\right)$$

$$= \sin\left(\frac{\pi}{2}\right)n + 2\sin(\pi)n + 2\cos\left(\frac{3\pi}{2}\right)n$$

but, $\sin(\pi n) = 0$

also $\frac{3\pi}{2} = 2\pi - \frac{\pi}{2}$

$$\therefore x(n) = \sin\left(\frac{\pi}{2}\right)n + 2\cos\left(2\pi - \frac{\pi}{2}\right)n$$

$$\therefore x(n) = \sin\left(\frac{\pi}{2}\right)n + \cos\left(\frac{-\pi}{2}\right)n$$

but,

$$\cos\left(\frac{\pi}{2}\right)n = \cos\left(\frac{-\pi}{2}\right)n = 0$$

$$\therefore x(n) = \sin\left(\frac{\pi}{2}\right)n$$

Ex.1.5.2 : A signal $x(t) = \sin(\omega t)$ of frequency 50 Hz is sampled using a sampling frequency of 80 Hz. Obtain the recovered signal if ideal reconstruction is used.

Soln. : Given :

$x(t) = \sin\omega t$ of frequency 50 Hz

$\therefore x(t) = \sin(2\pi \times 50t)$

Now, F_s = Sampling frequency = 80 Hz

Thus discrete signal is obtained by substituting

$$t = \frac{n}{F_s} = \frac{n}{80}$$

$$\therefore x(n) = \sin\left(2\pi 50\frac{n}{80}\right) = \sin\left(\frac{5\pi}{4}\right)n$$

Now, $\frac{5\pi}{4} = \left(2\pi - \frac{3\pi}{4}\right)$

$$\therefore x(n) = \sin\left(2\pi - \frac{3\pi}{4}\right)$$

$$x(n) = -\sin\left(\frac{3\pi}{4}\right)n$$

Since Ideal interpolation is used, analog signal is obtained by replacing n by t F_s.

$$\therefore n = 80t$$

$$\therefore x(t) = -\sin\left(\frac{3\pi}{4}80t\right)$$

$$\therefore x(t) = -\sin(60\pi t)$$

Hence a 50 Hz signal gets reconstructed as a 60 Hz signal.

Ex. 1.5.3 : What should be the sampling frequency to avoid aliasing for an analog signal represented as :

$x(t) = \cos(150\pi t) + 2\sin(300\pi t) - 4\cos(600\pi t)$

Obtain the discrete time signal if this sequence is sampled at $F_S = 400$ Hz.

Does aliasing occur ? If yes, calculate the aliased frequencies from the original frequencies.

Soln. :

Given :

$$x(t) = \cos(150\pi t) + 2\sin(300\pi t) - 4\cos(600\pi t) \quad \text{...(1)}$$

Here $2\pi F_1 t = 150\pi t$, $2\pi F_2 t = 300\pi t$ and $2\pi F_3 t = 600\pi t$

$$\therefore F_1 = 75\text{ Hz},\ F_2 = 150\text{ Hz, and } F_3 = 300\text{ Hz}$$

$\therefore$ Maximum frequency $F_{max} = 300$ Hz

Thus the sampling frequency, required to avoid aliasing is,

$$F_s \geq 2F_{max}$$

$$\therefore F_s \geq 600\text{ Hz}$$

The given signal is sampled at $F_s = 400$ Hz.

Thus discrete signal is obtained by substituting

$$t = \frac{n}{F_s} = \frac{n}{400} \text{ in Equation (1)}$$

$$\therefore t = \frac{n}{400}$$

$$\therefore \quad x(n) = \cos\left(150\pi\frac{n}{400}\right) + 2\sin\left(300\pi\frac{n}{400}\right) - 4\cos\left(600\pi\frac{n}{400}\right)$$

$$x(n) = \cos\left(\frac{3\pi}{8}\right)n + 2\sin\left(\frac{3\pi}{4}\right)n - 4\cos\left(\frac{3\pi}{2}\right)n \quad ...(2)$$

The third terms of $\frac{3\pi}{2}$ can be written as $-4\cos\left(2\pi - \frac{\pi}{2}\right)n$.

$$= -4\cos\left(-\frac{\pi}{2}\right)n$$

but, $\cos\left(\frac{\pi}{2}\right)n = \cos\left(\frac{-\pi}{2}\right)n = 0$

$$\therefore \quad x(n) = \cos\left(\frac{3\pi}{8}\right)n + 2\sin\left(\frac{3\pi}{4}\right)n$$

Hence aliasing does occur.

Ex.1.5.4 : An analog signal contains frequencies upto 10 kHz

(i) What range of sampling frequencies allows exact reconstruction of this signal from its samples.

(ii) If the signal is sampled with a sampling frequency $F_s = 8$ kHz

What is the folding frequency ?

Examine what happens to the frequency $F_1 = 5$ kHz

Examine what happens to the frequency $F_2 = 9$ kHz

Soln. :

(i) From the Nyquist criteria we know that,

$$F_s \geq 2W$$

$$\therefore \quad F_s \geq 2 \times 10$$

$$\therefore \quad F_s \geq 20 \text{ kHz}$$

This is the range of sampling frequencies which allows exact reconstruction.

(ii) The signal is sampled at 8 kHz.

Hence the folding frequency is,

$$F_{fold} = \frac{F_s}{2} = \frac{8 \text{ kHz}}{2} = 4 \text{ kHz}$$

$$F_1 = 5 \text{ kHz (given)}$$

Since F_{fold} is less than F_1 aliasing takes place. 5 KHz will look like 1kHz (5 – 4) = 1.

$$F_2 = 9 \text{ kHz (given)}$$

Since F_{fold} is less than F_1 aliasing takes place. 9KHz will look like 1 kHz (9 – (4+4)) = 1 kHz.

Ex. 1.5.5 : Consider the analog signal

$x_a(t) = 3\cos 2000\pi t + 5\sin 6000\pi t + 10\cos 12{,}000\pi t$.

(i) What is Nyquist rate for this signal ?

(ii) If this analog signal is sampled at $F_s = 5000$ samples/sec, what is the discrete-time signal obtained after sampling ?

(iii) What is the analog signal $y_a(t)$ we can reconstruct from the samples if we use ideal interpolation ?

Soln. :

Given :

$$x_a(t) = 3\cos 2000\pi t + 5\sin 6000\pi t + 10\cos 12000\pi t \quad ...(1)$$

(i) The given signal consists of following frequencies,

$$2\pi F_1 = 2000\pi t,\ 2\pi F_2 = 6000\pi t,\ 2\pi F_3 = 12000\pi t$$

$$\therefore \quad F_1 = 1000 \text{ Hz},\ F_2 = 3000 \text{ Hz and } F_3 = 6000 \text{ Hz}$$

$$\text{Thus } F_{max} = 6000 \text{ Hz} = 6 \text{ kHz}$$

$$\therefore \quad \text{Nyquist rate} = 2F_{max} = 12 \text{ kHz}$$

(ii) A discrete signal is obtained by substituting

$$t = \frac{n}{F_s} = \frac{n}{5000} \text{ in Equation (1)}$$

$$\therefore \quad x(n) = 3\cos\left(\frac{2000\pi n}{5000}\right) + 5\sin\left(\frac{6000\pi n}{5000}\right) + 10\cos\left(\frac{12000\pi n}{5000}\right)$$

$$\therefore \quad x(n) = 3\cos\left(\frac{2\pi}{5}\right)n + 5\sin\left(\frac{6\pi}{5}\right)n + 10\cos\left(\frac{12\pi}{5}\right)n$$

The second term $\left(\frac{6\pi}{5}\right)$ can be written as,

$$\left(\frac{6\pi}{5}\right) = \left(2\pi - \frac{4\pi}{5}\right)$$

$$\therefore \quad 5\sin\left(\frac{6\pi}{5}\right)n = 5\sin\left(2\pi - \frac{4\pi}{5}\right)n$$

Since $\sin(2\pi) = 0$, we obtain

$$5\sin\left(\frac{6\pi}{5}\right)n = 5\sin\left(-\frac{4\pi}{5}\right) = -5\sin\left(\frac{4\pi}{5}\right)$$

The third term, $\frac{12\pi}{5}$ can be written as,

$$\frac{12\pi}{5} = \left(2\pi + \frac{2\pi}{5}\right)$$

$$\therefore \quad 10\cos\left(2\pi + \frac{2\pi}{5}\right) = 10\cos\left(\frac{2\pi}{5}\right)$$

$$\therefore \quad x(n) = 3\cos\left(\frac{2\pi}{5}\right)n - 5\sin\left(\frac{4\pi}{5}\right)n + 10\cos\left(\frac{2\pi}{5}\right)$$

$$\therefore \quad x(n) = 13\cos\left(\frac{2\pi}{5}\right)n - 5\sin\left(\frac{6\pi}{5}\right)n$$

(iii) The reconstructed signal is obtained by substituting $n = t F_s$ in the discrete time signal.

Since $F_s = 5000$

$\therefore$ $n = 5000\,t$

$$\therefore \quad x(t) = 13 \cos\left(\frac{2\pi}{5}\right) 5000\,t - 5 \sin\left(\frac{4\pi}{5}\right) 5000\,t$$

$$\therefore \quad x(t) = 13 \cos 2000\,\pi t - 5 \sin 4000\pi t.$$

Ex. 1.5.6 : An analog signal is given by :

$x(t) = 3\cos(100\,\pi t) + 2\sin(300\,\pi t) - 4\cos(100\,\pi t)$

(1) What is the Nyquist rate of the sigal.

(2) Write the equation of the sampled signal

(3) If this analog signal is sampled at $F_s = 200$ samples/sec, what is the discrete-time signal obtained after sampling ?

Soln. : Given :

$$x(t) = 3\cos(100\,\pi t) + 2\sin(300\,\pi t) - 4\cos(100\,\pi t)$$

This can be written as,

$$x(t) = -\cos(100\,\pi t) + 2\sin(300\,\pi t) \qquad ...(1)$$

Here $2\pi F_1 t = 100\pi t$ and $2\pi F_2 t = 300\pi t$

$\therefore$ $F_1 = 50$ Hz and $F_2 = 150$ Hz,

$\therefore$ Maximum frequency $F_{max} = 150$ Hz

$\therefore$ Nyquist rate $= 2\,F_{max} = 300$ Hz

(2) The discrete signal is obtained by substituting

$$T = \frac{n}{F_s} = \frac{n}{300} \text{ in Equation (1)}$$

$$\therefore \quad x(n) = -\cos\left(100\pi \frac{n}{300}\right) + 2\sin\left(300\pi \frac{n}{300}\right)$$

$$\therefore \quad x(n) = -\cos\left(\frac{\pi}{3}\right) n + 2\sin(\pi\,n)$$

Since $\sin(\pi\,n) = 0$, we get

$$x(n) = -\cos\left(\frac{\pi}{3}\right) n$$

(3) Given : $F_s = 200$ Hz.

The discrete signal is obtained by substituting

$$t = \frac{n}{F_s} = \frac{n}{200} \text{ in Equation (1)}$$

$$\therefore \quad x(n) = -\cos\left(100\pi \frac{n}{200}\right) + 2\sin\left(300\pi \frac{n}{200}\right)$$

$$x(n) = -\cos\left(\frac{\pi}{2}\right) n + 2\sin\left(\frac{3\pi}{2}\right) n$$

Now $\frac{3\pi}{2} = 2\pi - \frac{\pi}{2}$

$$x(n) = -\cos\left(\frac{\pi}{2}\right) n + 2\sin\left(2\pi - \frac{\pi}{2}\right) n$$

Since $\sin(2\pi) = 0$, we obtain

$$x(n) = -\cos\left(\frac{\pi}{2}\right) n + 2\sin\left(-\frac{\pi}{2}\right) n$$

Ex. 1.5.7 : An analog signal is given as :

$x_a(t) = 15\cos(1250\,\pi t) + 17\cos(2170\,\pi t) + 33\cos(4750\,\pi t)$.

is converted to a discrete time signal. Determine the Nyquist rate, Folding frequency, resulting discrete time signal x(n) if the sampling frequency is 625 Hz. Also write the discrete frequencies in radians

Soln. :

Given

$x_a(t) = 15\cos(1250\,\pi t) + 17\cos(2170\,\pi t) + 33\cos(4750\,\pi t)$.

This signal consists of following frequencies,

$$2\pi F_1 = 1250\pi,\; 2\pi F_2 = 2170\pi \text{ and } 2\pi F_3 = 4750$$

$$\therefore F_1 = 625 \text{ Hz},\; F_2 = 1085 \text{ Hz, and } F_3 = 2375 \text{ Hz}$$

Thus the maximum frequency present in the signal is,

$$F_{max} = W = 2375 \text{ Hz}$$

$$\therefore \text{Nyquist rate} = 2\,F_{max} = 4750 \text{ Hz}$$

Given : The signal is being sampled at 625 Hz

$\therefore$ The folding frequency $= F_{fold} = \frac{F_s}{2} = \frac{625}{2} = 312.5$ Hz.

The discrete time signal is obtained by substituting

$$t = \frac{n}{F_s} = \frac{n}{625} \text{ in the given equation.}$$

$$\therefore \quad x(n) = 15\cos\left(1250\,\pi \frac{n}{625}\right) + 17\cos\left(2170\pi \frac{n}{625}\right) + 33\cos\left(4750\,\pi \frac{n}{625}\right)$$

$$\therefore \quad x(n) = 15\cos(2\pi)n + 17\cos(2\pi\;\; 1.736)\,n + 33\cos(2\pi\;\; 3.8)n$$

This equation is of the form $x(n) = A\cos(\omega_0 n)$

$\therefore$ $x(n) = 15\cos(2\pi)n + 17\cos(3.47\,\pi)n + 33\cos(7.6\,\pi)n$

Hence the discrete frequencies in terms of ω are,

$$\omega_1 = 2\pi$$

$$\omega_2 = 3.47\,\pi$$

$$\omega_3 = 7.6\,\pi$$

Ex. 1.5.8 : Consider the following analog signal

$x(t) = 2\sin(100\pi t)$

The signal is sampled at x(t) is sampled with a sampling rate $F_s = 50$ Hz. determine the discrete time signal.

MU - Dec. 2016, 10 Marks

Soln. : Given : $x(t) = 2 \sin (100\pi t)$ ...(1)

The discrete signal is obtained by substituting

$$t = \frac{n}{F_s} = \frac{n}{50} \text{ in Equation (1)}$$

$$\therefore \quad x(n) = 2 \sin \left(100\pi \frac{n}{50}\right)$$

$$\therefore \quad x(n) = 2 \sin 2\pi n$$

We vary n and realize that $x(n) = 0$ for all values of n

$$\therefore \quad x(n) = \{0,0,0,\ldots\ldots.\}$$

Ex. 1.5.9 : Consider the analog signal

$x(t) = 5 \cos 2\pi(1000)t + 10 \cos 2\pi(5000)t$ is to be sampled.

(i) What is Nyquist rate for this signal ?

(ii) If this analog signal is sampled at 4 kHz, will the signal be recovered from its samples ? **MU - May 2018, 10 Marks**

Soln. : (i)Nyquist rate

Given : $x(t) = 5 \cos 2\pi(1000)t + 10 \cos 2\pi(5000)t$ ...(1)

The given signal consists of following frequencies,

$$F_1 = 1000 \text{ Hz and}$$

$$F_2 = 5000 \text{ Hz}$$

$$\text{Thus } F_{max} = 5000 \text{ Hz} = 5 \text{ kHz}$$

$$\therefore \text{ Nyquist rate} = 2 F_{max} = 10 \text{ kHz}$$

(ii) Analog signal is sampled at 4 kHz .

A discrete signal is obtained by substituting

$$t = \frac{n}{F_s} = \frac{n}{4000} \text{ in Equation (1)}$$

$$\therefore \quad x(n) = 5 \cos 2\pi\left(\frac{1000\,n}{4000}\right) + 10 \cos 2\pi\left(\frac{5000\,n}{4000}\right)$$

$$\therefore \quad x(n) = 5 \cos\left(\frac{\pi}{2}\right) n + 10 \cos\left(\frac{5\pi}{2}\right) n$$

Now, in the second term $\left(\frac{5\pi}{2}\right)$ can be written as,

$$\left(\frac{5\pi}{2}\right) = \left(2\pi + \frac{\pi}{2}\right)$$

$$\therefore \quad 10 \cos\left(\frac{5\pi}{2}\right) n = 10 \cos\left(2\pi + \frac{\pi}{2}\right) n = 10 \cos\left(\frac{\pi}{2}\right) n$$

Hence we observe that $10 \cos\left(\frac{5\pi}{2}\right) n$ is identical to $10 \cos\left(\frac{\pi}{2}\right) n$. This is called aliasing.

$$\therefore \quad x(n) = 15 \cos\left(\frac{\pi}{2}\right) n$$

The continuous time signal had two different frequencies while the discrete time signal has only one.

(iii) The reconstructed signal is obtained by substituting $n = t F_s$ in the discrete time signal

Since $F_s = 4000$

$$\therefore \quad n = 4000\,t$$

$$\therefore \quad x(t) = 15 \cos\left(\frac{\pi}{2}\right) 4000t$$

$$\therefore \quad x(t) = 15 \cos 2\pi\, 1000t$$

We observe that the reconstructed signal has only one frequency of 1000 Hz.

1.6 Applications of DSP

Digital signal processing systems are used in a variety of applications. Look around and we realize that every electronic system has a DSP in it. Some of the applications are listed as follows :

1. Speech Recognition, Synthesis, Analysis.
2. Image processing applications.
3. Robotic vision.
4. Biomedical signal application like processing and analyzing ECG, EMG, CT, MRI, etc.
5. Data compression.
6. Military application like Radar signal processing, secure communications.
7. Industrial application like robotics, CNN.
8. Communication application like voice commands, cellular telephones.
9. Robotic vision, vibration analysis.

Summary

This chapter deals with the basics of Digital signal processing. A basic classification of signals is done. We have discussed the concept of frequency and explained the Nyquist criteria. Examples have been solved to understand Aliasing. A block diagram of a DSP system is discussed. The chapter ends with listing down the applications of DSP.

Review Questions

Q. 1 Explain advantages of Digital signal Processing over Analog Signal Processing

Q. 2 Explain aliasing effect.

Q. 3 Define : Aliasing

Q. 4 What is aliasing observed in sampling process ? How this can be avoided ?

Q. 5 Define a Nyquist rate.

Q. 6 Draw and explain basic elements of DSP.

Q. 7 Explain Concept of Frequency in Discrete Time Signals.

❑❑❑

CHAPTER 2

Discrete Time Signal and System

Syllabus :

Classification of Discrete-Time Signals, Classification of Discrete- Systems

Linear Convolution formulation for 1-D and 2-D signal (without mathematical proof), Auto and Cross Correlation formula evaluation, LTI system, Concept of Impulse Response and Step Response, Output of DT system using Time Domain Linear Convolution

2.1 Introduction

In chapter 1, we discussed the basics of digital signal processing. We also understood the method of obtaining a discrete time signal form an analog signal. In this chapter we discuss various aspects of discrete time signals and systems. We will also learn how to perform operations on discrete time signals.

2.1.1 Representation of Discrete Time Signals

There are various methods of representing a discrete time signal.

(1) Graphical representation

Consider a signal x(n) with values $x(-2) = 2$, $x(-1) = 1$, $x(0) = 1.5$, $x(1) = 1$, $x(2) = 2$, $x(3) = 3$, $x(4) = 1.5$. This signal can be represented graphically as shown in Fig. 2.1.1.

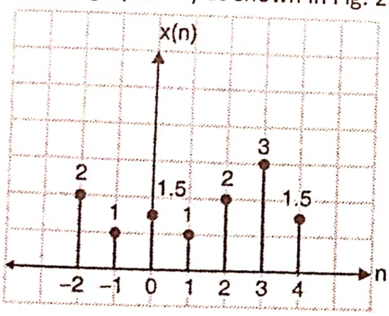


Fig. 2.1.1

(2) Sequence representation

A more concise way of representing the same signal is by representing it as a sequence shown as follows,

$$x(n) = \{2, 1, \underset{\uparrow}{1.5}, 1, 2, 3, 1.5\}$$

The arrow at the bottom indicates the origin.

If the arrow is not mentioned, then the first element is the origin.

Solved Example

Ex. 2.1.1 : Draw the graphical representation of the given signal $x(n) = \{1, 2, 2, 2, 3\}$

Soln. : Since there is no arrow, the first element is the origin.

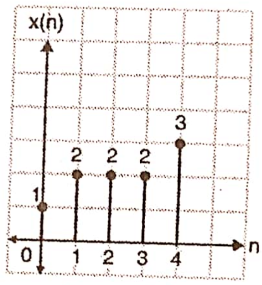


Fig. P. 2.1.1

An infinite sequence is represented by dots on both sides.

$$x(n) = \{\ldots, 1, 2, 4, 3, \ldots.\}$$

(3) Functional representation :

The discrete time signal can also be represented as,

$$x(n) = \begin{cases} 1 & n = 1,3 \\ -4 & n = 0 \\ 0 & \text{else where} \end{cases}$$

This can be graphically drawn as shown in Fig. 2.1.2.

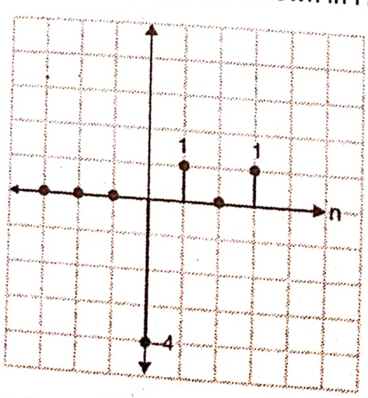


Fig. 2.1.2

Solved Example

Ex. 2.1.2 : Represent the following signals graphically :

(i) $x(n) = \{1, 2, \underset{\uparrow}{0}, -1, 1\}$

(ii) $x(n) = \{0, 0, -1, 2, \underset{\uparrow}{3}\}$

(iii) $x(n) = \{\underset{\uparrow}{0}, 1, -1, 1, -1\}$

Soln. : These signals are as shown in Figs. P.2.1.2(a), (b) and (c) respectively.

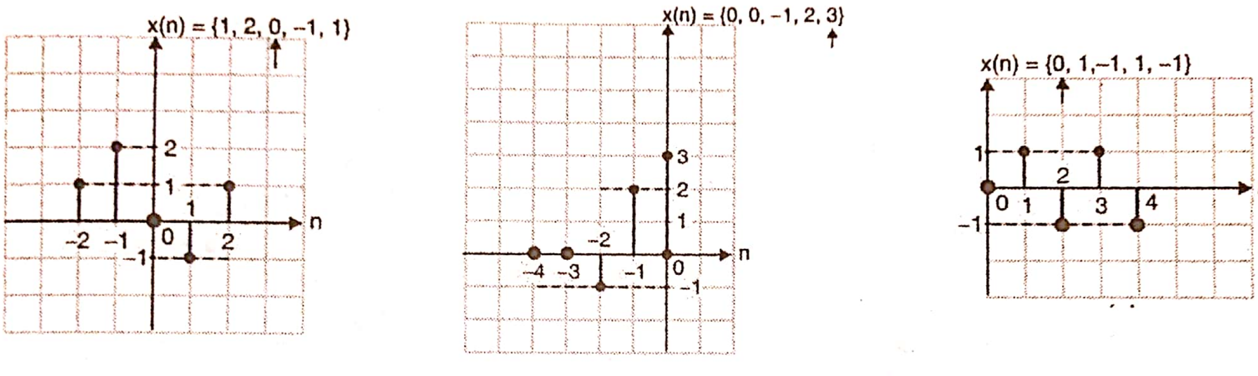


Fig. P. 2.1.2 : Graphical representation of given sequences

2.1.2 Elementary Discrete Time Signals

There are a number of basic signals that we encounter regularly and which play an important role in signal processing.

1. Delta or a unit impulse function

A unit impulse signal is denoted by $\delta(n)$ and is defined as,

$$\delta(n) = \begin{cases} 1 \ ; & n = 0 \\ 0 \ ; & \text{otherwise} \end{cases}$$

The graphical representation of the unit impulse signal is shown in Fig. 2.1.3.

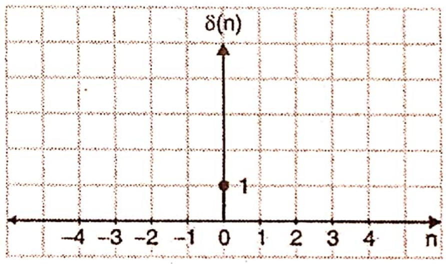


Fig. 2.1.3

A unit impulse function is a signal that is zero everywhere except at n = 0 where its value is unity.

2. Unit step signal

A unit step signal is denoted by $u(n)$ and is defined as,

$$u(n) = \begin{cases} 1 \ ; & n \geq 0 \\ 0 \ ; & \text{otherwise} \end{cases}$$

The graphical representation of the unit step signal is shown in Fig. 2.1.4.

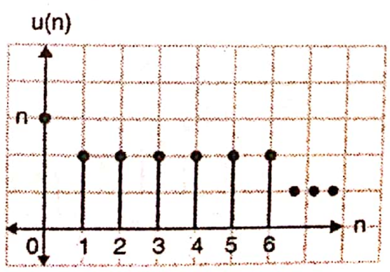


Fig. 2.1.4

3. Unit ramp signal

A unit ramp signal is denoted by $u_r(n)$ and is defined as,

$$u_r(n) = \begin{cases} n \; ; & n \geq 0 \\ 0 \; ; & \text{otherwise} \end{cases}$$

The graphical representation of the unit ramp signal is shown in Fig. 2.1.5.

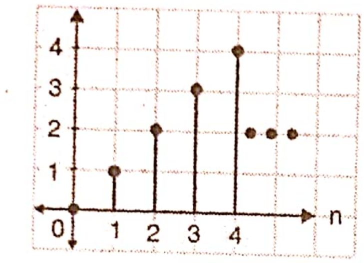


Fig. 2.1.5

4. Exponential signal

A exponential signal is of the form

$$x(n) = a^n \quad \text{for all n}$$

The graphical representation of the exponential signal is shown in Fig. 2.1.6. Four different ranges of 'a' are taken to study the change in the exponential pattern.

Fig. 2.1.6 illustrates different type of discrete-time exponential signals. When the value is $0 < a < 1$, the sequence decays exponentially, when value of $a > 1$, the sequence grows exponentially. Note that when $a < 0$, the discrete-time exponential signal takes alternating signs.

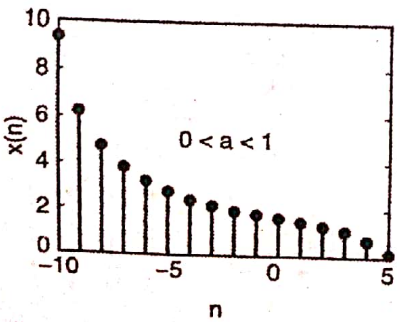
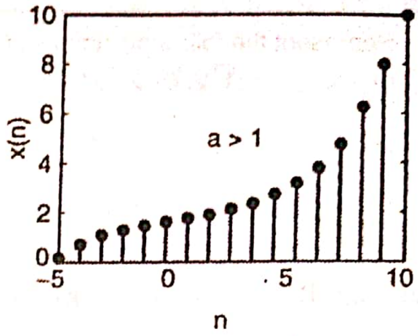
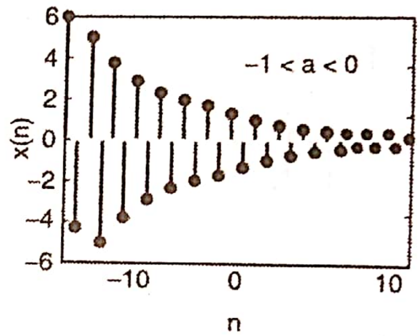


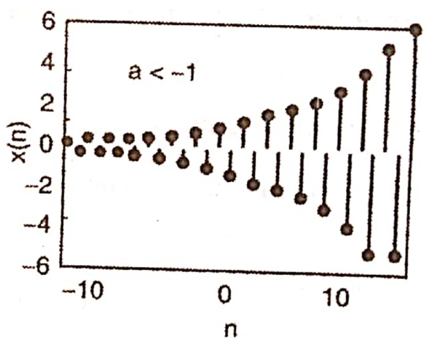
Fig. 2.1.6 (a)



(b)



(c)



(d)

Fig. 2.1.6 : Exponential sequences

5. Sinusoidal signal

A discrete time sinusoidal signal is given by the formula

$$x(n) = A \cos(\omega n + \phi)$$

Where A is the amplitude, ω is the frequency in radians and ϕ is the phase.

An example of discrete-time sinusoidal signal is shown in Fig. 2.1.7.

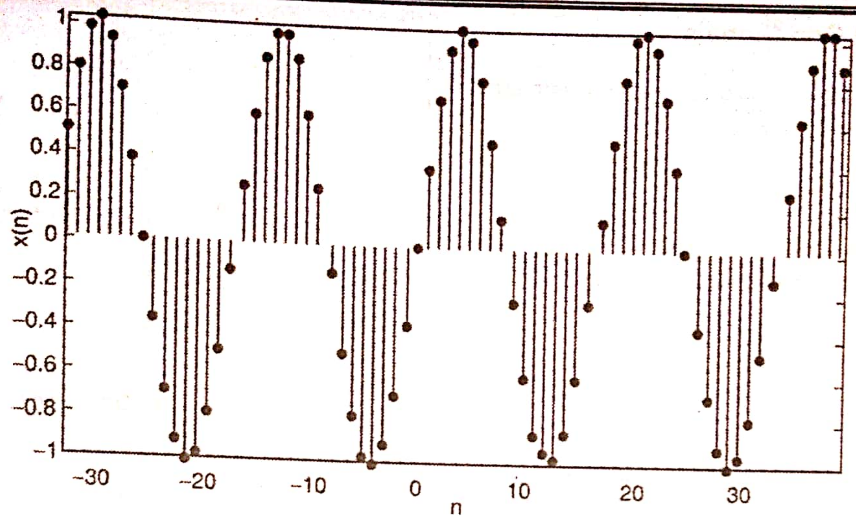


Fig. 2.1.7 : A sinusoidal sequence

Given below is the MATLAB code for generating various elementary discrete time signals. Run the above program and see the results.

```
%Plot Unit Step, Unit Ramp, Exponential and Sinusoidal
Functions

clc
clear all

%UNIT STEP
for n = 1 : 1 : 20% We take the length of the sequence to be
20
     x1(n) = 1;
end
figure(1)
stem(x1)

%UNIT RAMP
for n = 1 : 1 : 20
     x2 (n) = n ;
end
figure(2)
stem(x2)

% EXPONENTIAL SIGNAL
a = input ('enter the value of a for exponential signal :')
for n = 1 : 1 : 20
     x3(n) = a ^ n;
end
figure (3)
stem (x3)

%SINUSOIDAL SIGNAL
A = input ('enter the amplitude for sinusoidal function :')
w = input ('enter the value of frequency for sinusoidal
function :')
%Take w = 0.1*pi, 0.2*pi,....0.9* pi, pi
for n = 1 : 1 : 20
     x4(n) = A*cos (w*n) ;
end
figure(4)
stem(x4)
```

2.2 Classification of Discrete Time Signals

Discrete time signals have different characteristics. We now classify the discrete time signals based on these characteristics. The discrete time signals are classified as follows :

1. Even and odd signals (Symmetric and Anti-symmetric signals)
2. Periodic and Aperiodic signals
3. Energy and power signals

2.2.1 Even and Odd Signals

A discrete time signal is said to be even (symmetric) if it satisfies the following condition.

$$x(n) = x(-n) \qquad \ldots(2.2.1)$$

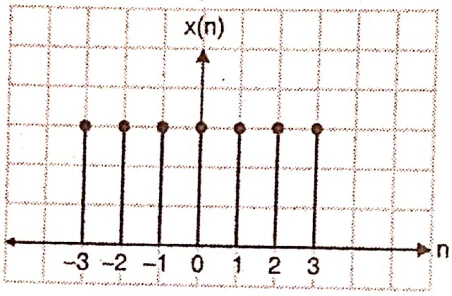
A discrete time signal is said to be odd (Anti-symmetric) if it satisfies the following condition.

$$x(n) = -x(-n) \qquad \ldots(2.2.2)$$

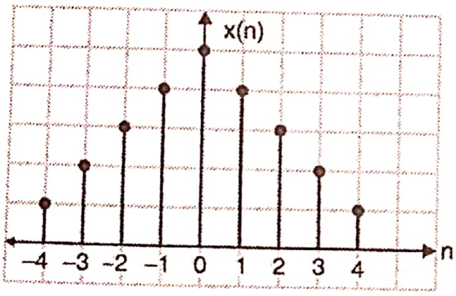
For an even signal, $x(1) = x(-1), x(2) = x(-2)\ldots$

While for a odd signal, $x(1) = -x(-1), x(2) = -x(-2)\ldots$

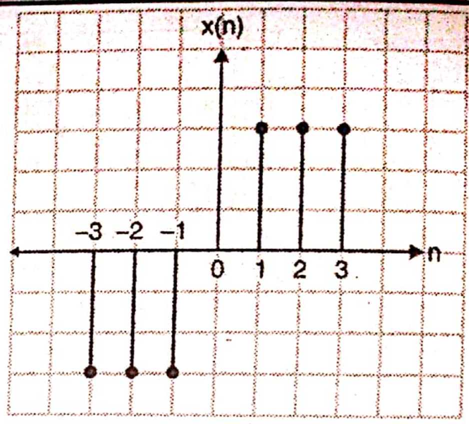
Given below are examples of even and odd signals.



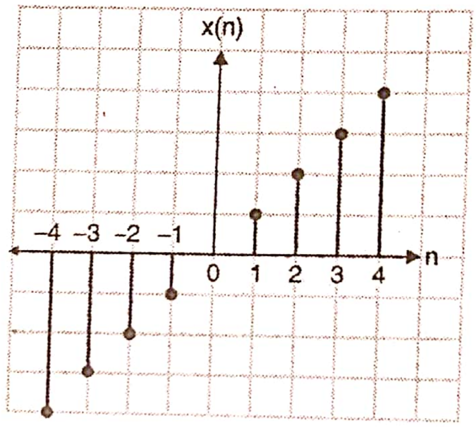
(a) Even signal



(b) Even signal



(c) Odd signal



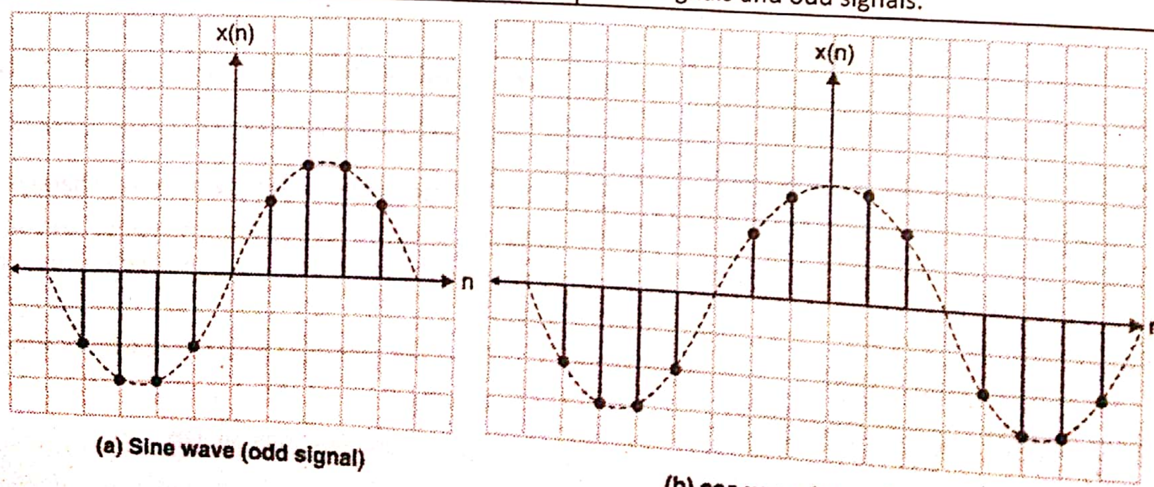
(d) Odd signal

Fig. 2.2.1

A sine wave is a odd signal while a cos wave is a even signal,

> **Note :** sin (50) = – sin (– 50)
> cos (50) = cos (– 50)

Most of the signals that we encounter are neither purely even or odd but are a combination of both. In other words any arbitrary signal can be expressed as a sum of even signals and odd signals.



(a) Sine wave (odd signal)

(b) cos wave (even signal)

Fig. 2.2.2

The even component is extracted by adding x(n) to x(– n) and dividing by 2 i.e.

$$x_e(n) = \frac{x(n) + x(-n)}{2} \quad \text{...(2.2.3)}$$

Similarly the even component is extracted by adding x(n) to – x(– n) and diving by 2.

$$X_o(n) = \frac{x(n) - x(-n)}{2} \quad \text{...(2.2.4)}$$

Clearly Equation (2.2.3) satisfies the symmetric condition in Equation (2.2.1) while Equation (2.2.4) satisfies the antisymmetric condition in Equation (2.2.2).

Let us solve an example to understand this better.

Note : x (– n) is simplify the mirror image of x (n).

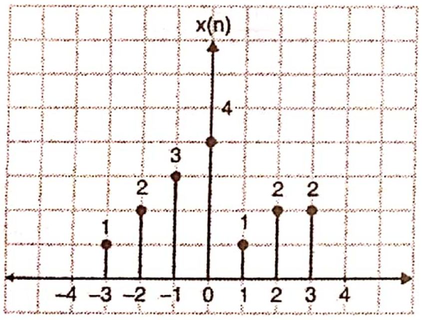
Solved Example

Ex. 2.2.1 : Find the even and odd components of the given discrete time signal

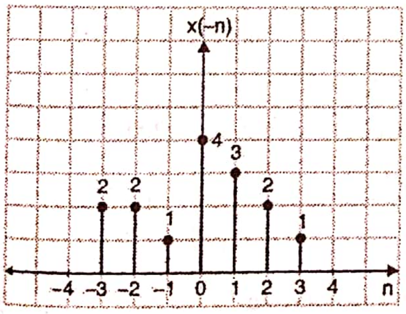
$$x(n) = \{1, 2, 3, \underset{\uparrow}{4}, 1, 2, 2\}$$

Soln. : We shall solve this example graphically. To find the even and odd components, we need x (n) as well as x (– n). We draw them one below the other.

As stated earlier, x (– n) is simply the mirror image of x (n).



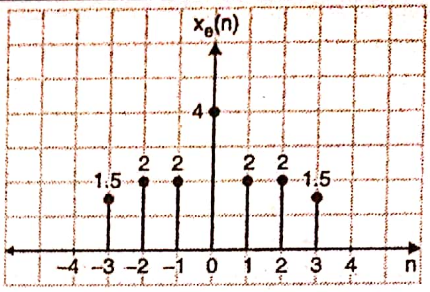
(a)



(b)

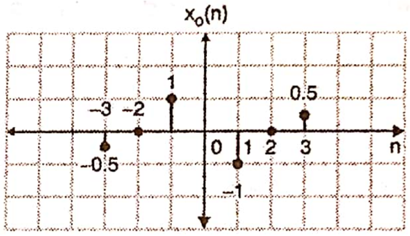
Fig. P. 2.2.1

Now, $x_e(n) = \frac{x(n) + x(-n)}{2}$



(c)

$$x_o(n) = \frac{x(n) - x(-n)}{2}$$



(d)

Fig. P. 2.2.1

Note : If we add x_e (n) and x_o (n), we get back the original signal ! !

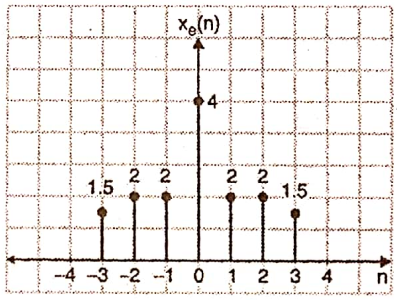


Fig. P. 2.2.1 (e)

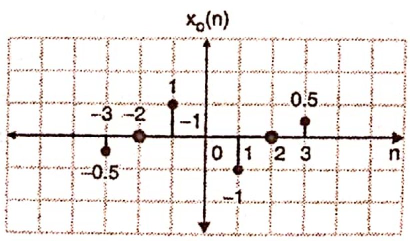


Fig. P. 2.2.1 (f)

$$x(n) = x_e(n) + x_o(n)$$

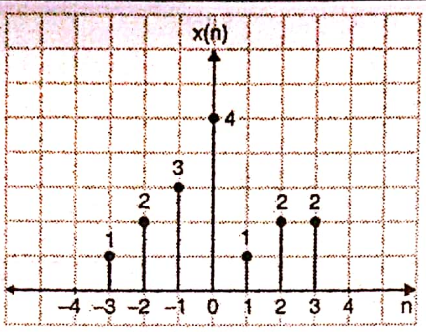


Fig. P.2.2.1 (g)

Take any sequence and this property will always work.

2.2.2 Periodic and Aperiodic Signals

A signal is periodic if there exist N such that

$$x(n+N) = x(n) \quad \ldots(2.2.5)$$

Let us take an example. Check whether $x(n) = A\cos(\omega n + \theta)$ is periodic.

$$x(n) = A\cos(\omega n + \theta)$$

$$\text{Here, } \omega = 2\pi f$$

$$\therefore \quad x(n) = A\cos(2\pi f n + \theta)$$

$$\text{Now,} \quad x(n+N) = A\cos(2\pi f(n+N) + \theta)$$

$$\text{For periodicity} \quad x(n+N) = x(n)$$

$$\therefore \quad A\cos(2\pi f(n+N) + \theta) = A\cos(2\pi f n + \theta)$$

$$\therefore \quad A\cos(2\pi f n + 2\pi f N + \theta) = A\cos(2\pi f n + \theta)$$

The above equation will be true only if

$$2\pi f N = 2\pi k \qquad \text{where, k is an integer.}$$

$$\therefore \quad f = \frac{k}{N}$$

Here, N and k are both integers.

Hence a discrete time signal is periodic only if it's frequency is a rational number (ratio of two integers).

If the input signal is expressed as a summation of two signals i.e., $x(n) = x_1(n) + x_2(n)$, we calculate f_1 and f_2 where, $f_1 = \frac{k_1}{N_1}$ and $f_2 = \frac{k_2}{N_2}$

The resultant signal x (n) is periodic if $\frac{N_1}{N_2}$ is a ratio of two integers.

The period of x (n) will be the LCM of N_1 and N_2.

A few examples would make things clear.

Solved Examples

Ex. 2.2.2 : Few discrete time sequences are given below

(i) $\cos(0.01\pi n)$ (ii) $\cos(3\pi n)$

(iii) $\sin(3n)$ (iv) $\cos\left(\frac{n}{8}\right)\cos\left(\frac{\pi n}{8}\right)$

Determine whether they are periodic or aperiodic. If a sequence is periodic, determine its fundamental period.

Soln. :

(i) Given :

$$x(n) = \cos(0.01\pi n) \quad \ldots(1)$$

We have the standard equation,

$$x(n) = \cos\omega n \quad \ldots(2)$$

Comparing Equations (1) and (2) we get,

$$\omega = 0.01\pi$$

$$\text{But} \quad \omega = 2\pi f$$

$$\therefore \quad 2\pi f = 0.01\pi$$

$$\therefore \quad f = \frac{0.01\pi}{2\pi} = \frac{0.01}{2}$$

$$\therefore \quad f = \frac{1}{200} \text{ cycles per sample} \quad \ldots(3)$$

Since frequency 'f 'is expressed as the ratio of two integers; this sequence is periodic. Now we have the condition of periodicity,

$$f = \frac{k}{N} \quad \ldots(4)$$

Here 'N' indicates, the fundamental period.

Comparing Equations (3) and (4)

Fundamental period = N = 200 samples

(ii) Given :

$$x(n) = \cos(3\pi n) \quad \ldots(5)$$

Comparing with Equation (2) we get,

$$\omega = 3\pi$$

$$\therefore \quad 2\pi f = 3\pi$$

$$\therefore \quad f = \frac{3}{2} \text{ cycles / sample} \quad \ldots(6)$$

Since 'f' is ratio of two integers; the given sequence is periodic. Comparing Equations (4) and (6) we get,

Fundamental period = N = 2 samples

(iii) Given sequence is,

$$x(n) = \sin 3n \quad ...(7)$$

Comparing with Equation (2) we get,

$$\omega = 3$$

$$\therefore \quad 2\pi f = 3$$

$$\therefore \quad f = \frac{3}{2\pi}$$

Here 2π is not an integer. That means 'f' cannot be expressed as the ratio of two integers. In this case f is irrational. Thus the given sequence is non-periodic.

(iv) Given sequence is,

$$x(n) = \cos\left(\frac{n}{8}\right)\cos\left(\frac{\pi n}{8}\right) \quad ...(8)$$

The standard equation can be expressed as,

$$x(n) = \cos\omega_1 n \cos\omega_2 n \quad ...(9)$$

Comparing Equations (8) and (9) we get,

$$\omega_1 = \frac{1}{8} \text{ and } \omega_2 = \frac{\pi}{8}$$

but, $\omega = 2\pi f$

$$\therefore \quad 2\pi f_1 = \frac{1}{8} \text{ and } 2\pi f_2 = \frac{\pi}{8}$$

$$\therefore \quad f_1 = \frac{1}{16\pi} \text{ and } f_2 = \frac{1}{16}$$

Hence f_1 is ratio of non-integer values. So it is non-periodic. While f_2 is ratio of two integer values. So it is periodic. But the total signal is multiplication of non-periodic and periodic signals. So the resultant signal x(n) is non-periodic.

Ex. 2.2.3 : Find if the following sequences are periodic or not. If yes find its fundamental time period.

(i) $x_1(n) = e^{j\left(\frac{\pi}{4}\right)n}$ (ii) $x_2(n) = 3\sin\left(\frac{1}{8}n\right)$

Soln. :

(i) Given : $$x_1(n) = e^{j\left(\frac{\pi}{4}\right)n} \quad ...(1)$$

A discrete time complex exponential is periodic if its relative frequency is a rational number. We have the standard equation of a discrete time complex exponential.

$$x(n) = e^{j\frac{2\pi k n}{N}} \quad ...(2)$$

Comparing Equations (1) and (2) we get,

$$\frac{2\pi k n}{N} = \frac{\pi}{4}n \qquad \therefore \frac{2k}{N} = \frac{1}{4}$$

$$\therefore \frac{k}{N} = \frac{1}{8}$$

Since it is ratio of two integers; the given sequence is periodic. It's fundamental period is N= 8 samples.

(ii) Given $$x_2(n) = 3\sin\left(\frac{1}{8}n\right) \quad ...(3)$$

We have the standard equation,

$$x(n) = A\sin\omega n \quad ...(4)$$

Comparing Equations (3) and (4) we get,

$$\omega = \frac{1}{8} \qquad \therefore 2\pi f = \frac{1}{8}$$

$$\therefore \quad f = \frac{1}{16\pi}$$

Here π is an irrational number hence f is not the ratio of two integers. Thus given sequence is non-periodic.

2.2.3 Energy and Power Signals

For a discrete time signal x (n), the energy is defined as,

$$E = \sum_{n=-\infty}^{+\infty} |x(n)|^2$$

The reason for using magnitude square is so that the definition applies to real valued as well as complex valued signals. If E is finite, then x (n) is called an **energy signal.** Many signals that possess infinite energy have finite power. The average power of a discrete time signal x (n) is defined as

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

A signal x (n) is said to be a **power signal** if the average power is finite.

Solved Example

Ex. 2.2.4 : Obtain energy for the signal $x(n) = a^n u(n)$ where $|a| < 1$.

Soln. : For D.T. signal, energy is given by,

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \quad ...(1)$$

The given equation of $x(n)$ is,

$$x(n) = a^n u(n) \quad ...(2)$$

Here $u(n)$ represents unit step. It's value is one for the range 0 to ∞. Hence multiplication by $u(n)$ does not change the amplitude of a^n; but it indicates that a^n is present only from $n = 0$ to $n = \infty$.

Thus Equation (1) reduces to,

$$E = \sum_{n=0}^{\infty} |a^n|^2 \quad ...(3)$$

Rearranging the term we get,

$$E = \sum_{n=0}^{\infty} [a^2]^n \quad ...(4)$$

We use the geometric series formula,

$$\sum_{n=0}^{\infty} A^n = 1 + A + A^2 + A^3 + ... = \frac{1}{1-A} \text{ if } |A| < 1.$$

Here $A = a^2$. Thus Equation (4) reduces to,

$$E = \frac{1}{1-a^2} \text{ if } |a^2| < 1$$

Ex. 2.2.5 : $x(n) = (0.5)^n u(n)$. State whether it is an energy or power signal. Justify.

Soln. : First we will calculate the energy of signal $x(n)$. It is given by,

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \quad ... (1)$$

The given signal is,

$$x(n) = (0.5)^n u(n) \quad ... (2)$$

Since it is multiplied by unit step; this signal is present from $n = 0$ to $n = \infty$. Thus Equation (1) becomes,

$$E = \sum_{n=0}^{\infty} [(0.5)^n]^2 = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2n}$$

$$E = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \quad ...(3)$$

We have the standard geometric series formula,

$$\sum_{n=0}^{\infty} A^n = 1 + A + A^2 + ... = \frac{1}{1-A}$$

$$\therefore \quad E = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

i.e. E is finite.

Note : If $0 < E < \infty$ then the signal is energy signal. Since the calculated value of energy is finite; the given signal is energy signal.

Ex. 2.2.6 : Determine the energy and power of signal given by,

$$x(n) = \left(\frac{1}{2}\right)^n \quad n \geq 0$$
$$= 3^n \quad n < 0$$

Soln. : The energy of signal is given by,

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \quad ... (1)$$

The given signal is, $x(n) = \left(\frac{1}{2}\right)^n$ for the range $n \geq 0$ and $x(n) = 3^n$ for $n < 0$. Thus Equation (1) becomes,

$$E = \sum_{n=0}^{\infty} \left[\left(\frac{1}{2}\right)^n\right]^2 + \sum_{n=-1}^{-\infty} [3^n]^2$$

$$\therefore \quad E = \sum_{n=0}^{\infty} \left[\left(\frac{1}{2}\right)^2\right]^n + \sum_{n=-1}^{-\infty} [3^2]^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n + \sum_{n=-1}^{-\infty} (9)^n \quad ...(2)$$

Consider the first summation term. Using geometric series formula we can express it as follows :

$$\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{1-\frac{1}{4}} = \frac{4}{3} \quad ...(3)$$

Now consider the second summation term. To make the limits of the summation positive, put $n = -m$. The limits change as follows.

When $n=-1 \Rightarrow -m=-1 \therefore m=1$

When $n=-\infty \Rightarrow -m=-\infty \therefore m=\infty$

$$\therefore \sum_{n=-1}^{-\infty} (9)^{n} = \sum_{m=+1}^{\infty} (9)^{-m} = \sum_{m=1}^{\infty} \left(\frac{1}{9}\right)^{m}$$

Now use the standard summation formula,

$$\sum_{n=1}^{\infty} A^{n} = \frac{A}{1-A}$$

$$\therefore \quad \sum_{m=1}^{\infty} \left(\frac{1}{9}\right)^{m} = \frac{1/9}{1-\frac{1}{9}} = \frac{\frac{1}{9}}{8/9} = \frac{1}{8} \quad \ldots(4)$$

Putting Equations (3) and (4) in Equation (2) we get,

$$E = \frac{4}{3} + \frac{1}{8}$$

$$\therefore \quad E = \frac{35}{24}$$

Since the energy is finite, the given signal is an Energy signal.

If the energy of signal is finite then its power is zero. Thus the power of given signal is zero.

$$\therefore \quad P = 0$$

Ex. 2.2.7 : Determine the energy and power of the signal given by,

$$x(n) = \left(\frac{1}{3}\right)^{n} u(n)$$

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Soln. :

$$x(n) = \left(\frac{1}{3}\right)^{n} u(n)$$

We know

$$E = \sum_{n=-\infty}^{+\infty} |x(n)|^{2}$$

$$= \sum_{n=-\infty}^{+\infty} \left|\left(\frac{1}{3}\right)^{n} u(n)\right|^{2}$$

Since there is u(n), we change the limits of summation

$$\therefore E = \sum_{n=0}^{\omega} \left(\frac{1}{3}\right)^{n2}$$

$$\therefore E = \sum_{n=0}^{\omega} \left(\frac{1}{9}\right)^{n}$$

From the geometric progression formula,

We know

$$\sum_{n=D}^{\omega} a^{n} = \frac{1}{1-a} \quad ; |a| < 1$$

$$\therefore E = \frac{1}{1-\frac{1}{9}} = 1.125$$

Since $E < \infty$, the given signal is a energy signal.

Since the signal has finite energy, the power of the signal is zero.

2.3 Operation on Signals

It is imperative for students to understand simple manipulation techniques on discrete time signals. Signal processing involves modifying the original signal. This modification is achieved by performing different operations on discrete time signals. The mathematical transformation from one signal to another is represented as,

$$y(n) = T[x(n)]$$

Here x (n) is the input, y (n) is the output and T is the operation that one performs.

The basic operations are :

1. Time shifting
2. Time reversal
3. Time scaling
4. Scalar multiplication
5. Signal addition and multiplication

We shall discuss each one in detail.

2.3.1 Time Shifting

The shift operation takes the input signal x (n) and shifts the signal, resulting in either a delay or an advancement of the signal.

Mathematically it is represented as,

$$y(n) = x(n-k)$$

Here k is the amount of shift required.

Let us see what this equation amounts to.

Solved Examples

Ex. 2.3.1 : $x(n) = \{1, 2, 3, 4, 5\}$ with ↑ under 2

Find $y(n) = x(n-2)$

Soln. : We draw x (n)

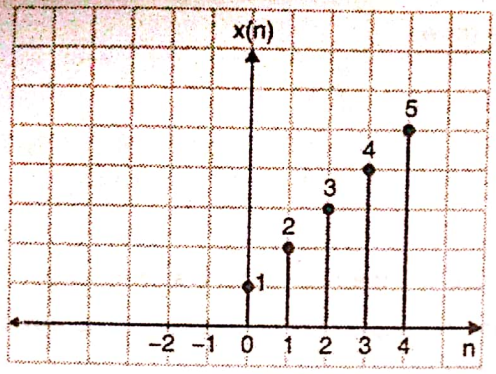

Fig. P.2.3.1

$$y(n) = T[x(n)]$$

$$\therefore \quad y(n) = x(n-2)$$

We take different values of n

$y(0) = x(-2) = 0$ ($\because$ the value of x(n) at n = −2 is 0)

$y(1) = x(-1) = 0$

$y(2) = x(0) = 1$ ($\because$ the value of x(n) at n = 0 is 1)

$y(3) = x(1) = 2$

$y(4) = x(2) = 3$

$y(5) = x(3) = 4$

$y(6) = x(4) = 5$

$y(7) = x(5) = 0$

We plot these values

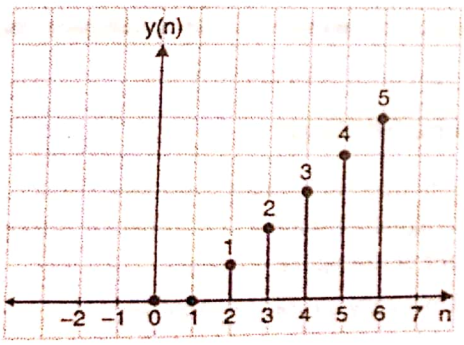

Fig. P. 2.3.1(a)

Hence we state that y (n) = x (n − 2) is a right shifted version of x (n) i.e., y (n) is a delayed version of x (n).

Ex. 2.3.2 : $x(n) = \{\underset{\uparrow}{1}, 2, 3, 4, 5\}$

Find $y(n) = x(n+2)$

Soln. :

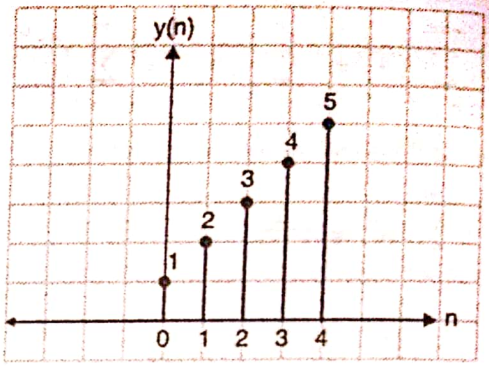

Fig. P.2.3.2

$$y(n) = T[x(n)]$$

$$\therefore \quad y(n) = x(n+2)$$

We take different values of n.

$y(-2) = x(0) = 1$

$y(-1) = x(1) = 2$

$y(0) = x(2) = 3$

$y(1) = x(3) = 4$

$y(2) = x(4) = 5$

$y(3) = x(5) = 0$

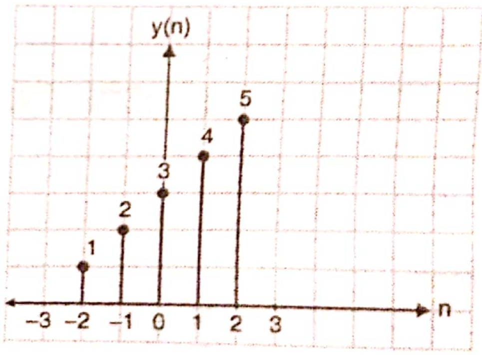

Fig. P. 2.3.2(a)

We hence note that y (n + 2) shifts the output to the left. This is known as time advance.

Hence for y (n − k), if k is > 1 → Right shift → Time delay

if k is < 1 → Left shift → Time advance

2.3.2 Time Reversal

This is also known as the folding operation. It is mathematically given by the equation,

$$y(n) = T[x(n)]$$

$$y(n) = x(-n)$$

This can be very easily understood by taking the same example.

Solved Example

Ex. 2.3.3 : $x(n) = \{\underset{\uparrow}{1}, 2, 3, 4, 5\}$ find y(n).

Soln. :

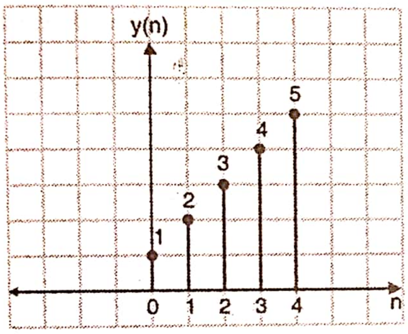

Fig. P. 2.3.3

$$y(n) = T[x(n)]$$

$$y(n) = x(-n)$$

We take different values of n

$$y(-4) = x(-(-4)) = x(4) = 5$$

$$y(-3) = x(3) = 4$$

$$x(-2) = x(2) = 3$$

$$x(-1) = x(1) = 2$$

$$y(0) = x(0) = 1$$

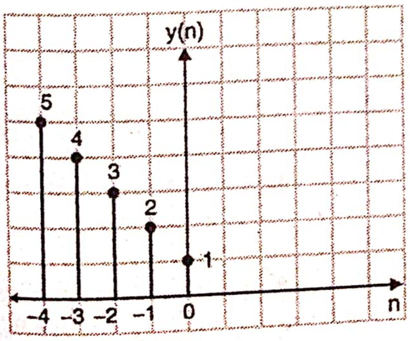

Fig. P.2.3.3(a)

Hence y (n) is a mirror image of x (n). It is a folded version of x (n).

We have used the time reversal operation while computing the odd and even components of a signal.

2.3.3 Time Scaling

This operation is related to change in time scale.

There are two types of time scaling operations :

(a) Down scaling
(b) Up scaling

(a) Down scaling : It is mathematically represented as,

$$y(n) = T[x(n)]$$

$$y(n) = x(2n)$$

We shall explain this with an example.

Solved Example

Ex. 2.3.4 : $x(n) = \{5, 4, 3, 2, \underset{\uparrow}{1}, 2, 3, 4, 5\}$

Find $y(n) = x(2n)$

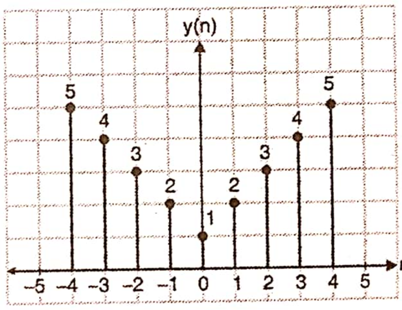

Fig. P. 2.3.4

Soln. :

$$y(n) = T[x(n)]$$

$$y(n) = x(2n)$$

We take different values of n

$$y(0) = x(0) = 1$$

$$y(1) = x(2) = 3$$

$$y(2) = x(4) = 5$$

$$y(3) = x(6) = 0$$

$$y(-1) = x(-2) = 3$$

$$y(-2) = x(-4) = 5$$

$$y(-3) = x(-6) = 0$$

$$\therefore \quad y(n) = \{5, 3, \underset{\uparrow}{1}, 3, 5\}$$

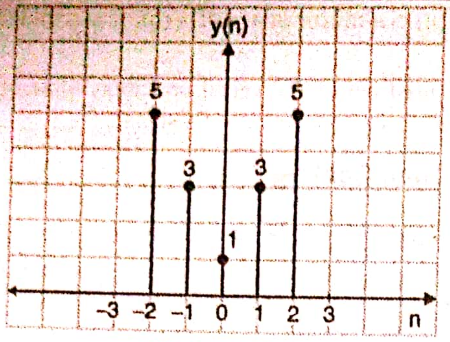

Fig. P.2.3.4(a)

Hence we note that the signal gets compressed. It is also important to note that the shape of x (n) and y (n) remain the same. This is known as down sampling.

(b) Up scaling : It is mathematically represented as,

$$y(n) = x(n/2)$$

We take the same example.

Solved Example

Ex. 2.3.5 : $x(n) = \{5, 4, 3, 2, \underset{\uparrow}{1}, 2, 3, 4, 5\}$ find y(n)

Soln. :

$$y(n) = T[x(n)]$$

$$y(n) = x(n/2)$$

We take different values of n

$$y(0) = x(0) = 1$$
$$y(1) = x(1/2) = \text{No sample}$$
$$y(2) = x(1) = 2$$
$$y(3) = x(3/2) = \text{No sample}$$
$$y(4) = x(2) = 3$$
$$y(5) = x(5/2) = \text{No sample}$$
$$y(6) = x(3) = 4$$
$$y(7) = x(7/2) = \text{No sample}$$
$$y(8) = x(4) = 5$$
$$y(-1) = x(-1/2) = \text{No sample}$$
$$y(-2) = x(-1) = 2$$
$$y(-3) = x(-3/2) = \text{No sample}$$
$$y(-4) = x(-2) = 3$$
$$y(-5) = x(-5/2) = \text{No sample}$$
$$y(-6) = x(-3) = 4$$
$$y(-7) = x(-7/2) = \text{No sample}$$
$$y(-8) = x(-4) = 5$$

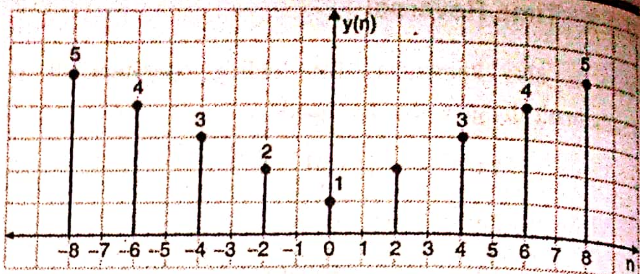

Fig. P. 2.3.5

y (n) is an expanded version of x (n). It is important to note that the shape of x(n) and y(n) remain the same. This is known as up-sampling.

2.3.4 Scalar Multiplication

In this operation, the entire signal is multiplied by a scalar. This is mathematically represented by,

$$y(n) = T[x(n)]$$

i.e., $$y(n) = \alpha \cdot x(n)$$

This results in amplification of the signal.

Solved Example

Ex. 2.3.6 : Let $x(n) = \{\underset{\uparrow}{1}, 2, 3, 4, 5\}$, find $y(n) = 2 \cdot x(n)$

Soln. : $$y(n) = T[x(n)]$$

$$\therefore \; y(n) = 2 \cdot x(n)$$

We take different values of n

$$y(0) = 2 \cdot x(0) = 2 \times 1 = 2$$
$$y(1) = 2 \cdot x(1) = 2 \times 2 = 4$$
$$y(2) = 2 \cdot x(2) = 2 \times 3 = 6$$
$$y(3) = 2 \cdot x(3) = 2 \times 4 = 8$$
$$y(4) = 2 \cdot x(4) = 2 \times 5 = 10$$

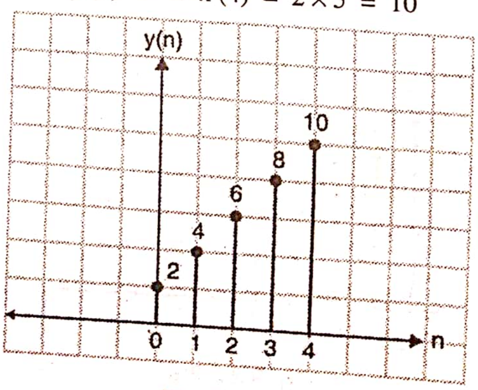

Fig. P. 2.3.6

2.3.5 Signal Addition and Multiplication

In these operations, signals are added or multiplied taking into account of their instances of time (their values of n).

Let us solve an examples to make things clear.

2.3.6 Solved Examples on Signals

Ex. 2.3.7 : Given two signals $x_1(n) = \{1, 1, \underset{\uparrow}{0}, 1, 1\}$ and $x_2(n) = \{2, 2, \underset{\uparrow}{0}, 2, 2\}$.

Perform (i) $x_1(n) + x_2(n)$ (ii) $x_1(n) \times x_2(n)$

Soln. : The first step is to place the two signals exactly one below the other.

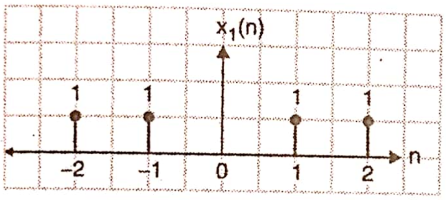

Fig. P.2.3.7

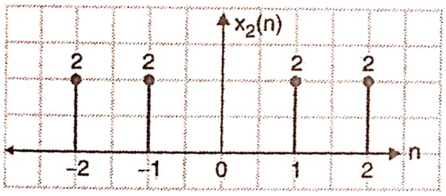

Fig. P.2.3.7(a)

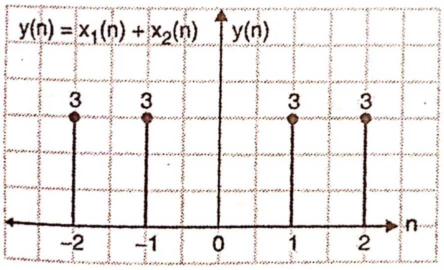

Fig. P.2.3.7(b)

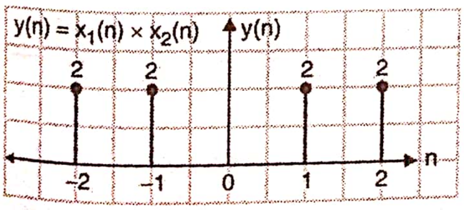

Fig. P. 2.3.7(c)

Bear in mind that we add / multiply only the corresponding terms.

All the operations discussed so far would be used to build up complex operations like the convolution operation.

With classification of signals and operation on signals complete, we now move to discuss various systems.

Ex. 2.3.8 : Draw $u(n) - u(n-5)$

Soln. :

$u(n)$ is a unit step while $u(n-5)$ is a unit step shifted by 5 to the right. We draw them one below the other and perform subtraction.

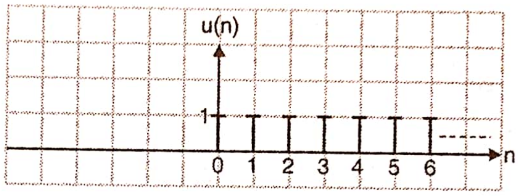

Fig. P.2.3.8

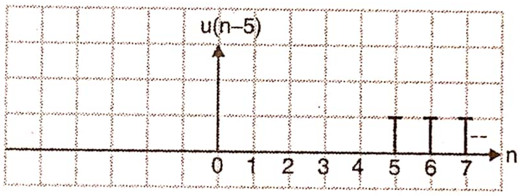

Fig. P.2.3.8(a)

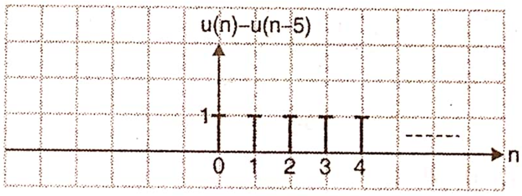

Fig. P. 2.3.8(b)

$$\therefore\ u(n) - u(n-5) = \{1, 1, 1, 1, 1\}$$

Ex. 2.3.9 :

For $x(n) = \{3, 2, 1, 6, 4, 5\}$, plot the following discrete time signals.

(i) $x(n+1)$

(ii) $x(-n)\ u(-n)$

(iii) $x(n-1)\ u(n-1)$

(iv) $x(n-1)\ u(n)$

(v) $x(n-2)$

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Soln. :

We begin with drawing $x(n)$.

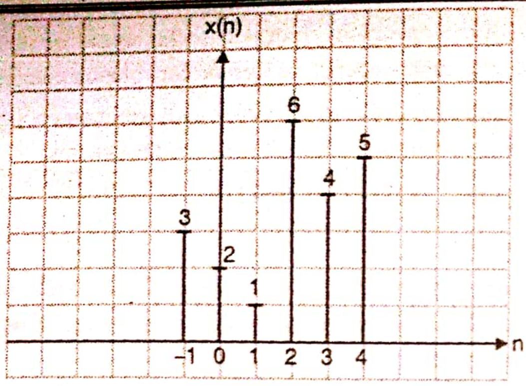

Fig. P. 2.3.9

(i) x(n+1)

This is x(n) shifted to the left by 1.

$$\therefore \; x(n+1) = \{3, 2, \underset{\uparrow}{1}, 6, 4, 5\}$$

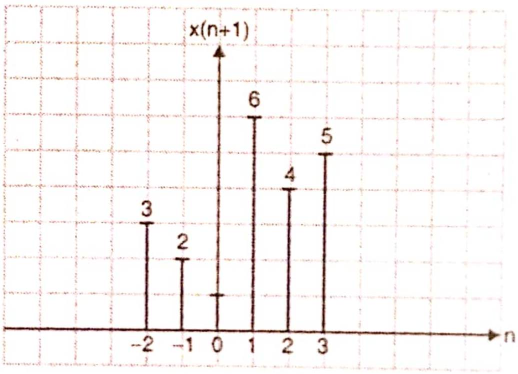

Fig. P.2.3.9(a)

(ii) x(–n) · u(–n)

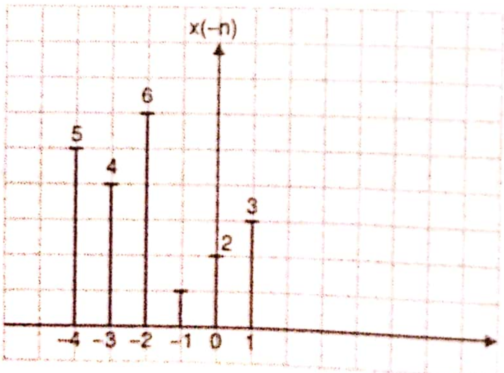

Fig. P.2.3.9(b)

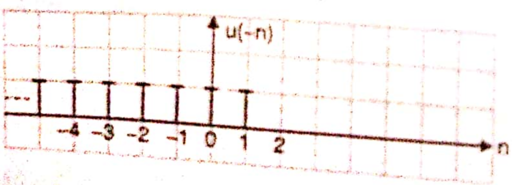

Fig. P.2.3.9(c)

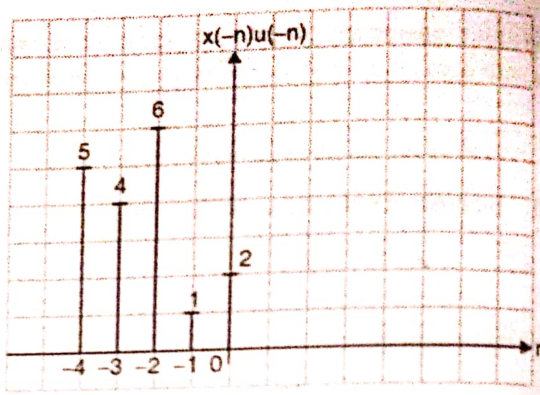

Fig. P.2.3.9(d)

(iii) x(n – 1) · u(–n – 1)

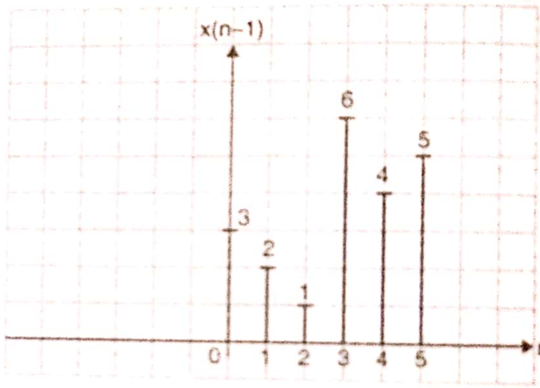

Fig. P.2.3.9(e)

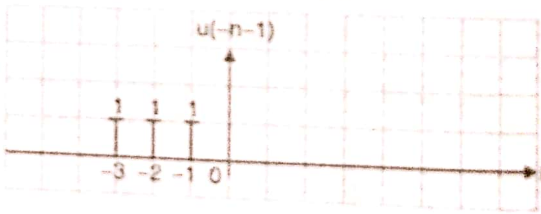

Fig. P.2.3.9(f)

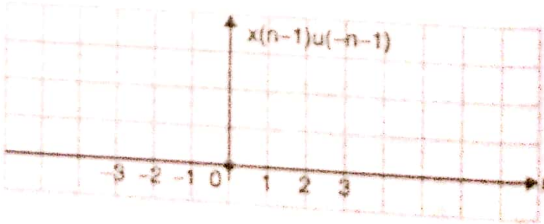

Fig. P.2.3.9(g)

All values are zero.

(iv) x(n – 1) u(n)

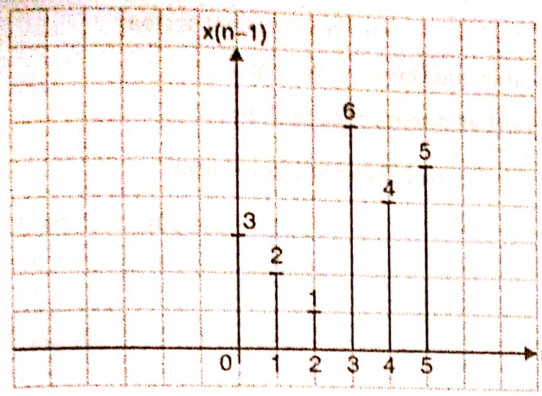

Fig. P.2.3.9(h)

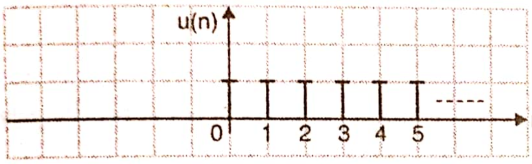

Fig. P.2.3.9(i)

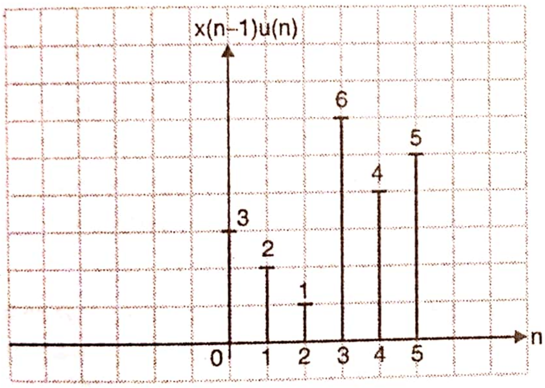

Fig. P.2.3.9(j)

(v) x(n – 2)

This is x(n) shifted to the right by 2.

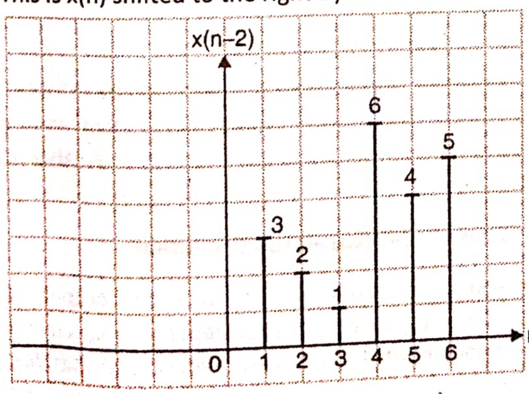

Fig. P.2.3.9(k)

Ex. 2.3.10 : For x(n) = {1, 2, –1, 5, 0, 4}, plot the following discrete time signals.

(i) x(n + 3) (ii) x(–n – 2)

(iii) x(n – 2) δ(n – 2) (iv) x(2n)

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Soln. : We begin with drawing x(n)

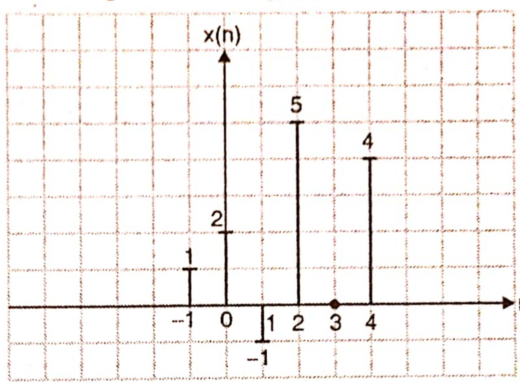

Fig. P.2.3.10

(i) x(n + 3)

This is x(n) shifted to the left by 3.

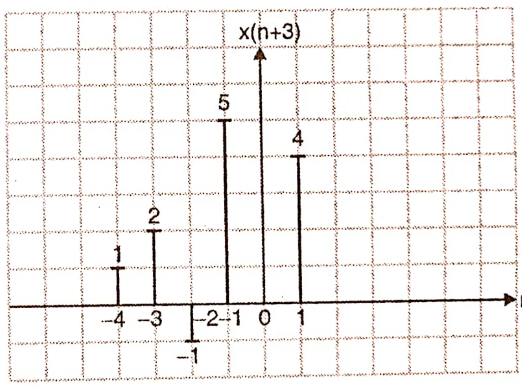

Fig. P.2.3.10(a)

(ii) x(–n – 2)

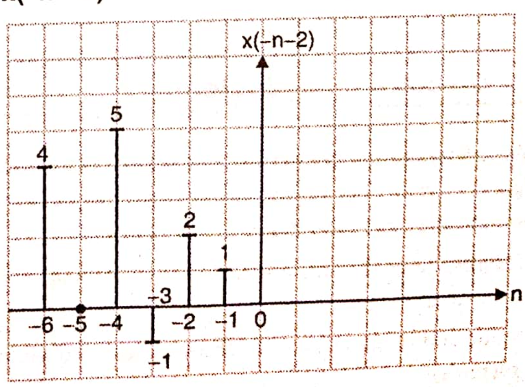

Fig. P.2.3.10(b)

(iii) x(n – 2) δ(n – 2)

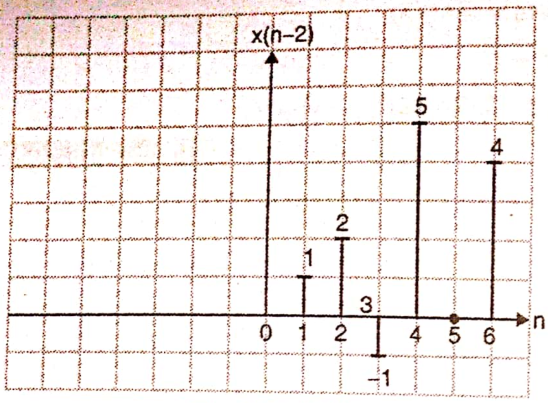

Fig. P.2.3.10(c)

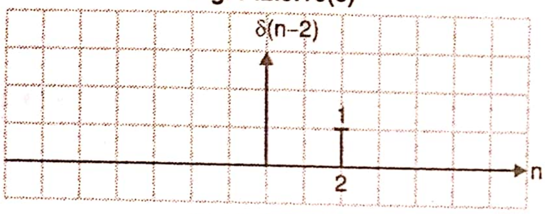

Fig. P.2.3.10(d)

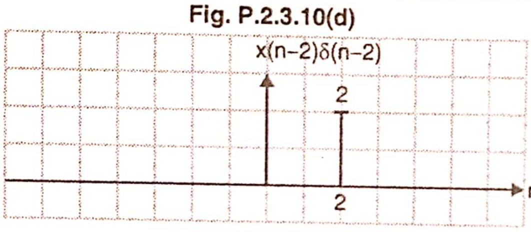

Fig. P.2.3.10(e)

(iv) x(2n)

This is time compression also known as down sampling.

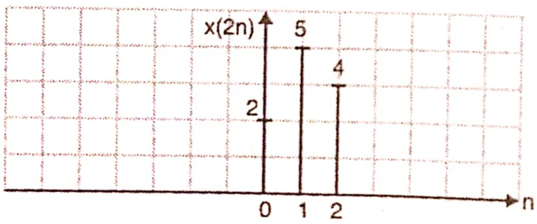

Fig. P.2.3.10(f)

2.4 Discrete Time Systems

As defined earlier, a discrete time system is a device or an algorithm that operates on a discrete time signal. It is mathematically represented as y (n) = T [x (n)]

Discrete time systems are classified according to their general characteristics.

1. Static and Dynamic system (Memory less and Memory systems)
2. Causal and Anti-causal system
3. Linear and Non-linear system
4. Time Variant and Time invariant system
5. Stable and Unstable system

We shall discuss each of them.

1. Static and Dynamic systems

- A system is said to be static if the output of the system depends only on the present input and not on the past or future input.
- Given below are a few examples of static system,

$$y(n) = T[x(n)]$$

(i) $y(n) = 4x(n)$

(ii) $y(n) = \log x(n)$

(iii) $y(n) = A \cos x(n)$

- In each case, y (n) requires only the present value of input x (n)

i.e. from $y(n) = \log x(n)$, we write

$$y(1) = \log x(1)$$

$$y(2) = \log x(2)$$

- Static system are also called memoryless system as we do not need to store pervious input values
- A dynamic system is one in which the output y (n) depends on the present as well as past inputs.

For example, $y(n) = x(n) + x(n-2)$

- In this case, to find out y (4), we need x (4) which is the present input as well as x (4 – 2) = x (2) which is the past input. Hence to find out entire y (n), we need present inputs as well as part inputs.

Some examples of Dynamic systems are :

1. $y(n) = \dfrac{x(n) + x(n-12)}{2}$

2. $y(n) = x(n) - x(n-1)$

- Since past input values need to be stored and retrieved, the system requires a certain amount of memory. Hence Dynamic systems are also called memory systems.

2. Causal and Non-causal system

- A system is said to be causal if the output of the system y (n) at any time n depends only on present and past inputs but does not depend on future inputs.

i.e. $y(n) = x(n) + x(n-2) + x(n-16)$

- If the output of a system also depends on future inputs it known as a Non-Casual systems.

For example, $y(n) = x(n) + x(n+1)$

- In this case, to find out say y (4), we need x (4) which is the present input as well as x (5) which is the future input.
- Hence in an non-causal system, we need to predict the future inputs to get the present input. Given below are a few examples of Causal and Non-Causal systems.

1. $y(n) = \dfrac{x(n) + x(n-1) + x(n-2)}{3}$ → causal
2. $y(n) = \log(x(n))$ → causal
3. $y(n) = x(2n)$ → Non-causal
4. $y(n) = x(n) - x(n+2)$ → Non-causal

3. Linear and Non-Linear Systems

- A system is said to be linear if it satisfies the superposition principle.
- Superposition principle states that the response of the system to a weighted sum of signals is equal to the corresponding weighted sum of output of the system to individual input signals.

i.e. $T[ax_1(n) + bx_2(n)] = aT[x_1(n)] + bT[x_2(n)]$

A block diagram will make things clearer.

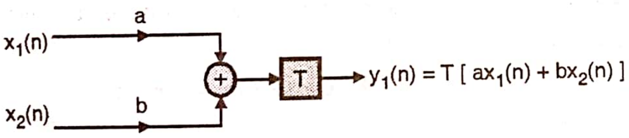

Fig. 2.4.1

and

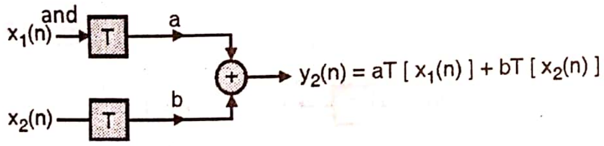

Fig. 2.4.2

Note : The system T is said to be linear if $y_1(n) = y_2(n)$.

Solved Example

Ex. 2.4.1 : Check whether the following systems are linear or non linear.

1. $y(n) = n\,x(n)$
2. $y(n) = e^{x(n)}$
3. $y(n) = x^2(n)$

Soln. : 1. $y(n) = n\,x(n)$

We know

$$y(n) = T[x(n)]$$
$$y(n) = n[x(n)]$$

Here the system multiplies the input x(n) with n.

We use two inputs, $x_1(n)$ and $x_2(n)$

We check if $T[x_1(n) + x_2(n)] = T[x_1(n)] + T[x_2(n)]$

(1) $T[x_1(n) + x_2(n)]$

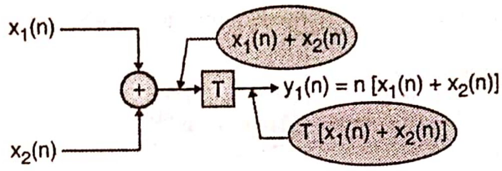

Fig. P. 2.4.1

$$\therefore \quad y_1(n) = n[x_1(n) + x_2(n)]$$

Now,

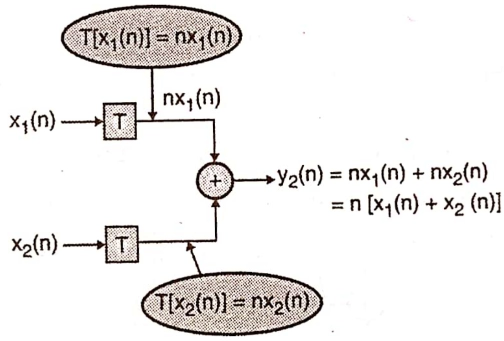

Fig. P. 2.4.1(a)

$$\therefore \quad y_2(n) = n[x_1(n) + x_2(n)]$$

$$\therefore \quad y_1(n) = y_2(n), \text{ hence system is linear.}$$

(2) $y(n) = e^{x(n)}$

We know,

$$y(n) = T[x(n)]$$

Hence the system in this case computes the exponential of the input.

We use two inputs, $x_1(n)$ and $x_2(n)$

We check for $T[x_1(n) + x_2(n)] = T[x_1(n)] + T[x_2(n)]$

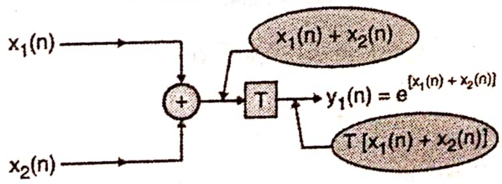

Fig. P.2.4.1(b)

$$y_1(n) = e^{[x_1(n) + x_2(n)]}$$

Now,

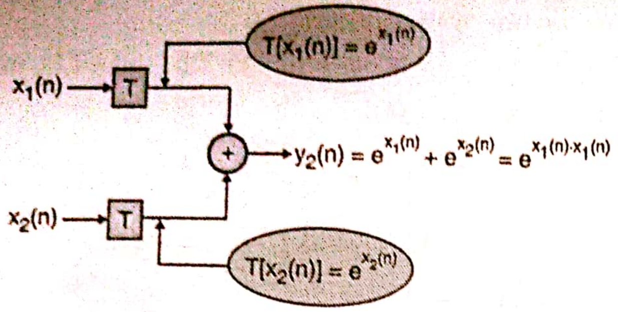

Fig. P.2.4.1(c)

$$y_2(n) = e^{x_1(n) \cdot x_2(n)}$$

Hence $y_1(n) \neq y_2(n)$; Hence system is unstable.

(3) $y(n) = x^2(n)$

We know $y(n) = T[x(n)]$

Hence the system computes the square of the input $x(n)$.

We use to inputs $x_1(n)$ and $x_2(n)$ and check it

$$T[x_1(n) + x_2(n)] = T[x_1(n)] + T[x_2(n)]$$

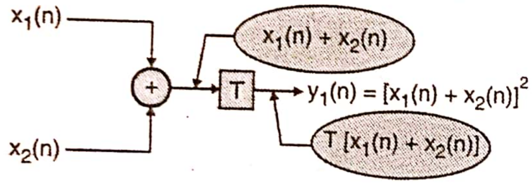

Fig. P. 2.4.1(d)

$$\therefore \quad y_1(n) = [x_1(n) + x_2(n)]^2$$

Now,

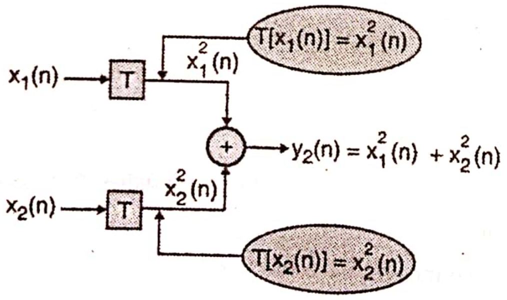

Fig. P. 2.4.1(e)

$$y_2(n) = x_1^2(n) + x_2^2(n)$$

$$\therefore \quad y_1(n) \neq y_2(n)$$

Hence the system is Non-Linear.

4. Time Variant and Time Invariant Systems

A system is Time invariant (Also called shift invariant) if its input-output characteristics do not change with time. If y (n) is the response of the system to input x (n), then y (n – k) will be the response of the system to input x (n – k) i.e. if the input gets shifted by k samples, then the output also shifts by k samples.

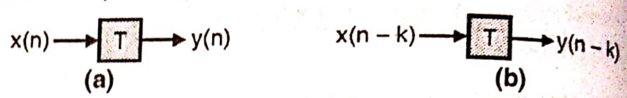

Fig. 2.4.3

To check whether the system in Time invariant or not, we perform the following steps :

Step 1 : Delay input x (n) by k samples and observe the output. We call this output y (n, k).

Step 2 : Delay the output y (n) by k samples i.e., in the given equation y (n), replace n by (n – k). We call this output y (n – k).

If y (n, k) = y (n – k) the system is said to be time invariant or else it is called time variant.

Solved Example

Ex. 2.4.2 : Check whether the following systems are time invariant or time variant.

1. $y(n) = e^{x(n)}$
2. $y(n) = x(n) + x(n-1)$
3. $y(n) = n\,x(n)$

Soln. :

1. $y(n) = e^{x(n)}$

We know y (n) = T [x (n)]

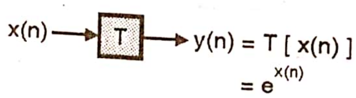

Fig. P.2.4.2

Step 1 : Delay the input

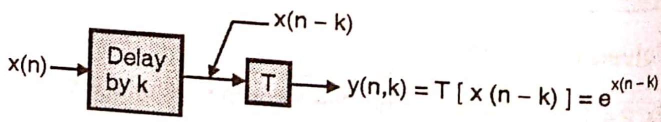

Fig. P.2.4.2(a)

$$\therefore \quad y(n, k) = e^{x(n-k)}$$

Step 2 : Delay the output

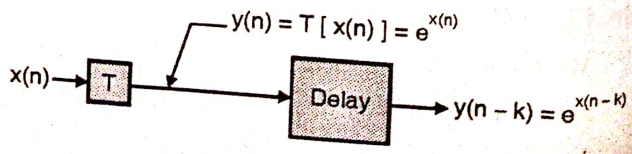

Fig. P.2.4.2(b)

$\therefore \; y(n-k) = e^{x(n-k)}$

Since y (n, k) = y (n – k), system is Time Invariant.

2. y (n) = x (n) + x (n – 1)

$$y(n) = T[x(n)]$$

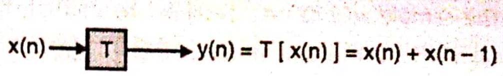

Fig. P.2.4.2(c)

Step 1 : Delay the input

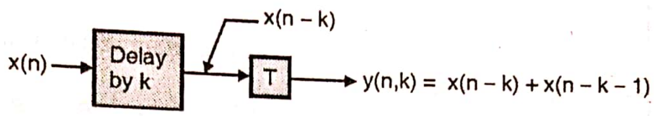

Fig. P.2.4.2(d)

Step 2 : Delay the output

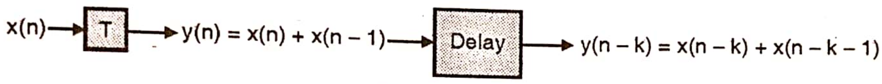

Fig. P.2.4.2(e)

Since, y (n, k) = y (n – k), the system is Time Invariant.

3. y (n) = n x (n)

$$y(n) = T[x(n)]$$

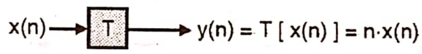

Fig. P.2.4.2 (f)

Step 1 : Delay the input

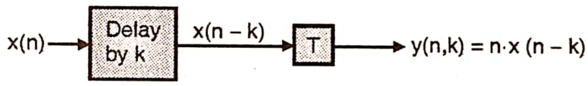

Fig. P.2.4.2 (g)

$$y(n, k) = n \cdot x(n-k)$$

Step 2 : Delay the output

$$y(n-k) = (n-k)\, x(n-k)$$

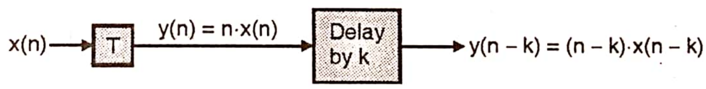

Fig. P.2.4.2 (h)

Since y (n, k) ≠ y (n – k), the system is Time Variant.

5. Stable and Unstable systems

A system is said to be stable if for a Bounded input, the system produces a bounded output.

Let M_x be a finite number i.e., $M_x < \infty$. Input is said to be bounded if $|x(n)| \le M_x < \infty$.

Similarly let M_y is finite number i.e., $M_y < \infty$. Output is said to be bounded if $|y(n)| \le M_y < \infty$.

Solved Example

Ex. 2.4.3: Check whether the following systems are stable.

i) $y(n) = e^{x(n)}$ ii) $y(n) = x(2n)$

Soln. :

In each of the cases, if x (n) is finite, y (n) will also be finite. Hence both systems are stable.

2.4.1 Representation of a Signal in Terms of Weighted Sum of Shifted Discrete Impulse

Any arbitrary signal can be represented as a summation of shifted and scaled impulses, δ (n). Let us show this with a simple example. Consider an input signal x(n).

$$x(n) = \{0.5, 2, 1, \underset{\uparrow}{0.5}, 1, 2, 0.5\}$$

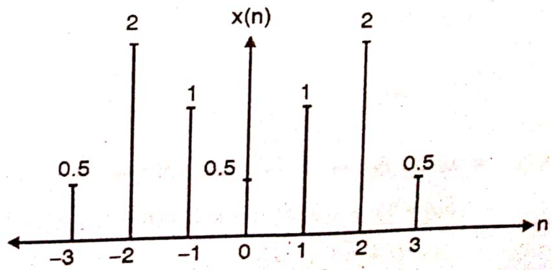

Fig. 2.4.4

The sample x(0) can be obtained by multiplying x(0), the magnitude, with a unit impulse function δ(n).

$$\text{i.e., } x(0) \cdot \delta(n) = \begin{cases} x(0); & \text{for } n = 0 \\ 0; & \text{for } n \neq 0 \end{cases}$$

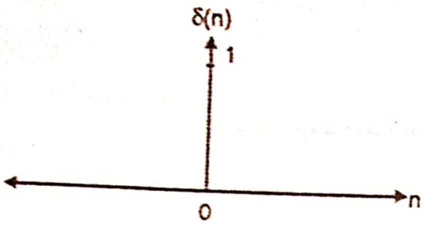

Fig. 2.4.5

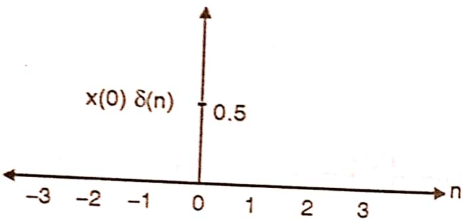

Fig. 2.4.6

We could get the other components of x(n) by shifting the delta function.

$$\text{i.e., } x(-1) \cdot \delta(n+1) = \begin{cases} x(-1); & \text{for } n = -1 \\ 0; & \text{for } n \neq -1 \end{cases}$$

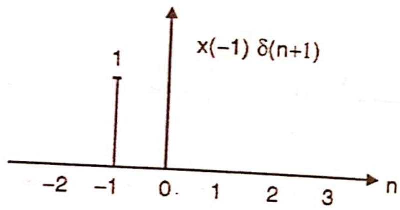

Fig. 2.4.7

In a similar manner

$$x(-2) \cdot \delta(n+2) = \begin{cases} x(-2); & \text{for } n = -2 \\ 0; & \text{for } n \neq -2 \end{cases}$$

$$x(-3) \cdot \delta(n+3) = \begin{cases} x(-3); & \text{for } n = -3 \\ 0; & \text{for } n \neq -3 \end{cases}$$

$$x(1) \cdot \delta(n-1) = \begin{cases} x(1); & \text{for } n = 1 \\ 0; & \text{for } n \neq 1 \end{cases}$$

$$x(2) \cdot \delta(n-2) = \begin{cases} x(2); & \text{for } n = 2 \\ 0; & \text{for } n \neq 2 \end{cases}$$

$$x(3) \cdot \delta(n-3) = \begin{cases} x(3); & \text{for } n = 3 \\ 0; & \text{for } n \neq 3 \end{cases}$$

$$\therefore\ x(n) = x(-3)\,\delta(n+3) + x(-2)\,\delta(n+2) + x(-1)\,\delta(n+1) + x(0)\,\delta(n) + x(1)\,\delta(n-1) + x(2)\,(n-2) + x(3)\,(n-3)$$

∴ In general, we have

$$x(n) = \sum_{k=-\infty}^{\infty} x(n)\,\delta(n-k)$$

Hence a signal can be viewed as a summation of scaled and shifted impulses. This equation will be used in deriving an important equation known as the Convolution sum.

2.5 Impulse Response and Convolution

– As discussed earlier, a simple block diagram of a system is shown in Fig. 2.5.1.

$$x(n) \rightarrow \boxed{T} \rightarrow y(n), \quad y(n) = T\,[\,x(n)\,]$$

Fig. 2.5.1

– If the input x (n) is a unit impulse δ (n), then the output of the system is known as the impulse response of the system and in denoted as h (n).

$$\delta(n) \rightarrow \boxed{T} \rightarrow y(n) = h(n)$$

Fig. 2.5.2

$$\therefore\ h(n) = T\,[\delta(n)]$$

– The impulse response completely characterizes the system.

– We have shown in the earlier section that any arbitrary input x (n) can be represented as a weighted sum of discrete impulses.

$$\text{i.e., } x(n) = \sum_{k=-\infty}^{+\infty} x(k) \cdot \delta(n-k)$$

We know,

$$y(n) = T\,[x(n)]$$

$$y(n) = T\left[\sum_{k=-\infty}^{+\infty} x(k) \cdot \delta(n-k)\right] \qquad \ldots(2.5.1)$$

– If we assume the system to be linear, then Equation (2.5.1) can be written as,

$$y(n) = \sum_{k=-\infty}^{+\infty} x(k)\,T\,[\,\delta(n-k)\,]$$

$$\text{Let, } T\,[\,\delta(n-k)\,] = h(n, k)$$

i.e., Let the response to the shifted impulse sequence be denoted by h(n, k).

$$\therefore \quad y(n) = \sum_{k=-\infty}^{+\infty} x(k) \cdot h(n, k) \quad \text{...(2.5.2)}$$

- If we now assume the system to be time invariant,

$$h(n, k) = h(n-k) \quad \text{...(2.5.3)}$$

- Substituting Equation (2.5.3) in Equation (2.5.2), we get ,

$$y(n) = \sum_{k=-\infty}^{+\infty} x(k) \cdot h(n-k)$$

- This equation is called the convolution sum and is one of the most important formula in Digital Signal Processing and Image Processing applications.
- It states that for a Linear Time-Invariant (LTI) system, if the input sequence x(n) and impulse response h(n) are known, y(n) can be found out from the convolution sum.

The convolution sum is represented as,

$$y(n) = x(n) * h(n)$$ where, * denotes the convolution operation.

2.5.1 Computation of Linear Convolution

There are two methods of computing linear convolution.

1. Graphical method
2. Tabular method

We will begin with discussing the graphical method and then move on to the tabular method which is very easy to perform.

2.5.2 Linear Convolution using Graphical Method

Graphical method gives us a pictorial representation of how complex the convolution operation is Linear convolution is given by the formula.

$$y(n) = \sum_{k=-\infty}^{+\infty} x(k)\, h(n-k) \quad \text{...(2.5.4)}$$

It involves the folding operation, shifting operation, multiplication operation and the summation operation.

In the original formula of convolution, the range of k is $-\infty \leq k + \infty$ and the range of n is $-\infty \leq n \leq +\infty$.

However this ranges depend on the length of x(n) and h(n).

Range of n

Consider the following notations,

y_l = Lowest range of y(n)

y_h = Highest range of y(n)

x_l = Lowest range of x(n)

x_h = Highest range of x(n)

h_l = Lowest range of h(n)

h_h = Highest range of h(n)

The range of n is y(n) will be equal to

$$y_l = x_l + h_l \text{ and } y_h = x_h + h_h$$

Let us consider an examples to understand this better.

2.5.2(A) Solved Examples on Graphical Method

Ex. 2.5.1 : What will be the range of y(n) if $x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$

$h(n) = \{1, \underset{\uparrow}{2}, 2, -1\}$

Soln. :

Here $x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$

x(0) x(1) x(2) x(3)

$\therefore x_l = 0, x_h = 3$

Similarly,

$h(n) = \{1, \underset{\uparrow}{2}, 2, -1\}$

h(–1) h(0) h(1) h(2)

$\therefore h_l = -1, h_h = 2$

Now $y_l = x_l + h_l$

$= 0 - 1 = -1$

and $y_h = x_h + h_h$

$= 3 + 2 = 5$

Hence the range of n will be $-1 \leq n \leq 5$.

$\therefore$ The output will have values,

$$\{y(-1), y(0), y(1), y(2), y(3), y(4), y(5)\}$$

Range of k

The range of k will be the same as that of the sequence x(n). In the above example, since $x_l = 0$ and $x_h = 3$, the range of k will be $0 \leq k \leq 3$.

Hence for the above example the formula of convolution reduces to,

$$y(n) = \sum_{k=0}^{3} x(k)\, h(n-k); \qquad -1 \leq n \leq 5$$

Ex 2.5.2 : Consider an audio signal x(n) = {1, 2, 4}. This signal is filtered using a low pass filter having an impulse response h(n) = {1, 1, 1}. Obtain the filtered output signal.

Soln. : Filtering a signal implies the convolution operation. Always remember this. The filtered output is given by the formula.

$$y(n) = x(n) * h(n)$$

i.e.

$$y(n) = \sum_{k=-\infty}^{+\infty} x(k)\, h(n-k)$$

When n = 0, the formula reduces to

$$y(0) = \sum_{k=-\infty}^{+\infty} x(k)\, h(-k)$$

i.e. we need x(k) and h(– k)

The steps involved are as follows :

(1) Draw x(k) and h(– k) one below the other.

(2) Multiply corresponding elements of x(k) and h (– k) and add the resultants.

(3) Shift h(– k) to the right and repeat step (2). Continue shifting to the right till there is no overlap.

(4) Shift h(– k) to the left and repeat step (2). Continue shifting to the left till there is no overlap.

x(k) = {1, 2, 4}, h(k) = {1, 1, 1}

Since there are no arrows, the first elements of x(k) and h(k) are taken as the origin.

(1) Draw x(k) and h (– k)

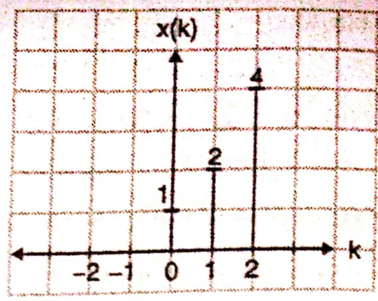

Fig. P. 2.5.2

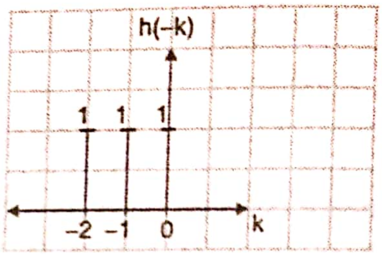

Fig. P. 2.5.2(a)

$$y(0) = \sum x(k)\, h(-k) = 1 \times 1 = 1$$

(2) Shift h(– k) to the right

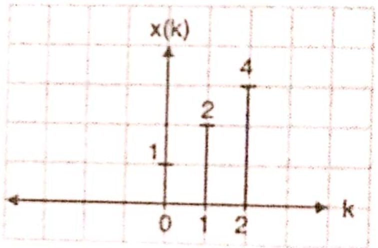

Fig. P. 2.5.2(b)

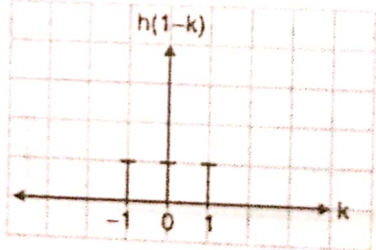

Fig. P. 2.5.2(c)

$$y(1) = \sum x(k)\, h(1-k)$$

$$y(1) = 1(1) + 2(1) = 3$$

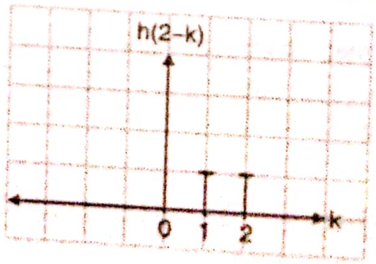

Fig. P. 2.5.2(d)

$$y(2) = \sum x(k)\, h(2-k)$$

$$y(2) = 1(1) + 2(1) + 4(1) = 7$$

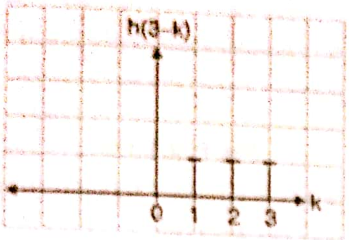

Fig. P. 2.5.2(e)

$$y(3) = \sum x(k)\, h(3-k)$$

$$y(3) = 1(0) + 2(1) + 4(1) = 6$$

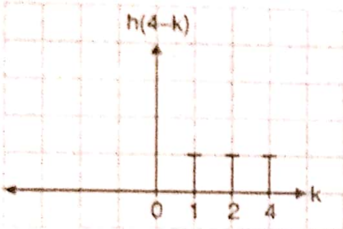

Fig. P. 2.5.2(f)

$$y(4) = \sum x(k)\, h(4-k)$$

$$y(4) = 1(0) + 2(0) + 4(1) = 4$$

Now for the 5thshift we have

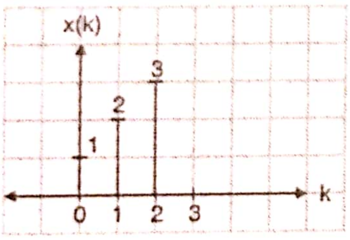

Fig. P. 2.5.2(g)

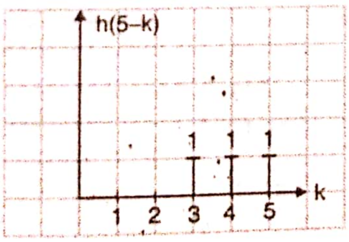

Fig. P. 2.5.2(h)

$$y(5) = \sum x(k)\, h(5-k)$$

We see there is no overlap between x(k) and h(5 – k) hence the resultant is 0. All subsequent right shifts will give in zero values.

We now shift h(–k) to the left

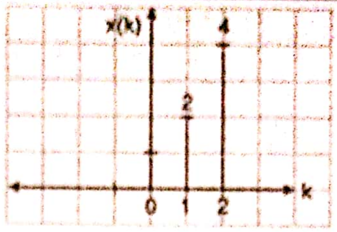

Fig. P. 2.5.2(i)

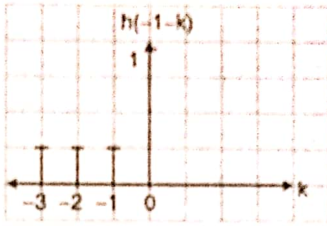

Fig. P. 2.5.2(j)

$$y(-1) = 0$$

$$y(-1) = \sum x(k)\, h(-1-k) = 0$$

We see there is no overlap between x(k) and h(–1 – k) hence the resultant is 0. All subsequent left shifts will give us zero values.

$$\therefore \quad y(n) = \{y(0), y(1), y(2), y(3), y(4)\}$$

$$\therefore \quad y(n) = \{1, 3, 7, 6, 4\}$$

Hence the filtered audio signal is y(n) = {1, 3, 7, 6,4}.

There are two ways to check the result.

(i) Length of y(n) = Length of x(n) + Length of h(n) – 1

$$= 3 + 3 - 1 = 5$$

We observe that length of y(n) obtained is also 5.

(ii) Sum of elements of x(n) × sum of elements of h(n) = sum of elements of y(n).

i.e. $\Sigma x(n) \times \Sigma h(n) = \Sigma y(n)$

in our case

$$\Sigma x(n) = 1 + 2 + 4 = 7$$

$$\Sigma h(n) = 1 + 1 + 1 = 3$$

$$\therefore \quad \Sigma x(n) \times \Sigma h(n) = 7 \times 3 = 21$$

$$\text{Now} \quad \Sigma y(n) = 1 + 3 + 7 + 6 + 4 = 21$$

$$\therefore \quad \Sigma x(n) \times \Sigma h(n) = \Sigma y(n)$$

Hence the result obtained is correct.

Ex 2.5.3 : Compute linear convolution of the following sequences $x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$, $h(n) = \{1, \underset{\uparrow}{2}, 2, -1\}$

Soln. :

$$y(n) = \sum_{k=-\infty}^{+\infty} x(k)\, h(n-k)$$

We draw x(k) and h(– k) and obtain y(0).

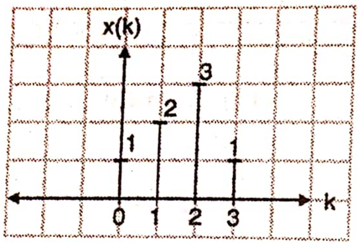

Fig. P. 2.5.3

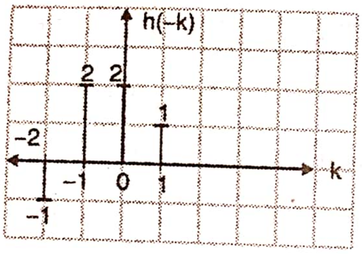

Fig. P. 2.5.3(a)

$$y(0) = 1(2) + 2(1) = 4$$

We shift h(–k) to the right till there is no overlay

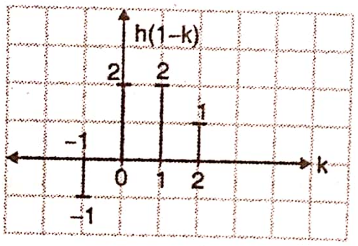

Fig. P. 2.5.3(b)

$$y(1) = 1(2) + 2(2) + 3(1) = 9$$

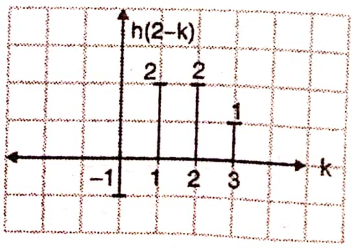

Fig. P. 2.5.3(c)

$$y(2) = 1(-1) + 2(2) + 3(2) + 1(1) = 10$$

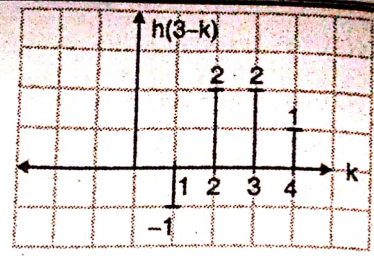

Fig. P. 2.5.3(d)

$$y(3) = 2(-1) + 3(2) + 1(2) = 6$$

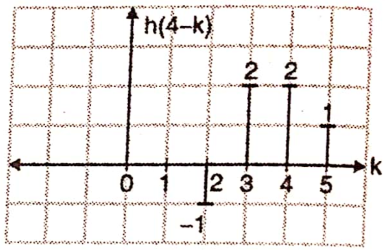

Fig. P. 2.5.3(e)

$$y(4) = 3(-1) + 1(2) = -1$$

Fig. P. 2.5.3(f)

$$y(5) = 1(-1) = -1$$

beyond this there will be no overlap between x(k) and h(n – k). Hence we stop shifting h(– k) to the right.

We now shift h(–k) to the left.

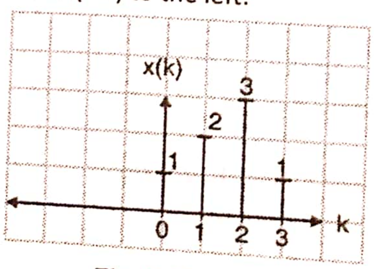

Fig. P. 2.5.3(g)

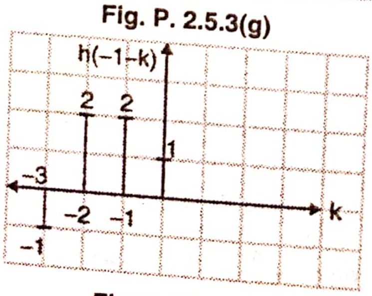

Fig. P. 2.5.3(h)

$$y(-1) = 1(1) = 1$$

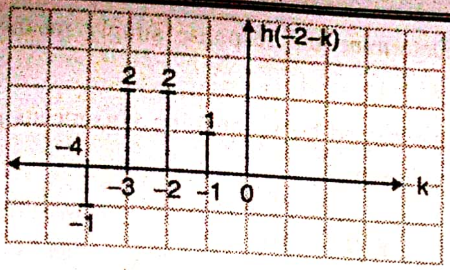

Fig. P. 2.5.3(i)

$$y(-2) = 0$$

Since there is no overlap, we stop shifting h(–k) to the left.

$$\therefore \quad y(n) = \{1, \underset{\uparrow}{4}, 9, 10, 6, -1, -1\}$$

We will check if this result is correct

$$\text{length of } y(n) = \text{length of } x(n) + \text{length of } h(n) - 1$$

$$= 4 + 4 - 1 = 7$$

We observe that y(n) obtained has 7 elements

Also

$$\Sigma x(n) \times \Sigma h(n) = \Sigma y(n)$$

$$\Sigma x(n) = 1 + 2 + 3 + 1 = 7$$

$$\Sigma h(n) = 1 + 2 + 2 - 1 = 4$$

$$\Sigma x(n) \times \Sigma h(n) = 7 \times 4 = 28$$

Now $\quad \Sigma y(n) = 1 + 4 + 9 + 10 + 6 - 1 - 1 = 28$

Hence the result obtained is correct.

2.5.3 Linear Convolution using Tabular Method

We have learnt how to perform linear convolution using the graphical method. There is a much easier way to performing linear convolution. This is the tabular method.

In this method we follow the given steps.

Step 1 : Form a matrix as shown in Fig. 2.5.3.

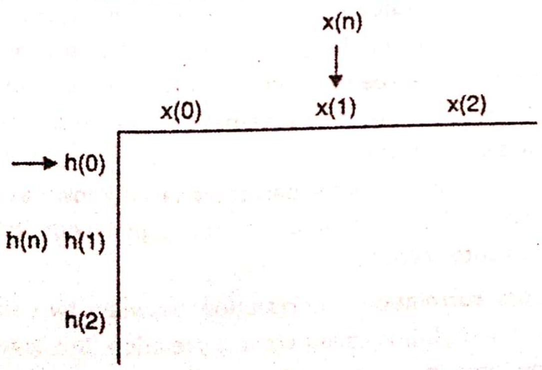

Fig. 2.5.3

We assume arrows are at x(1) and h(0).

We could interchange the positions of x(n) and h(n).

Step 2 : Multiply corresponding elements of x(n) and h(n)

	x(0)	x(1) ↓	x(2)
→ h(0)	x(0) h(0)	x(1) h(0)	x(2) h(0)
h(1)	x(0) h(1)	x(1) h(1)	x(2) h(1)
h(2)	x(0) h(2)	x(1) h(2)	x(2) h(2)

Fig. 2.5.4

Step 3 : We separate the elements diagonally

We shade the section where the arrows intersect. In this case the arrows intersect at x(1)h(0) hence we have shaded the entire diagonal section.

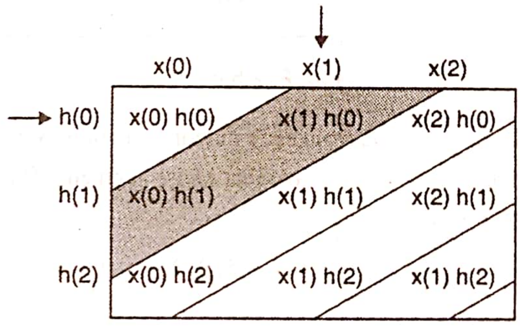

Fig. 2.5.5

Step 4 : We add the elements in each section to obtain y(n). The sum of elements in the shaded section gives us y(0)

$$\therefore \quad y(-1) = x(0) \cdot h(0)$$

$$y(0) = x(0)\,h(1) + x(1)\,h(0)$$

$$y(1) = x(0)\,h(2) + x(1)\,h(1) + x(2)h(0)$$

$$y(2) = x(1)\,h(2) + x(2)\,h(1)$$

$$y(3) = x(2) - h(2)$$

$$\therefore \quad y(n) = \{y(-1), \underset{\uparrow}{y(0)}, y(1), y(2), y(3)\}$$

We will solve a couple of example to understand the tabular method better.

2.5.3(A) Solved Examples on Tabular Method

Ex. 2.5.4 : Perform linear convolution of x(n) = {1,1,0,1, 1}
↑ (under 0)

h (n) = {1, – 2, – 3, 4}
↑ (under 4)

Soln. : We form a matrix of x(n) and h(n)

h(n) \ x(n)	1	1	0 ↓	1	1
1	1 × 1	1 × 1	1 × 0	1 × 1	1 × 1
–2	–2 × 1	–2 × 1	–2 × 0	–2 × 1	–2 × 1
–3	–3 × 1	–3 × 1	–3 × 0	–3 × 1	–3 × 1
→ 4	4 × 1	4 × 1	4 × 0	4 × 1	4 × 1

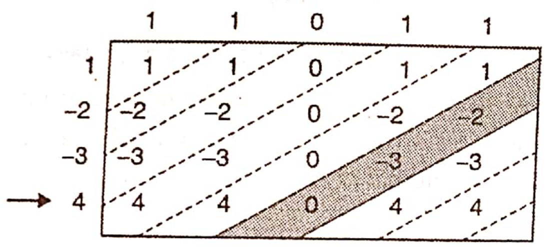

Fig. P. 2.5.4

We add elements from each diagonal section. Since the arrows intersect at 0, we shade that entire segment. The sum of this section will be the origin value.

$$\therefore \quad y(n) = \{1, -1, -5, 2, 3, \underset{\uparrow}{-5}, 1, 4\}$$

We check whether this result is correct.

Length of y(n) = Length of x(n) + Length of h(n) – 1

$= 5 + 4 - 1 = 8$

The length of y(n) obtained = 8.

Also, $\Sigma x(n) \times \Sigma h(n) = \Sigma y(n)$

$\Sigma x(n) = 1 + 1 + 0 + 1 + 1 = 4$

$\Sigma h(n) = 1 - 2 - 3 + 4 = 0$

$\therefore \quad \Sigma x(n) \times \Sigma h(n) = 4 \times 0 = 0$

Now $\Sigma y(n) = 1 - 1 - 5 + 2 + 3 - 5 + 1 + 4$

$= 0$

∴ Our result is correct.

Ex. 2.5.5 : The impulse response of a linear time invariant system is h(n) = {1, 2, 1, – 1}. Determine the response of the system to the input x(n) = {1, 2, 3, 1}
↑ (under 2 of h(n))

Soln. : Determine the response simply means obtaining y(n).

Given x(n) and h(n), obtaining y(n) means performing linear convolution.

$$y(n) = x(n) * h(n)$$

We use the tabular method.

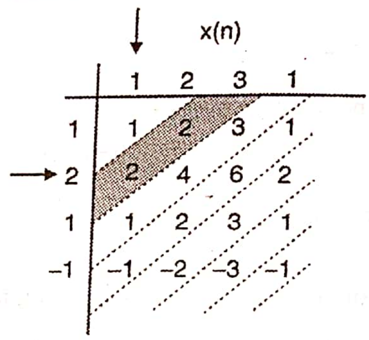

Fig. P. 2.5.5

$$\therefore \quad y(n) = \{1, 4, 8, 8, 3, -2, -1\}$$

This is the output response of the system.

Length of y(n) = Length of x(n) + Length of h(n) – 1

$= 4 + 4 - 1 = 7$

Also, $\Sigma x(n) \times \Sigma h(n) = \Sigma y(n)$

$\Sigma x(n) = 1 + 2 + 3 + 1 = 7$

$\Sigma h(n) = 1 + 2 + 1 - 1 = 3$

$\therefore \quad \Sigma x(n) \times \Sigma h(n) = 7 \times 3 = 21$

Now $\Sigma y(n) = 1 + 4 + 8 + 8 + 3 - 2 - 1$

$= 21$

∴ Our result is correct.

2.6 Correlation

Correlation is basically used when we want to compare two signals. It is a very important operation in signal processing and image processing. Correlation is a measure of the degree to which two signals are similar.

Correlation as a mathematical operation closely resembles convolution.

Correlation of two separate signals is known as cross correlation, while correlation of the signal with itself is known as auto correlation.

1. **Cross correlation :** Correlation between two signals x(n) and y(n), is called cross correlation. It is given by the formula

$$r_{xy}(l) = \sum_{n=-\infty}^{+\infty} x(n) \cdot y(n-l)$$

2. **Auto correlation :** Many times it is necessary to correlate the signal with itself for example determination of time delays between transmitted and received signal.

   Hence the two signals are generated from the same source. It is given by the formula

$$r_{xx}(l) = \sum_{n=-\infty}^{+\infty} x(n)\, x(n-l)$$

Simple method to calculate correlation

We will use one of the important properties of correlation. It is as follows :

$$r_{xy}(l) = x(n) * y(-n) \qquad \ldots(2.6.1)$$

Observe Equation (2.6.1), The L.H.S. indicates cross correlation of x (n) and y (n). R.H.S. term indicates the convolution ('*') operation. It is the convolution of x (n) and y (– n). Here y (– n) is the folded version of y (n).

So this equation implies :

(i) Take signal x (n) as it is.

(ii) Fold sequence y (n) to obtain y (– n).

(iii) Obtain the convolution of x (n) and y (– n).

(iv) The result of convolution will be same as the correlation.

Let us solve the same example using the method.

2.6.1 Solved Examples on Correlation

Ex. 2.6.1 : $x(n) = \{1, 1, \underset{\uparrow}{0}, 1\}$ and $y(n) = \{4, -3, \underset{\uparrow}{-2}, 1\}$

Soln. : In the tabular form simply flip y (n).

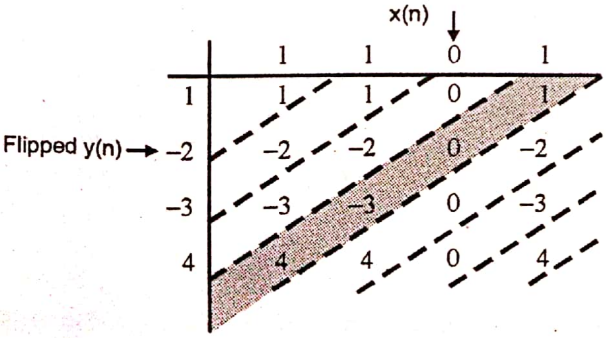

Fig. P. 2.6.1

$$\therefore\ r_{xy} = \{1, -1, -5, \underset{\uparrow}{2}, 2, -3, 4\}$$

Thus we get

$$r_{xy}(l) = x(n) * y(-n)$$

$$= \{1, -1, -5, \underset{\uparrow}{2}, 2, -3, 4\}$$

Ex. 2.6.2 : Determine the auto-correlation of the following signal :

$$x(n) = \{\underset{\uparrow}{1}, 2, 1, 1\}$$

Soln. :

Let $x_1(n) = \{\underset{\uparrow}{1}, 2, 1, 1\}$,

$x_2(n) = \{\underset{\uparrow}{1}, 2, 1, 1\}$

We use the tabular method.

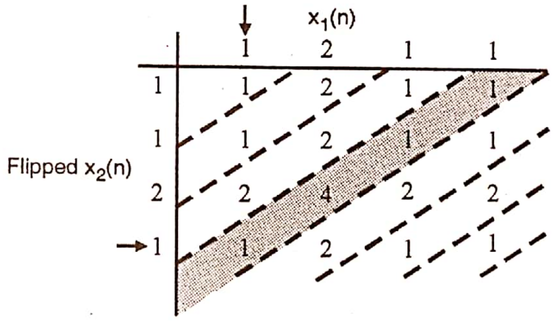

Fig. P.2.6.2

$$\therefore\ r_{xx} = \{1, 3, 5, \underset{\uparrow}{7}, 5, 3, 1\}$$

Summary

In this chapter we discussed representation of Discrete time signals. We classified discrete time signals as well as discrete time systems. How to operate on discrete time signals was understood. In this chapter we derived one the most important formula which is known as convolution. Two methods of computing the convolution were explained and examples were solved. We also discussed correlation in this chapter

Review Questions

Q. 1 Define discrete time system. Explain any three properties with suitable example.

Q. 2 Define a periodic signal

Q. 3 Define : Even signal and Odd signal

Q. 4 Define power signal.

Q. 5 Sketch and define two standard DT signals δ(n) and u(n).

Q. 6 With example explain the time scaling and time reversal operations performed on discrete time signals.

Q. 7 With example, explain how linear convolution can be obtained using graphical method.

Q. 8 Explain in brief different properties of convolution.

Q. 9 Explain why the result of linear and circular convolution is not same.

Q. 10 How N point circular convolution can be obtained using linear convolution ?

❑❑❑

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