CHAPTER 3

Discrete Fourier Transform (DFT)

Syllabus :

Introduction to DTFT, DFT, Relation between DFT and DTFT, IDFT, Properties of DFT without mathematical proof (Scaling and Linearity, Periodicity, Time Shift and Frequency Shift, Time, Reversal, Convolution Property and Parsevals' Energy Theorem). DFT computation using DFT properties. Transfer function of DT System in frequency domain using DFT. Linear and Circular Convolution using DFT, Convolution of long sequences, Introduction to 2-D DFT Circular Convolution (without mathematical proof), Linear convolution using Circular Convolution. Spectral Analysis using FFT

3.1 The Fourier Transform

– Fourier transform is named after the French mathematician Jean Baptiste Joseph Fourier who was born in 1768 in Auxerre, France. The Fourier transform is one of the most important transforms in engineering. Most of the branches of engineering owe their growth to the wonderful Fourier transform. The Fourier transform gave a new direction and understanding to the field of Digital Signal Processing.

So what exactly is the Fourier transform?

– To understand this, Let us look at a simple experiment that we have all conducted in school. Consider Fig. 3.1.1.

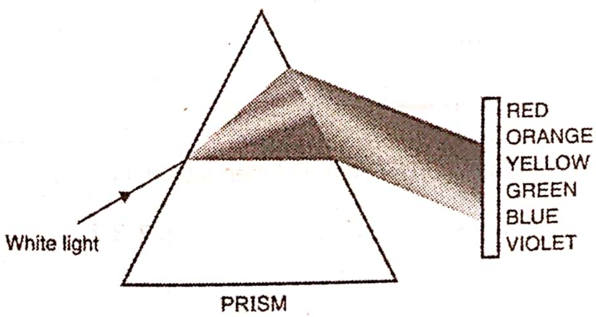


Fig. 3.1.1

– White light when passed through a prism gets split into different wavelengths (colours). Isaac Newton who was born in 1643, in England submitted a paper in 1672 to the Royal Society where he used the word Spectrum to describe the continuous band of colours produced by this apparatus. Newton was the first to note that white light can be split into its constituent colours using a prism.

– Since colours are nothing but wavelengths and wavelengths are related to frequency, splitting of white light into different colours is actually a form of frequency analysis. Newton then placed another prism upside down and showed that constituent colours blended back to give white light again. Fig. 3.1.2.

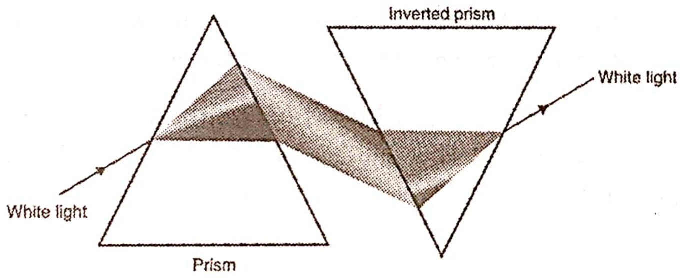


Fig. 3.1.2

Let us summarize what is stated.

– The first prism splits white light into different colours (wavelengths) and the second prism takes these different colours and reproduces white light.

– Joseph Fourier studied the works of Isaac Newton and showed that what the prism does to light, the Fourier transform does to signals.

– White light is replaced by a signal, the first prism by the Fourier transform and the second inverted prism by the Inverse Fourier transform.

– The Fourier transform breaks up a signal of any shape into pure sinusoidal functions. The inverse Fourier transform combines the pure sinusoidal functions to give back the original signal. To put it simply, the Fourier transform tells us the frequencies that are present in a given signal.

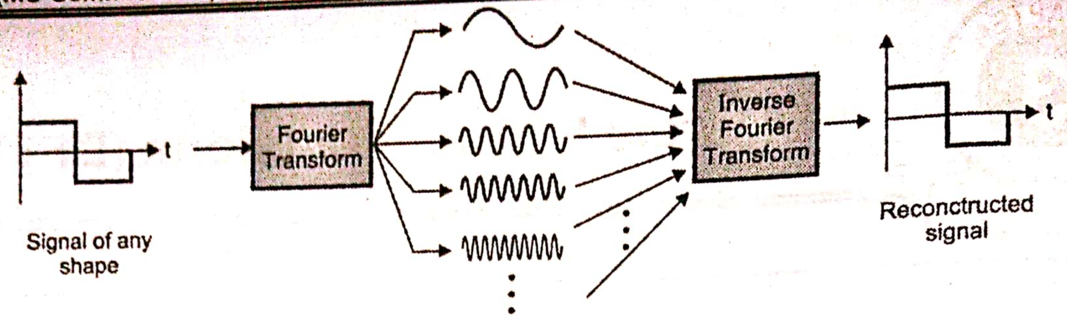


Fig. 3.1.3

3.2 Discrete Time Fourier Transform

- Discrete time Fourier transform, as the name suggest is used when the signal is discrete in time and is a periodic. It is given by the formula.

$$X(\omega) = \sum_{n=\infty}^{+\infty} x(n)\, e^{-j\omega n}$$

- Since x(n) is discrete, we use the summation (Σ) operation.
- Hence x(n) is the discrete time signal and X(ω) are the frequency components present in the signal.
- We can obtain x(n) from X (ω) by using the Inverse Discrete Time Fourier Transform (I.D.T.F.T), It is given by the formula,

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(\omega)\, e^{+j\omega n}\, d\omega$$

- Here we use the integral operation because X(ω) is continuous.
- The DTFT does not exist for every periodic sequence. A sufficient condition for the existence of the DTFT for a periodic signal x(n) is.

$$\sum_{n=-\infty}^{+\infty} |x(n)| < \infty$$

- In other words, The DTFT exists only if the signal x(n) is absolutely summable.

3.2.1 Solved Examples on DTFT of Standard Signals

Ex. 3.2.1 : Obtain DTFT of unit impulse, δ (n)

Soln. : A unit impulse is shown in Fig. P. 3.2.1.

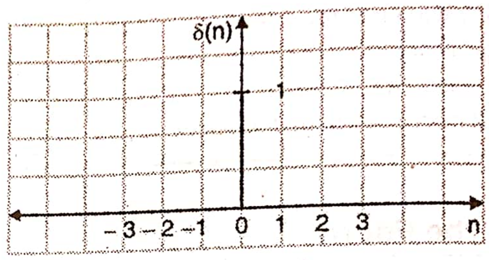


Fig. P. 3.2.1

$$\delta(n) = \begin{cases} 1, & ;\ n = 0 \\ 0 & ;\ \text{otherwise} \end{cases}$$

The DTFT is given by the formula,

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}$$

$$\therefore\ X(\omega) = \sum_{n=-\infty}^{+\infty} \delta(n)\, e^{-j\omega n}$$

Since δ (n) exists only at n = 0.

$$\therefore\ X(\omega) = \sum_{n=0} 1.\ e^{-j\omega n} = 1 \times e^{j\omega o}$$

$$\therefore\ X(\omega) = 1$$

$$\therefore\ \delta(n) \xleftrightarrow{FT} 1$$

Ex. 3.2.2 : Obtain DTFT of a unit step signal u(n)

Soln. : A unit step signal is shown in Fig. P. 3.2.2.

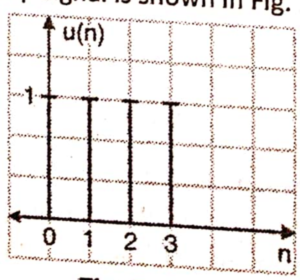


Fig. P. 3.2.2

$$u(n) = \begin{cases} 1 & ;\ n \geq 0 \\ 0 & ;\ \text{otherwise} \end{cases}$$

The DTFT is given by the formula,

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}$$

Since u(n) exists only for $n \geq 0$, we change the limits of the summation.

$$X(\omega) = \sum_{n=0}^{\infty} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (e^{-j\omega})^n$$

We know from the geometric progression formula,

$$\therefore \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \qquad |a| < 1$$

$$\therefore x(\omega) = \frac{1}{1-e^{-j\omega}}$$

$$\therefore u(n) \xrightarrow{FT} \frac{1}{1-e^{-j\omega}}$$

Ex. 3.2.3 : Obtain DTFT of rectangular pulse

$x(n) = A$

$0 \leq n \leq L-1 = 0$; otherwise

Soln. : The rectangular pulse is an shown in Fig. P. 3.2.3.

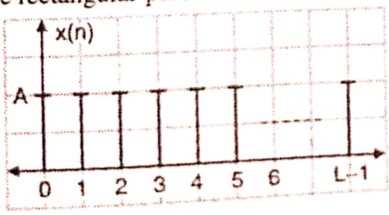


Fig. P. 3.2.3

This is similar to a unit step except that a unit step is infinite in length.

The DTFT is given by the formula,

$$\therefore X(\omega) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}$$

We change the limits of the summation,

$$\therefore X(\omega) = \sum_{n=0}^{L-1} A\, e^{-j\omega n}$$

$$\therefore A \sum_{n=0}^{\infty} e^{-j\omega n}$$

$$= A \sum_{n=0}^{L-1} (e^{-j\omega})^n$$

We know from the geometric of progression formula.

$$\sum_{K=N_1}^{N_2} a^n = \frac{a^{N1} - a_0^{N_2 - 1}}{1-a}$$

Here $N_1 = 0$, and $N_2 = L-1$

$$\therefore X(\omega) = A\left[\frac{1-e^{j\omega L}}{1-e^{-j\omega}}\right]$$

$$\therefore \text{Rectangular pulse } A \xleftrightarrow{FT} A\left[\frac{1-e^{-j\omega L}}{1-e^{-j\omega}}\right]$$

Ex. 3.2.4 : Obtain DTFT and sketch the magnitude spectrum for,

$$X(n) = u(n) - u(n-4)$$

Soln. :

Here u(n) is a step input while u(n – 4) is a unit step shifted by 4.

Hence,

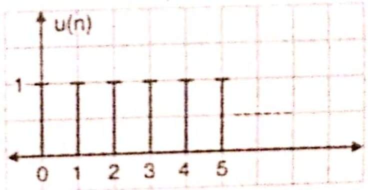


Fig. P. 3.2.4

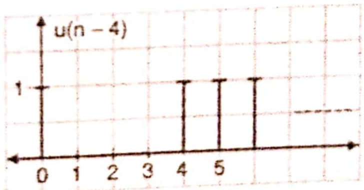


Fig. P. 3.2.4(a)

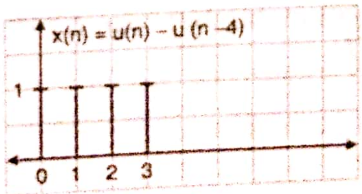


Fig. P. 3.2.4(b)

$$\therefore x(n) = \{\underset{\uparrow}{1}, 1, 1, 1\}$$

The DTFT is given by the formula,

$$\therefore \quad X(\omega) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n}$$

In our case

$$\therefore \quad x(\omega) = \sum_{n=0}^{3} 1 . e^{-j\omega n}$$

Since the limits of the summation are small, we open up the summation.

$$\therefore \quad X(\omega) = e^{-j\omega 0} + e^{-j\omega 1} + e^{-j\omega 2} + e^{-j\omega 3}$$

$$\therefore \quad X(\omega) = 1 + e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega}$$

We need to sketch the magnitude spectrum,

$$e^{-j\omega} = \cos\omega - j\sin\omega$$

$$\therefore \quad |e^{-j\omega}| = \sqrt{\cos^2\omega + \sin^2\omega} = 1$$

Similarly,

$$|e^{-j2\omega}| = \sqrt{\cos^2 2\omega + \sin^2 2\omega} = 1$$

and

$$|e^{-j3\omega}| = \sqrt{\cos^2 3\omega + \sin^2 3\omega} = 1$$

$$\therefore \quad |X(\omega)| = 1 + 1 + 1 + 1 = 4$$

$$\therefore \quad |X(\omega)| = 4 \qquad \text{...for all } \omega$$

∴ The Magnitude Spectrum is shown in Fig. P. 3.2.4(c).

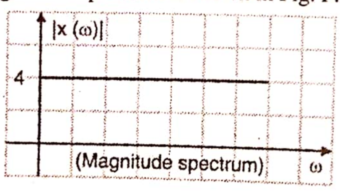


Fig. P. 3.2.4(c)

> **Note :** The spectrum is continuous.

3.3 Relationship between DTFT and DFT

The DTFT is given by the formula,

$$X(k) = \sum_{n=\infty}^{+\infty} x(n)\, e^{-j\omega n} \qquad \text{... (3.3.1)}$$

Since the DTFT is continuous, we sample the ω axis to discrete.

We sample ω such that

$$\omega = \frac{2\pi k}{N} \qquad \text{... (3.3.2)}$$

Where, N = Number of sample.

Substituting Equation (3.3.2) is Equation (3.3.1), we get,

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}} \qquad \text{... (3.3.3)}$$

N is normally equal to the length of the input signal x (n)

Equation (3.3.3) is called the DFT and is one of the most important transforms in engineering.

3.4 Discrete Fourier Transform (DFT)

- The DFT is a modification of the Fourier transform so as to use it for discrete time signals and to obtain a discrete frequency spectrum. We shall now proceed to explain the DFT.
- Remember, the objective of the DFT is to obtain the frequencies present in the signal.
- Consider a discrete time input signal x(n) of length N. The DFT of x(n) is denoted by X(k) and is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$$

$$k = 0, 1, 2, \dots N-1 \qquad \text{...(3.4.1)}$$

- Since the input signal x(n) is of length N, the summation varies from 0 to N – 1 and Equation (3.4.1) is called as N-point DFT.
- As mentioned in the earlier section, Equation (3.4.1) acts as the prism, breaking down the input signal into various frequency components.
- We can obtain x(n) from X(k) using the Inverse Discrete Fourier Transform IDFT which is given by the formula,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{\frac{+j2\pi kn}{N}}$$

$$n = 0, 1, 2, \dots N-1 \qquad \text{...(3.4.2)}$$

Equation (3.4.2) is called as N–point IDFT.

- This acts as the inverted prism which gives us the original signal x(n) by combining the constituent frequencies.

Note : $\frac{1}{N}$ in Equation (3.4.2) is a scaling function.

- This book uses the scaling function in the IDFT formula while other books choose to use it in the DFT formula. Some other books use $\frac{1}{\sqrt{N}}$ in both DFT and IDFT formulae.
- MATLAB uses $\frac{1}{N}$ in the IDFT formula and hence we too choose to use it there.
- Unless specified the length of x(n) and X(k) are the same.

3.4.1 Solved Examples on DFT of Simple Signals

In most of the examples, we begin by changing the limits of the summation depending on the input signal.

Ex. 3.4.1 : Obtain DFT of a unit impulse signal δ (n).

Soln.: Here $x(n) = \delta(n)$

The DFT is given by the equation,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \quad ; k = 0, 1, \ldots\ldots N-1$$

$$\therefore \quad X(k) = \sum_{n=0}^{N-1} \delta(n)\, e^{\frac{-j2\pi kn}{N}}$$

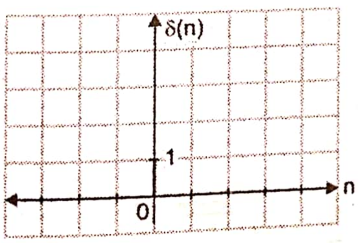


Fig. P.3.4.1

Since δ (n) exists only at n = 0, for all other values of the summation the result will be equal to zero.

$$\therefore \quad X(k) = \delta(0)\, e^{\frac{-j2\pi k \cdot 0}{N}} = \delta(0) \cdot e^{0}$$

$$(\because \delta(0) = 1)$$

$$\therefore \quad X(k) = 1$$

Hence, $\quad \delta(n) \xleftrightarrow{\text{DFT}} 1$

Ex. 3.4.2 : Obtain DFT of $\delta(n - n_0)$.

Soln.: Here $x(n) = \delta(n - n_0)$

$\delta(n - n_0)$ is basically δ (n) shifted to the right by n_0.

$\therefore$ $\delta(n - n_0)$ is 1 at n_0 and 0 at all other values of n.

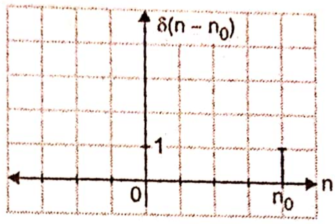


Fig. P.3.4.2

The DFT is given by the equation,

$$\therefore \quad X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \quad ; k = 0, 1, \ldots N-1$$

$$\therefore \quad X(k) = \sum_{n=0}^{N-1} \delta(n - n_0)\, e^{\frac{-j2\pi kn}{N}}$$

Since $\delta(n - n_0)$ exist at $n = n_0$, for all other values of the summation, the result will be equal to zero.

$$\therefore \quad X(k) = \delta(n - n_0)\, e^{\frac{-j2\pi k n_0}{N}}$$

$$\therefore \quad X(k) = 1 \cdot e^{\frac{-j2\pi k n_0}{N}} \qquad (\because \delta(n - n_0) = 1)$$

$$\therefore \delta(n - n_0) \xleftrightarrow{\text{DFT}} e^{\frac{-j2\pi k n_0}{N}}$$

Similarly,

$$\therefore \delta(n + n_0) \xleftrightarrow{\text{DFT}} e^{\frac{+j2\pi k n_0}{N}}$$

Ex. 3.4.3 : Obtain DFT of u (n) – u (n – 4).

Soln.: Here $x(n) = u(n) - u(n-4)$

Let us draw u (n) – u (n – 4) to understand what the input is.

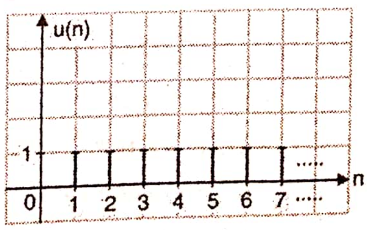


Fig. P.3.4.3

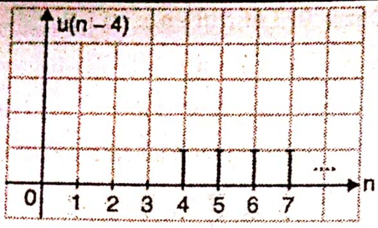


Fig. P.3.4.3(a)

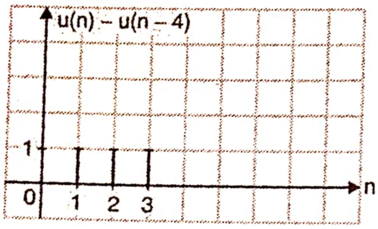


Fig. P.3.4.3(b)

u(n) is a unit step, while u (n – 4) is a unit step shifted to the right by 4.

From the Fig. P. 3.4.3(a) it is clear that,

$$x(n) = \{ \underset{\uparrow}{1}, 1, 1, 1\}$$

The DFT is given by the equation,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j\,2\pi kn}{N}} \quad ; k = 0, 1, 2, \ldots N-1$$

We now change the limits of the summation as x(n) exists only from 0 to 3.

$\therefore$ N = 4

$\therefore$ Range of k is also 0 to 3.

$$\therefore \quad X(k) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j\,2\pi kn}{4}} \quad ; k = 0, 1, 2, 3,$$

For each value of k, we vary n from 0 to 3.

(i) k = 0

$$X(0) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j2\pi 0\cdot n}{4}}$$

$$= x(0)\, e^{\frac{-j2\pi 0\cdot 0}{4}} + x(1)\, e^{\frac{-j2\pi 0\cdot 1}{4}} + x(2)\, e^{\frac{-j2\pi 0\cdot 2}{4}} + x(3)\, e^{\frac{-j2\pi 0\cdot 3}{4}}$$

$$= x(0)\, e^0 + x(1)\, e^0 + x(2)\, e^0 + x(3)\, e^0$$

$$= x(0) + x(1) + x(2) + x(3)$$

$$= 1 + 1 + 1 + 1$$

$$\therefore \quad X(0) = 4$$

(ii) k = 1

$$X(1) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j2\pi 1\cdot n}{4}}$$

$$= x(0)\, e^{\frac{-j2\pi 1\cdot 0}{4}} + x(1)\, e^{\frac{-j2\pi 1\cdot 1}{4}} + x(2)\, e^{\frac{-j2\pi 1\cdot 2}{4}} + x(3)\, e^{\frac{-j2\pi 1\cdot 3}{4}}$$

$$= x(0)\, e^0 + x(1)\, e^{\frac{-j\pi}{2}} + x(2)\, e^{-j\pi} + x(3)\, e^{\frac{-j3\pi}{2}}$$

$$= 1 + 1\cdot e^{\frac{-j\pi}{2}} + 1\cdot e^{-j\pi} + 1\cdot e^{\frac{-j3\pi}{2}}$$

$$= 1 + 1\left(\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right) + 1\,(\cos\pi - j\sin\pi) + 1\left(\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right)$$

$$= 1 + (0 - j) + (-1 - 0) + (0 + j)$$

$$= 1 - j - 1 + j$$

$$\therefore X(1) = 0$$

(iii) k = 2

$$X(2) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j2\pi 2\cdot n}{4}}$$

$$= x(0)\, e^{\frac{-j2\pi 2\cdot 0}{4}} + x(1)\, e^{\frac{-j2\pi 2\cdot 1}{4}} + x(2)\, e^{\frac{-j2\pi 2\cdot 2}{4}} + x(3)\, e^{\frac{-j2\pi 2\cdot 3}{4}}$$

$$= x(0)\, e^0 + x(1)^{-j\pi} + x(2)\, e^{-j2\pi} + x(3)\, e^{-j3\pi}$$

$$= 1 + 1\,(\cos\pi - j\sin\pi) + 1\cdot(\cos 2\pi - j\sin 2\pi) + 1\,(\cos 3\pi - j\sin 3\pi)$$

$$= 1 + (-1 - 0) + (1 - 0) + (-1 - 0) = 0$$

$$\therefore \quad X(2) = 0$$

(iv) k = 3

$$X(3) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j2\pi 3\cdot n}{4}}$$

$$= x(0)\, e^{\frac{-j2\pi 3\cdot 0}{4}} + x(1)\, e^{\frac{-j2\pi 3\cdot 1}{4}} + x(2)\, e^{\frac{-j2\pi 3\cdot 2}{4}} + x(3)\, e^{\frac{-j2\pi 3\cdot 3}{4}}$$

$$= x(0)\, e^0 + x(1)\, e^{\frac{-j3\pi}{2}} + x(2)\, e^{-j3\pi} + x(3)\, e^{\frac{-j9\pi}{2}}$$

$$= 1 + 1\left(\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right) + 1\cdot(\cos 3\pi - j\sin 3\pi) + 1\left(\cos\frac{9\pi}{2} - j\sin\frac{9\pi}{2}\right)$$

$= 1 + 1 \cdot (0 + j) + 1(-1 - 0) + (0 - j)$

$= 1 + j - 1 - j$

$= 0$

$\therefore \quad X(3) = 0$

$\therefore \quad X(k) = \{X(0),\ X(1),\ X(2),\ X(3)\}$

$\therefore \quad X(k) = \{4, 0, 0, 0\}$

Hence, $\{1, 1, 1, 1\} \xleftrightarrow{DFT} \{4, 0, 0, 0\}$

Let us draw the two sequences obtained.

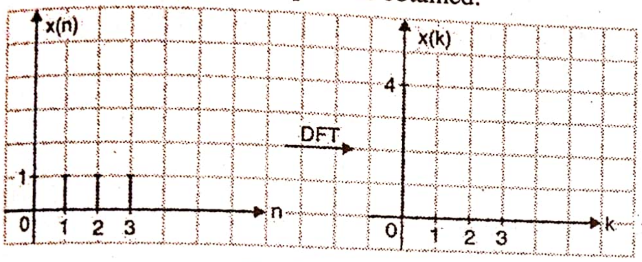


Fig. P.3.4.3(c)

Ex. 3.4.4 : Compute the DFT of the sequence given as $x(n) = (-1)^n$ for

(i) $N = 3$

(ii) $N = 4$

Soln.:

(i) N = 3

The range of 'n' is $n = 0$ to $N - 1$ that means $n = 0$ to 2.

Given sequence is , $x(n) = (-1)^n$.

For $\quad n = 0 \Rightarrow x(0) = (-1)^0 = 1$

$n = 1 \Rightarrow x(1) = (-1)^1 = -1$

$n = 2 \Rightarrow x(2) = (-1)^2 = 1$

$\therefore \quad x(n) = \{\underset{\uparrow}{1}, -1, 1\}$

According to the definition of DFT

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \qquad \ldots k = 0 \text{ to } N-1$$

$$\therefore \quad X(k) = \sum_{n=0}^{2} x(n)\, e^{\frac{-j2\pi kn}{3}} \qquad \ldots k = 0 \text{ to } 2$$

For k = 0

$$X(0) = \sum_{n=0}^{2} x(n)\, e^{\frac{-j2\pi 0 \cdot n}{3}}$$

$X(0) = x(0)\, e^0 + x(1)\, e^0 + x(2)\, e^0$

$\therefore \quad X(0) = x(0) + x(1) + x(2) = 1 - 1 + 1$

$\therefore \quad$ **X (0) = 1**

For k = 1

$$X(1) = \sum_{n=0}^{2} x(n)\, e^{\frac{-j2\pi n}{3}}$$

$\therefore \quad X(1) = x(0)\, e^0 + x(1)\, e^{\frac{-j2\pi}{3}} + x(2)\, e^{\frac{-j4\pi}{3}}$

$\therefore \quad X(1) = 1 - 1 \cdot e^{\frac{-j2\pi}{3}} + 1\, e^{\frac{-j4\pi}{3}}$

$\therefore \quad X(1) = 1 - (-0.5 - j\,0.866) + (-0.5 + j\,0.866)$

$\therefore \quad$ **X (1) = 1**

For k = 2

$$X(2) = \sum_{n=0}^{2} x(n)\, e^{\frac{-j2\pi \cdot 2n}{3}}$$

$\therefore \quad X(2) = x(0)\, e^0 + x(1)\, e^{\frac{-j4\pi}{3}} + x(2)\, e^{\frac{-j8\pi}{3}}$

$\therefore \quad X(2) = 1 - (-0.5 + j\,0.866) + (-0.5 - j\,0.866)$

$\therefore \quad$ **X (2) = 1**

Thus DFT is, $X(k) = \{1, 1, 1\}$

(ii) N = 4

The range of 'n' is n = 0 to 3.

The sequence x(n) can be written as,

$$x(n) = \{\underset{\uparrow}{1}, -1, 1, -1\}$$

According to the definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \qquad \ldots k = 0 \text{ to } N-1$$

$$\therefore \quad X(k) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j2\pi kn}{4}} \qquad \ldots k = 0 \text{ to } 3$$

$$\therefore \quad X(k) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j\pi kn}{2}}$$

For k = 0 $\quad X(0) = \sum_{n=0}^{3} x(n)\, e^0 = \sum_{n=0}^{3} x(n)$

$\therefore \quad X(0) = x(0) + x(1) + x(2) + x(3)$

$= 1 - 1 + 1 - 1$

$\therefore \quad$ **X (0) = 0**

For k = 1

$$X(1) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j\pi n}{2}}$$

$$\therefore \quad X(1) = x(0)\,e^{0} + x(1)\,e^{\frac{-j\pi}{2}} + x(2)\,e^{-j\pi} + x(3)\,e^{\frac{-j3\pi}{2}}$$

$$\therefore \quad X(1) = 1 - (-0 - j\,1) + (-1 - j\,0) - (0 + j\,1)$$

$$\therefore \quad \mathbf{X(1) = 0}$$

For k = 2

$$X(2) = \sum_{n=0}^{3} x(n)\, e^{-j\pi n}$$

$$\therefore \quad X(2) = x(0)\,e^{0} + x(1)\,e^{-j\pi} + x(2)\,e^{-j2\pi} + x(3)\,e^{-j3\pi}$$

$$\therefore \quad X(2) = 1 - (-1 - j\,0) + (1)(1 - j\,0) - 1\,(-1 - j\,0)$$

$$\therefore \quad \mathbf{X(2) = 4}$$

For k = 3

$$X(3) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j3\pi n}{2}}$$

$$\therefore \quad X(3) = x(0)\,e^{0} + x(1)\,e^{\frac{-j3\pi}{2}} + x(2)\,e^{-j3\pi} + x(3)\,e^{\frac{-j9\pi}{2}}$$

$$\therefore \quad X(3) = 1 - (0 + j\,1) + (-1 - j\,0) - (0 - j\,1)$$

$$\therefore \quad \mathbf{X(3) = 0}$$

Thus DFT is,

$$\mathbf{X(k) = \{0, 0, 4, 0\}}$$

Ex. 3.4.5 : Obtain the DFT of the sequence $x(n) = \delta(n) + 3\,\delta(n-2) + 2\,\delta(n-3)$

Soln. : The given sequence is shown below. The length of the sequence is from 0 to 3.

$$\therefore \quad N = 4$$

We know, $\delta(n) \xleftrightarrow{\text{DFT}} 1$

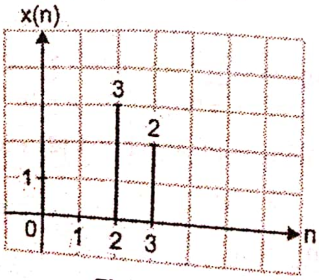


Fig. P.3.4.5

And $\quad \delta(n - n_0) \xleftrightarrow{\text{DFT}} e^{\frac{-j2\pi k n_0}{N}}$

$$\therefore \quad 3\,\delta(n-2) \xleftrightarrow{\text{DFT}} 3\,e^{\frac{-j2\pi k\cdot 2}{4}} = 3e^{-j\pi k}$$

$$\text{and} \quad 2\,\delta(n-3) \xleftrightarrow{\text{DFT}} 2\,e^{\frac{-j2\pi k\cdot 3}{4}} = 2e^{\frac{-j3\pi k}{2}}$$

$\therefore$ DFT of the given sequence is,

$$\mathbf{X(k) = 1 + 3e^{-j\pi k} + 2\,e^{-j3\pi k}}$$

Ex. 3.4.6 : Compute DFT of $x(n) = \cos\left(\frac{2\pi}{N} k_0\, n\right)$.

Soln. :

We have, $\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$

Thus given sequence can be written as,

$$x(n) = \frac{1}{2}\left[e^{\frac{+j2\pi k_0 n}{N}} + e^{\frac{-j2\pi k_0 n}{N}}\right]$$

According to the definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi k n}{N}}$$

$$\therefore \quad X(k) = \frac{1}{2}\sum_{n=0}^{N-1}\left[e^{\frac{+j2\pi k_0 n}{N}} + e^{\frac{-j2\pi k_0 n}{N}}\right]\cdot e^{\frac{-j2\pi k n}{N}}$$

$$\therefore \quad X(k) = \frac{1}{2}\cdot\sum_{n=0}^{N-1} e^{\frac{+j2\pi k_0 n}{N}}\cdot e^{\frac{-j2\pi k n}{N}} + \frac{1}{2}\cdot\sum_{n=0}^{N-1} e^{\frac{-j2\pi k_0 n}{N}}\cdot e^{\frac{-j2\pi k n}{N}}$$

$$\therefore \quad X(k) = \frac{1}{2}\sum_{n=0}^{N-1}\left[e^{\frac{j2\pi(k_0 - k)}{n}}\right]^{n} + \frac{1}{2}\sum_{n=0}^{N-1}\left[e^{\frac{-j2\pi(k_0 + k)}{N}}\right]^{n}$$

Now we have standard summation formula,

$$\sum_{n=N_1}^{N_2} a^n = \begin{cases} \dfrac{a^{N_1} - a^{N_2+1}}{1-a}, & a \neq 1 \\ N_2 - N_1 + 1, & a = 1 \end{cases}$$

Here $N_1 = 0$, $N_2 = N - 1$.

$$\therefore X(k) = \frac{1}{2}\left[\frac{\left(e^{\frac{j2\pi(k_0-k)}{N}}\right)^0 - \left(e^{\frac{j2\pi(k_0-k)}{N}}\right)^{N-1+1}}{1-e^{\frac{j2\pi(k_0-k)}{N}}}\right] + \frac{1}{2}\left[\frac{\left(e^{\frac{-j2\pi(k_0+k)}{N}}\right)^0 - \left(e^{\frac{-j2\pi(k_0+k)}{N}}\right)^{N-1+1}}{1-e^{\frac{-j2\pi(k_0+k)}{N}}}\right]$$

$$= \frac{1}{2}\left[\frac{1-e^{j2\pi(k_0-k)}}{1-e^{\frac{j2\pi}{N}(k_0-k)}}\right] + \frac{1}{2}\left[\frac{1-e^{-j2\pi(k_0+k)}}{1-e^{\frac{-j2\pi}{N}(k_0+k)}}\right]$$

Ex. 3.4.7 : Compute the DFT of $x(n) = \{1, 2, 3, 4\}$. Draw the corresponding magnitude and phase spectrum.

Soln.: $x(n) = \{1, 2, 3, 4\}$

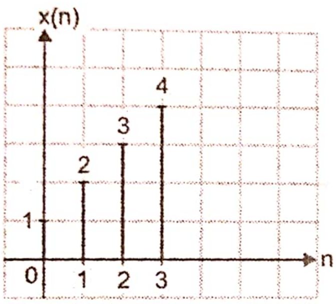


Fig. P.3.4.7

DFT is given by the equation,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi k \cdot n}{4}} \quad ; k = 0, 1, \ldots, N-1$$

Since the length of x(n) is 4, N = 4

$$\therefore X(k) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j2\pi kn}{4}} \quad ; k = 0, 1, 2, 3,$$

For each value of k, we vary n from 0 to 3.

(i) k = 0

$$\therefore X(0) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j2\pi 0 \cdot n}{4}}$$

$$= x(0)\, e^{\frac{-j2\pi 0\cdot 0}{4}} + x(1)\, e^{\frac{-j2\pi 0\cdot 1}{4}} + x(2)\, e^{\frac{-j2\pi 0\cdot 2}{4}} + x(3)\, e^{\frac{-j2\pi 0\cdot 3}{4}}$$

$$= e^0 + 2\cdot e^0 + 3\cdot e^0 + 4\cdot e^0$$

$$= 1+2+3+4$$

$$\therefore \mathbf{X(0) = 10}$$

(ii) k = 1

$$X(1) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j2\pi 1 \cdot n}{4}}$$

$$= x(0)\, e^{\frac{-j2\pi 1\cdot 0}{4}} + x(1)\, e^{\frac{-j2\pi 1\cdot 1}{4}} + x(2)\, e^{\frac{-j2\pi 1\cdot 2}{4}} + x(3)\, e^{\frac{-j2\pi 1\cdot 3}{4}}$$

$$= x(0)\, e^0 + x(1)\, e^{\frac{-j\pi}{2}} + x(2)\, e^{-j\pi} + x(3)\, e^{\frac{-j3\pi}{2}}$$

$$= 1\cdot 1 + 2\left[\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right] + 3\,[\cos\pi - j\sin\pi] + 4\left[\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right]$$

$$= 1 + 2\,(0-j) + 3\,(-1-0) + 4\,(0+j)$$

$$= 1 - 2j - 3 + 4j$$

$$\therefore \mathbf{X(1) = -2+2j}$$

(iii) k = 2

$$X(2) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j2\pi 2 \cdot n}{4}}$$

$$= x(0)\, e^{\frac{-j2\pi 2\cdot 0}{4}} + x(1)\, e^{\frac{-j2\pi 2\cdot 1}{4}} + x(2)\, e^{\frac{-j2\pi 2\cdot 2}{4}} + x(3)\, e^{\frac{-j2\pi 2\cdot 3}{4}}$$

$$= x(0)\, e^0 + x(1)\, e^{-j\pi} + x(2)\, e^{-j2\pi} + x(3)\, e^{-j3\pi}$$

$$= 1\cdot 1 + 2\,(\cos\pi - j\sin\pi) + 3\,(\cos 2\pi - j\sin 2\pi) + 4\,(\cos 3\pi - j\sin 3\pi)$$

$$= 1 + 2\,(-1-0) + 3\,(1-0) + 4\,(-1-0)$$

$$= 1 - 2 + 3 - 4$$

$$\therefore \mathbf{X(2) = -2}$$

(iv) k = 3

$$X(3) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j2\pi 3 \cdot n}{4}}$$

$$= x(0)\, e^{\frac{-j2\pi 3\cdot 0}{4}} + x(1)\, e^{\frac{-j2\pi 3\cdot 1}{4}} + x(2)\, e^{\frac{-j2\pi 3\cdot 2}{4}} + x(3)\, e^{\frac{-j2\pi 3\cdot 3}{4}}$$

$$= x(0)\, e^0 + x(1)\, e^{-j\frac{3\pi}{2}} + x(2)\, e^{-j3\pi} + x(3)\, e^{-j\frac{9\pi}{2}}$$

$$= 1\cdot 1 + 2\left(\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right) + 3\,(\cos 3\pi - j\sin 3\pi) + 4\left(\cos\frac{9\pi}{2} - j\sin\frac{9\pi}{2}\right)$$

$$= 1 + 2\,(0+j) + 3\,(-1-0) + 4\,(0-j)$$

$$= 1 + 2j - 3 - 4j$$

$$\therefore \mathbf{X(3) = -2-2j}$$

Hence the final DFT of x(n) is,

$$X(k) = \{10, -2+2j, -2, -2-2j\}$$

As is evident, the DFT is complex. (As in most practical cases) and hence can be split into magnitude and phase we know,

$$\text{Magnitude} = \sqrt{(\text{Real})^2 + (\text{Imaginary})^2} \text{ and}$$

$$\text{Phase} = \tan^{-1}\left(\frac{\text{Imaginary}}{\text{Real}}\right)$$

$$\therefore \quad |X(k)| = \{10, 2.83, 2, 2.83\}$$

$$\angle X(k) = \{0, 135°, 0°, -135°\}$$

We now draw the magnitude and phase. They are known as magnitude spectrum and phase spectrum.

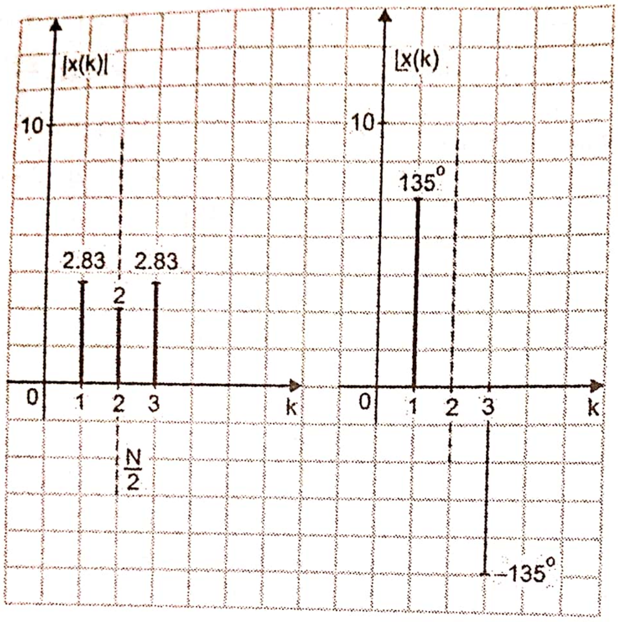


Fig. 3.4.7(a)

Note : We have drawn a dotted line at $\frac{N}{2}$ (since $N = 4, \frac{N}{2} = 2$) which we will explain later.

Ex. 3.4.8 : Find N = 5 point DFT for x(n) = { 1,0,1,0,1 }.

Soln. :

Given : $x(n) = \{1, 0, 1, 0, 1\}$ and $N = 5$

This is a five point DFT.

According to the definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}, \quad k = 0 \text{ to } N-1$$

$$\therefore \quad X(k) = \sum_{n=0}^{4} x(n)\, e^{\frac{-j2\pi kn}{5}}, \quad k = 0 \text{ to } 4$$

For k = 0

$$X(0) = \sum_{n=0}^{4} x(n)\, e^0 = \sum_{n=0}^{4} x(n)$$

$$\therefore \quad X(0) = x(0) + x(1) + x(2) + x(3) + x(4)$$

$$= 1 + 0 + 1 + 0 + 1$$

$$\therefore \quad \mathbf{X(0) = 3}$$

For k = 1

$$X(1) = \sum_{n=0}^{4} x(n)\, e^{\frac{-j2\pi n}{5}}$$

$$\therefore \quad X(1) = x(0)\, e^0 + x(1)\, e^{\frac{-j2\pi}{5}} + x(2)\, e^{\frac{-j4\pi}{5}} + x(3)\, e^{\frac{-j6\pi}{5}} + x(4)\, e^{\frac{-j8\pi}{5}}$$

$\therefore X(1) = 1 + 0 + [-0.81 - j0.59] + 0 + [0.31 + j0.95]$

$\therefore$ **X (1) = 0.5 + j0.36**

For k = 2

$$X(2) = \sum_{n=0}^{4} x(n)\, e^{\frac{-j4\pi n}{5}}$$

$$\therefore X(2) = x(0)\,e^{0} + x(1)\,e^{\frac{-j4\pi}{5}} + x(2)\,e^{\frac{-j8\pi}{5}} + x(3)\,e^{\frac{-j12\pi}{5}} + x(4)\,e^{\frac{-j16\pi}{5}}$$

$$\therefore X(2) = 1 + 0 + [0.31 + j0.95] + 0 + [-0.81 + j0.59]$$

$$\therefore \mathbf{X(2) = 0.5 + j1.54}$$

For k = 3

$$X(3) = \sum_{n=0}^{4} x(n)\, e^{\frac{-j6\pi n}{5}}$$

$$\therefore X(3) = x(0)\,e^{0} + x(1)\,e^{\frac{-j6\pi}{5}} + x(2)\,e^{\frac{-j12\pi}{5}} + x(3)\,e^{\frac{-j18\pi}{5}} + x(4)\,e^{\frac{-j24\pi}{5}}$$

$$\therefore X(3) = 1 + 0 + [0.31 - j0.95] + 0 + [-0.81 - j0.59]$$

$$\therefore \mathbf{X(3) = 0.5 - j1.54}$$

For k = 4

$$X(4) = \sum_{n=0}^{4} x(n)\, e^{\frac{-j8\pi n}{5}}$$

$$\therefore X(4) = x(0)\,e^{0} + x(1)\,e^{\frac{-j8\pi}{5}} + x(2)\,e^{\frac{-j16\pi}{5}} + x(3)\,e^{\frac{-j24\pi}{5}} + x(4)\,e^{\frac{-j32\pi}{5}}$$

$$\therefore X(4) = 1 + 0 + [-0.81 + j0.59] + 0 + [0.31 - j0.59]$$

$$\therefore \mathbf{X(4) = 0.5 - j0.36}$$

$\therefore$ The 5 point DFT of the given sequence is

$$X(k) = \{3,\ 0.5 + j0.36,\ 0.5 + j1.54,\ 0.5 - j1.54,\ 0.5 - j\,0.36\}$$

Ex. 3.4.9 : Find the DFT of the following finite duration sequence of length L.

$$x(n) = \begin{cases} A & \text{For } 0 \le n \le L-1 \\ 0 & \text{Otherwise} \end{cases}$$

Soln. :

According to the definition of DFT, We have

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$$

$$\therefore X(k) = \sum_{n=0}^{L-1} A \cdot e^{\frac{-j2\pi kn}{N}}$$

$$= A \sum_{n=0}^{L-1} \left(e^{\frac{-j2\pi k}{N}}\right)^{n}$$

We have standard summation formula,

$$\sum_{k=N_1}^{N_2} a^k = \frac{a^{N_1} - a^{N_2+1}}{1-a}$$

Here $N_1 = 0$, $N_2 = L - 1$ and $a = e^{\frac{-j2\pi k}{N}}$

$$\therefore X(k) = A\left[\frac{\left(e^{\frac{-j2\pi k}{N}}\right)^{0} - \left(e^{\frac{-j2\pi k}{N}}\right)^{L-1+1}}{1 - e^{\frac{-j2\pi k}{N}}}\right]$$

$$\therefore X(k) = A\left[\frac{1 - e^{\frac{-j2\pi kL}{N}}}{1 - e^{\frac{-j2\pi k}{N}}}\right]$$

Ex. 3.4.10 : Derive DFT of the sample data sequence x(n) = {1, 1, 2, 2, 3, 3} and compute magnitude and phase spectrum.

Soln. : According to the DFT equation,

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi kn}{N}} \qquad k = 0, 1, 2, \ldots N^{-1}$$

Hence N = 6,

$$X(k) = \sum_{n=0}^{5} x(n) \cdot e^{-j\frac{2\pi kn}{6}} \quad ; k = 0, 1, 2, 3, 4, 5$$

$$X(k) = x(0)\cdot e^{-0} + x(1)\cdot e^{-j\frac{2\pi k}{6}} + x(2)\cdot e^{-j\frac{4\pi k}{6}} + x(3)\cdot e^{-j\frac{6\pi k}{6}} + x(4)\cdot e^{-j\frac{8\pi k}{6}} + x(5)\cdot e^{-j\frac{10\pi k}{6}}$$

For k = 0

$$X(0) = x(0)\cdot e^0 + x(1)\cdot e^0 + x(2)\cdot e^0 + x(3)\cdot e^0 + x(4)\cdot e^0 + x(5)\cdot e^0$$

$$\therefore \quad X(0) = 1+1+2+2+3+3 = 12$$

If k = 1

$$X(1) = x(0) + x(1)\cdot e^{-j\frac{2\pi}{6}} + x(2)\cdot e^{-j\frac{4\pi}{6}} + x(3)\cdot e^{-j\frac{6\pi}{6}} + x(4)\cdot e^{-j\frac{8\pi}{6}} + x(5)\cdot e^{-j\frac{10\pi}{6}}$$

$$X(1) = 1 + 1\cdot\left(\cos\frac{\pi}{3} - j\sin\frac{\pi}{3}\right) + 2\cdot\left(\cos\frac{2\pi}{3} - j\sin\frac{2\pi}{3}\right) + 2\cdot(\cos\pi - j\sin\pi) + 3\left(\cos\frac{4\pi}{3} - j\sin\frac{4\pi}{3}\right) + 3\left(\cos\frac{5\pi}{3} - j\sin\frac{5\pi}{3}\right)$$

$$\therefore \quad \mathbf{X(1) = -1.5 + j\,2.6}$$

For k = 2

$$X(2) = x(0) + x(1)\cdot e^{-j\frac{4\pi}{6}} + x(2)\cdot e^{-j\frac{8\pi}{6}} + x(3)\cdot e^{-j\frac{12\pi}{6}} + x(4)\cdot e^{\frac{-16\pi}{6}} + x(5)\cdot e^{\frac{-20\pi}{6}}$$

$$X(2) = 1 + 1\left(\cos\frac{2\pi}{3} - j\sin\frac{2\pi}{3}\right) + 2\left(\cos\frac{4\pi}{3} - j\sin\frac{4\pi}{3}\right) + 2(\cos 2\pi - j\sin 2\pi) + 3\left(\cos\left(\frac{8\pi}{3}\right) - j\sin\frac{8\pi}{3}\right) + 3\left(\cos\frac{10\pi}{3} - j\sin\frac{10\pi}{3}\right)$$

$$\therefore \quad \mathbf{X(2) = -1.5 + j\,0.86}$$

For k = 3

$$X(3) = x(0) + x(1)\cdot e^{-j\frac{6\pi}{6}} + x(2)\cdot e^{-j\frac{12\pi}{6}} + x(3)\cdot e^{-j\frac{18\pi}{6}} + x(4)\cdot e^{-j\frac{24\pi}{6}} + x(5)\cdot e^{-j\frac{30\pi}{6}}$$

$$X(3) = 1 + 1(\cos\pi - j\sin\pi) + 2(\cos 2\pi - j\sin 2\pi) + 2(\cos 3\pi - j\sin 3\pi) + 3(\sin 5\pi - j\sin 5\pi) + 3(\cos 4\pi - j\sin 4\pi)$$

$$\therefore \quad \mathbf{X(3) = 0}$$

For k = 4

$$X(4) = x(0) + x(1)\cdot e^{-j\frac{8\pi}{6}} + x(2)\cdot e^{-j\frac{16\pi}{6}} + x(3)\cdot e^{-j\frac{24\pi}{6}} + x(4)\cdot e^{-j\frac{32\pi}{6}} + x(5)\cdot e^{-j\frac{40\pi}{6}}$$

$$X(4) = 1 + 1\left(\cos\left(\frac{4\pi}{3}\right) - j\sin\frac{4\pi}{3}\right) + 2\left(\cos\frac{8\pi}{3} - j\sin\frac{8\pi}{3}\right) + 2(\cos 4\pi - j\sin 4\pi) + 3\left(\cos\frac{16\pi}{3} - j\sin\frac{16\pi}{3}\right) + 3\left(\cos\frac{20\pi}{3} - j\sin\frac{20\pi}{3}\right)$$

$$\therefore \quad \mathbf{X(4) = -0.5 - j\,0.86}$$

For k = 5

$$X(5) = x(0) + x(1)\cdot e^{-j\frac{10\pi}{6}} + x(2)\cdot e^{-j\frac{20\pi}{6}} + x(3)\cdot e^{-j\frac{30\pi}{6}} + x(4)\cdot e^{-j\frac{40\pi}{6}} + x(5)\cdot e^{-j\frac{50\pi}{6}}$$

$$X(5) = 1 + 1\left(\cos\left(\frac{5\pi}{3}\right) - j\sin\frac{5\pi}{3}\right) + 2\left(\cos\frac{10\pi}{3} - j\sin\frac{10\pi}{3}\right) + 2(\cos 5\pi - j\sin 5\pi) + 3\left(\cos\frac{20\pi}{3} - j\sin\frac{20\pi}{3}\right) + 3\left(\cos\frac{25\pi}{3} - j\sin\frac{25\pi}{3}\right)$$

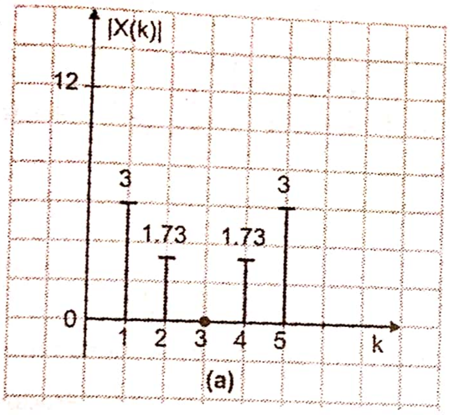
$$\therefore \quad \mathbf{X(5) = -1.5 - j\,2.6}$$

$\therefore \quad X(k) = \{12, -1.5 + j\,2.6, -1.5\ j\,0.86, 0, -1.5 - j\,0.86, -1.5 - j\,2.6\}$

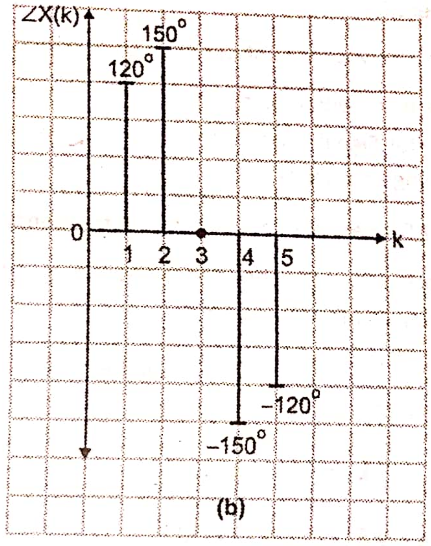
Magnitude $\rightarrow |X(k)| = \{12, 3, 1.73, 0, 1.73, 3\}$

Phase $\rightarrow \angle X(k) = \{0, 120°, 150°, 0, -150°, -120°\}$

We plot the magnitude and phase diagram as shown in Fig. P.3.4.10.



(a) Magnitude spectrum



(b) Phase spectrum

Fig. P. 3.4.10

Note : Observing the magnitude and phase plot of Ex. 3.4.7 and Ex. 3.4.10 we note that

1. The magnitude plot is symmetric about the $\frac{N}{2}$ point. If N is odd, then it is symmetric about the $\frac{N+1}{2}$ point.
2. The phase plot is anti-symmetric about the $\frac{N}{2}$ point. If N is odd, then it is anti-symmetric about the $\frac{N+1}{2}$ point.

This is always true.

3.5 The Fourier Spectrum

At this stage it is extremely important to understand what the spectrum means. We stated that the Fourier Transform gives us the frequency components present in a signal. How do we get these.

We know how to solve a DFT example.

Let us take a 8 point example :

Solved Examples

Ex. 3.5.1 : Given x(n) = {8, 2, 1, 5, 6, 2, 4, 3}

The DFT of this signal is

$X(k) = \{31, 0.58 + 1.58\,j, 9 + 4j, 3.41 - 4.4\,j, 7, 3.41 + 4.4\,j, 9 - 4\,j, 0.58 - 1.58j\}$

Verify this by a program or by using the fft(x) command in MATLAB.

Soln. : Since X(k) is complex, it has magnitude and phase. We plot its Magnitude.

$$|X(k)| = \{31, 1.69, 9.84, 5.58, 7, 5.58, 9.84, 1.69\}$$

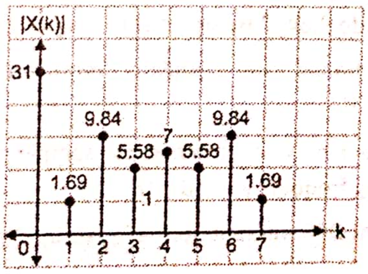


Fig. P.3.5.1

The X-axis is the frequency axis. The following questions come to our mind

(1) What does 0, 1, 2, ..., 7 represent ?

(2) Is the X axis in Hz ?

(3) Does it mean that the original signal x(n) has frequencies 0 Hz, 1 Hz,...., 6 Hz, 7 Hz ?

- These are very important questions that need to be answered.
- Make sure you understand the next few lines.
- As we have seen in earlier chapters, the entire subject of Digital Signal Processing carries no meaning if the Sampling Frequency (F_s) is not known.
- The DFT is basically sampling of the frequency axis.
- Suppose x(n) was obtained by sampling x(t) with a sampling frequency of 8000 Hz.
- A 8 point DFT will split this sampling frequency into 8 equal parts.

$$k \cdot \frac{F_s}{N} \qquad k = 0, 1, 2, \ldots, 7$$

- Since it is a 8 point DFT, N = 8.

∴ We get 0 Hz, 1 kHz, 2 kHz, 3 kHz,....,7 kHz

- Hence the X-axis for the given example is,

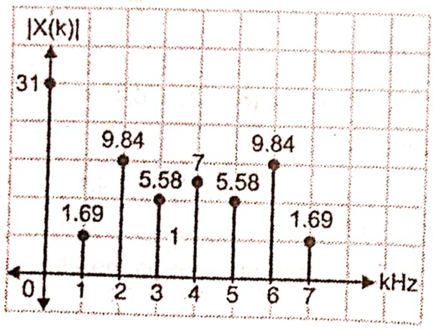


Fig. P.3.5.1(a)

- Hence we state that x(n) is made up of a DC value of 31, 1 Hz signal with magnitude 1.69, 2 Hz signal with magnitude 9.84, 7 Hz signal with magnitude 1.69.
- If we change the sampling frequency, the X-axis changes !!
- Suppose x(n) was obtained by sampling x(t) with a sampling frequency of 10 kHz.
- A 8-point DFT will split up the X-axis by the formula,

$$k \cdot \frac{F_s}{N}; \qquad k = 0,1,2, \ldots, 7$$

Since we take a 8-point DFT, N = 8

∴ We get 0 Hz, 1250 Hz, 2500 Hz, 3750 Hz, 5000 Hz, 6250 Hz, 7500 Hz and 8750 Hz.

Hence the spectrum would be

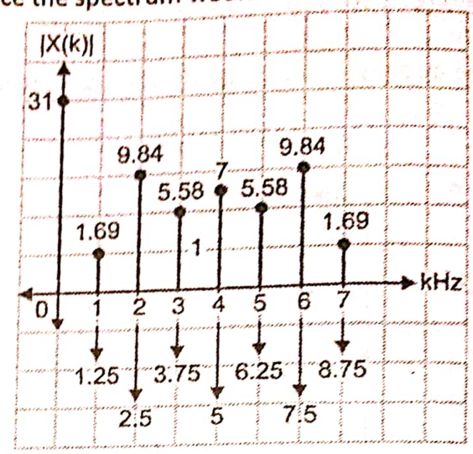


Fig. P. 3.5.1(b)

- Hence we state what has been stated earlier.
- The entire area of Digital Signal Processing makes sense only if we know the sampling frequency. Only if we know the sampling frequency, will we be able to find out the frequency components present in the signal.

3.6 Inverse Discrete Fourier Transform (IDFT)

We shall now learn how to compute the IDFT. It is similar to the DFT except that the negative exponential is replaced by a positive one. The IDFT is given by the equation.

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{\frac{+j\,2\pi kn}{N}} \quad ; \qquad n = 0, 1, 2, \ldots N-1$$

Solved Example

Ex. 3.6.1 : Compute the IDFT of X(k) = {4, 0, 0, 0}

Soln.: IDFT is given by the equation,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{\frac{+j\,2\pi kn}{N}} \quad ; n = 0, 1, \ldots N-1$$

Since length of X(k) is 4, N = 4

$$\therefore \; x(n) = \frac{1}{4} \sum_{k=0}^{3} X(k)\, e^{\frac{+j\,2\pi kn}{4}} \quad ; n = 0, 1, 2, 3$$

For each value of n, we vary k from 0 to 3

(i) n = 0

$$x(0) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, e^{\frac{+j2\pi k\cdot 0}{4}}$$

$$= \frac{1}{4}\Big[X(0)\, e^{\frac{+j2\pi 0\cdot 0}{4}} + X(1)\, e^{\frac{+j2\pi 1\cdot 0}{4}} + X(2)\, e^{\frac{+j2\pi 2\cdot 0}{4}} + X(3)\, e^{\frac{+j2\pi 3\cdot 0}{4}}\Big]$$

$$= \frac{1}{4}[X(0)\, e^0 + X(1)\, e^0 + X(2)\, e^0 + X(3)\, e^0]$$

$$= \frac{1}{4}[4+0+0+0]$$

$$= 1$$

$$\therefore\ x(0) = 1$$

(ii) n = 1

$$x(1) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, e^{\frac{+j2\pi k\cdot 1}{4}}$$

$$= \frac{1}{4}\Big[X(0)\, e^{\frac{+j2\pi 0\cdot 1}{4}} + X(1)\, e^{\frac{+j2\pi 1\cdot 1}{4}} + X(2)\, e^{\frac{+j2\pi 2\cdot 1}{4}} + X(3)\, e^{\frac{+j2\pi 3\cdot 1}{4}}\Big]$$

$$= \frac{1}{4}\Big[4e^0 + 0\cdot e^{\frac{j2\pi}{4}} + 0\, e^{+j\pi} + 0\, e^{\frac{j3\pi}{2}}\Big]$$

$$\therefore\ x(1) = 1$$

(iii) n = 2

$$x(2) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, e^{\frac{+j2\pi k\cdot 2}{N}}$$

$$= \frac{1}{4}\Big[X(0)\, e^{\frac{+j2\pi 0\cdot 2}{4}} + X(1)\, e^{\frac{+j2\pi 1\cdot 2}{4}} + X(2)\, e^{\frac{+j2\pi 2\cdot 2}{4}} + X(3)\, e^{\frac{+j2\pi 3\cdot 2}{4}}\Big]$$

$$= \frac{1}{4}[4e^0 + 0\cdot e^{j\pi} + 0\, e^{j2\pi} + 0\, e^{j3\pi}] = 1$$

$$\therefore\ x(2) = 1$$

(iv) n = 3

$$x(3) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, e^{\frac{+j2\pi k\cdot 3}{4}}$$

$$= \frac{1}{4}\Big[X(0)\, e^{\frac{+j2\pi 0\cdot 3}{4}} + X(1)\, e^{\frac{+j2\pi 1\cdot 3}{4}} + X(2)\, e^{\frac{+j2\pi 2\cdot 3}{4}} + X(3)\, e^{\frac{+j2\pi 3\cdot 3}{4}}\Big]$$

$$= \frac{1}{4}[4e^0 + 0\cdot e^{\frac{j3\pi}{2}} + 0\, e^{j3\pi} + 0\, e^{j9\pi/2}] = 1$$

$$\therefore\ x(3) = 1$$

Hence, we have,

$$x(n) = [x(0), x(1), x(2), x(3)]$$

$$\therefore\quad x(n) = \{1, 1, 1, 1\}$$

If we compare this example with Ex. 3.4.3, we can observe that,

$$X(k) = \{4, 0, 0, 0\} \overset{\text{IDFT}}{\longleftrightarrow} x(n) = \{1, 1, 1, 1\}$$

3.7 Computing DFT by Matrix Method

We will now learn how to compute the DFT of a sequence by using the matrix method. This is a much easier way as compared to the earlier technique which involved opening up the summation sign for each value of X(k).

The DFT is given by the equation,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \quad ; k = 0, 1, 2, \ldots, N-1$$

"We define a new term W_N which is called the twiddle factor."

$$W_N = e^{\frac{-j2\pi}{N}}$$

Substituting this is the DFT equation we get,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \quad ; k = 0, 1, 2, \ldots N-1 \quad \ldots(3.7.1)$$

Similarly the IDFT which is given by the equation,

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{\frac{+j2\pi kn}{N}} \quad ; n = 0, 1, \ldots N-1$$

can be written in terms of W_N as,

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn} \quad \ldots(3.7.2)$$

Equation (3.7.1) can be represented in the matrix form as

$$\text{DFT} \rightarrow X(k) = [W_N]\, x(n) \qquad \ldots(3.7.3)$$

where,
$$x(n) = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}_{N \times 1} ; \; X(k) = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix}_{N \times 1}$$

and

$$W_N^{nk} = \begin{array}{c} \\ k\downarrow \\ 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \end{array} \begin{array}{c} n \rightarrow \\ \begin{array}{ccccc} 0 & 1 & 2 & \cdots & N-1 \end{array} \\ \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & & \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}_{N \times N} \end{array}$$

$$\ldots(3.7.4)$$

Each value of W_N^{nk} matrix is obtained by multiplying n and k. W_N is called a N × N DFT matrix.

> **Note :** Changing n and k co-ordinates will not make any difference.

Similarly, Equation (3.7.2) can be written as,

$$\text{IDFT} \rightarrow x(n) = \frac{1}{N}\, [W_N^*]\, X(k) \qquad \ldots(3.7.5)$$

Before we start solving examples of DFT using the matrix method, let us try to generate W_N matrix.

3.7.1 Solved Examples of DFT

Ex. 3.7.1 : Generate a 4 × 4 DFT matrix.

Soln.: Since we require a 4 × 4 matrix N = 4.

Therefore both, n and k will vary from 0 to 3.

$$W_4^{nk} = \begin{array}{c} k\downarrow \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} W_4^{0.0} & W_4^{0.1} & W_4^{0.2} & W_4^{0.3} \\ W_4^{1.0} & W_4^{1.1} & W_4^{1.2} & W_4^{1.3} \\ W_4^{2.0} & W_4^{2.2} & W_4^{2.2} & W_4^{2.3} \\ W_4^{3.0} & W_4^{3.1} & W_4^{3.2} & W_4^{3.3} \end{bmatrix} \end{array} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}$$

We know $W_N = e^{\frac{-j2\pi}{N}}$

We calculate each value of the matrix,

$$W_4^0 = e^{-j\frac{2\pi 0}{4}} = 1$$

$$W_4^1 = e^{-j\frac{2\pi}{4}\cdot 1} = -j$$

$$W_4^2 = e^{-j\frac{2\pi}{4}\cdot 2} = -1$$

$$W_4^3 = e^{-j\frac{2\pi}{4}\cdot 3} = +j$$

$$W_4^4 = e^{-j\frac{2\pi}{4}\cdot 4} = +1$$

$$W_4^6 = e^{-j\frac{2\pi}{4}\cdot 6} = -1$$

$$W_4^9 = e^{-j\frac{2\pi}{4}\cdot 9} = -j$$

$$\therefore \quad W_4^{nk} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & +1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \qquad \ldots(1)$$

Hence a 4 × 4 DFT matrix is shown in Equation (1).

As mentioned earlier, W_N is called the Twiddle factor.

The word twiddle means "Turn in a twisting or spinning motion". In other words, to move in a turning (circular) fashion. A lot of us have a habit of twiddling our tumbs while waiting for an oral exam.

Now W_N is called a twiddle factor because it is cyclic in nature.

Shown below are W_4 values mapped on a circle.

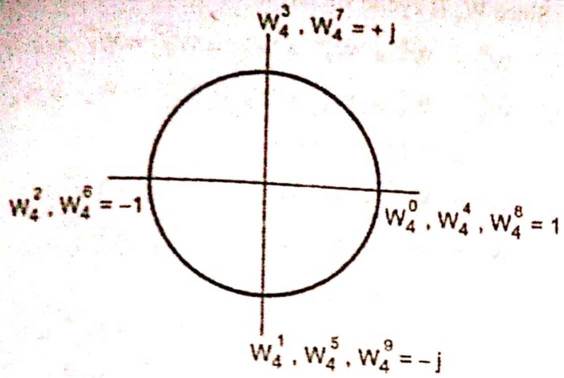


Fig. P. 3.7.1(a)

$$\therefore \quad \begin{aligned} W_4^0 &= W_4^4 = W_4^8 = 1 \\ W_4^1 &= W_4^5 = W_4^9 = -j \\ W_4^2 &= W_4^6 = -1 \\ W_4^3 &= W_4^7 = +j \end{aligned}$$

Similarly for W_8, we can draw the values of the 8×8 matrix using a circle.

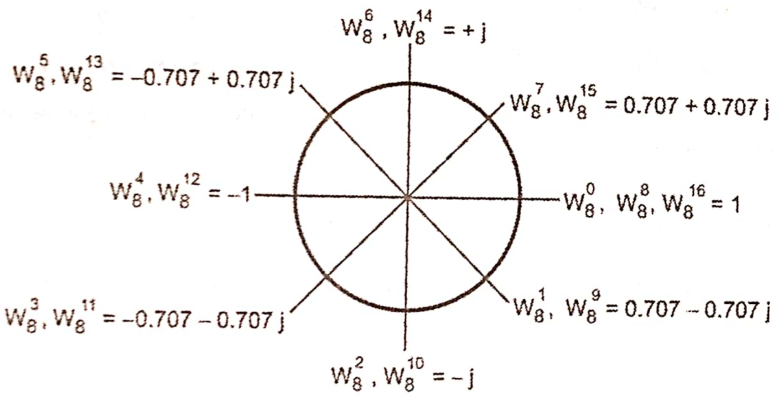


Fig. P. 3.7.1(b)

Let us calculate each of these values and verify

$$\therefore \quad W_8^0 = e^{-j\frac{2\pi}{8}\cdot 0} = \cos 0 - j \sin 0 = 1$$

$$\therefore \quad W_8^1 = e^{-j\frac{2\pi}{8}\cdot 1} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = 0.707 - 0.707\, j$$

$$\therefore \quad W_8^2 = e^{-j\frac{2\pi}{8}\cdot 2} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

$$\therefore \quad W_8^3 = e^{-j\frac{2\pi}{8}\cdot 3} = \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4} = -0.707 - 0.707\, j$$

$$\therefore \quad W_8^4 = e^{-j\frac{2\pi}{8}\cdot 4} = \cos\pi - j\sin\pi = -1$$

$$\therefore \quad W_8^5 = e^{-j\frac{2\pi}{8}\cdot 5} = \cos\frac{5\pi}{4} - j\sin\frac{5\pi}{4} = -0.707 + 0.707\, j$$

$$\therefore \quad W_8^6 = e^{-j\frac{2\pi}{8}\cdot 6} = \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} = +j$$

$$\therefore \quad W_8^7 = e^{-j\frac{2\pi}{8}\cdot 7} = \cos\frac{7\pi}{4} - j\sin\frac{7\pi}{4} = 0.707 + 0.707\, j$$

$$\therefore \quad W_8^8 = e^{-j\frac{2\pi}{8}\cdot 8} = \cos 2\pi - j\sin 2\pi = +1$$

$$\therefore \quad W_8^9 = e^{-j\frac{2\pi}{8}\cdot 9} = \cos\frac{9\pi}{4} - j\sin\frac{9\pi}{4}$$

$$= 0.707 - 0.707\, j$$

$$\therefore \quad W_8^{10} = e^{-j\frac{2\pi}{8}\cdot 10} = \cos\frac{5\pi}{2} - j\sin\frac{5\pi}{2} = -j$$

$$\therefore \quad W_8^{11} = e^{-j\frac{2\pi}{8}\cdot 11} = \cos\frac{11\pi}{4} - j\sin\frac{11\pi}{4}$$

$$= -0.707 + 0.707\, j$$

$$\therefore \quad W_8^{12} = e^{j\frac{2\pi}{8}\cdot 12} = \cos 3\pi - j\sin 3\pi = -1$$

$$\therefore \quad W_8^{13} = e^{j\frac{2\pi}{8}\cdot 13} = \cos\frac{13\pi}{4} - j\sin\frac{13\pi}{4}$$

$$= -0.707 + 0.707\, j$$

$$\therefore \quad W_8^{14} = e^{j\frac{2\pi}{8}\cdot 14} = \cos\frac{7\pi}{2} - j\sin\frac{7\pi}{2} = +j$$

$$\therefore \quad W_8^{15} = e^{j\frac{2\pi}{8}\cdot 15} = \cos\frac{15\pi}{4} - j\sin\frac{15\pi}{4}$$

$$= 0.707 + 0.707\, j$$

$$\therefore \quad W_8^{16} = e^{j\frac{2\pi}{8}\cdot 16} = \cos 4\pi - j\sin 4\pi = 1$$

Using these values we can easily generalize a 8×8 DFT matrix.

$$W_8^{nk} = \begin{array}{c|cccccccc} k \downarrow \backslash n \rightarrow & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ 1 & W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ 2 & W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^8 & W_8^{10} & W_8^{12} & W_8^{14} \\ 3 & W_8^0 & W_8^3 & W_8^6 & W_8^9 & W_8^{12} & W_8^{15} & W_8^{18} & W_8^{21} \\ 4 & W_8^0 & W_8^4 & W_8^8 & W_8^{12} & W_8^{16} & W_8^{20} & W_8^{24} & W_8^{28} \\ 5 & W_8^0 & W_8^5 & W_8^{10} & W_8^{15} & W_8^{20} & W_8^{25} & W_8^{30} & W_8^{35} \\ 6 & W_8^0 & W_8^6 & W_8^{12} & W_8^{18} & W_8^{24} & W_8^{30} & W_8^{36} & W_8^{42} \\ 7 & W_8^0 & W_8^7 & W_8^{14} & W_8^{21} & W_8^{28} & W_8^{35} & W_8^{42} & W_8^{49} \end{array}$$

Since W_8 is cyclic we have,

$$\therefore \quad W_8^0 = W_8^8 = W_8^{16} = W_8^{24} = W_8^{32} = W_8^{40} = 1$$

$$\therefore \quad W_8^1 = W_8^9 = W_8^{17} = W_8^{25} = W_8^{33} = W_8^{41} = W_8^{49}$$

$$= 0.707 - 0.707\,j$$

$$\therefore \quad W_8^2 = W_8^{10} = W_8^{18} = W_8^{26} = W_8^{34} = W_8^{42} = j$$

$$\therefore \quad W_8^3 = W_8^{11} = W_8^{19} = W_8^{27} = W_8^{35} = W_8^{43}$$

$$= -0.707 \;\; 0.707\,j$$

$$\therefore \quad W_8^4 = W_8^{12} = W_8^{20} = W_8^{28} = W_8^{36} = W_8^{44} = 1$$

$$\therefore \quad W_8^5 = W_8^{13} = W_8^{21} = W_8^{29} = W_8^{37} = W_8^{45}$$

$$= -0.707 + 0.707\,j$$

$$\therefore \quad W_8^6 = W_8^{14} = W_8^{22} = W_8^{30} = W_8^{38} = W_8^{46} = j$$

$$\text{and} \quad W_8^7 = W_8^{15} = W_8^{23} = W_8^{31} = W_8^{39} = W_8^{47}$$

$$= 0.707 + 0.707\,j$$

Using these values we generate a 8×8 DFT matrix.

$$W_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0.707 - j\,0.707 & -j & -0.707 - j\,0.707 & -1 & -0.707 + j\,0.707 & j & 0.707 + j\,0.707 \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & -0.707 - j\,0.707 & j & 0.707 - j\,0.707 & -1 & 0.707 + j\,0.707 & -j & -0.707 + j\,0.707 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -0.707 + j\,0.707 & -j & 0.707 + j\,0.707 & -1 & 0.707 - j\,0.707 & j & -0.707 - j\,0.707 \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & 0.707 + j\,0.707 & j & -0.707 + j\,0.707 & -1 & -0.707 - j\,0.707 & -j & 0.707 - j\,0.707 \end{bmatrix}$$

Let us now solve a few examples by using the DFT matrix method.

Ex. 3.7.2 : Compute the DFT of the sequence $x(n) = \{1, 1, 1, 1\}$

Soln.: This is the same sequence which was solved in Ex. 3.4.3.

The DFT is given by the equation,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn},$$

Where, $$W_N^{nk} = e^{\frac{-j\,2\pi kn}{N}}$$

Since the length of $x(n) = 4$, the DFT equation reduces to,

$$X(k) = \sum_{n=0}^{3} x(n)\, W_4^{kn} \; ; \; k = 0, 1, 2, 3$$

In matrix form it is written as,

$$X(k) = [W_4]\, x(n)$$

Where, $x(n) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$\therefore \quad X(k) = \overset{W_4}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}} \overset{x(n)}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}$$

$$\therefore \quad X(k) = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence, $X(k) = [\ 4,\ 0,\ 0,\ 0\ ]$

We note the answer is the same as that obtained in Example 3.4.3.

Ex. 3.7.3 : Compute the DFT of $x(n) = \{1, 2, 3, 4\}$

Soln. : This is the same as Example 3.5.7.

Since x(n) is of length 4, N = 4 and we generate a DFT matrix of size 4×4.

$$\therefore \quad X(k) = [W_4]_{4 \times 4}\, x(n)$$

$$\therefore \quad X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$X(k) = \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$$\therefore \quad X(k) = \{\ 10,\ -2+2j,\ -2,\ -2-2j\ \}$$

This is the same as obtained in Example 3.5.7.

You must have realized that the matrix method is much easier that the conventional method of opening the summation.

> **Note :** The DFT matrix depends only on the length of x(n) and not the values of x(n). Because of this for every 4 point input, we use the same 4×4 DFT matrix.

A 4×4 DFT matrix can be easily memorized using the 5 steps.

Step 1 : $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & & & \\ 1 & & & \\ 1 & & & \end{bmatrix}$

Step 2 : $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & & \\ 1 & & & \\ 1 & & & -j \end{bmatrix}$

Step 3 : $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & & +j \\ 1 & & & \\ 1 & +j & & -j \end{bmatrix}$

Step 4 : $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & & j \\ 1 & & +1 & \\ 1 & j & & -j \end{bmatrix}$

Step 5 : $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & +1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix}$

Ex. 3.7.4 : Compute the IDFT of $X(k) = \{10, -2+2j, -2, -2-2j\}$.

Soln. : The IDFT is given by the equation

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-nk} \qquad ; n = 0, 1, 2, \ldots N-1$$

In matrix form it is written as,

$$x(n) = \frac{1}{N} [W_N^*]\, X(k)$$

Since X(k) is of length 4, N = 4 and we generate a IDFT matrix of size 4×4.

$$\therefore \quad x(n) = \frac{1}{4} \overset{W_4^*}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & +1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}} \overset{X(k)}{\begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}}$$

$$x(n) = \frac{1}{4} \begin{bmatrix} 4 \\ 8 \\ 12 \\ 16 \end{bmatrix}$$

$$x(n) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\therefore \quad x(n) = \{ 1 \quad 2 \quad 3 \quad 4 \}$$

Ex. 3.7.5 : Compute the DFT of x(n) = {1, 1, 0, 0}

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Soln. :

Since x(n) is of length 4, N = 4 and we generate a DFT matrix of size 4 × 4.

$$\therefore \quad X(k) = [W_4]_{4\times4}\, x(n)$$

$$\therefore \quad X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$X(k) = \begin{bmatrix} 2 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}$$

$$\therefore \quad X(k) = \{ 2, 1-j, 0, 1+j\}$$

Ex. 3.7.6 : Compute the IDFT of X(k) = {2, 1 – j, 0, 1 + j}.

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Soln. : The IDFT is given by the equation

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-nk} \qquad ; n = 0, 1, 2, \ldots N-1$$

In matrix form it is written as,

$$x(n) = \frac{1}{N}[W_N^*]\, X(k)$$

Since X(k) is of length 4, N = 4 and we generate a IDFT matrix of size 4 × 4.

$$\therefore \quad x(n) = \frac{1}{4}\overset{W_4^*}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & +1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}}\overset{X(k)}{\begin{bmatrix} 2 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}}$$

$$x(n) = \frac{1}{4}\begin{bmatrix} 4 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

$$x(n) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad x(n) = \{ 1 \quad 1 \quad 0 \quad 0 \}$$

Ex. 3.7.7 : For a continuous time signal equation, s(t) = sin [2 π 1000 t] + 0.5 sin [2 π 2000 t] Sample the given signal at 8000 samples/sec and find out 8 point DFT. Plot magnitude and phase response.

Soln. :

Given : Sampling rate F_s = 8000 sample / sec. Discrete signal is obtained by replacing t by n/F_s

$$\therefore \quad x(n) = \sin\left[2\pi\left(\frac{1000}{8000}\right)\right] n + 0.5 \sin\left[2\pi\left(\frac{2000}{8000}\right) n\right] = \sin\left(\frac{\pi}{4}\right) n + 0.5 \sin\left(\frac{\pi}{2}\right) n$$

For $n = 0$ to 7

$$x(0) = \sin 0 + 0.5 \sin 0 = 0$$

$$x(1) = \sin\frac{\pi}{4} + 0.5 \sin\frac{\pi}{2} = 1.207$$

$$x(2) = \sin\frac{\pi}{2} + 0.5 \sin \pi = 1$$

$$x(3) = \sin\frac{3\pi}{4} + 0.5 \sin\frac{3\pi}{2} = 0.207$$

$$x(4) = \sin \pi + 0.5 \sin 2\pi = 0$$

$$x(5) = \sin\frac{5\pi}{4} + 0.5 \sin\frac{5\pi}{2} = -0.207$$

$$x(6) = \sin\frac{3\pi}{2} + 0.5 \sin 3\pi = -1$$

$$x(7) = \sin\frac{7\pi}{4} + 0.5 \sin\frac{7\pi}{2} = -1.207$$

$$\therefore \quad x(n) = \{0, 1.207, 1, 0.207, 0, -0.207 -1, -1.207\}$$

Since the length of x(n) is equal to 8, we use a 8 × 8 DFT matrix,

$$\therefore \quad X(k) = [w_8]\, x(n)$$

X(k) = [illegible]

$$= \begin{bmatrix} 0 \\ 0.5j \\ -2j \\ 0.006j \\ 0 \\ 0.5j \\ 2j \\ -0.006j \end{bmatrix} = \begin{bmatrix} 0 \\ -j3.99 \\ -j2 \\ j0.0003 \\ 0 \\ -j0.0003 \\ j2 \\ j3.99 \end{bmatrix}$$

$X(k) = \{0, -j3.99, -j2, j0.0003, 0, -j0.0003, j2, j3.99\}$

$|X(k)| = \{0, 3.99, 2, 0.0003, 0, 0.0003, 2, 3.99\}$

$\angle X(k) = \{0, -90°, -90°, 0, -90°, 90°, 90°\}$

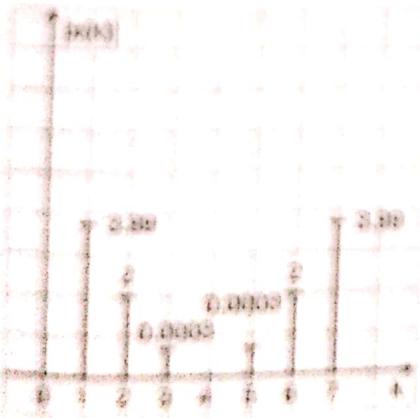

Fig. P. 3.7.7

Fig. P. 3.7.7(a)

Ex. 3.7.8 : Compute the DFT of the following frequency

$x(n) = \cos \frac{n\pi}{4}$ $n = 0, 1, 2, 3$

Soln. :

$$x(n) = \cos\left(\frac{n\pi}{4}\right) n = 0, 1, 2, 3$$

We substitute values of n

$$x(0) = \cos\left(\frac{0 \cdot \pi}{4}\right) = 1$$

$$\therefore\ x(1) = \cos\left(\frac{\pi}{4}\right) = 0.707$$

$$\therefore\ x(2) = \cos\left(\frac{2\pi}{4}\right) = 0$$

$$\therefore\ x(3) = \cos\left(\frac{3\pi}{4}\right) = -0.707$$

$$\therefore\ x(n) = \{1, 0.707, 0, -0, 707\}$$

A 4 – point DFT is matrix form is given by the equation

$$X(k) = [W_4]\, x(n)$$

$$y(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 0.707 \\ 0 \\ -0.707 \end{bmatrix}$$

$$\therefore\ y(k) = \{1, 1-j\,1.414, 1, 1+j\,1.414\}$$

Ex. 3.7.9 : Compute circular convolution of $x_1(n) = \{1,1,2,2\}$ $x_2(n) = \{1, 2, 3, 4\}$

Soln. :

$$y(n) = x_1(n) \circledast x_2(n)$$

We generate a circular matrix of $x_2(n)$

$$\therefore \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 15 \\ 17 \\ 15 \\ 13 \end{bmatrix}$$

$$\therefore\ y(n) = \{15, 17, 15, 13\}$$

Ex. 3.7.10 : Compute the 8-point DFT of the sequence $x(n) = \{0, 1, 2, 3\}$. Also draw the magnitude and phase plot.

Soln. : Since we require a 8-point DFT, we append four zeros to the original input sequence.

$$\therefore\ x(n) = \{0, 1, 2, 3, 0, 0, 0, 0\}$$

As N = 8, we generate a 8 × 8 DFT matrix.

$$X(k) = [W_8]\, x(n)$$

$$X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0.707-j0.707 & -j & -0.707-j0.707 & -1 & -0.707+j0.707 & j & 0.707+j0.707 \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & -0.707-j0.707 & j & 0.707-j0.707 & -1 & 0.707+j0.707 & -j & -0.707+j0.707 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -0.707+j0.707 & -j & 0.707+j0.707 & -1 & 0.707-j0.707 & j & -0.707-j0.707 \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & 0.707+j0.707 & j & -0.707+j0.707 & -1 & -0.707-j0.707 & -j & 0.707-j0.707 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving the above matrix operation, we obtain

$$\therefore\ X(k) = \begin{bmatrix} 6 \\ -1.414-j\,4.83 \\ -2+j2 \\ 1.414-j\,0.83 \\ -2 \\ 1.414+j\,0.83 \\ -2-j2 \\ -1.414+j\,4.83 \end{bmatrix}$$

$\therefore$ $X(k) = \{6, -1.414 - j\,4.83, -2 + j2, 1.414 - j\,0.83,$
$-2, 1.414 + j\,0.83, -2 - j2, -1.414 + j\,4.83\}$

Since X(k) is complex, we plot the magnitude and phase.

$$|X(k)| = \sqrt{(\text{Real})^2 + (\text{Imaginary})^2}$$

$$\therefore \quad |X(k)| = \{6, 5.03, 2.83, 1.64, 2, 1.64, 2.83, 5.03\}$$

$$\angle X(k) = \tan^{-1}\left(\frac{\text{Imaginary}}{\text{Real}}\right)$$

$$\therefore \quad \angle X(k) = \{0°, -106.31°, 135°, -30.4°, 0°, 30.4°, -135°, 106.31°\}$$

We now draw the magnitude and phase spectrum.

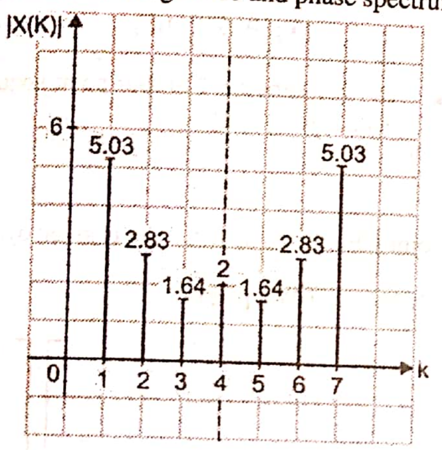

Fig. P.3.7.10

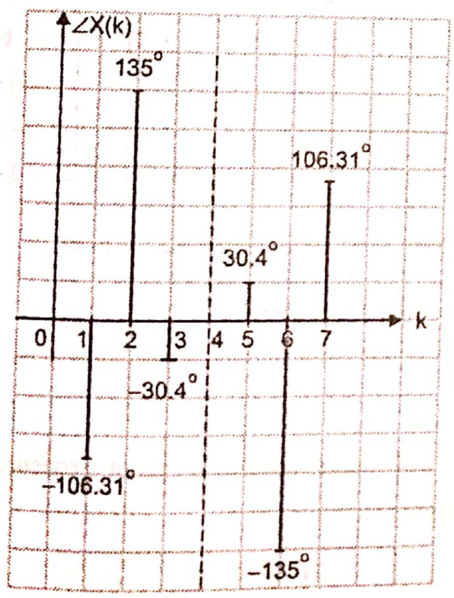

Fig. P.3.7.10(a)

A dotted line is drawn at $\frac{N}{2} = 4$.

Notice that the magnitude spectrum is symmetric about the $\frac{N}{2}$ point while the phase spectrum is anti-symmetric about the $\frac{N}{2}$ point. This is always true.

Ex. 3.7.11 : Determine 2-point and 4-point DFT of a sequence,

$x(n) = u(n) - u(n-2)$

Sketch the magnitude of DFT in both the cases.

Soln. :

We begin with Identify the signal

$x(n) = u(n) - u(n-2)$ is shown in Fig. P.3.7.11.

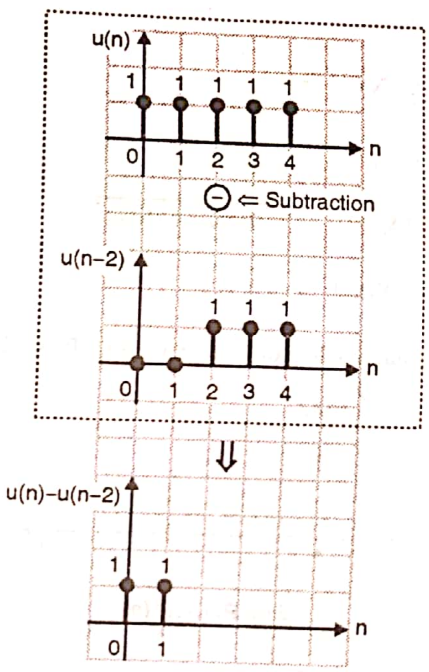

Fig. P. 3.7.11

$$\therefore \quad x(n) = \{1, 1\}$$

(i) Two point DFT

Since N = 2, we use a 2×2

DFT matrix

$$\therefore \quad X(k) = [W_2]_{2\times 2}\, x(n) \qquad \dots (1)$$

Generation of W_2 is shown below

$$W_2^{kn} = \begin{array}{c|cc} & n=0 & n=1 \\ \hline k=0 & W_2^0 & W_2^0 \\ k=1 & W_2^0 & W_2^1 \end{array}$$

Now, $W_2^0 = e^{\frac{-j2\pi}{2}\cdot 0} = 1$

$$W_2^1 = e^{\frac{-j2\pi}{2}\cdot 1} = e^{-j\pi} = -1$$

$$\therefore \quad W_2^{nk} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

From Equation (1) we write $W_2 X(n)$

$$X(k) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \quad X(k) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\therefore \quad X(k) = \{2,0\}$$

$$|X(k)| = \sqrt{(\text{Real})^2 + (\text{Img})^2}$$

$$\therefore \quad |X(k)| = \sqrt{2^2} = 2$$

$\therefore$ Magnitude plot is shown in Fig. P. 3.7.11(a).

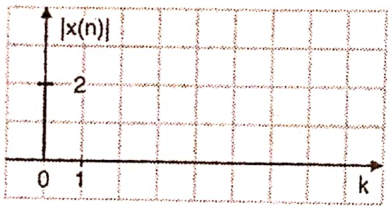

Fig. P.3.7.11(a)

(ii) Four point DFT

Since $N = 4$, we use 4×4 DFT matrix

We append two zeros to x (n) to get $x(n) = \{1,1,0,0\}$

We have already learnt how to generate a 4×4 DFT matrix

$$\therefore \quad X(k) = [W_4]_{4\times 4}\, x(n)$$

$$\therefore \quad X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad X(k) = \begin{bmatrix} 2 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}$$

$$\therefore \quad X(k) = \{2, 1-j, 0, 1+j\}$$

$$|X(k)| = \sqrt{(\text{Real})^2 + (\text{Imaginary})^2}$$

$$\therefore \quad |X(k)| = \{2, 1.414, 0, 1.414\}$$

The magnitude plot is shown in Fig. P. 3.7.11(b).

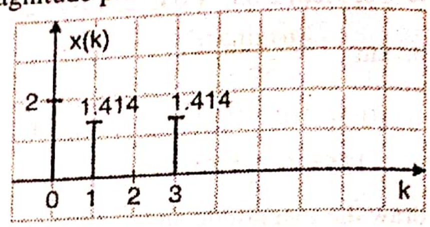

Fig. P. 3.7.11(b)

Ex. 3.7.12 : Compute the DFT of four point sequence $x(n) = \{0, 1, 2, 3\}$

Soln. :

The four point DFT in the matrix form is given by,

$$X(k) = [W_4]\cdot x(n)$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\therefore \quad X(k) = \begin{bmatrix} 0+1+2+3 \\ 0-j-2+3j \\ 0-1+2-3 \\ 0+j-2-3j \end{bmatrix} = \begin{bmatrix} 6 \\ -2j+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$$\therefore \quad X(k) = \{6, -2+2j, -2, -2-2j\}$$

Ex. 3.7.13 : Calculate 8 point DFT of $x(n) = \{1, 2, 1, 2\}$

Soln. :

First we will make length of given sequence '8' by appending zeros to x(n).

$$\therefore \quad x(n) = \{1, 2, 1, 2, 0, 0, 0, 0\}$$

Now

$$X(k) = [W_8]\, x(n)$$

We have already learnt – how to compute W_8, which is a 8 × 8 DFT matrix.

$$X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0.707 - j\,0.707 & -j & -0.707 - j\,0.707 & -1 & -0.707 + j\,0.707 & j & 0.707 + j\,0.707 \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & -0.707 - j\,0.707 & j & 0.707 - j\,0.707 & -1 & 0.707 + j\,0.707 & -j & -0.707 + j\,0.707 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -0.707 + j\,0.707 & -j & 0.707 + j\,0.707 & -1 & 0.707 - j\,0.707 & j & -0.707 - j\,0.707 \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & 0.707 + j\,0.707 & j & -0.707 + j\,0.707 & -1 & -0.707 - j\,0.707 & -j & 0.707 - j\,0.707 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore X(k) = \begin{bmatrix} 1 + 2 + 1 + 2 + 0 + 0 + 0 + 0 \\ 1 + 1.414 - j\,1.414 - j - 1.414 - j\,1.414 + 0 + 0 + 0 + 0 \\ 1 - j\,2 - 1 + j\,2 + 0 + 0 + 0 + 0 \\ 1 - 1.414 - j\,1.414 + j + 1.414 - j\,1.414 + 0 + 0 + 0 + 0 \\ 1 - 2 + 1 - 2 + 0 + 0 + 0 + 0 \\ 1 - 1.414 + j\,1.414 - j + 1.414 + j\,1.414 + 0 + 0 + 0 + 0 \\ 1 + j\,2 - 1 - j\,2 + 0 + 0 + 0 + 0 \\ 1 + 1.414 + j\,1.414 + j - 1.414 + j\,1.414 + 0 + 0 + 0 + 0 \end{bmatrix}$$

$$\therefore \quad X(k) \quad = \begin{bmatrix} 6 \\ 1 - j3.828 \\ 0 \\ 1 - j\,1.828 \\ -2 \\ 1 + j\,1.828 \\ 0 \\ 1 + j3.828 \end{bmatrix}$$ This is the required DFT.

∴ The 8 point DFT of the sequence is

$\therefore$ X (k) ={6,1 – j3.828, 0, 1 – j 1.828, – 2, 1 + j 8. 28, 0, 1 + j 3.828}

Ex. 3.7.14 : Compute 8 point DFT of the sequence x (n) = {0, 1, 2, 3}. Sketch the magnitude and phase plot also

Soln. : Given sequence is, x (n) = {0, 1, 2, 3}

It is asked to calculate 8 point DFT.

We append four zero to x (n) to make its length equal to 8,

$$\therefore x(n) = \{0, 1, 2, 3, 0, 0, 0, 0\}$$

Now $X(k) = [W_8]\,x(n)$

$$X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0.707 - j0.707 & -j & -0.707 - j0.707 & -1 & -0.707 + j0.707 & j & 0.707 + j0.707 \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & -0.707 - j0.707 & j & 0.707 - j0.707 & -1 & 0.707 + j0.707 & -j & -0.707 + j0.707 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -0.707 + j0.707 & -j & 0.707 + j0.707 & -1 & 0.707 - j0.707 & j & -0.707 - j0.707 \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & 0.707 + j0.707 & j & -0.707 + j0.707 & -1 & -0.707 - j0.707 & -j & 0.707 - j0.707 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore X(k) = \begin{bmatrix} 6 \\ -1.414 - j\,4.828 \\ -2 + j\,2 \\ 1.414 - j\,0.828 \\ -2 \\ 1.414 + j\,0.828 \\ -2 - j\,2 \\ -1.414 + j\,4.828 \end{bmatrix}$$

$$\text{Magnitude response} = |X(k)| = \sqrt{(\text{Real})^2 + (\text{Imaginary})^2}$$

$$\therefore |X(k)| = \{6, 5.03, 2.83, 1.64, 2, 1.64, 2.83, 5.03\}$$

$$\text{Phase response} = \tan^{-1}\left\{\frac{\text{Imaginary}}{\text{Real}}\right\}$$

$$\angle X(k) = \{0°, -106.3°, 135°, -30.36°, 0°, 30.36°, -135°, 106.32°\}$$

The magnitude plot and phase plot are shown in Fig. P. 3.7.14.

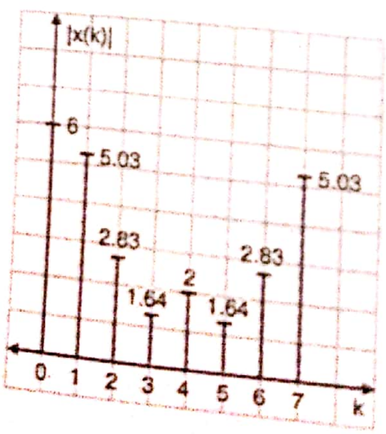

Fig. P. 3.7.14

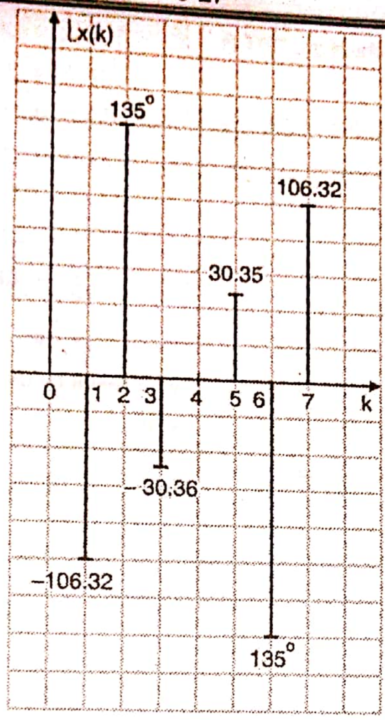

Fig. P. 3.7.14(a)

Ex. 3.7.15 : Plot the magnitude and phase spectrum of the sampled data sequence = {2, 0, 0, 1} which was obtained using a sampling frequency of 20 Hz.

Soln. : To obtain magnitude and phase response we will calculate DFT as follows,

$$X(k) = [W_4]\, x(n)$$

$$\therefore\ X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2+j \\ 1 \\ 2-j \end{bmatrix}$$

$$\therefore\ X(k) = \{\,3,\ 2+j,\ 1,\ 2-j\,\}$$

Magnitude plot

$$|X(k)| = \sqrt{R_e^2 + I_m^2}$$

$$\therefore\ |X(k)| = \{3, 2.24, 1, 2.24\}$$

$$\angle X(k) = \tan^{-1}\left(\frac{\text{Img}}{\text{Real}}\right)$$

$$\angle X(k) = \{0°, 26.56°, 0°, -26.56°\}$$

We draw the, magnitude and phase plot

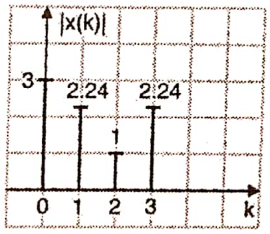

Fig. P. 3.8.15

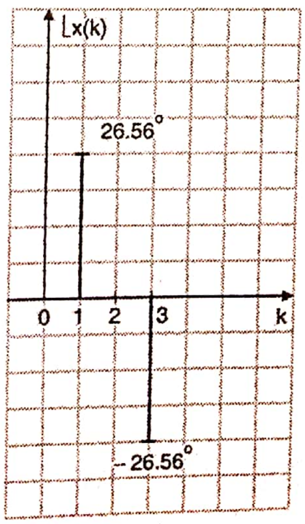

Fig. P. 3.7.15(a)

Ex. 3.7.16 : Compute the sequence x(n) for which the DFT is {20, – 4 + 4j, – 4, – 4 – 4j}

Soln. : Since we need to find the sequence x(n), we compute the IDFT.

The IDFT is given by the equation

$$\therefore \quad x(n) = \frac{1}{N}\,[W_4^*]\,X(k)$$

Since N = 4, we use a 4 × 4 IDFT matrix.

$$x(n) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & +1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 20 \\ -4+4j \\ -4 \\ -4-4j \end{bmatrix}$$

$$\therefore x(n) = \frac{1}{4}\begin{bmatrix} 8 \\ 16 \\ 24 \\ 32 \end{bmatrix}$$

$$\therefore x(n) = \{ 2, 4, 6, 8 \}$$

Given below is a MATLAB code to compute the DFT of a sequence of any length.

Program to Compute the DFT using Matrix Method

```
clc
clear all
x=[0 1 2 3 0 0 0 0];
N=length (x);

for k = 0 : 1 : N–1
    for n = 0 : 1 : N–1
        W (n+1, k+1) =exp (–j*2*pi*k*n/N);
    end
end
X=W*x'
X_Mag=abs (X);
X_Ph=rad2deg (angle (X));
subplot (2, 2, 1)
stem (X_Mag)
title ('Magnitude plot')
subplot (2, 2, 2)
stem (X_Ph)
title ('Phase plot')

X =
    6.0000 + 0.0000i
   –1.4142 – 4.8284i
   –2.0000 + 2.0000i
    1.4142 – 0.8284i
   –2.0000 – 0.0000i
    1.4142 + 0.8284i
   –2.0000 – 2.0000i
   –1.4142 + 4.8284i
```

Ex. 3.7.17 : Find the DFT of following sequence and check your result by using IDFT x (n) = {0, 1, 2, 3}

Soln. :

We have already obtained DFT of x (n) of Ex. 2.7.2

$$\text{It is } X(k) = \{6, 2j-2, -2, -2j-2\}$$

Calculation of IDFT :

$$x(n) = \frac{1}{N}\left[W_N^*\right] X_N$$

$$\therefore \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 6 \\ 2j-2 \\ -2 \\ -2j-2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 6+2j-2-2-2j-2 \\ 6-2-2j+2-2+2j \\ 6-2j+2-2+2j+2 \\ 6+2+2j+2+2-2j \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 \\ 4 \\ 8 \\ 12 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\therefore \quad x(n) = \{0, 1, 2, 3\}$$

Ex. 3.7.18 : Show that the basis matrix for DFT is unitary. Consider N = 4.

Soln. : A matrix is said to be unitary is

$$W_N = W_N^{*T} = I$$

We need to check is

$$W_4 \cdot W_N^{*T} = I$$

We have,

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$\text{and } W_4^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

Now

$$W_4 \cdot W_4^{*T} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$$4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \therefore 4\,[I]$$

Hence DFT matrix is unitary

3.8 Discrete Fourier Transform (DFT) Properties

MU - May 2016, Dec. 2016, May 2017

> **Q.** State any five DFT properties.
> **(May 2016, Dec. 2016, May 2017, 10 Marks)**

Like any other transform, the DFT has several important properties. These properties help us in computing the DFT easily.

The properties that we study here are listed below.

1. Linearity
2. Periodicity
3. Circular time shift
4. Circular frequency shift
5. Time reversal
6. Symmetry property
7. Complex Conjugate property
8. Parseval's theorem
9. Multiplication
10. Circular Convolution property

Though the Proof of each of the properties is not a part the syllabus, we shall prove them for the sake of continuity. Students can choose to ignore them.

We will also solve examples using them.

3.8.1 Linearity

MU - Dec. 2015, Dec. 2017, May 2018

> **Q.** State the DFT Properties : Linearity.
> **(Dec. 2015, Dec. 2017, May 2018, 2 Marks)**

$$\text{If} \quad x_1(n) \overset{\text{DFT}}{\longleftrightarrow} X_1(k) \quad \text{and}$$

$$x_2(n) \overset{\text{DFT}}{\longleftrightarrow} X_2(k) \quad \text{then}$$

$$a\,x_1(n) + bx_2(n) \overset{\text{DFT}}{\longleftrightarrow} a\,X_1(k) + bX_2(k)$$

Proof : According to the definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}}$$

$$\text{Let} \quad x(n) = ax_1(n) + bx_2(n)$$

$$\therefore \quad X(k) = \sum_{n=0}^{N-1} [ax_1(n) + bx_2(n)]\, e^{\frac{-j2\pi nk}{N}}$$

$$= \sum_{n=0}^{N-1} a\,x_1(n)\, e^{\frac{-j2\pi nk}{N}} + \sum_{n=0}^{N-1} bx_2(n)\, e^{\frac{-j2\pi nk}{N}}$$

$$= a \sum_{n=0}^{N-1} x_1(n)\, e^{\frac{-j2\pi nk}{N}} + b \sum_{n=0}^{N-1} x_2(n)\, e^{\frac{-j2\pi nk}{N}}$$

$$= a\,X_1(k) + b\,X_2(k)$$

Hence

$$\text{If} \quad x_1(n) \overset{\text{DFT}}{\longleftrightarrow} X_1(k)$$

$$\text{and} \quad x_2(n) \overset{\text{DFT}}{\longleftrightarrow} X_2(k), \quad \text{then}$$

$$ax_1(n) + bx_2(n) \overset{\text{DFT}}{\longleftrightarrow} aX_1(k) + bX_2(k)$$

Solved Example

Ex. 3.8.1 : $x_1(n) = \{1, 2, 3, 4\}$ and $x_2(n) = \{5, 6, 7, 8\}$

Compute the DFT of the sequence $x_3(n) = 2x_1(n) + 3x_2(n)$

Soln. : From the Linearity property we know

$$\text{If} \quad x_1(n) \overset{\text{DFT}}{\longleftrightarrow} X_1(k)$$

$$x_2(n) \overset{\text{DFT}}{\longleftrightarrow} X_2(k)$$

$$\text{then,} \quad a\,x_1(n) + bx_2(n) \overset{\text{DFT}}{\longleftrightarrow} aX_1(k) + bX_2(k)$$

$$\text{Now,} \quad x_3(n) = 2x_1(n) + 3x_2(n)$$

$$\therefore \quad \text{DFT}\{x_3(n)\} = \text{DFT}\{a\,x_1(n)\} + \text{DFT}\{b\,x_2(n)\}$$

is $x_3(n) \xleftrightarrow{DFT} aX_1(k) + bX_2(k)$

We calculate DFT of $x_1(n)$ and $x_2(n)$

$$X_1(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\therefore \quad X_1(k) = \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

Similarly $$X_2(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$$

$$\therefore \quad X_2(k) = \begin{bmatrix} 26 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$$\therefore \quad \text{DFT}\{x_3(n)\} = X_3(k) = 2X_1(k) + 3X_2(k)$$

$$= 2\begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix} + 3\begin{bmatrix} 26 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$$= \begin{bmatrix} 20 \\ -4+4j \\ -4 \\ -4-4j \end{bmatrix} + \begin{bmatrix} 78 \\ -6+6j \\ -6 \\ -6-6j \end{bmatrix}$$

$$\therefore \quad X_3(k) = \begin{bmatrix} 98 \\ -10+10j \\ -10 \\ -10-10j \end{bmatrix}$$

$$\therefore \quad X_3(k) = \{98, -10+10j, -10, -10-10j\}$$

3.8.2 Periodicity

MU - Dec. 2015, Dec. 2017, May 2018

Q. State the DFT Properties : Periodicity.
(Dec. 2015, Dec. 2017, May 2018, 2 Marks)

If $$x(n) \xleftrightarrow{DFT} X(k)$$

then $$X(k+N) = X(k)$$

i.e., The DFT is periodic with a period N.

Proof : According to the definition of the DFT

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}}$$

We replace k by (k + N)

$$\therefore \quad X(k+N) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi n(k+N)}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}} \cdot e^{\frac{-j2\pi nN}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}} \cdot e^{-j2\pi n}$$

Now, $e^{-j2\pi n} = \cos 2\pi n - j\sin 2\pi n = 1$

$$\therefore \quad X(k+N) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}}$$

$$= X(k)$$

$$\therefore \quad X(k+N) = X(k)$$

Hence the DFT is periodic with period N.

Circular shift of a sequence :

Before we move to the next properties, let us understand how to visualize a periodic signal.

Consider a sequence x(n),

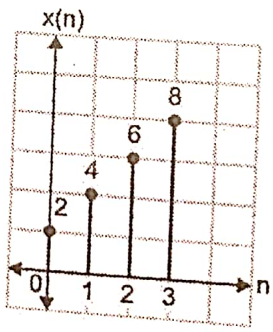

Fig. 3.8.1(a)

A periodic extension of a sequence x(n) can be written as,

$$x_p(n) = \sum_{l=-\infty}^{+\infty} x(n - lN); \quad \text{where N is the period.}$$

This is shown in Fig. 3.8.1 (b).

A simple way of visualizing a periodic signal is to imagine x(n) on a circle in the counter clockwise direction. The Fig. 3.8.1(b) will make this clear.

$x_p(n) = X((n))_N$ is the periodic extension of x(n).

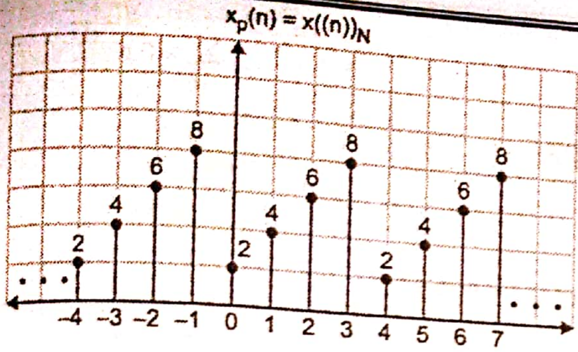

Fig. 3.8.1(b)

As stated earlier, the periodic signal x_p (n) can be viewed as lying on a circle in the counter-clockwise direction.

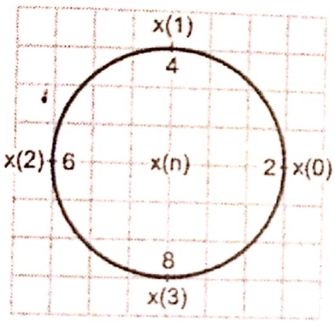

Fig. 3.8.1(c)

As we move around the circle in the counter-clockwise direction we get,

x(0), x(1), x(2), x(3), x(0), x(1), x(2), x(3), ...

$\therefore \quad x_p(n) = x[(n \text{ modulo } N)] = x_p((n))_N$

If we now shift the periodic signal by say 2.

We get,

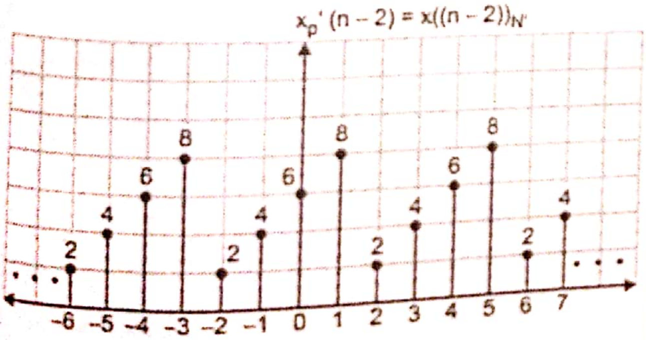

Fig. 3.8.1(d)

This is nothing but a circular shift of the circle as shown in Fig. 3.8.1(e)

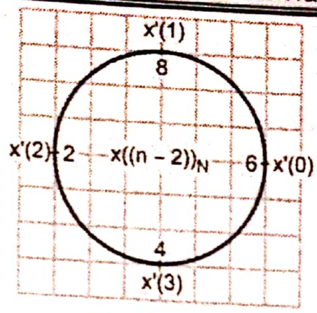

Fig. 3.8.1(e)

Here $((n))_N$ denotes n modulo N.

$\therefore$ $x((n-2))_N$ denotes x(2 modulo N)

A general way of representing a periodic signal is,

$\therefore \quad x_p(n) = x[(n \text{ modulo } N)]$ or $x((n))_N$.

If n is between 0 and N − 1, then leave it as it is. If not, then add or subtract multiples of N from n until the result is between 0 and N − 1.

Example : $x(-3 \text{ Modulo } 4) = x((-3))_4 = x(1)$

$x(11 \text{ Modulo } 8) = x((11))_8 = x(2)$

The shifted version of $x_p(n)$ shown in Fig. 3.8.1(d) can be written as,

$$x'_p(n) = x_p(n-k) = \sum_{l=-\infty}^{+\infty} x(n - lN - k)$$

In general if

$$x(n) = \{x(0),\ x(1),\ x(2),\ \ldots\ldots,\ x(N-1)\}$$

then $x((n-1))_N = \{x(N-1),\ x(0),\ x(1),\ \ldots,\ x(N-2)\}$

$$x((n-2))_N = \{x(N-2),\ x(N-1),\ x(0),\ x(1),\ \ldots,\ x(N-3)\}$$

$$\vdots \qquad \vdots$$

$$x((n-N))_N = \{x(0),\ x(1),\ \ldots\ldots,\ x(N-1)\}$$

$\therefore \quad x((n-N))_N = x(n)$

In a similar way,

$$x((n-m))_N = x(N-m+n)$$

We now prove the circular shift property of the DFT.

3.8.3 Circular Time Shift

MU - Dec. 2015, Dec. 2017, May 2018

Q. State the DFT Properties : Time Shift.
(Dec. 2015, Dec. 2017, May 2018, 2 Marks)

If $$x(n) \xleftrightarrow{DFT} X(k)$$

then $$x((n-m))N \xleftrightarrow{DFT} ee^{\frac{-j2\pi mk}{N}} \cdot X(k)$$

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}}$$

$$\therefore DFT\{x((n-m))_N\} = \sum_{n=0}^{m-1} x((n-m))_N\, e^{\frac{-j2\pi nk}{N}} + \sum_{n=m}^{N-1} x((n-m))_N\, e^{\frac{-j2\pi nk}{N}} \quad \ldots(3.8.1)$$

We see that the above equation is made up of two parts. We work on the first part.

Since, $x((n-m))_N = x(N-m+n)$, we have

$$\sum_{n=0}^{m-1} x((n-m))_N\, e^{\frac{-j2\pi nk}{N}} = \sum_{n=0}^{m-1} x(N-m+n)\, e^{\frac{-j2\pi nk}{N}}$$

Let $N-m+n = l$

$$n = l-N+m$$

$$\therefore \sum x((n-m))_N\, e^{\frac{-j2\pi nk}{N}} = \sum_{l=N-m}^{N-1} x(l)\, e^{\frac{-j2\pi k(-N+m+l)}{N}}$$

$$= \sum_{l=N-m}^{N-1} x(l)\, e^{\frac{-j2\pi k(l+m)}{N}} \cdot e^{+j2\pi k}$$

Since $e^{+j2\pi k} = 1$ for $k = 0, 1, 2, \ldots$ we have,

$$\sum x((n-m))_N\, e^{\frac{-j2\pi nk}{N}} = \sum_{l=N-m}^{N-1} x(l)\, e^{\frac{-j2\pi k(l+m)}{N}} \quad \ldots(3.8.2)$$

Similarly for the second part of Equation (3.8.1) we have,

$$\sum_{n=m}^{N-1} x((n-m))_N\, e^{\frac{-j2\pi nk}{N}} = \sum_{l=0}^{N-1-m} x(l)\, e^{\frac{-j2\pi k(m+l)}{N}} \quad \ldots(3.8.3)$$

Substituting Equations (3.8.2) and (3.8.3) in Equation (3.8.1) we get,

$$DFT\{x((n-m))_N\} = \sum_{l=N-m}^{N-1} x(l)\, e^{\frac{-j2\pi k(l+m)}{N}} + \sum_{l=0}^{N-m-1} x(l)\, e^{\frac{-j2\pi k(m+l)}{N}}$$

$$= e^{\frac{-j2\pi mk}{N}} \sum_{l=0}^{N-1} x(l)\, e^{\frac{-j2\pi k}{N}}$$

$$DFT\{x((n-m))_N\} = e^{-j2\pi km/N} \cdot X(k)$$

$$\therefore \quad x((n-m))_N \xleftrightarrow{DFT} e^{\frac{-j2\pi mk}{N}} \cdot X(k)$$

Solved Examples

Ex. 3.8.2 : (i) Given $x(n) = \{1, 2, 3, 4\}$

Find the DFT of x(n).

(ii) Using results obtained in (i) and not otherwise, obtain the DFT of the following sequences.

$x_1(n) = \{4, 1, 2, 3\}$

$x_2(n) = \{3, 4, 1, 2\}$

Soln. :

(i) We first find the DFT of x(n) using the matrix notation

$$X = W_N^{nk} \cdot x$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\therefore \quad X(k) = \{10, -2+2j, -2, -2-2j\}$$

(ii) We now assume that x(n) is periodic, we plot x(n)

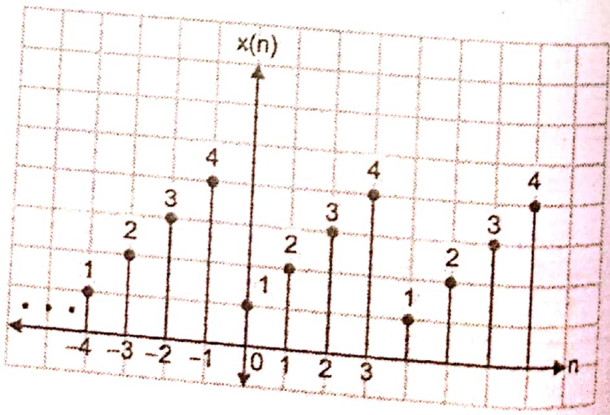

Fig. P. 3.8.2

Now $x_1(n)$ is x(n) shifted to the right by one.

Note, $x_1(n)$ can also be x(n) shifted to the left by 3.

i.e. $x_1(n) = x(n-1)$ OR $x_1(n) = x(n+3)$

Let's take $x_1(n) = x(n-1)$

(Both will give same answer)

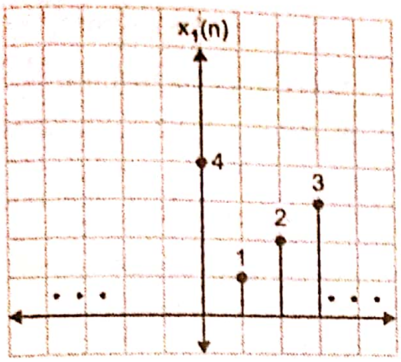

Fig. P. 3.8.2(a)

We know

If $\quad x(n) \xleftrightarrow{DFT} X(k)$

$$x(n-p) \xleftrightarrow{DFT} e^{\frac{-j2\pi pk}{N}} \cdot X(k)$$

i.e., $\quad x(n-p) \xleftrightarrow{DFT} W_N^{pk}\ X(k)$

$\therefore\ x(n-1) \xleftrightarrow{DFT} W_N^{k} \cdot X(k)$

$\therefore\ x_1(n) = x(n-1) \xleftrightarrow{DFT} W_N^{k} \cdot X(k)$

We already know $X(k) = \{10, -2+2j, -2, -2-2j\}$

i.e., $\quad X_1(k) = W_4^k\ X(k)$

$$X_1(0) = W_4^0\ X(0) = (1)(10)$$

$$= 10$$

$$X_1(1) = W_4^1\ X(1) = (-j)(-2+2j) = 2+j2$$

$$X_1(2) = W_4^2\ X(2) = (-1)(-2) = 2$$

$$X_1(3) = W_4^3\ X(3) = (j)(-2-2j) = 2-j2$$

$\therefore\quad$ DFT $\{4, 1, 2, 3\} = \{10, 2+j2, 2, 2-j2\}$

$\therefore\quad X_1(k) = \{10, 2+j2, 2, 2-j2\}$

Similarly for $x_2(n)$, we observe that $x_2(n)$ is simply $x(n)$ shifted to the left by 2 or to the right by 2.

$\therefore\quad x_2(n) = x(n+2) = x(n-2)$

Let us take $x_2(n) = x(n+2)$

(Both will give the same answer)

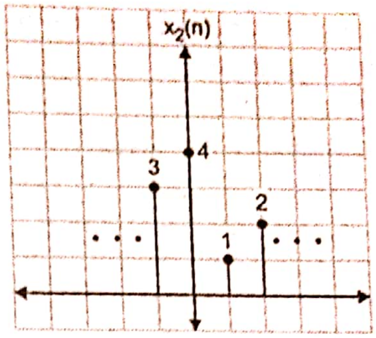

Fig. P. 3.8.2(b)

Since $\quad x(n-p) \xleftrightarrow{DFT} W_N^{pk} \cdot X(k)$

$\therefore\ x(n+2) \xleftrightarrow{DFT} W_4^{2k}\ X(k)$

$$X_2(k) = W_4^{2k}\ X(k)$$

$$X_2(0) = W_4^0\ X(k) = (1)\cdot(10) = 10$$

$$X_2(1) = W_4^2\ X(k) = (-1)(-2+2j) = 2-2j$$

$$X_2(2) = W_4^4\ X(k) = (1)(-2) = -2$$

$$X_2(3) = W_4^6\ X(k) = (-1)(-2-2j) = 2+2j$$

$\therefore\quad$ DFT $\{3, 4, 1, 2\} = \{10, 2-j2, -2, 2+j2\}$

$\therefore\quad X_2(k) = \{10, 2-2j, -2, 2+2j\}$

Ex. 3.8.3 :

(i) $x(n) = \{1, 2, 3, 4\}$ find DFT $X(k)$

(ii) Using results obtained in part (i) and not otherwise find the DFT of following sequences

$x_1(n) = \{4, 1, 2, 3\}$ $\quad x_2(n) = \{2, 3, 4, 1\}$

$x_3(n) = \{3, 4, 1, 2\}$ $\quad x_4(n) = \{4, 6, 4, 6\}$

Soln. :

(i)
$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+3+4 \\ 1-2j-3+4j \\ 1-2+3-4 \\ 1+2j-3-4j \end{bmatrix} = \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$\therefore\quad X(k) = \{10, -2+2j, -2, -2, -2j\}$

(ii) (a) Given $x_1(n) = \{4, 1, 2, 3\}$ and $x(n) = \{1, 2, 3, 4\}$. Thus $x_1(n)$ is obtained by delaying $x(n)$ by one position

$\therefore \quad x_1(n) = x(n-1)$

According to circular time shift property

$$x(n-l) \xleftrightarrow{DFT} X(k)\, e^{\frac{-j2\pi kl}{N}}$$

Here $\quad l = 1$

$$\therefore \quad X_1(k) = X(k)\, e^{\frac{-j2\pi k}{N}}$$

For $k = 0 \Rightarrow \quad X_1(0) = X(0)\, e^0 = 10$

$$k = 1 \Rightarrow \quad X_1(1) = X(1)\, e^{\frac{-j2\pi}{4}}$$

$$= (-2+2j)\left[\cos\frac{2\pi}{4} - j\sin\frac{2\pi}{4}\right]$$

$\therefore \quad X_1(1) = (-2+2j)(-j)$

$\therefore \quad X_1(1) = 2 + j2$

$$\text{For } k = 2 \Rightarrow \quad X_1(2) = X(2)\, e^{\frac{-j4\pi}{4}} = X(2)\, e^{-j\pi}$$

$\therefore \quad X_1(2) = (-2)[\cos\pi - j\sin\pi]$

$= (-2)(-1)$

$\therefore \quad X_1(2) = 2$

$$\text{For } k = 3 \Rightarrow \quad X_1(3) = X(3)\, e^{\frac{-j6\pi}{4}}$$

$$= (-2-2j)\left[\cos\frac{6\pi}{4} - j\sin\frac{6\pi}{4}\right]$$

$\therefore \quad X_1(3) = (-2-2j)(+j)$

$\therefore \quad X_1(3) = 2 - j2$

$\therefore \quad X_1(k) = \{10, 2+j2, 2, 2-j2\}$

(b) Given $x_2(n) = \{2, 3, 4, 1\}$ and $x(n) = \{1, 2, 3, 4\}$. Thus $x_2(n)$ is obtained by advancing $x(n)$ by ± position

$\therefore \quad x_2(n) = x(n+1)$

According to circular time shifting property,

$$x(n-l) \xleftrightarrow{DFT} X(k)\, e^{\frac{-j2\pi kl}{N}}$$

$$x(n+1) \xleftrightarrow{DFT} X(k)\, e^{\frac{-j2\pi k}{N}}$$

$$X_2(k) = X(k)\, e^{\frac{-j2\pi k}{N}}$$

For $k = 0 \Rightarrow \quad X_2(0) = X(0)\, e^{\circ} = 10$

$$\text{For } k = 1 \Rightarrow \quad X_2(1) = X(1)\, e^{\frac{-j2\pi}{4}}$$

$$= (-2+2j)\left[\cos\frac{2\pi}{4} + j\sin\frac{2\pi}{4}\right]$$

$\therefore \quad X_2(1) = (-2+2j)(0+j)$

$\therefore \quad X_2(1) = -2 - j2$

$$\text{For } k = 2 \Rightarrow \quad X_2(2) = X(2)\, e^{\frac{-j4\pi}{4}}$$

$= (-2)[\cos\pi + j\sin\pi]$

$\therefore \quad X_2(2) = (-2)[-1]$

$\therefore \quad X_2(2) = 2$

$$\text{For } k = 3 \Rightarrow \quad X_2(3) = X(3)\, e^{\frac{-j6\pi}{4}}$$

$$= (-2-2j)\left[\cos\frac{6\pi}{4} + j\sin\frac{6\pi}{4}\right]$$

$\therefore \quad X_2(3) = (-2-2j)[-j]$

$\therefore \quad X_2(3) = -2 + 2j$

$\therefore \quad X_2(k) = \{10, -2, -j2, 2, -2+j2\}$

(c) Given $x_3(n) = \{3, 4, 1, 2\}$ and $x(n) = \{1, 2, 3, 4\}$. That means $x_3(n)$ is obtained by delaying $x(n)$ by 2 positions.

$\therefore \quad x_3(n) = x(n-2)$

According to circular time shifting property,

$$x(n-l) \xleftrightarrow{DFT} X(k)\, e^{\frac{-j2\pi kl}{N}}$$

$$x(n-2) \xleftrightarrow{DFT} X(k)\, e^{\frac{-j4\pi k}{4}}$$

$\therefore \quad X_3(k) = X(k)\, e^{-j\pi k}$

For $k = 0 \Rightarrow X_3(0) = X(0)\, e^{\circ} = 10$

For $k = 1 \Rightarrow X_3(1) = X(1)\, e^{-j\pi} = (-2+2j)[\cos - j\sin\pi]$

$\therefore \quad X_3(1) = (-2+2j)(-1)$

$\therefore \quad X_3(1) = 2 - 2j$

For $k = 2 \Rightarrow X_3(2) = X(2)\, e^{-j2\pi}$

$= (-2)[\cos 2\pi - j\sin 2\pi]$

$\therefore \quad X_3(2) = (-2)(1)$

$\therefore \quad X_3(2) = -2$

For $k = 3 \Rightarrow X_3(3) = X(3)\, e^{-j3\pi}$

$= (-2-j2)[\cos 3\pi - j\sin 3\pi]$

$\therefore \quad X_3(3) = (-2-j2)[-1]$

$\therefore \quad X_3(3) = 2 + j2$

$\therefore \quad X_3(k) = \{10, 2-j2, -2, 2+j2\}$

(d) Given $x_4(n) = \{4, 6, 4, 6\}$ and $x(n) = \{1, 2, 3, 4\}$

Thus $x_4(n)$ and $x(n)$ are related as,

$$x_4(n) = x(n) + x(n \mp 2)$$

Using half period shift property,

$$X_4(k) = X(k) + (-1)^k X(k) \quad \ldots [\because \; x(n \mp 2) \leftrightarrow (-1)^k X(k)]$$

For $k = 0 \Rightarrow X_4(0) = X(0) + (-1)^0 X(0) = 10 + 10$

$$X_4(0) = 20$$

For $k = 1 \Rightarrow X_4(1) = X(1) + (-1)^1 X(1) = -2 + 2j + 2 - 2j$

$$\therefore \quad X_4(1) = 0$$

For $k = 2 \Rightarrow X_4(2) = X(2) + (-1)^2 X(2) = -2 + (-2)$

$$\therefore \quad X_4(2) = -4$$

For $k = 3 \Rightarrow X_4(3) = X(3) + (-1)^3 X(3) = -2 - 2j + 2 + 2j$

$$\therefore \quad X_4(3) = 0$$

$$\therefore \quad X_4(k) = \{20, 0, -4, 0\}$$

Ex. 3.8.4 : For a given sequence $x(n) = \{2, 0, 0, 1\}$, perform following operations :

(i) Find out the 4 point DFT of $x(n)$

(ii) Plot $x(n)$, its periodic extension $x_p(n)$ and $x_p(n-3)$

(iii) Find out 4 point DFT of $x_p(n-3)$

(iv) Add phase angle in (i) with factor $-\left[\frac{2\pi rk}{N}\right]$

where $N = 4$, $r = 3$ $k = 0, 1, 2, 3$

(v) Comment to the result you had in point (i) and (ii)

Soln. :

Given : $x(n) = \{2, 0, 0, 1\}$

(i) $X(k) = [W_4]\, x_4$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0+0+1 \\ 2+0+0+j \\ 2+0+0-1 \\ 2+0+0-j \end{bmatrix} = \begin{bmatrix} 3 \\ 2+j \\ 1 \\ 2-j \end{bmatrix}$$

$$\therefore \quad X(k) = \{3, 2+j, 1, 2-j\}$$

(ii) The plot of $x(n)$, $x_p(n)$ and $x_p(n-3)$ are shown in Figs. P. 3.8.4(a), P. 3.8.4(b) and P. 3.8.4(c) respectively.

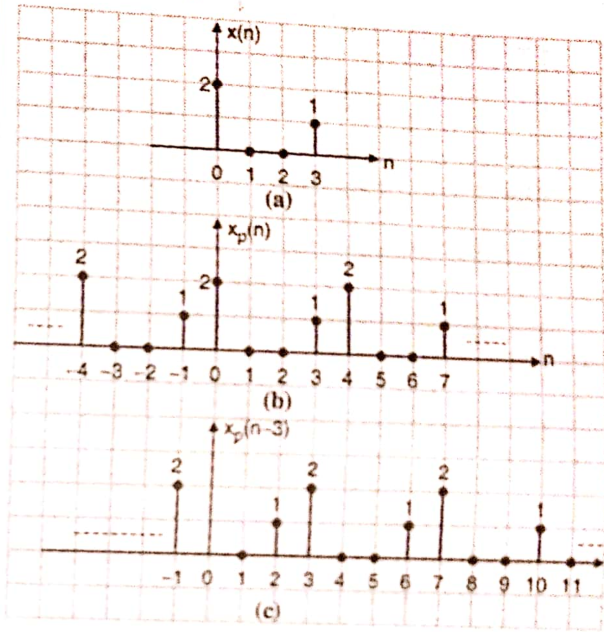

Fig. P. 3.8.4

(iii) In Fig. P. 3.8.4, consider the sequence in the basic range; that means n = 0 to 3.

$$\therefore \quad x'(n) = \{0, 0, 1, 2\}$$

Comparing this to the original sequence,

$x(n) = \{2, 0, 0, 1\}$ we can write,

$$x'(n) = x((n-3))$$

According to circular time shifting property,

$$x(n-l)_N \longleftrightarrow X(k)\, e^{\frac{-j2\pi kl}{N}}$$

Here $l = 3$, $N = 4$

and $X(k) = \{3, 2+j, 1, 2-j\}$

So, DFT of $x'(n)$ is,

$k = 0 \Rightarrow X(0)\, e^{-j0} = 3$

$k = 1 \Rightarrow X(1)\, e^{\frac{-j2\pi\, 1\, 3}{4}} = (2+j) \cdot e^{\frac{-j3\pi}{2}}$

$$= (2+j)(0+j) = -1+j2$$

$$k = 2 \Rightarrow X(2) \cdot e^{\frac{-j2\pi\, 2\, 3}{4}} = 1 \cdot e^{-j3\pi} = -1$$

$$k = 3 \Rightarrow X(3)\, e^{\frac{-j2\pi\, 3\, 3}{4}}$$

$$= (2-j) \cdot e^{\frac{-j9\pi}{2}} = (2-j)(0-j) = -1-j2$$

$$\therefore \quad X'(k) = \{3, -1+j2, -1, -1-j2\}$$

(iv) We have, $X(k) = \{3, 2+j, 1, 2-j\}$

The added phase angle is $e^{\frac{-j2\pi rk}{N}}$; $N = 4$, $r = 3$ and $k = 0, 1, 2, 3$

That means phase angle is $e^{\frac{-j2\pi 3k}{4}} = e^{\frac{-j3\pi k}{2}}$

Due to this added phase angle DFT will be same as DFTOF (ii).

(v) The DFTOF $x_p(n)$ is same as that of $x(n)$; because DFT is periodic in nature.

Ex. 3.8.5 : Consider the finite length sequence x(n) shown in Fig. P. 3.8.5. The five point DFT of x (n) is denoted by X (k). Plot the sequence whole DFT is

$$Y(k) = e^{-\frac{4\pi k}{5}} X(k)$$

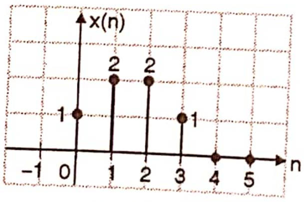

Fig. P. 3.8.5

Soln. : The given sequence contains 5 points. It can be written as,

$$x(n) = \{\underset{\uparrow}{1}, 2, 2, 1, 0\}$$

According to circular time shifting property,

$$x((n-l))_N \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)\, W_N^{kl}$$

We have $W_N = e^{\frac{-j2\pi}{N}}$ Here $N = 5$

$$\therefore \quad x((n-l))_5 \overset{\text{DFT}}{\underset{5}{\longleftrightarrow}} X(k)\, e^{\frac{-j2\pi kl}{5}} \quad \ldots(1)$$

Given : $$Y(k) = e^{\frac{-j4\pi 4}{5}} X(k) \quad \ldots(2)$$

Comparing Equations (1) and (2) we note $l = 2$

$$\therefore \quad y(n) = x(n-2)_5$$

This simply means y(n) is obtained by circularly delaying x(n) by 2 position. We arrange the values of x(n) on a circle in the counter otherwise direction.

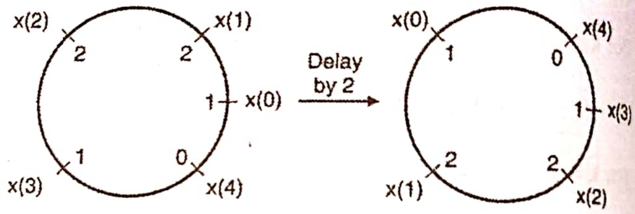

Fig. P. 3.8.5(a)

$$\therefore \quad y(n) = \{1, 0, 1, 2, 2\}$$

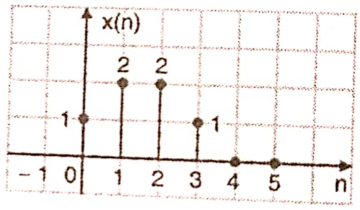

Fig. P. 3.8.5(b)

Ex. 3.8.6 :

$$x[n] = \begin{cases} 1 & 0 \le n \le 3 \\ 0 & 4 \le n \le 7 \end{cases}$$

(i) Find DFT X [k]

(ii) Using the result obtained in (i) find the DFT of the following sequences.

$$x_1[n] = \begin{cases} 1 & n = 0 \\ 0 & 1 \le n \le 4 \\ 1 & 5 \le n \le 7 \end{cases}$$

and $$x_2[n] = \begin{cases} 0 & 0 \le n \le 1 \\ 1 & 2 \le n \le 5 \\ 0 & 6 \le n \le 7 \end{cases}$$

Soln. :

(i) **Given,** $$x(n) = \begin{cases} 1, & 0 \le n \le 3 \\ 0, & 4 \le n \le 7 \end{cases}$$

$$\therefore \quad x(n) = \{1, 1, 1, 1, 0, 0, 0, 0\}$$

The DFT of x(n) is calculated as follows.

$$\therefore X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0.707-j0.707 & -j & -0.707-j0.707 & -1 & -0.707+j0.707 & j & 0.707+j0.707 \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & -0.707-j0.707 & j & 0.707-j0.707 & -1 & 0.707+j0.707 & -j & -0.707+j0.707 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -0.707+j0.707 & -j & 0.707+j0.707 & -1 & 0.707-j0.707 & j & -0.707-j0.707 \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & 0.707+j0.707 & j & -0.707+j0.707 & -1 & -0.707-j0.707 & -j & 0.707-j0.707 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore X(k) = \begin{bmatrix} 1+1+1+1+0+0+0+0 \\ 1+0.707-j0.707-j-0.707-j0.707+0+0+0+0 \\ 1-j-1+j+0+0+0+0 \\ 1-0.707-j0.707+j+0.707-j0.707+0+0+0+0 \\ 1-1+1-1+0+0+0+0 \\ 1-0.707+j0.707-j+0.707+j0.707 \\ 1+j-1-j+0+0+0+0 \\ 1+0.707+j0.707+j-0.707+j0.707+0+0+0+0 \end{bmatrix}$$

$$\therefore X(k) = \{4,\ 1-j2.414,\ 0,\ 1-j0.414,\ 0,\ 1+j0.414,\ 0,\ 1+j2.414\}$$

(ii) Given $x_1(n) = \begin{cases} 1, & n=0 \\ 0, & 1 \le n \le 4 \\ 1, & 5 \le n \le 7 \end{cases}$

$$x_1(n) = \{1, 0, 0, 0, 0, 1, 1, 1\}$$

Comparing with x(n), we can write

$$x_1(n) = x((n-5))$$

According to circular time shifting property,

$$x_1((n-1)) \underset{N}{\overset{DFT}{\leftrightarrow}} X(k) \cdot W_N^{kl}$$

$$\therefore \quad x_1(n) = x((n-5)) \underset{N}{\overset{DFT}{\leftrightarrow}} X(k) \cdot W_N^{k.5}$$

Now $\quad W_N^k = e^{\frac{-j2\pi k}{N}}$

$$\therefore W_N^{5k} = e^{\frac{-j2\pi}{8} \times 5k}$$

$$= e^{-j\frac{10\pi k}{8}} = -0.707 + j0.707$$

$X_1(0) = e^0 \cdot X(0) = 4$

$X_1(1) = e^{-j\frac{10\pi}{8}} X(1)$

$= (-0.707 + j0.707)(1 - j2.414) = 1 + j\,2.41$

$X_1(2) = e^{-j10\frac{\pi}{8} \times 2}\ X(2) = e^{-j\frac{20\pi}{8}}\ X(2) = 0$

$X_1(3) = e^{-j\frac{10\pi}{8} \times 3}\ X(3) = e^{-j\frac{30\pi}{8}}\ X(3)$

$= (0.707 + j0.707)(1 - j0.414) = 1 + j0.414$

$X_1(4) = e^{-j\frac{10\pi}{8} \times 4}\ X(4) = 0$

$X_1(5) = e^{-j\frac{10\pi}{8} \times 5}\ X(5)$

$= (0.707 - j0.707)(1 + j0.414) = 1 - j0.414$

$X_1(6) = e^{-j\frac{10\pi}{8} \times 6}\ X(6) = 0$

$X_1(7) = e^{-j\frac{10\pi}{8} \times 7}\ X(7)$

$= e^{\frac{-j70\pi}{8}}\ (1 + j2.414) = 1 - j\,2.41$

$\therefore \quad X_1(k) = \{4, 2.41 - j, 0, 1 + j0.414, 0,$
$1 - j0.414, 0, 2.41 + j\}$

Given $x_2(n) = \begin{cases} 0, & 0 \le n \le 1 \\ 1, & 2 \le n \le 5 \\ 0, & 6 \le n \le 7 \end{cases}$

$$\therefore x_2(n) = \{0, 0, 1, 1, 1, 1, 0, 0\}$$

$$\text{Thus} \quad x_2(n) = x((n+2))$$

Using circular time shifting property

$$\therefore x_2(n) = x((n+2)) \underset{N}{\overset{DFT}{\leftrightarrow}} X(k) \cdot W_N^{-kl}$$

$$\therefore X_2(k) = X(k) \cdot W_8^{-2k}$$

$$\text{Now } W_8^{2k} = e^{\frac{-j2\pi.2k}{8}} = e^{\frac{-j\pi k}{2}}$$

$$\therefore X_2(k) = X(k) \cdot e^{-\frac{j\pi k}{2}}$$

$$\therefore X_2(0) = x(0) \cdot e^0 = 4$$

$$X_2(1) = x(1) \cdot e^{-\frac{j\pi}{2}} = (1 - j2.414)(0 + j1)$$

$$= -2.414 - j$$

$$X_2(2) = X(2) \cdot e^{j\pi} = 0$$

$$X_2(3) = X(3)\, e^{\frac{j3\pi}{2}} = (1 - j0.414)(0 - j) = 0.414 + j$$

$$X_2(4) = X(4)\, e^{j2\pi} = 0$$

$$X_2(5) = X(5)\, e^{\frac{j5\pi}{2}} = (1 + j0.414)(0 + j) = 0.414 - j$$

$$X_2(6) = X(6)\, e^{j3\pi} = 0$$

$$X_2(7) = X(7)\, e^{\frac{j7\pi}{2}} = (1 + j2.414)(0 - j) = -2.414 - j$$

$$X_2(k) = \{4, -2.414 - j, 0, 0.414 + j, 0, 0.414 - j, 0 - 2.414 + j\}$$

3.8.4 Circular Frequency Shift

$$\text{If} \quad x(n) \overset{DFT}{\longleftrightarrow} X(k)$$

$$\text{then} \quad x(n)\, e^{\frac{+j2\pi mn}{N}} \overset{DFT}{\longleftrightarrow} X((k-m))_N$$

Proof :

$$DFT\{x(n)\} = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}}$$

$$\therefore DFT\{x(n)\, e^{\frac{+j2\pi mn}{N}}\}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{\frac{+j2\pi mn}{N}} \cdot e^{\frac{-j2\pi nk}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi n(k-m)}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi n(N+k-m)}{N}}$$

$$= X(N + k - m)$$

$$= X((k-m))_N$$

$$DFT\{x(n)\, e^{+j2\pi mn/N}\} = X((k-m))_N$$

This is the circular frequency shift property.

Solved Example

Ex. 3.8.7 : Given signal $x(n) = \{2, 2, 1, 4\}$ has a 4-point DFT $X(k)$. Without performing DFT or IDFT, find out the sequence $x_1(n)$ which would have a 4 point DFT $X(k-1)$.

Soln. : This is what the problem states

$$x(n) \overset{DFT}{\longleftrightarrow} X(k)$$

then

$$?? \overset{DFT}{\longleftrightarrow} X(k-1)$$

The output in the frequency domain is shifted. Hence we use the circular frequency shift property.

$$\text{i.e.,} \quad \text{if} \quad x(n) \overset{DFT}{\longleftrightarrow} X(k)$$

$$\text{then} \quad x(n)\, e^{\frac{j2\pi m \cdot n}{N}} \overset{DFT}{\longleftrightarrow} X(k-m)$$

In this case $m = 1$

$$\therefore \quad x(n)\, e^{\frac{j2\pi n}{N}} \overset{DFT}{\longleftrightarrow} X(k-1)$$

$$\therefore \quad x_1(n) = x(n)\, e^{\frac{j2\pi n}{4}}$$

$$x_1(0) = x(0)\, e^{\frac{j2\pi 0}{4}} = x(0) = 2$$

$$x_1(1) = x(1) e^{\frac{j2\pi.1}{4}} = x(1) \cdot (+j) = 2j$$

$$x_1(2) = x(2)\, e^{\frac{j2\pi.2}{4}} = x(2) \cdot (-1) = -1$$

$$x_1(3) = x(3)\, e^{\frac{j2\pi 3}{4}} = x(3) \cdot (-j) = -4j$$

$$\therefore \quad x_1(n) = \{2, 2j, -1, -4j\}$$

Ex. 3.8.8 : For a discrete time sequence $x(n) = (1, 2, 3, 4)$, DFT is given by $X(k) = \{10, -2+2j, -2, -2-2j\}$. Compute the DFT of $\hat{x}(n) = \{3, 4, 1, 2\}$ using circular time shift property of DFT.

Soln. :

Given : $x(n) = \{1, 2, 3, 4\}$ and

$\hat{x}(n) = \{3, 4, 1, 2\}$.

That means $\hat{x}(n)$ is obtained by circularly delaying x (n) by 2 positions. It is as shown in Fig. P. 3.8.8.

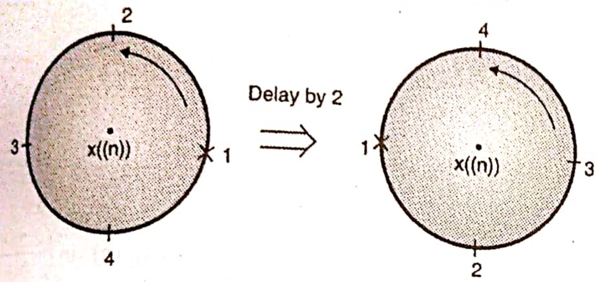

Fig. P. 3.8.8

$$\therefore \hat{x}(n) = x((n-2))$$

According to circular time shifting property,

$$x((n-l))_N \underset{N}{\overset{DFT}{\longleftrightarrow}} X(k)\, W_N^{kl}$$

Here, $N = 4$ and $l = 2$

$$\therefore \hat{x}(n) = x((n-2))_4 \underset{4}{\overset{DFT}{\longleftrightarrow}} X(k)\, W_4^{2k}$$

but, $W_N = e^{\frac{-j2\pi}{N}}$

$$\therefore W_4 = e^{\frac{-j2\pi}{4}} = e^{\frac{-j\pi}{2}}$$

Let, DFT of $\hat{x}(n) = \hat{X}(k)$

$$\therefore \hat{X}(k) = X(k) \cdot W_4^{2k} = X(k) \cdot e^{\frac{-j\pi}{2} \cdot 2k}$$

$$\therefore \hat{X}(k) = X(k)\, e^{-j\pi k}$$

Given, $X(k) = \{10, -2+2j, -2, -2-2j\}$

for $k = 0 \Rightarrow \hat{X}(0) = X(0)\, e^0 = X(0) = 10$

for $k = 1 \Rightarrow \hat{X}(1) = X(1)\, e^{-j\pi}$

$= (-2+2j)(\cos\pi - j\sin\pi)$

$= (-2+2j)(-1) = 2-2j$

for $k = 2 \Rightarrow \hat{X}(2) = X(2)\, e^{-j2\pi}$

$= -2(\cos 2\pi - j\sin 2\pi) = -2$

and for $k = 3 \Rightarrow \hat{X}(3) = X(3)\, e^{-j3\pi}$

$= (-2-2j)(\cos 3\pi - j\sin 3\pi)$

$= (-2-2j)(-1) = 2+2j$

$$\therefore \hat{X}(k) = \{10, 2-2j, -2, 2+2j\}$$

3.8.5 Time Reversal

MU - Dec. 2015, Dec. 2017, May 2018

Q. State the DFT Properties : Time Reversal

(Dec. 2015, Dec. 2017, May 2018, 2 Marks)

If x(n) is periodic,

$$x((-n))_N = x(N-m) \qquad 0 \le n \le N-1$$

If $x(n) \overset{DFT}{\longleftrightarrow} X(k)$

$$x(N-n) \overset{DFT}{\longleftrightarrow} X(N-k)$$

OR

$$x((-n))_N \overset{DFT}{\longleftrightarrow} X(N-k)$$

Proof : $\text{DFT}\{x(m)\} = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}}$

$$\therefore \text{DFT}\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n)\, e^{\frac{-j2\pi nk}{N}}$$

Let, $N - n = m$ $\qquad \therefore n = N - m$

$$\therefore \text{DFT}\{x(N-n)\} = \sum_{m=0}^{N-1} x(m)\, e^{\frac{j\pi k(N-m)}{N}}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{-j2\pi k \cancel{N}}{\cancel{N}}} \cdot e^{\frac{j2\pi km}{N}}$$

but,

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{j\pi km}{N}}$$

We now insert $e^{\frac{-j2\pi mN}{N}}$ since $e^{\frac{-j2\pi mN}{N}}$

$\therefore$ We get,

$$\text{DFT}\{x(N-m)\} = \sum_{m=0}^{N-1} x(m)\, e^{\frac{-j2\pi km}{N}} \cdot e^{\frac{-j2\pi mN}{N}}$$

$$= \sum x(m)\, e^{\frac{-j2\pi m(N-k)}{N}}$$

Comparing this with the standard DFT equation, we have

$$\{x(N-m)\} \overset{DFT}{\longleftrightarrow} X(N-k)$$

OR $\qquad x((-m))_N \overset{DFT}{\longleftrightarrow} X(N-k)$

$$x((-n))_N \overset{DFT}{\longleftrightarrow} X((-k))_N$$

Solved Example

Ex. 3.8.9 : Find the DFT of the given sequence x(n)= {1, 2, 3, 4}. Using the results obtained in (i) and not otherwise find the DFT of the signal $x_1(n) = \{1, 4, 3, 2\}$

Soln. :

(i) We find the DFT using the matrix method

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}}$$

$$X(k) = [W_N] \cdot x \quad ; \quad W_N \text{ is the DFT matrix.}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\therefore \quad X(k) = \{10, -2+2j, -2, -2-2j\}$$

(ii) Here $x_1(n) = \{1, 4, 3, 2\}$

On careful observation, we note that

$$x_1(n) = x(-n) \quad \text{(We assume x (n) to be periodic)}$$

i.e., if $x(n) \overset{DFT}{\longleftrightarrow} X(k)$

then $x(-n) \overset{DFT}{\longleftrightarrow} X(-k)$

We have already shown that

$$X(k) = \{10, -2+2j, -2, -2-2j\}$$

$$\therefore \quad X(-k) = \{10, -2-2j, -2, -2+2j\}$$

$$\therefore \quad \text{DFT}\{x_1(n)\} = \{10, -2-2j, -2, -2+2j\}$$

3.8.6 Symmetry Property

- The symmetry properties of DFT are derived in the similar way of DTFT symmetry properties. We know that DFT of sequence x (n) is denoted by X (k).
- Now if x (n) and X (k) are complex valued sequence then it can be represented as follows :

$$x(n) = x_R(n) + j\,x_I(n),\ 0 \le n \le N-1 \qquad \ldots(3.8.4)$$

and $$X(k) = X_R(k) + j\,X_I(k),\ 0 \le k \le N-1 \qquad \ldots(3.8.5)$$

Here 'R' stands for real part and 'I' stands for imaginary part.

According to the definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \qquad \ldots(3.8.6)$$

Putting Equation (3.8.5) in Equation (3.8.6)

$$X(k) = \sum_{n=0}^{N-1} [x_R(n) + j\,x_I(n)]\, e^{\frac{-j2\pi kn}{N}} \qquad \ldots(3.8.7)$$

But according to Euler's identify,

$$e^{\frac{-j2\pi kn}{N}} = \cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right)$$

Putting this value in Equation (3.8.7) we get,

$$X(k) = \sum_{n=0}^{N-1} [x_R(n) + j\,x_I(n)] \left[\cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right)\right]$$

$$\therefore\ X(k) = \sum_{n=0}^{N-1} \left[x_R(n)\cdot\cos\left(\frac{2\pi kn}{N}\right) - j\,x_R(n)\sin\left(\frac{2\pi kn}{N}\right) + j\,x_I(n)\cdot\cos\left(\frac{2\pi kn}{N}\right) - j^2\,x_I(n)\sin\left(\frac{2\pi kn}{N}\right)\right]$$

Here $j^2 = -1$; and writing summation for real and imaginary parts separately we get,

$$X(k) = \sum_{n=0}^{N-1}\left[x_R(n)\cos\left(\frac{2\pi kn}{N}\right) + x_I(n)\sin\left(\frac{2\pi kn}{N}\right)\right] - j\sum_{n=0}^{N-1}\left[x_R(n)\sin\left(\frac{2\pi kn}{N}\right) - x_I(n)\cos\left(\frac{2\pi kn}{N}\right)\right] \qquad \ldots(3.8.8)$$

Comparing Equations (3.8.8) and (3.8.5) we can write,

$$X_R(k) = \sum_{n=0}^{N-1}\left[x_R(n)\cos\left(\frac{2\pi kn}{N}\right) + x_I(n)\sin\left(\frac{2\pi kn}{n}\right)\right] \qquad \ldots(3.8.9)$$

and

$$X_I(k) = -\sum_{n=0}^{N-1}\left[x_R(n)\sin\left(\frac{2\pi kn}{N}\right) - x_I(n)\cos\left(\frac{2\pi kn}{N}\right)\right] \qquad \ldots(3.8.10)$$

Equations (3.8.9) and (3.8.10) are obtained by using definition of DFT. Similarly we can obtain real and imaginary parts of x (n) using definition of IDFT.

$$X_R(n) = \frac{1}{n}\sum_{k=0}^{N-1}\left[X_R(k)\cos\left(\frac{2\pi kn}{N}\right) - X_I(k)\sin\left(\frac{2\pi kn}{N}\right)\right] \qquad \ldots(3.8.11)$$

and

$$x_I(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k)\sin\left(\frac{2\pi kn}{N}\right) + X_I(k)\cos\left(\frac{2\pi kn}{N}\right)\right] \quad \ldots(3.8.12)$$

Now we will consider different cases as follows :

Case (i) : When x (n) is real valued

Statement : If x (n) is real valued then

$$X(N-k) = X(-k) = X^*(k)$$

Proof : According to the definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \quad \ldots(3.8.13)$$

Replacing k by N – k,

$$X(N-k) = \sum_{n=0}^{N-1} x(n)\, W_N^{(N-k)\,n}$$

$$\therefore\ X(N-k) = \sum_{n=0}^{N-1} x(n)\, W_N^{Nn}\, W_N^{-kn} \quad \ldots(3.8.14)$$

Now we have, twiddle factor $W_N = e^{-\frac{j2\pi}{N}}$

$$\therefore\ W_N^{Nn} = \left(e^{-\frac{j2\pi}{N}}\right)^{Nn}$$

$$= e^{-j2\pi n} = \cos 2\pi n - j\sin 2\pi n$$

Since n is an integer, $\cos 2\pi n = 1$ and $\sin 2\pi n = 0$

$$\therefore\ W_N^{Nn} = 1$$

Thus Equation (3.8.14) becomes,

$$X(N-k) = \sum_{n=0}^{N-1} x(n)\, W_N^{-kn} \quad \ldots(3.8.15)$$

Comparing Equation (3.8.15) with definition of DFT (Equation (3.8.13)) we get,

$$X(N-k) = X(-k) \quad \ldots(3.8.16)$$

Now using Equation (3.8.13) we can write,

$$X^*(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{-kn} \quad \ldots(3.8.17)$$

Thus, from Equations (3.8.16) and (3.8.17) we get,

$$X(N-k) = X(-k) = X^*(k)$$

Case (ii) : When x (n) is real and even

Statement : When x (n) is real and even which means,

$$x(n) = x(N-n) \text{ then DFT becomes,}$$

$$X(k) = X_R(k)$$

Proof : Since imaginary part is zero; putting $x_I(n) = 0$ in Equation (3.8.9) we get,

$$X_R(k) = \sum_{n=0}^{N-1} X_R(n)\cos\left(\frac{2\pi kn}{N}\right)$$

Similarly IDFT can be written by putting $X_I(k) = 0$ in Equation (3.8.11).

$$\therefore\ X_R(n) = \frac{1}{N}\sum_{n=0}^{N-1} X_R(k)\cos\left(\frac{2\pi kn}{N}\right)$$

Case (iii) : When x (n) is real and odd

Statement : When x (n) is real and odd which means,

$$x(n) = -x(N-n) \text{ then the DFT becomes,}$$

$$X(k) = -j\sum_{n=0}^{N-1} x_R(n)\sin\left(\frac{2\pi kn}{N}\right)$$

Proof : Since x (n) is real, we will put $x_I(n) = 0$ in Equation (3.8.8). Similarly x (n) is odd and 'cos' is even function so we can write, $\cos\left(\frac{2\pi kn}{N}\right) = 0$. Thus first summation in Equation (3.8.8) becomes zero. In the second summation of Equation (3.8.8), putting $x_I(n) = 0$ we get,

$$X(k) = -j\sum_{n=0}^{N-1} x_R(n)\sin\left(\frac{2\pi kn}{N}\right)$$

Similarly IDFT can be written as,

$$x(n) = j\frac{1}{N}\sum_{k=0}^{N-1} X_R(k)\sin\left(\frac{2\pi kn}{N}\right)$$

Case (iv) : When x (n) is purely imaginary sequence

When x (n) is purely imaginary which means $x_R(n) = 0$ and $x(n) = j\,x_I(n)$ then putting $x_R(n) = 0$ in Equation (3.8.9) we get,

$$X_R(k) = \sum_{n=0}^{N-1} x_I(n)\sin\left(\frac{2\pi kn}{N}\right)$$

And putting $x_R(n) = 0$ in Equation (3.8.10) we get,

$$X_I(k) = \sum_{n=0}^{N-1} x_I(n)\cos\left(\frac{2\pi kn}{N}\right)$$

Symmetry properties can be summarized as shown in Fig. 3.8.2.

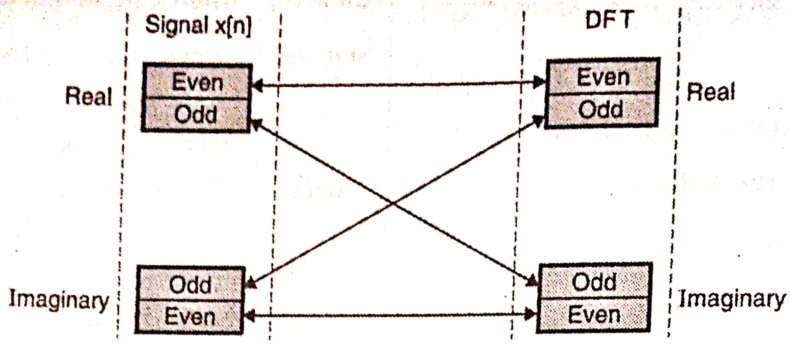

Fig. 3.8.2 : Summary of symmetry property

This summary is shown in Table 3.8.1.

Table 3.8.1

N point sequence $x[n]\ 0 \le n \le N-1$	N point DFT
$x^*[n]$	$X^*[N-k]$
$x^*[N-n]$	$X^*[k]$
$x_R[n]$	$X_{ce}[k] = \frac{1}{2}\left[X[k] + X^*[N-k]\right]$
$j\,x_I[n]$	$X_{ce}[k] = \frac{1}{2}\left[X[k] - X^*[N-k]\right]$
$x_{ce}[n] = \frac{1}{2}[x[n] + x^*[N-n]]$	$X_R[k]$
$x_{co}[n] = \frac{1}{2}\left[x[n] + x^*[N-n]\right]$	$j\,X_I[k]$

Solved Example

Ex. 3.8.10 : The first five points of the 8 point DFT of a real valued sequence are,
{0.25, 0.125 – j0.3018, 0, 0.125 – j 0.0518, 0} Determine the remaining three points.

Soln. : Given DFT points are :

$$X(0) = 0.25$$
$$X(1) = 0.125 - j\,0.3018$$
$$X(2) = 0$$
$$X(3) = 0.125 - j\,0.0518$$
$$X(4) = 0$$

Given sequence is a real valued sequence. According to the symmetry property we have,

$$X^*(k) = X(N-k)$$
$$\text{or } X(k) = X^*(N-k) \quad \ldots(1)$$

This is 8 point DFT. Thus N = 8

$$\therefore\ X(k) = X^*(8-k) \quad \ldots(2)$$

Now we want remaining three samples namely X (5) X (6) and X (7). Putting k = 5 in Equation (2),

$$X(5) = X^*(8-5) = X^*(3)$$

We have $X(3) = 0.125 - j\,0.0518$

$$\therefore\ X^*(3) = 0.125 + j\,0.0518$$
$$\therefore\ X(5) = 0.125 + j\,0.0518$$

Putting k = 6 in Equation (2),

$$X(6) = X^*(8-6) = X^*(2)$$

We have $X(2) = 0$, Thus $X^*(2) = 0$

$$\therefore\ X(6) = 0$$

Similarly putting k = 7 in Equation (2) we get,

$$X(7) = X^*(8-7) = X^*(1)$$

We have $X(1) = 0.125 - j\,0.3018$

$$\therefore\ X(7) = 0.125 + j\,0.3018$$

Ex. 3.8.11 : The first five DFT points of real and even sequence x(n) of length eight are given below. Find remaining three points.

$$X(k) = \{5, 1, 0, 2, 3, \ldots\}$$

Soln. : Given DFT points are :

$$X(0) = 5, X(1) = 1, X(2) = 0,$$
$$X(3) = 2 \text{ and } X(4) = 3.$$

According to symmetry property we have,

$$X^*(k) = X(N-k)$$
$$\therefore X(k) = X^*(N-k)$$

This is 8 point DFT. Thus N = 8

$$\therefore X(k) = X^*(8-k)$$
$$\therefore X(5) = X^*(8-5) = X^*(3)$$
$$\therefore X(5) = 2$$
$$X(6) = X^*(8-6) = X^*(2)$$
$$\therefore X(6) = 0$$
$$X(7) = X^*(8-7) = X^*(1)$$
$$\therefore X(7) = 1$$

Ex. 3.8.12 : Compute DFT of the sequence $x_1(n) = \{1, 2, 4, 2\}$ using property and not otherwise compute DFT of $x_2(n) = \{1+j, 2+2j, 4+4j, 2+2j\}$

Soln. : $x_1(n) = \{1, 2, 4, 2\}$

$$\therefore X_1(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \\ 2 \end{bmatrix}$$

$$\therefore X_1(k) = \begin{bmatrix} 1+2+4+2 \\ 1-2j-4+2j \\ 1-2+4-2 \\ 1+2j-4-2j \end{bmatrix} = \begin{bmatrix} 9 \\ -3 \\ 1 \\ -3 \end{bmatrix}$$

Now $x_2(n) = \{1+j, 2+2j, 4+4j, 2+2j\}$

$$\therefore x_2(n) = x_1(n) + j\, x_1(n)$$
$$\therefore X_2(k) = X_1(k) + j\, X_1(k)$$
$$\therefore X_2(k) = \{9+9j, -3-3j, 1+j, -3-3j\}$$

Ex. 3.8.13 : For a 8-point DFT, the first five DFT coefficients are X(k) = X { 10, 2 + 3j, 1 + 2j, j, 4} Find the remaining coefficients.

Soln. :

Given :

$$X(0) = 10, \quad X(1) = 2+3j, \quad X(2) = 1+2j,$$
$$X(3) = j, \quad X(4) = 4$$

We know, $X(k) = X^*(N-k)$

In this case, N = 8

$$\therefore X(5) = X^*(8-5) = X^*(3) = +j$$
$$X(6) = X^*(8-6) = X^*(2) = 1-2j$$
$$X(7) = X^*(8-7) = X^*(1) = 2-3j$$

Therefore the entire DFT sequence is,

$$X(k) = \{10, 2+3j, 1+2j, j, 4, j, 1-2j, 2-3j\}$$

> **Note :** The magnitude of the DFT is symmetric about the N/2 point. While the phase of the DFT is antisymmetric about the N/2 point.

Ex. 3.8.14 : The first five points of the 8-point DFT of a real valued sequence are,

X(k) = {10, 2.2 – j 0.3, 0, 1.2 – j0.13, 0}

Determine the remaining three points.

Soln. : (We use the symmetry property here)

i.e. $X(k) = X^*(N-k)$

We have,

$$X(0) = 10, \quad X(1) = 2.2 - j0.3, \quad X(2) = 0,$$
$$X(3) = 1.2 - j\,0.13, \quad X(4) = 0$$

Since this is a 8-point DFT, N = 8

$$\therefore X(k) = X^*(8-k)$$
$$\therefore X(5) = X^*(8-5) = X^*(3)$$
$$X(6) = X^*(8-6) = X^*(2)$$
$$X(7) = X^*(8-7) = X^*(1)$$
$$\therefore X(5) = 1.2 + j0.13$$
$$X(6) = 0$$
$$X(7) = 2.2 + j0.3$$

$\therefore$ The final 8-point DFT sequence is,

$$X(k) = \{10, 2.2 - j\,0.3, 0, 1.2 - j0.13, 0, 2.2 + j0.3\}$$

Ex. 3.8.15 : A sequence is given as $x(n) = \{1 + 2j, 1 + 3j, 2 + 4j, 2 + 2j\}$. From basic definition. Find X (k).

If $x_1(n) = \{1, 1, 2, 2\}$

$x_2(n) = \{2, 3, 4, 2\}$

Find $X_1(k)$ and $X_2(k)$ by using DFT only.

Soln. :

Given : $x(n) = \{1 + 2j, 1 + 3j, 2 + 4j, 2 + 2j\}$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1+2j \\ 1+3j \\ 2+4j \\ 2+2j \end{bmatrix}$$

$$= \begin{bmatrix} 1+2j+1+3j+2+4j+2+2j \\ 1+2j-j+3-2-4j+2j-2 \\ 1+2j-1-3j+2+4j-2-2j \\ 1+2j+j-3-2-4j-2j+2 \end{bmatrix}$$

$$= \begin{bmatrix} 6+11j \\ -j \\ +j \\ -2-3j \end{bmatrix}$$

$$\therefore \quad X(k) = \{6++11j, -j, j, -2-3j\}$$

$$\text{Now, } x = x_1(n) + j\,x_2(n)$$

$$x^*(n) = x_1(n) - j\,x_2(n)$$

$$x(n) + x^*(n) = 2\,x_1(n)$$

$$x_1(n) = \frac{x(n) + x^*(n)}{2}$$

$$\therefore \quad X_1(k) = \frac{X(k) + X^*(-k)}{2}$$

$$\text{And} \quad X_2(k) = \frac{X(k) - X^*(-k)}{2j}$$

(a) $$X_1(k) = \frac{X(k) + X^*(-k)}{2}$$

$$X(k) = \{6+11j, -j, +j, -2-3j\}$$

$$X^*(k) = \{6-11j, +j, -j, -2+j3\}$$

$$X^*(-k) = \{6-11j, -2, +j3, -j+j\}$$

$$\text{Hence,} \quad X_1(0) = \frac{X(0) + X^*(0)}{2}$$

$$= \frac{6+11j+6-11j}{2} = 6$$

$$X_1(1) = \frac{X(1) + X^*(-1)}{2}$$

$$= \frac{-j-2+j3}{2} = -1+j$$

$$X_1(2) = \frac{X(2) + X^*(-2)}{2} = \frac{j-j}{2} = 0$$

$$X_1(3) = \frac{X(3) + X^*(-3)}{2}$$

$$= \frac{-2-j3+j}{2} = -1-j$$

$$X_1(k) = \{X_1(0), X_1(1), X_1(2), X_1(3)\}$$

$$= \{6, -1+j, 0, -1-j\}$$

(b) $$X_2(k) = \frac{X(k) - X^*(-k)}{2j}$$

$$X_2(0) = \frac{X(0) - X^*(0)}{2j}$$

$$= \frac{6+11j-6+11j}{2j} = 11$$

$$X_2(1) = \frac{X(1) - X^*(-1)}{2j}$$

$$= \frac{-j+2-j3}{2j} = \frac{2-j4}{2j}$$

$$= \frac{j}{j} \times \frac{2-j4}{2j} = \frac{2j+4}{-2} = -2-j$$

$$X_2(2) = \frac{X(2) - X^*(-2)}{2j} = \frac{j+j}{2j} = 1$$

$$X_2(3) = \frac{X(3) - X^*(-3)}{2j}$$

$$= \frac{-2-j3-j}{2j} = \frac{j}{j}\left(\frac{-2-j4}{2j}\right) = -2+j$$

$$X_2(k) = \{X_2(0), X_2(1), X_2(2), X_3(3)\}$$

$$= \{11, -2-j, 1, -2+j\}$$

3.8.7 Complex Conjugate Property

If $\quad x(n) \xleftrightarrow{\text{DFT}} X(k)$

then $\quad x^*(n) \xleftrightarrow{\text{DFT}} X^*(N-k) = X^*((-k))_N$

Proof : From the DFT equation we have,

$$\text{DFT}\{x(n)\} = \sum_{N=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}}$$

$$\therefore \quad \text{DFT}\{x^*(n)\} = \sum_{n=0}^{N-1} x^*(n)\, e^{\frac{-j2\pi nk}{N}}$$

$$= \sum_{n=0}^{N-1} [x(n)\, e^{j2\pi nk/N}]^*$$

$$= \sum_{n=0}^{N-1} [x(n)\, e^{-j2\pi n(N-k)/N}]^*$$

$$= X^*(N-k)$$

$$\therefore \quad \text{DFT}[x^*(n)] = X^*(N-k) = X^*((-k))_N$$

3.8.8 Parseval's Theorem

If $x(n) \xleftrightarrow{DFT} X(k)$

Then Energy of the signal is,

$$E = \sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

Proof : Energy of a signal is given by the equation,

$$E = \sum_{n=0}^{N-1} |x(n)|^2 \quad \ldots(3.8.18)$$

$$E = \sum_{n=0}^{N-1} x(n)\, X^*(n) \quad \ldots(3.8.19)$$

From the IDFT equation we have,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{+nK}$$

$$\therefore \quad x^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X^*(k)\, W_N^{+nK} \quad \ldots(3.8.20)$$

We substitute Equation (3.8.20) in Equation (3.8.19)

$$\therefore \quad E = \sum_{k=0}^{N-1} x(n) \left[\frac{1}{N} \sum_{n=0}^{W-1} X^*(k)\, W_N^{+nk} \right]$$

Rearranging the terms, we have,

$$E = \frac{1}{N} \sum_{k=0}^{N-1} X^*(k) \cdot \left[\sum_{k=0}^{N-1} x(n)\, W_N^{nk} \right]$$

$$\therefore \quad E = \frac{1}{N} \sum_{k=0}^{N-1} X^*(k) \cdot X(k)$$

$$E = \frac{1}{N} \Sigma |X(k)|^2 \quad \ldots(3.8.21)$$

$\therefore$ We have,

$$E = \sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

Solved Example

Ex. 3.8.16 : $X(k) = \{10, -2, 0, 2\}$.

Compute the energy of the signal x(n).

Soln. : From Parseval's Energy theorem, we have,

$$E = \sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

$$\therefore \quad E = \frac{1}{4} \{ |X(0)|^2 + |X(1)|^2 + |X(2)|^2 + |X(3)|^2 \}$$

$$= \frac{1}{4} \{ 100 + 4 + 0 + 4 \}$$

$$E = 27$$

Ex. 3.8.17 : Verify Parseval's Theorem for sequence,

$$x(n) = \left(\frac{1}{2}\right)^n u(n)$$

Soln. : According to Parseval's Theorem,

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2 \quad \ldots(1)$$

Step I : Need to prove this equality.

Consider value of N = 4. That means n = 0 to N – 1

Let, $\therefore$ n = 0 to 3

Thus sequence x (n) is generated as follows,

For $n = 0 \Rightarrow \quad x(0) = \left(\frac{1}{2}\right)^0 = 1$

$n = 1 \Rightarrow \quad x(1) = \left(\frac{1}{2}\right)^1 = \frac{1}{2}$

$n = 2 \Rightarrow \quad x(2) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$n = 3 \Rightarrow \quad x(3) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

$$\therefore \quad x(n) = \left\{ \underset{\uparrow}{1}, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \right\}$$

Step II : Consider L.H.S. of Equation (1).

$$\text{L.H.S} = \sum_{n=0}^{N-1} |x(n)|^2 = \sum_{n=0}^{3} |x(n)|^2$$

$$\text{L.H.S} = |x(0)|^2 + |x(1)|^2 + |x(2)|^2 + |x(3)|^2$$

$$\therefore \quad \text{L.H.S} = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{8}\right)^2$$

$$\therefore \quad \text{L.H.S} = 1.328125 \quad \ldots(2)$$

Step III : Consider R.H.S term,

$$\text{R.H.S} = \frac{1}{N} \sum_{n=0}^{N-1} |X(k)|^2$$

First we will calculate DFT X (k) using matrix method.

$$X(k) = [W_4]\, X_N$$

$$\therefore \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{8} \end{bmatrix}$$

$$\therefore \; X(k) = \{1.875, \; 0.75 - j0.375, \; 0.625, \; 0.75 + j0.375\}$$

Now are will calculate $|X(k)|$.

$$\therefore \; |X(k)| = \{\underset{\uparrow}{1.875}, 0.838525, 0.625, 0.838525\}$$

Step IV :

$$\text{R.H.S} = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

$$\therefore \; \text{R.H.S} = \frac{1}{4} \sum_{k=0}^{3} |X(k)|^2$$

$$\therefore \; \text{R.H.S} = \frac{1}{4} \{|X(0)|^2 + |X(1)|^2 + |X(2)|^2 + |X(3)|^2\}$$

$$\therefore \; \text{R.H.S} = 1.328125$$

Since L.H.S= R.H.S ; Parseval's theorem is verified.

Ex. 3.8.18 : State and prove Parseval's theorem for the following sequence $x(n) = \{1, 2, 3, 4\}$.

Soln. :

Given :

$$x(n) = \{1, 2, 3, 4\}$$

According to parseval's theorem,

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

Step I :

Here $N = 4$

$$\text{L.H.S.} = \sum_{n=0}^{3} |x(n)|^2$$

$$= |x(0)|^2 + |x(1)|^2 + |x(2)|^2 + |x(3)|^2$$

$$= 1 + (2)^2 + (3)^2 + (4)^2$$

$$= 1 + 4 + 9 + 16 = 30$$

Step II : First we will calculate DFT of x (n) that is X (k)

$$X(k) = [W_4] \, x_N$$

$$\therefore \; X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$$\therefore \; X(k) = \{10, -2+2j, -2, -2-2j\}$$

Step III :

$$\text{R.H.S.} = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

Now $|X(k)| = \{\underset{\uparrow}{10}, 2.828, 2, 2.828\}$

$$\therefore \; \text{R.H.S.} = \frac{1}{4} \sum_{k=0}^{3} |X(k)|^2$$

$$= \frac{1}{4} [|X(0)|^2 + |X(1)|^2 + |X(2)|^2 + |X(3)|^2]$$

$$= \frac{1}{4} [100 + 7.997584 + 4 + 7.997584]$$

R.H.S. = 30

Step IV : Since L.H.S. = R.H.S, Parseval's theorem is verified.

3.8.9 Multiplication of Two Sequences

If $x_1(n) \xleftrightarrow{\text{DFT}} X_1(k)$

and $x_2(n) \xleftrightarrow{\text{DFT}} X_2(k)$

then $x_1(n) \cdot x_2(n) \xleftrightarrow{\text{DFT}} \frac{1}{N} [X_1(k) \circledast X_2(k)]$

We shall explain each one in detail.

3.9.1 Concentric Circle Method

Given two periodic sequences $x_1(n)$ and $x_2(n)$

$$y(n) = x_1(n) \circledast x_2(n)$$

Step 1 : Plot the samples of $x_1(n)$ evenly around the outer circle in the *counter clockwise* direction.

Step 2 : Plot the samples of $x_2(n)$ evenly around the inner circle in the *clockwise*. Make sure the positions of $x_1(0)$ and $x_2(0)$ coincide.

Step 3 : Multiply all the corresponding samples of $x_1(n)$ and $x_2(n)$ and add. This will give us $y_1(0)$.

Step 4 : Rotate the inner circle in the direction of the outer circle (*counter clockwise*) by one step and repeat step 3.

Step 5 : Repeat step 4 till $x_1(0)$ and $x_2(0)$ again coincide.

We shall take an example to understand these steps.

3.9.1(A) Solved Example on Concentric Circle Method

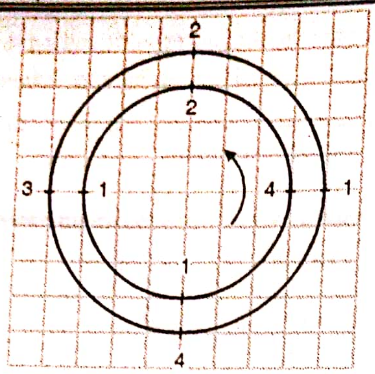

Fig. P. 3.9.1(a)

We multiply the corresponding samples and add. This gives us y(0).

$\therefore \quad y(0) = 4 \times 1 + 2 \times 2 + 1 \times 3 + 1 \times 4 = 15$

We now rotate the inner circle in the anticlock-wise direction by one step. This gives us,

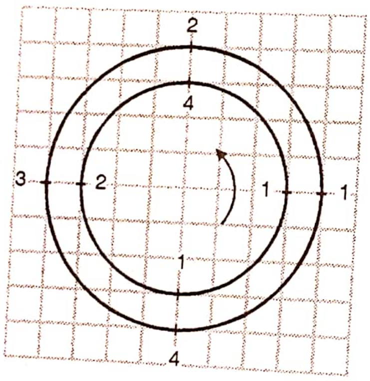

Fig. P.3.9.1(b)

$\therefore \quad y(1) = 1 \times 1 + 4 \times 2 + 2 \times 3 + 1 \times 4 = 19$

We continue rotating the inner circle in steps of one.

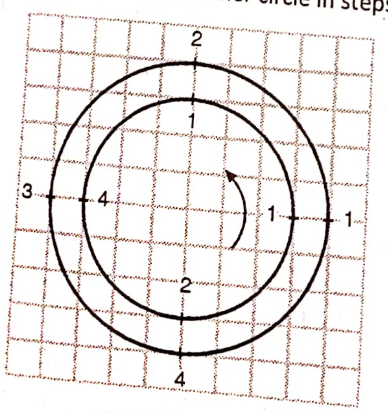

Fig. P.3.9.1(c)

$y(2) = 1 \times 1 + 1 \times 2 + 4 \times 3 + 2 \times 4 = 23$

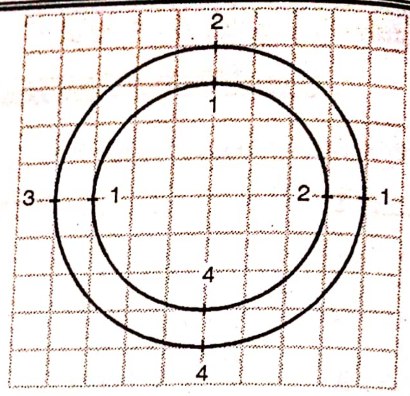

Fig. P. 3.9.1(d)

$y(3) = 2 \times 1 + 1 \times 2 + 1 \times 3 + 4 \times 4 = 23$

If we now rotate the inner circle, we come back to the starting point. Hence we stop here.

$\therefore \quad y(n) = \{15, 19, 23, 23\}$

The second method of performing circular convolution is using the Matrix method.

As is evident circular convolution can be performed only when $x_1(n)$ and $x_2(n)$ are of the same length.

If they are not of the same length, then we need to zero pad the shorter sequence and make it equal to the longer sequence.

Ex. 3.9.2 : Find the circular convolution of two sequences, $x_1(n) = \{1, -1, 2, -4\}$ and $x_2(n) = \{1, 2\}$

Soln. : Since $x_2(n)$ is shorter than $x_1(n)$, we zero pad it so that their lengths become equal

$\therefore \quad x_2(n) = \{1, 2, 0, 0\}$

We now perform circular convolution.

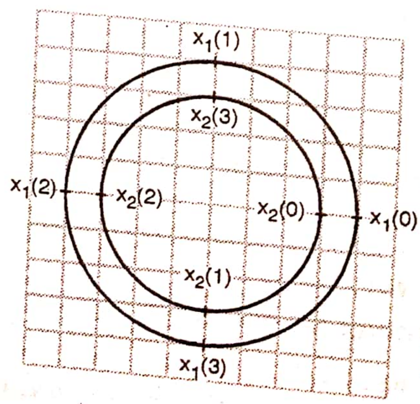

Fig. P. 3.9.2

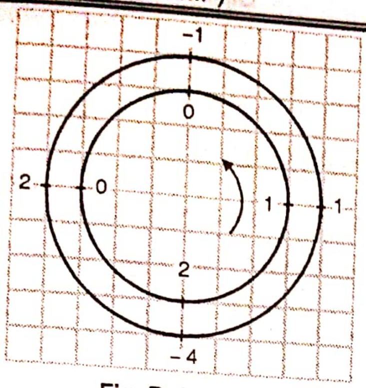

Fig. P. 3.9.2(a)

$$y(0) = 1\times 1+0\times -1+0\times -2+2\times -4=-7$$

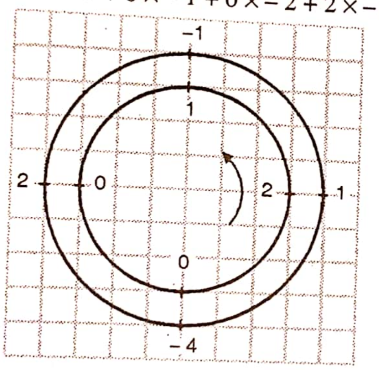

Fig. P. 3.9.2(b)

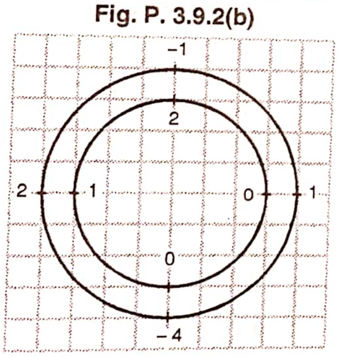

Fig. P. 3.9.3(c)

$$y(1) = 2\times 1+1\times -1+0\times 2+0\times 4=1$$

$$y(2) = 0\times 1+2\times -1+1\times 2+0\times -4=0$$

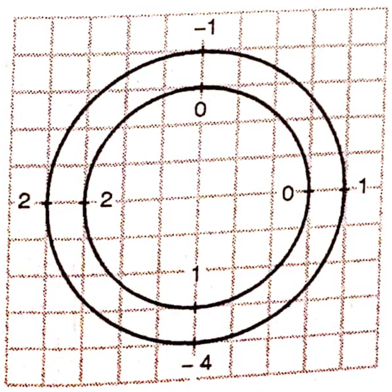

Fig. P. 3.9.2(d)

$$y(3) = 0\times 1+0\times -1+2\times 2+1\times -4$$

$$= 0$$

$$\therefore \quad y(n) = \{-7, 1, 0, 0\}$$

Ex. 3.9.3 : Determine

$$y(n) = x(n) \text{ (N) } h(n)$$

Where

$$x(n) = \{1, 2, 3, 1\};\ h(n) = \{4, 3, 2, 2\}$$

Soln. :

We generate a circular matrix of h(n)

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 & 3 \\ 3 & 4 & 2 & 2 \\ 2 & 3 & 4 & 2 \\ 2 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 17 \\ 19 \\ 22 \\ 19 \end{bmatrix}$$

$$\therefore \quad y(n) = \{17, 19, 22, 19\}$$

Ex. 3.9.4 : For the following sequences

$$x_1(n) = \begin{cases} 1 & 0 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$x_2(n) = \begin{cases} 1 & 0 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Compute linear convolution using circular convolution.

Soln. :

Given : $x_1(n) = \begin{cases} 1 & 0 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$

and $x_2(n) = \begin{cases} 1 & 0 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$

$$\therefore \quad x_1(n) = \{1, 1, 1\} \text{ and } x_2(n) = \{1, 1, 1\}$$

Here, $L_1 = L_2 = 3$

To result of linear convolution would have a length $L_1 + L_2 - 1$

$$\therefore \quad 3+3-1 = 5$$

We now append zeros to $x_1(n)$ and $x_2(n)$ to make their lengths equal to 5.

$$\therefore \quad x_1(n) = \{1,1,1,0,0\} \text{ and}$$

$$x_2(n) = \{1,1,1,0,0\}$$

The circular convolution is performed as follows.

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\therefore \quad y(n) = \{1, 2, 3, 2, 1\}$$

This is the result that we would have got had we used circular convolution.

Let us verify the result by performing linear convolution. We use the tabular method.

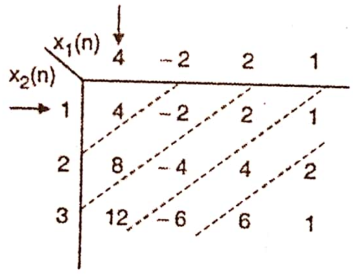

Fig. P. 3.9.4

$$\therefore \quad y(n) = \{1, 2, 3, 2, 1\}$$

Hence we get the same result.

Ex. 3.9.5 : Using the DFT method, obtain the circular convolution of the following :

$x_1(n) = [1 \ 2 \ 1 \ -2]$

$x_2(n) = [3 \ -2 \ 1 \ -3]$

Verify your result using the graphical method.

Soln. : From the DFT property we know,

$$x_1(n) \circledast x_2(n) \xleftrightarrow{\text{DFT}} X_1(k) \cdot X_2(k)$$

We begin with computing the DFT of $x_1(n)$ and $x_2(n)$

Step I :

Given :

$$x_1(n) = \{1, 2, 1, -2\}$$

Now, $X_1(k) = [W_4] X_{1N}$

$$\therefore X_1(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+1-2 \\ 1-2j-1-2j \\ 1-2+1+2 \\ 1+2j-1+2j \end{bmatrix} = \begin{bmatrix} 2 \\ -4j \\ 2 \\ 4j \end{bmatrix}$$

Similarly,

$$X_2(k) = [W_4] X_{2N}$$

$$\therefore X_2(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3-2+1-3 \\ 3+2j-1-3j \\ 3+2+1+3 \\ 3-2j-1+3j \end{bmatrix} = \begin{bmatrix} -1 \\ 2-j \\ 9 \\ 2+j \end{bmatrix}$$

Step II :

Let $Y(k) = X_1(k) \cdot X_2(k)$

$$= \{2, -4j, 2, 4j\} \cdot \{-1, 2-j, 9, 2+j\}$$

$$\therefore Y(k) = \{-2, -4-8j, 18, -4+8j\}$$

Step III :

The result of circular convolution is obtained by performing IDFT of Y (k) as follows,

$$y(n) = \frac{1}{N}[W_4^*]\, Y(k)$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} -2 \\ -4-8j \\ 18 \\ -4+8j \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -2-4-8j+18-4+8j \\ -2-4j+8-18+4j+8 \\ -2+4+8j+18+4-8j \\ -2+4j-8-18-8-4j \end{bmatrix} = \frac{1}{4}\begin{bmatrix}8\\-4\\24\\-36\end{bmatrix} = \begin{bmatrix}2\\-1\\6\\-9\end{bmatrix}$$

$$\therefore \; y(n) = \{2, -1, 6, -9\}$$

This is the final Result we verify the result by performing direct circular convolution. We generate a circular matrix of $x_1(n)$

$$\begin{bmatrix} y(0)\\ y(1)\\ y(2)\\ y(3)\end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 & 2\\ 2 & 1 & -2 & 1\\ 1 & 2 & 1 & -2\\ -2 & 1 & 2 & 1\end{bmatrix}\begin{bmatrix}3\\-2\\1\\-3\end{bmatrix}$$

$$\begin{bmatrix} y(0)\\ y(1)\\ y(2)\\ y(3)\end{bmatrix} = \begin{bmatrix}2\\-1\\6\\-9\end{bmatrix}$$

$$\therefore \; y(n) = 2, -1, 6, -9\}$$

Which is the same as obtained using the DFT.

3.9.2 Matrix Method

In this method, we generate a circular matrix of $x_2(n)$ and multiply it by $x_1(n)$.

$$\begin{bmatrix} y(0)\\ y(1)\\ y(2)\\ \vdots\\ y(N-1)\end{bmatrix} = \begin{bmatrix} x_2(0) & x_2(N-1) & x_2(N-2) \cdots\cdots & x_2(1)\\ x_2(1) & x_2(0) & x_2(N-1) \cdots\cdots & x_2(2)\\ x_2(2) & x_2(1) & x_2(0) \quad \cdots\cdots & x_2(3)\\ \vdots & \vdots & x_2(1) \quad \cdots\cdots & \vdots\\ x_2(N-2) & \vdots & \vdots & \vdots\\ x_2(N-1) & x_2(N-2) & & x_2(0)\end{bmatrix}\begin{bmatrix} x_1(0)\\ x_1(1)\\ x_1(2)\\ \vdots\\ x_1(N-1)\end{bmatrix}$$

Note : Since the next column of the circular matrix will be the same as the first column we stop. The x_2 matrix is called a circular matrix.

3.9.2(A) Solved Example on Matrix Method

Ex. 3.9.6 : Perform circular convolution on the given two sequence $x_1(n) = \{1, 2, 3, 4\}$, $x_2(n) = \{4, 1, 1, 2\}$.

Soln. :

Soln. : We generate a circular matrix of $x_2(n)$ and multiply by $x_1(n)$.

$$y(n) = x_1(n) \circledast x_2(n)$$

$$\begin{bmatrix} y(0)\\ y(1)\\ y(2)\\ y(3)\end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 & 1\\ 1 & 4 & 2 & 1\\ 1 & 1 & 4 & 2\\ 2 & 1 & 1 & 4\end{bmatrix}\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$$

$$\therefore \text{ We get } \begin{bmatrix} y(0)\\ y(1)\\ y(2)\\ y(3)\end{bmatrix} = \begin{bmatrix}15\\19\\19\\23\end{bmatrix}$$

$$\text{i.e., } y(n) = \{15, 19, 19, 23\}$$

This is the same result as obtained using the concentric circle method.

Ex. 3.9.7 : Given the two sequences of length 4 are,

$x(n) = \{0,1,2,3\}$, $h(n) = \{2,1,1,2\}$

Find circular convolution

Soln. : The lengths of x (n) and h (n) are equal. We plot x (n) on the outer circle in counter clock wise direction, and h(n) on the inner circle in the clock wise direction. We multiply and add corresponding elements.

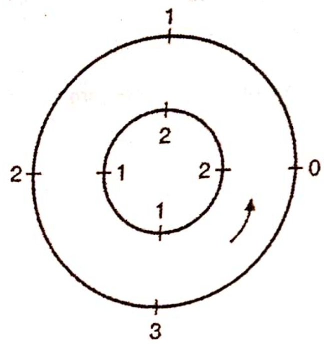

Fig. P. 3.9.7(a)

$\therefore \; y(0) = (0 \times 2) + (1 \times 2) + (2 \times 1) + (3 \times 1) = 7$

We now rotate the inner circle and continue to multiply and add the corresponding elements. Hence we have

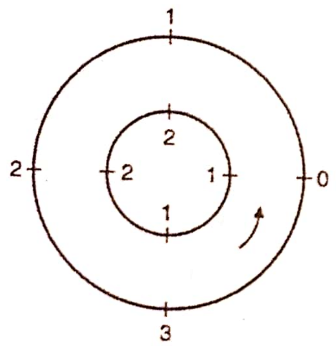

Fig. P. 3.9.7(b)

$\therefore \; y(1) = (0 \times 1) + (1 \times 2) + (2 \times 2) + (3 \times 1) = 9$

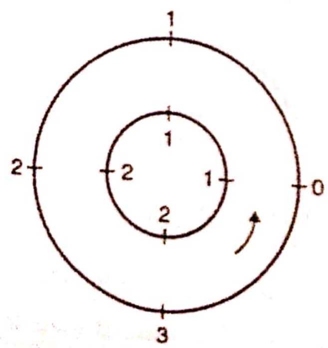

Fig. P. 3.9.7(c)

$\therefore \; y(2) = (0 \times 1) + (1 \times 1) + (2 \times 2) + (3 \times 2) = 11$

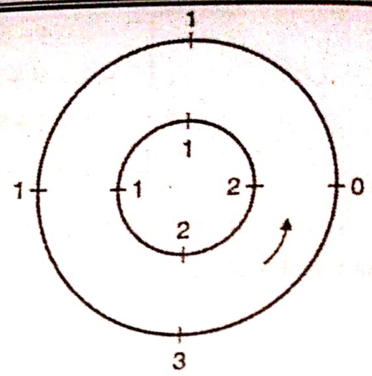

Fig. P. 3.9.7(d)

$\therefore \; y(3) = (0 \times 2) + (1 \times 1) + (2 \times 1) + (3 \times 2) = 9$

If we rotate one more time, we come back to the starting position. Hence we stop the process.

$\therefore \; y(n) = \{7, 9, 11, 9\}$

Ex. 3.9.8 : Find circular convolution for the following sequence using graphical method.

(i) $x_1(n) = \delta(n) + \delta(n-1) + \delta(n-2)$

$x_2(n) = 2\,\delta(n) - \delta(n-1) + 2\,\delta(n-2)$

(ii) $x_1(n) = \delta(n) + \delta(n-2) - \delta(n-2) + \delta(n-3)$

$x_2(n) = \delta(n) - \delta(n-2) + \delta(n-4)$

Soln. :

(i) $x_1(n)$ and $x_2(n)$ can be written is the form

$x_1(n) = \{\underset{\uparrow}{1}, 1, 1\}$ and $x_2(n) = \{\underset{\uparrow}{2}, -1, 2\}$

Since both $x_1(n)$ and $x_2(n)$ are of equal lengths, we place $x_1(n)$ on the outer circle in the counter clockwise direction and $x_2(n)$ on the inner circular in the clockwise direction.

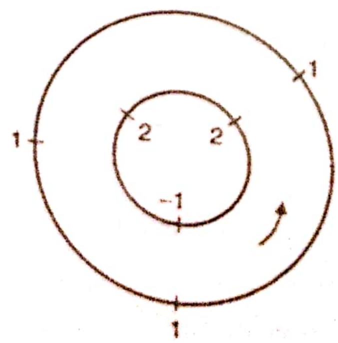

Fig. P. 3.9.8(a)

$y(0) = (1 \times 2) + (1 \times 2) + (1 \times (-1)) = 3$

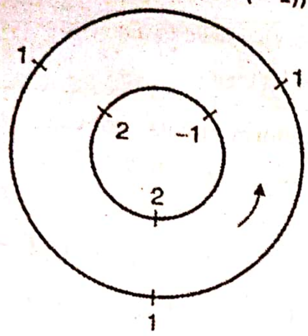

Fig. P. 3.9.8(b)

$y(1) = (1 \times (-1)) + (1 \times 2) + (1 \times 2) = 3$

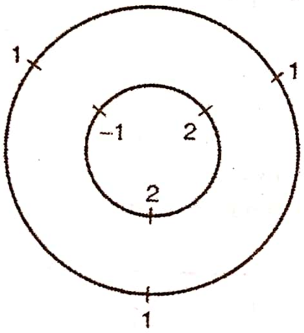

Fig. P. 3.9.8(c)

$$y(2) = (1 \times 2) + (1 \times (-1)) + 1\,(2) = 3$$

If we now rotate the inner circle, we come back to the starting point. Hence we stop the process.

$$\therefore \; y(n) = \{3, 3, 3\}$$

(ii) $x_1(n)$ and $x_2(n)$ can be written is the form

$$x_1(n) = \{1, 1, -1, -1\}$$

and $$x_2(n) = \{1, 0, 1, 0, 1\}$$

We append a zero to $x_1(n)$ to make the lengths of $x_1(n)$ and $x_2(n)$ equal.

$$\therefore \; x_1(n) = \{1, 1, -1, -1, 0\} \text{ and}$$

$$x_2(n) = \{1, 0, 1, 0, 1\}$$

We place $x_1(n)$ on the outer circle, and $x_2(n)$ on the inner circle

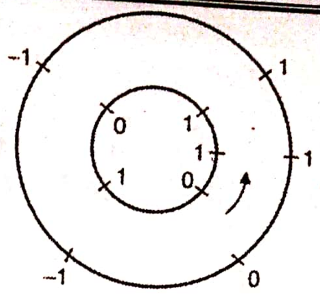

Fig. P. 3.9.8(d)

$y(0) = (1 \times 1) + (1 \times 1) + (-1 \times 0) + (-1 \times 1) + (0 \times 0) = 1$

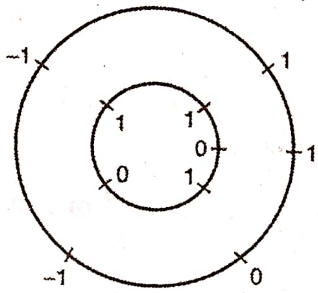

Fig. P. 3.9.8(e)

$y(1) = (1 \times 0) + (1 \times 1) + (-1 \times 1) + (-1 \times 0) + (0 \times 1) = 0$

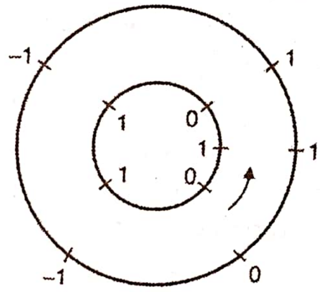

Fig. P. 3.9.8(f)

$y(2) = (1 \times 1) + (1 \times 0) + (-1 \times 1) + (-1 \times 1) + (0 \times 0) = -1$

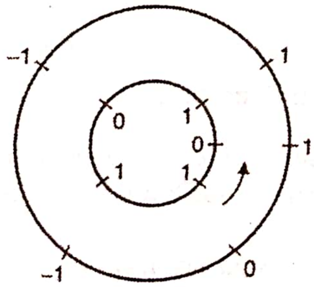

Fig. P. 3.9.8(g)

$y(3) = (1 \times 0) + (1 \times 1) + (-1 \times 0) + (-1 \times 1) + (0 \times 1) = 0$

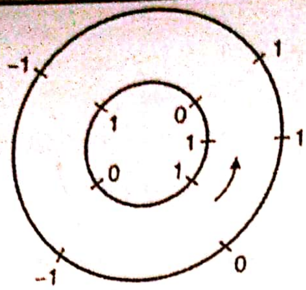

Fig. P. 3.9.8(h)

$\therefore\ y(4) = (1\times 1) + (1\times 0) + (-1\times 1) + (-1\times 0) + (0\times 1) = 0$

$\therefore\ y(n) = \{1, 0, -1, 0, 0\}$

> **Note :** Unlike Linear convolution, where the length of the output signal is $L_1 + L_2 - 1$, the length of the output signal in circular convolution is max (L_1, L_2)

3.9.3 Getting Linear Convolution from Circular Convolution

- We can get the linear convolution result using circular convolution by making a small change to the method described in the earlier example.
- The main difference between linear and circular convolution is the length.
- Given two sequences $x_1(n)$ and $x_2(n)$ of lengths L_1 and L_2 respectively, the length of $y(n)$ using linear convolution would be $L_1 + L_2 - 1$ while the length of the result of circular convolution will be max(L_1, L_2).
- What we do here is append zeros to both $x_1(n)$ and $x_2(n)$ so that the lengths of both the sequences are $L_1 + L_2 - 1$. After this if we perform circular convolution, we get the result of linear convolution.
- Let us take an example to prove the above statement.

3.9.4 Solved Examples

Ex. 3.9.9 : Perform circular convolution on the following sequences so that the result is the same as that of Linear convolution

$x_1(n) = \{1, 2, 3, 1\}$, $x_2(n) = \{1, 1, 1\}$

Soln. : Here the length of $x_1(n) = 4$ and length of $x_2(n) = 3$. Linear convolution would give us a result of length $L_1 + L_2 - 1$ i.e., $4 + 3 - 1 = 6$

We now append zeros to both $x_1(n)$ and $x_2(n)$ so that they are both of length 6.

$$\therefore\quad x_1(n) = \{1, 2, 3, 1, 0, 0\}$$

$$x_2(n) = \{1, 1, 1, 0, 0, 0\}$$

> **Note :** In this case we need to append two zeros to $x_1(n)$ and three zeros to $x_2(n)$.

We now perform circular convolution. We shall use the matrix method for its simplicity.

$$y(n) = x_1(n) \circledast x_2(n)$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore\quad y(n) = \{1, 3, 6, 6, 4, 1\}$$

We now check if this is same as what we would have got had we performed linear convolution. We shall again use the matrix method $y(n) = x_1(n) * x_2(n)$

	1	2	3	1
1	1	2	3	1
1	1	2	3	1
1	1	2	3	1

$$\therefore\quad y(n) = \{1, 3, 6, 6, 4, 1\}$$

Hence we see that the two results are the same.

Go ahead try out your own sums. This technique will always work. Given below is a simple program to implement circular convolution.

CIRCULAR CONVOLUTION

```
Clear all
clc

x= [1 2 6 4 7 3 4];
h= [4 3 2 2];
N2= length (x);
N3= length (h);
N = max (N2,N3);
%% Append Zeros to the shorter signal and make the two
equal
if N2< N
        x=[x zeros(1,N-N2) ];
        else
        h=[h zeros (1,N-N3) ];
end
b=h' ; c=b ;
for n=1:1:N - 1
```

```
b=[b(N);b(1:N-1,:) ];
c=[c b];
end
CIR_CONV=c*x'
```

Ex. 3.9.10 : Compute circular convolution of

$x_1(n) = \{1,1,2,2\}$

$x_2(n) = \{1, 2, 3, 4\}$

Soln. :

$$y(n) = x_1(n) \circledast x_2(n)$$

We generate a circular matrix of $x_2(n)$

$$\therefore \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 15 \\ 17 \\ 15 \\ 13 \end{bmatrix}$$

$$\therefore \quad y(n) = \{15, 17, 15, 13\}$$

Ex. 3.9.11: Compute DFT of $x(n) = 1,\ 0 \le n \le 2 = 0$ Otherwise $N = 4$, Find $|X(k)|$ and $\angle X(k)$

Soln. : Since $N = 4$, we have

$$x(n) = \{1,1,1,0\}$$
$$\uparrow$$

A 4 – point DFT in matrix form is given by the equation

$$X(k) = [W_4]\, x(n)$$

$$X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore X(k) = \{3, -j, 1, j\}$$

Now,

$$\text{Magnitude } |X(k)| = \{3,1,1,1\}$$

$$\text{Phase } \angle X(k) = \tan^{-1}\left\{\frac{\text{Imaginary}}{\text{Real}}\right\}$$

$$\therefore \angle X(k) = \{0, -90°, 0, 90°\}$$

Ex. 3.9.12 : Use the four point DFT and IDFT to determine the circular convolution of the given two sequences.

$x_1(n) = \{1, 2, 3, 1\}$

$x_2(n) = \{4, 3, 2, 2\}$

Soln. : We use the **circular convolution** property

i.e. $x_1(n) \circledast x_2(n) \xleftrightarrow{\text{DFT}} X_1(k) \cdot X_2(k)$

We find the DFT's of $x_1(n)$ and $x_2(n)$

$$X_1 = [W_N] \cdot x_1$$

$$\begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

$$\therefore X_1(k) = \{7, -2-j, 1, -2+j\}$$

Similarly

$$X_2 = [W_N] \cdot x_2$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$

$$\therefore \quad X_2(k) = \{11, 2-j, 1, 2+j\}$$

$$X_3 = X_1(k) \cdot X_2(k) = \{77, -5, 1, -5\}$$

Since this is in the Fourier domain, we finally calculate the IDFT.

$$X_3(k) = \frac{1}{N}[W_N]^* \cdot X_3$$

$$\begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ x_3(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 77 \\ -5 \\ 1 \\ -5 \end{bmatrix}$$

$$\therefore \quad x_3 = \{17, 19, 22, 19\}$$

This is the result that we would obtain if we circularly convolved $x_1(n)$ and $x_2(n)$

$$\therefore \quad x_1(n) \circledast x_2(n) = \{17, 19, 22, 19\}$$

Let us check our result by performing circular convolution of $x_1(n)$ and $x_2(n)$

$$\therefore \quad x_3(n) = x_1(n) \circledast x_2(n)$$

$$\begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ x_3(3) \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 & 3 \\ 3 & 4 & 2 & 2 \\ 2 & 3 & 4 & 2 \\ 2 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

$$\therefore \quad x_3(n) = \{17, 19, 22, 19\}$$

Hence we get the same result as what we got earlier.

Ex. 3.9.13 : Compute circular convolution of $x_1(n) = \{1,2,3,4\}$ and $x_2(n) = \{2,1,2,1\}$

Soln. :

$$x_1(n) = \{1,2,3,4\}$$

$$x_2(n) = \{2,1,2,1\}$$

$$y(n) = x_1(n) \circledast x_2(n)$$

OR $$y(n) = x_1(n) \textcircled{N} x_2(n)$$

We generate a circular matrix of $x_2(n)$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \\ 14 \\ 16 \end{bmatrix}$$

$$\therefore \quad y(n) = \{14, 16, 14, 16\}$$

Ex. 3.9.14 : Perform circular convolution of

$x_1(n) = \{2, 1, 2, 1\}$

$x_2(n) = \{4, 3, 2, 1\}$

Soln. : We generate a circular matrix or $x_2(n)$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 16 \\ 14 \\ 16 \\ 14 \end{bmatrix}$$

$$\therefore \quad y(n) = \{16,14,16,14\}$$

Ex. 3.9.15 : Find linear and circular convolution of the following sequences

$x_1[n] = [4, -2, 2, 1]$

$x_2[n] = [1, 2, 3]$

Soln. :

Given :

(i) Linear convolution

$$x_1(n) = [4, -2, 2, 1]$$

$$x_2(n) = [1, 2, 3]$$

Number of sample in $x_1(n)$ is $L_1 = 4$

Number of sample in $x_2(n)$ is $L_2 = 3$

So from linear convolution number of samples will be,

$$L_2 + L_2 - 1 = 6$$

$$x(n) = x_1(n) * x_2(n)$$

We use the tabular method,

We assume the first value as the origin,

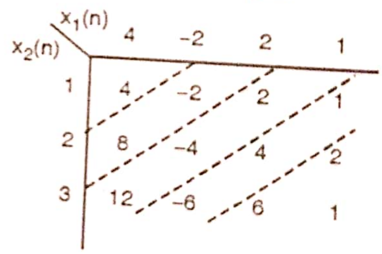

Fig. P. 3.9.15

$$\therefore \quad y(n) = \{4, 6, 10, -1, 8, 1\}$$

(ii) Circular Convolution

For circular convolution, the length of both the signals should be equal.

We append one zero to $x_2(n)$

$$\therefore \quad x_1(n) = \{4, -2, 2, 1\}$$

$$x_2(n) = \{1, 2, 3, 0\}$$

We generate a circular matrix of $x_2(n)$

$$y_1(n) = x_1(n) \circledast x_2(n)$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 2 & 1 & 0 & 3 \\ 3 & 2 & 1 & 0 \\ 0 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 12 \\ 9 \\ 10 \\ -1 \end{bmatrix}$$

$$\therefore \; y(n) = \{12, 9, 10, 1\}$$

Ex. 3.9.16 : Find circular convolution of two finite duration signals

$$x_1(n) = \{1, -1, -2, 3, -1\}$$

$$x_2(n) = \{1, 2, 3\}$$

Soln. : For circular convolution, length of both signals should be equal. We append two zeros to $x_2(n)$.

$$\therefore \; x_1(n) = \{1, -1, -2, 3, -1\} \text{ and}$$

$$x_2(n) = \{1, 2, 3, 0, 0\}$$

Now,

$$y(n) = x_1(n) \circledast x_2(n)$$

We generate a circular matrix of $x_2(n)$.

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ -1 \\ -4 \\ -1 \end{bmatrix}$$

$$\therefore \; y(n) = \{8, -2, -1, -4, -1\}$$

Ex. 3.9.17 : Use the four point DFT and IDFT to determine the circular convolution of sequences

$$x_1(n) = (\underset{\uparrow}{1}, 2, 3, 1) \qquad x_2(n) = (\underset{\uparrow}{4}, 3, 2, 2)$$

Soln. :

Step I : The four point DFT of $x_1(n)$ is $X_1(k)$ and it is given by,

$$X_1(k) = [W_4]\, x_1$$

We have,

$$[W_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$\therefore \; X_1(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ -2-j \\ 1 \\ -2+j \end{bmatrix}$$

$$\therefore \; X_1(k) = \{7, -2-j, 1, -2+j\} \qquad \ldots(1)$$

Similarly, $X_2(k) = [W_4]\, x_{2N}$

$$\therefore \; X_2(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$

$$\therefore \; X_2(k) = \begin{bmatrix} 4+3+2+2 \\ 4-3j-2+2j \\ 4-3+2-2 \\ 4+3j-2-2j \end{bmatrix} \begin{bmatrix} 11 \\ 2-j \\ 1 \\ 2+j \end{bmatrix}$$

$$\therefore \; X_2(k) = \{11, 2-j, 1, 2+j\}$$

Step II : Now according to property of circular convolution,

$$x_1(n) \circledast x_2(n) = X_1(k) \cdot X_2(k) = Y(k)$$

$$\therefore \; Y(k) = \{7, -2-j, 1, -2+j\} \cdot \{11, +2-j, 1, 2+j\}$$

$$\therefore \; Y(k) = \{77, -5, 1, -5\}$$

Step III : Let the result of $x_1(n) \circledast x_2(n)$ be sequence $y(n)$. It is obtained by computing IDFT of $Y(k)$.

According to the definition of IDFT we have,

$$y(n) = \frac{1}{N}\,[W_N^*] \cdot X_N$$

$$= \frac{1}{4}[W_4^*]\cdot Y_N$$

$$\therefore\ y(n) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & +j \end{bmatrix}\begin{bmatrix} 77 \\ -5 \\ 1 \\ -5 \end{bmatrix}$$

$$y(n) = \frac{1}{4}\begin{bmatrix} 68 \\ 76 \\ 88 \\ 76 \end{bmatrix} = \begin{bmatrix} 17 \\ 19 \\ 22 \\ 19 \end{bmatrix}$$

$$y(n) = \{17, 19, 22, 19\}$$

Ex. 3.9.18 : Find the response of FIR filter with impulse response h(n) = {1, 2, 4} to the input sequence x (n) = {1, 2} using circular convolution.

Soln. :

Step I :

$$x(n) = \{1, 2\},\ h(n) = \{1, 2, 4\}$$

Here $L_1 = 2$ and $L_2 = 3$

Hence the length of linear convolution result will be

$$L_1 + L_2 = -1 = 2 + 3 - 1 = 4$$

We append zero to make the length of x(n) and h (n) = 4

$$\therefore\ x(n) = \{1,2,0,0\} \text{ and } h(n) = \{1,2,4,0\}$$

We compute the DFT of x(n) and h (n) and use the property,

$$x(n) \circledast h(n) \xleftrightarrow{\text{DFT}} X(k)\cdot H(k)$$

$$\therefore\ X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2 \\ 1-2j \\ 1-2 \\ 1+2j \end{bmatrix}$$

$$\therefore\ X(k) = \{3, 1-2j, 3-3+2j\}$$

$$\text{And}\quad H(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 4 \\ 0 \end{bmatrix}$$

$$\therefore\ H(k) = \{7, -3-2j, 3, -3+2j\}$$

$$\text{Now } Y(k) = X(k)\,.\,H(k)$$

We perform element by element multiplication

$$\therefore\ Y(k) = \{21, -7+4j, -3, -7-4j\}$$

We finally compute the IDFT of y (k),

$$\text{Now}\quad y(n) = \frac{1}{N}[W_4^*]\,Y_N$$

$$\therefore\ y(n) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & -j \end{bmatrix}\begin{bmatrix} 21 \\ -7+4j \\ -3 \\ -7-4j \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 21 - 7 + 4j - 3 - 7 - 4j \\ 21 - 7j + 4j^2 + 3 + 7j + 4j^2 \\ 21 + 7 - 4j - 3 + 7 + 4j \\ 21 + 7j - 4j^2 + 3 - 7j - 4j^2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 16 \\ 32 \\ 32 \end{bmatrix}$$

$$\therefore\ y(n) = \{\underset{\uparrow}{1}, 4, 8, 8\}$$

This is the final response of the FIR filter.

Verify.

Let us, Verify the result by performing linear convolution using tabular method

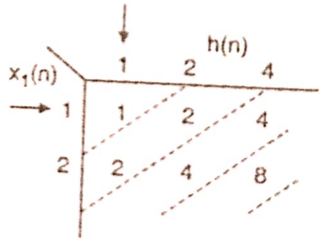

Fig. P. 3.9.18

$$y(n) = \{1, 4, 8, 8\}$$

Hence we get the same result.

Ex. 3.9.19 : If x(n) = {1, 2, 3, 2}, h (n) = {1, 0, 2, 0}

(i) Find circular convolution using time domain method.

(ii) Find linear convolution using circular convolution.

Soln. :

(i) We will solve it using the graphical method and then test our result using the matrix method.

$$y(n) = x_1(n) \circledast x_2(n)$$

Since the length's of both $x_1(n)$ and $x_2(n)$ are the same, we do not need to append zeros to either of the signals.

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We take two concentric circles. Steps involved are as follows :

(a) Plot x_1 (n) on outer circle in counter clockwise direction.

(b) Plot x_2 (n) on inner circle in clockwise direction.

(c) Ensure the origins of the two signals are aligned. Multiply and add corresponding elements.

(d) Move inner circle in the clockwise direction and repeat step (c).

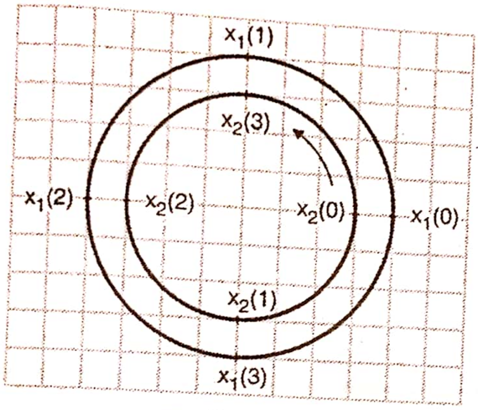

Fig. P. 3.9.19

Hence we have,

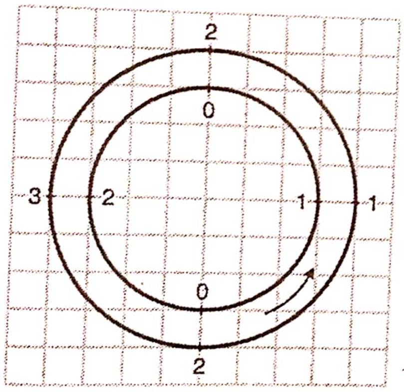

Fig. P. 3.9.19(a)

$\therefore$ $y(0) = (1 \times 1) + (2 \times 0) + (3 \times 2) + (2 \times 0) = 7$

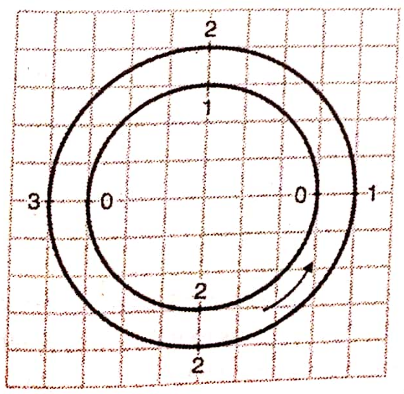

Fig. P. 3.9.19(b)

$\therefore$ $y(0) = (1 \times 0) + (2 \times 1) + (3 \times 0) + (2 \times 2) = 6$

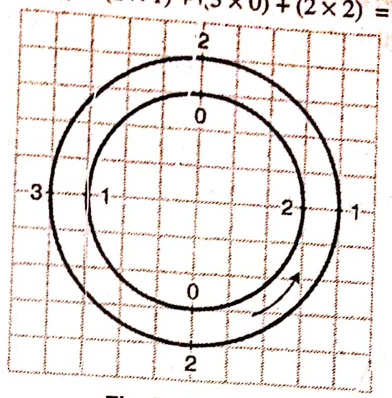

Fig. P. 3.9.19(c)

$\therefore$ $y(0) = (1 \times 2) + (2 \times 0) + (3 \times 1) + (2 \times 0) = 5$

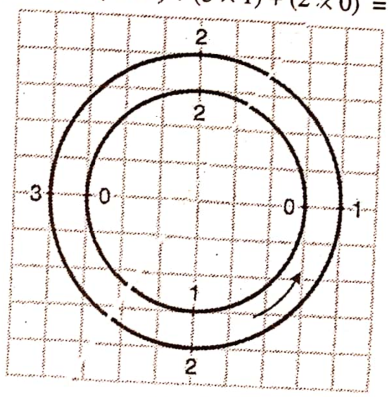

Fig. P. 3.9.19(d)

$\therefore$ $y(0) = (1 \times 0) + (2 \times 2) + (3 \times 0) + (2 \times 1) = 6$

If we now rotate the inner circle, we come back to the starting position. Hence we stop the process.

$\therefore \quad y(n) = \{7, 6, 5, 6\}$

$\therefore \quad x_1(n) \circledast x_2(n) = \{7, 6, 5, 6\}$

We verify this result using the matrix method. We form a circular matrix of x_2 (n).

$$\therefore \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} x_2(0) & x_2(3) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & x_2(3) \\ x_2(3) & x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \end{bmatrix}$$

$$\therefore \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 5 \\ 6 \end{bmatrix}$$

$\therefore \quad y(n) = \{7, 6, 5, 6\}$

which is the same as obtained using the graphical method.

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(ii) Find linear convolution using circular convolution. Since length of x_1 (n) = 4 and length of x_2 (n) = 4, linear convolution would give us a result of length

$L_1 + L_2 - 1 = 4 + 4 - 1 = 7$

We now append zeroes to both x_1 (n) and x_2 (n). So that they are both of length 7.

$$\therefore \quad x_1(n) = \{1, 2, 3, 2, 0, 0, 0\}$$

$$x_2(n) = \{1, 0, 2, 0, 0, 0, 0\}$$

$$y(n) = x_1(n) \circledast x_2(n)$$

We will now perform circular convolution using the matrix method.

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 2 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad y(n) = \{1, 2, 5, 6, 6, 4, 0\}$$

We now verify this result by performing regular convolution.

$$y(n) = x_1(n) * x_2(n)$$

	1	0	2	0
1	1	0	2	0
2	2	0	4	0
3	3	0	6	0
2	2	0	4	0

Fig. P. 3.9.19(e)

$$\therefore \quad y(n) = \{1, 2, 5, 6, 6, 4, 0\}$$

Hence, we observe that we get the same result using linear convolution.

3.10 Filtering of Long Duration Signals

- Filtering of a signal is required for various reasons like removal of noise, boosting certain frequencies while attenuating the rest, to name a few.
- Linear filtering is basically linear convolution that we have studied earlier.
- Hence if x(n) is our signal (song, ecg, etc) which has been corrupted by noise, and h (n) is the impulse response (filter), then the filtered output signal y (n) is given by the linear convolution operation.

$$y(n) = x(n) * h(n)$$

$$\text{i.e.} \quad y(n) = \sum_{k=-\infty}^{+\infty} x(k)\, h(n-k)$$

- In the real world, due to high sampling rates, x(n) is very long and usually runs into thousands of samples. Because of its length, it is difficult to perform regular linear convolution at one time.
- The way devised to deal with this issue is to break x(n) into smaller blocks of size N. Computation on each of these blocks is then done separately. Each of these processed blocks are then, appended (connected) to each other to obtain the final filtered output y (n).
- There are two methods of performing this kind of filtering

1. Overlap - save method.
2. Overlap - add method.

We shall discuss both these methods in detail.

3.10.1 Overlap - Save Method

As mentioned earlier, we break x(n) into smaller blocks of length N. (x_1 (n), x_2 (n), ...). It is important to note that while the length N can be our choice, it should be longer than the length of h (n).

$$\text{i.e.} \quad N > \text{length } h(n).$$

We use the following notations.

Where, N = Length of each block

M = Length of h (n)

L = Values from current x(n)

$$\therefore \quad N = (M - 1) + L$$

- Each block x_i (n) of length N is formed such that it contains (M – 1) values from the previous block and the remaining L values from the current x(n).
- Since there is an overlap of (M – 1) values from the previous block, it known as overlap- save method.
- As the 1st block does not have a previous block, we append (M – 1) zeros to it at the beginning. A pictorial representation would help us understand this better.

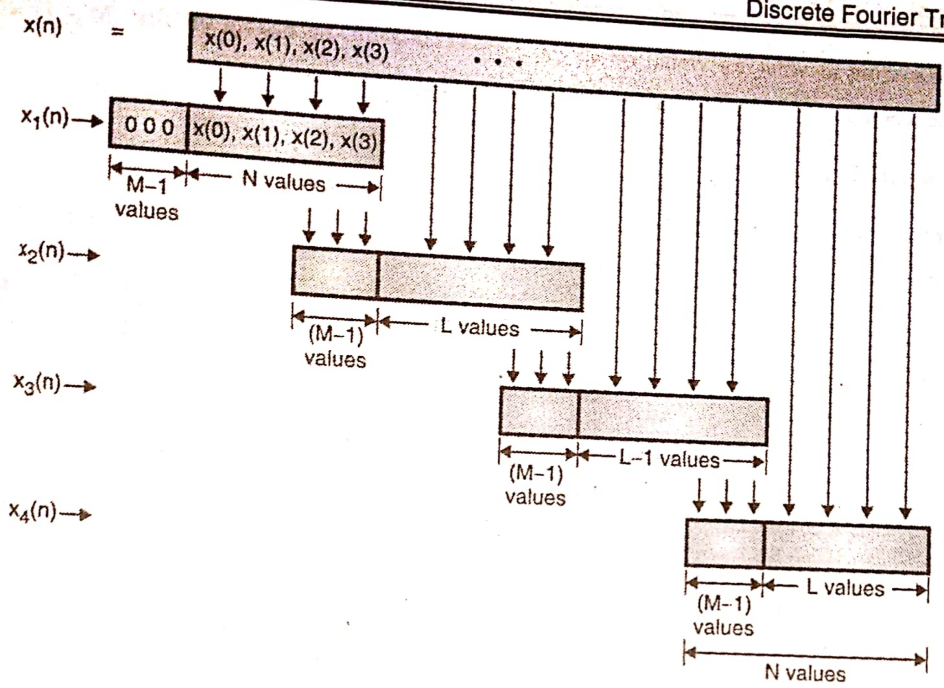

Fig. 3.10.1

Once these block are created, each of these are "circularly convolved" with h (n).

$$\text{i.e.} \quad y_1(n) = x_1(n) \circledast h(n)$$

$$y_2(n) = x_2(n) \circledast h(n)$$

$$\vdots \qquad \vdots \text{ etc.}$$

The final filtered output y (n) is obtained by discarding the first (M – 1) values of each result. $y_1(n)$, $y_2(n)$, ... and appending the remaining values. A pictorial representation is shown in Fig. 3.10.2.

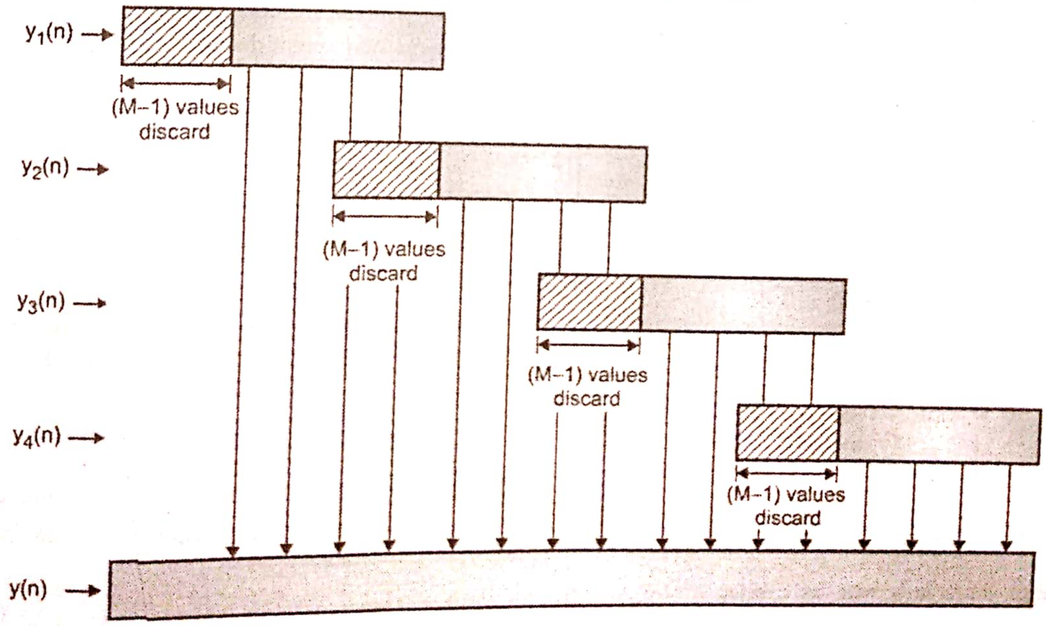

Fig. 3.10.2

This y (n) is the same as what we would obtain by performing linear convolution

i.e. $y(n) = x(n) \circledast h(n)$

It is important to note that we obtained y (n) by performing circular convolution of x(n) with individual blocks.

Let us solve an example to understand the method explained.

Solved Example

Ex. 3.10.1 : Let $x(n) = \{1, 2, 3, 4, 5, 6, 7\}$

and $h(n) = \{1, 0, 2\}$

Perform convolution using overlap save method.

Soln. : We choose a block size of N. In this method, each block contains (M – 1) data points of the previous block followed by L new data points. For the first block, the first (M – 1) values are set to zero. In our case M – 1 = 2. We choose N = 5 and L = 3.

$$x(n) = \{1, 2, 3, 4, 5, 6, 7\}$$

$$\therefore x_1(n) = \{0, 0, 1, 2, 3\}$$

$$x_2(n) = \{2, 3, 4, 5, 6\}$$

$$x_3(n) = \{5, 6, 7, 0, 0\}$$

Now $h(n) = \{1, 0, 2\}$

Since the block size = 5, we append 2 zeros to h(n)

$$\therefore \quad h(n) = \{1, 0, 2, 0, 0\}$$

We now perform circular convolution.

$$\therefore \quad y_1(n) = x_1(n) \circledast h(n) = \{4, 6, 1, 2, 5\}$$

$$\therefore \quad y_2(n) = x_2(n) \circledast h(n) = \{12, 15, 8, 11, 14\}$$

$$\therefore \quad y_3(n) = x_3(n) \circledast h(n) = \{5, 6, 17, 12, 14\}$$

We arrange the results with an overlap of (M – 1) values as shown and discard the first (M – 1) values of each result. In short, we discard the first (M – 1) = 2 values of each result.

$$\underbrace{4 \quad 6}_{\text{Discard}} \quad 1 \quad 2 \quad 5$$

$$\underbrace{12 \quad 15}_{\text{Discard}} \quad 8 \quad 11 \quad 14$$

$$\underbrace{5 \quad 6}_{\text{Discard}} \quad 17 \quad 12 \quad 14$$

$$\therefore y(n) = \{1, 2, 5, 8, 11, 14, 17, 12, 14\}$$

We check the result by performing linear convolution

$$y(n) = x(n) \circledast h(n)$$

	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0
2	2	4	6	8	10	12	14

$$y(n) = \{1, 2, 5, 8, 11, 14, 17, 12, 14\}$$

3.10.2 Overlap - Add Method

We use the same notations used in overlap–save method. The overlap - add method is similar to the overlap–save method except the way the block are created.

In this method, each block is created by taking L values from x(n) and appending (M – 1) zeros at the end of each block.

The size of each block is N = L + (M – 1).

A pictorial representation would help us understand this better.

Once these blocks are created, each of these are "Circularly convolved" with h (n), just like overlap–save method.

i.e. $y_1(n) = x_1(n) \circledast h(n)$

$y_2(n) = x_2(n) \circledast h(n)$

The final filtered output y (n), is obtained by overlapping and adding the last (M– 1) values of result $y_1(n)$, $y_2(n)$, ...etc.

A pictorial representation is shown in Fig. 3.10.3.

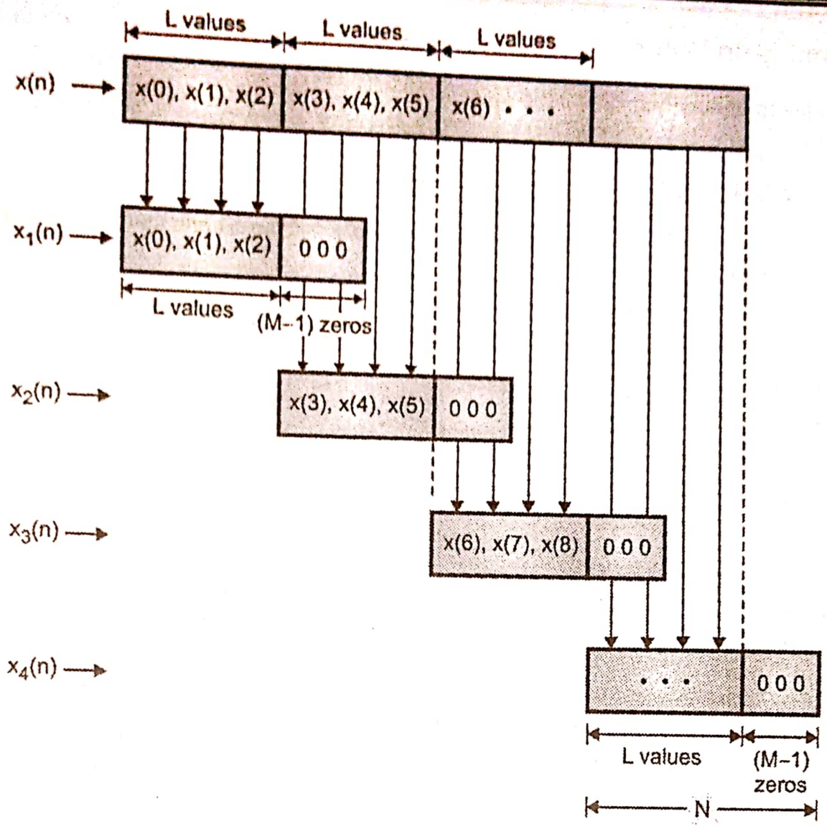

Fig. 3.10.3

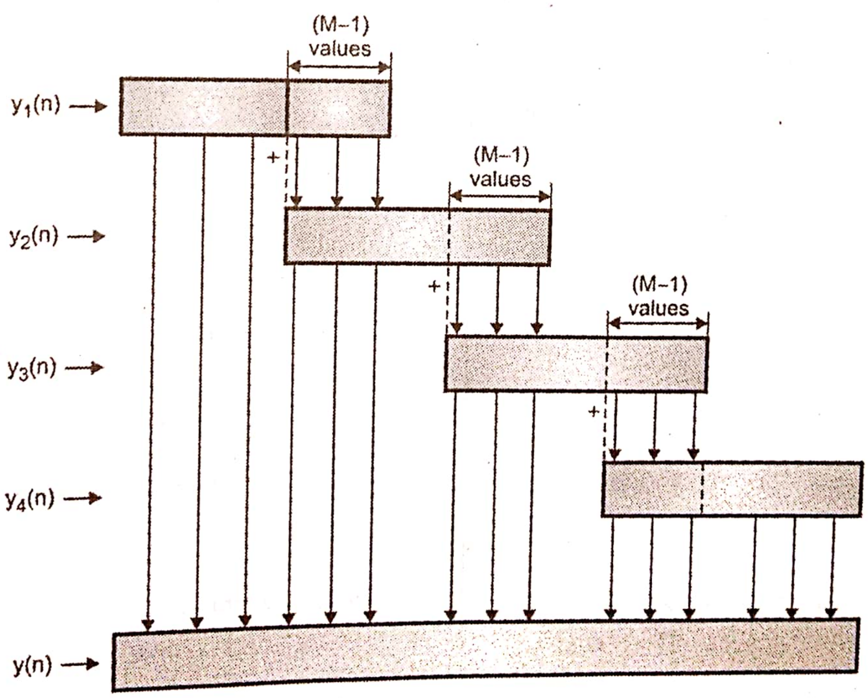

Fig. 3.10.4

3.10.3 Solved Examples on Method

Let us solve an example to understand the method just discussed.

Ex. 3.10.2 : Let $x(n) = \{1, 2, 3, 4, 5, 6, 7\}$

and $h(n) = \{1, 0, 2\}$

Perform convolution using overlap add method.

Soln. : Overlap–add and overlap save methods are used to perform convolution on long sequences. In both these methods, the input is divided into blocks. Circular convolution is then performed on each of the blocks.

Here, we assume that the length of input, $x(n)$ is L and the length of the impulse response, $h(n) = M$.

Overlap - add method

In this, the input is divided into blocks of data of size L and $(M - 1)$ zeros are appended to it. This makes the size of data blocks $N = L + (M - 1)$.

In our example we choose a block size of $N = 5$.

Since $h(n)$ has a length of 3, we append $M - 1 = (3 - 1) = 2$ zeros to L. The length of $L = 3$.

$$\therefore \quad x(n) = \{1, 2, 3, 4, 5, 6, 7\}$$
$$x_1(n) = \{1, 2, 3, 0, 0\}$$
$$x_2(n) = \{4, 5, 6, 0, 0\}$$
$$x_3(n) = \{7, 0, 0, 0, 0\}$$

We increase the size of $h(n)$ by appending zeros so that it is equal to $N = 5$.

$$\therefore \quad h(n) = \{1, 0, 2, 0, 0\}$$

We now perform circular convolution

$$\therefore \quad y_1(n) = x_1(n) \circledast h(n) = \{1, 2, 5, 4, 6\}$$
$$\therefore \quad y_2(n) = x_1(n) \circledast h(n) = \{4, 5, 14, 10, 12\}$$
$$\therefore \quad y_3(n) = x_3(n) \circledast h(n) = \{7, 0, 14, 0, 0\}$$

These results are placed as shown below and added. The last two terms are added. Since we had appended 2 zeros to $h(n)$.

```
y  =  1  2  5  4  6
               ↓  ↓      Add
               4  5  14 10 12
                      ↓  ↓      Add
                      7  0  14  0  0
```

$$\therefore \quad y(n) = \{1, 2, 5, 8, 11, 14, 17, 12, 14\}$$

We would have got the same answer had we performed linear convolution as shown earlier.

Ex. 3.10.3 : Find linear convolution using overlap-add and overlap-save method of the following sequences :

$x[n] = \{1, 2, -1, 2, 3, -2, -3, -1, 1, 1, 2, -1\}$;

$h[n] = \{1, 2, 3\}$

Compare the results and state the usage of each method.

Soln. :

Overlap save method

We will form blocks of length 4.

Here $\quad N = 4$

Length of $h = M = 3$

$\therefore \quad M - 1 = 2$

Now to make each block we need $L = 2$ data from $x(n)$. The different blocks are formed as follows :

For the 1[st] block, we append $M - 1 = 2$ zeros

$$x_1(n) = \{0, 0, 1, 2\}$$

$$\therefore x_1(n) = \{\underbrace{0, 0}_{\substack{M-1 \\ \text{zeros}}}, \underbrace{1, 2}_{\substack{2\text{ data} \\ \text{from } x(n)}}\}$$

$$x_2(n) = \{\underbrace{1, 2}_{\substack{M-1 \\ \text{from} \\ x(n)}}, \underbrace{-1, 2}_{\substack{\text{New} \\ \text{data}}}\}$$

Similarly other blocks can be formed,

$$x_3(n) = \{-1, 2, 3, -2\}$$
$$x_4(n) = \{3, -2, -3, -1\}$$
$$x_5(n) = \{-3, -1, 1, 1\}$$
$$x_6(n) = \{1, 1, 2, -1\}$$
$$x_7(n) = \{2, -1, 0, 0\}$$

Given $\quad h(n) = \{1, 2, 3\}$

We append zeros to $h(n)$ so that it is equal to $N = 4$

$$\therefore \quad h(n) = \{1, 2, 3, 0\}$$

The different output blocks are calculated as follows :

$$y_1(n) = x_1(n) \circledast h(n)$$

We create a circular of h(n)

$$\therefore \quad y_1(n) = \begin{bmatrix} 1\,0\,3\,2 \\ 2\,1\,0\,3 \\ 3\,2\,1\,0 \\ 0\,3\,2\,1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+3+4 \\ 0+0+0+6 \\ 0+0+1+0 \\ 0+0+2+2 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 1 \\ 4 \end{bmatrix}$$

$$\therefore \quad y_1(n) = \{7, 6, 1, 4\} \qquad \text{....(1)}$$

$$y_2(n) = x_2(n) \circledast h(n)$$

$$\therefore \quad y_2(n) = \begin{bmatrix} 1\,0\,3\,2 \\ 2\,1\,0\,3 \\ 3\,2\,1\,0 \\ 0\,3\,2\,1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0-3+4 \\ 2+2+0+6 \\ 3+4-1+0 \\ 0+6-2+2 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \\ 6 \\ 6 \end{bmatrix}$$

$$\therefore \quad y_2(n) = \{2, 10, 6, 6\} \qquad \text{...(2)}$$

$$y_3(n) = x_3(n) \circledast h(n)$$

$$\therefore \quad y_3(n) = \begin{bmatrix} 1\,0\,3\,2 \\ 2\,1\,0\,3 \\ 3\,2\,1\,0 \\ 0\,3\,2\,1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} -1+0+9-4 \\ -2+2+0-6 \\ -3+4+3+0 \\ 0+6+6-2 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 4 \\ 10 \end{bmatrix}$$

$$\therefore \quad y_3(n) = \{4, -6, 4, 10\} \qquad \text{...(3)}$$

$$y_4(n) = x_4(n) \circledast h(n)$$

$$\therefore \quad y_4(n) = \begin{bmatrix} 1\,0\,3\,2 \\ 2\,1\,0\,3 \\ 3\,2\,1\,0 \\ 0\,3\,2\,1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3+0-9-2 \\ 6-2+0-3 \\ 9-4-3+0 \\ 0-6-6-1 \end{bmatrix} = \begin{bmatrix} -8 \\ 1 \\ 2 \\ -13 \end{bmatrix}$$

$$\therefore \quad y_4(n) = \{-8, 1, 2, -13\} \qquad \text{...(4)}$$

$$y_5(n) = x_5(n) \circledast h(n)$$

$$\therefore \quad y_5(n) = \begin{bmatrix} 1\,0\,3\,2 \\ 2\,1\,0\,3 \\ 3\,2\,1\,0 \\ 0\,3\,2\,1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3+0+3+2 \\ -6-1+0+3 \\ -9-2+1+0 \\ 0-3+2+1 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ -10 \\ 0 \end{bmatrix}$$

$$\therefore \quad y_5(n) = \{2, -4, -10, 0\} \qquad \text{...(5)}$$

$$y_6(n) = x_6(n) \circledast h(n)$$

$$\therefore \quad y_6(n) = \begin{bmatrix} 1\,0\,3\,2 \\ 2\,1\,0\,3 \\ 3\,2\,1\,0 \\ 0\,3\,2\,1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+6-2 \\ 2+1+0-3 \\ 3+2+2+0 \\ 0+3+4-1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 7 \\ 6 \end{bmatrix}$$

$$\therefore \quad y_6(n) = \{5, 0, 7, 6\} \qquad \text{...(6)}$$

$$y_7(n) = x_7(n) \circledast h(n)$$

$$\therefore \quad y_7(n) = \begin{bmatrix} 1\,0\,3\,2 \\ 2\,1\,0\,3 \\ 3\,2\,1\,0 \\ 0\,3\,2\,1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0+0+0 \\ 4-2+0+0 \\ 6-2+0+0 \\ 0-3+0+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \\ -3 \end{bmatrix}$$

$$\therefore \quad y_7(n) = \{2, 2, 4, -3\} \qquad \text{...(7)}$$

The final output y (n) is obtained by discarding initial 2 samples from each output block and then by connecting all output blocks one after other.

$$\therefore \; y(n) = \{1, 4, 6, 6, 4, 10, 2, -13, -1, 0, 0, 7, 6, 4, -3\}$$

Let us verify the result obtained by using regular linear convolution we use the tabular method.

	1	2	−1	2	3	−2	−3	−1	1	1	2	−1
1	1	2	−1	2	3	−2	−3	−1	1	1	2	−1
2	2	4	−2	4	6	−4	−6	−2	2	2	4	−2
3	3	6	−3	6	9	−6	−9	−3	3	3	6	−3

$\therefore y(n) = \{1, 4, 6, 6, 4, 10, 2, -13, -1, 0, 0, 7, 6, 4, -3\}$

3.11 Summary of DFT properties

1. Linearity	$ax_1(n) + b\,x_2(n) \xleftrightarrow{\text{DFT}} a\,X_1(k) + b\,X_2(k)$
2. Periodicity	$X(k) = X(k+N)$
3. Circular time shift	$x(n-m)_N \xleftrightarrow{\text{DFT}} e^{-j2\pi km/N}X(k)$
4. Circular frequency shift	$x(n)\,e^{-j2\pi km/N} \xleftrightarrow{\text{DFT}} X(k-m)_N$
5. Time reversal	$x((-n))_N \xleftrightarrow{\text{DFT}} x((-k))_N$
6. Complex conjugate	$x^*((n))_N \xleftrightarrow{\text{DFT}} x((-k))_N$ $x(n) \xleftrightarrow{\text{DFT}} X(N-K)$
7. Parsevals theorem	$\sum_{n=0}^{N-1} \lvert x(n)\rvert^2 = \frac{1}{N}\sum_{n=0}^{N-1} \lvert X(k)\rvert^2$
8. Circular convolution	$x_1(n) \circledast x_2(n) \xleftrightarrow{\text{DFT}} X_1(k)\cdot X_2(k)$

3.12 2 D- Discrete Fourier Transform (2-DFT)

1-D DFT is useful while dealing with 1-Dimensinal signals. Images are 2 Dimensional signals and we need to use 2-D DFT to compute their frequency components. The 2-D DFT is given by the formula,

$$F(u, v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x, y)\, e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)} \quad \ldots (3.12.1)$$

We use the separability property while computing the 2-D DFT.

3.12.1 The Separability Property

This property states that a 2-D DFT can be separated into two 1-D DFTs.

We know (assume a square image)

$$F(u, v) = \sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x, y)\, e^{-j2\pi\left(\frac{ux}{N}+\frac{vy}{N}\right)} \quad \ldots (3.12.2)$$

This can be split up as,

$$F(u, v) = \sum_{x=0}^{N-1} e^{-j2\pi\frac{ux}{N}} \sum_{y=0}^{N-1} f(x, y)\, e^{\frac{-j2\pi vy}{N}}$$

$$\text{Let } F(x, v) = \sum_{y=0}^{N-1} f(x, y)\, e^{-j2\pi\frac{vy}{N}}$$

$$\therefore F(u, v) = \sum_{y=0}^{N-1} f(x, v)\, e^{-j2\pi\frac{ux}{N}}$$

As stated, the principle advantage of the separability property is that the 2-D DFT can be obtained in two steps by successive application of 1-D DFT.

F(u, v) can be obtained by applying 1-D DFT along the rows and then along the columns. One could also take the 1-D DFT first along the columns and then along the rows.

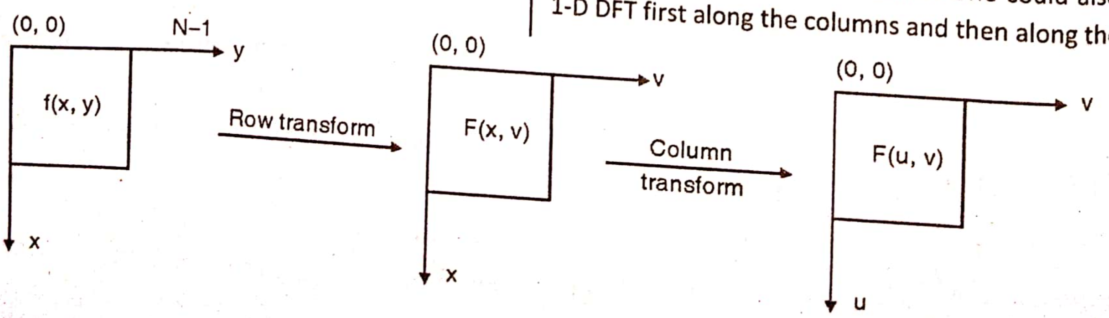

Fig. 3.12.1

3.12.2 Solved Examples on 2- DFT

Ex. 3.12.1 : Find the DFT of the image :

0	1	2	1
1	2	3	2
2	3	4	3
1	2	3	2

Soln. : We have studied the 1-D DFT matrix. We shall use the DFT along the rows and then along the columns.

Length of each row is N = 4 $\therefore$ we need a 4×4 DFT matrix.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 0 \\ -2 \end{bmatrix} \quad \rightarrow \text{DFT of } 1^{st} \text{ row}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ 0 \\ -2 \end{bmatrix} \quad \rightarrow \text{DFT of } 2^{nd} \text{ row}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ -2 \\ 0 \\ -2 \end{bmatrix} \quad \rightarrow \text{DFT of } 3^{rd} \text{ row}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ 0 \\ -2 \end{bmatrix} \quad \rightarrow \text{DFT of } 4^{th} \text{ row}$$

Hence we have an intermediate stage $\begin{bmatrix} 4 & -2 & 0 & -2 \\ 8 & -2 & 0 & -2 \\ 12 & -2 & 0 & -2 \\ 8 & -2 & 0 & 2 \end{bmatrix}$

Now using the 1-D DFT along the columns of this intermediate image we get

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 12 \\ 8 \end{bmatrix} = \begin{bmatrix} 32 \\ -8 \\ 0 \\ -8 \end{bmatrix} \quad \rightarrow \text{DFT of } 1^{st} \text{ column}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} -2 \\ -2 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} -8 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow \text{DFT of } 2^{nd} \text{ column}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow \text{DFT of } 3^{rd} \text{ column}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -i & 1 & -i \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} -2 \\ -2 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} -8 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow \text{DFT of } 4^{th} \text{ column}$$

$\therefore$ The final DFT of the entire image is

32	−8	0	−8
−8	0	0	0
0	0	0	0
−8	0	0	0

Another way of computing the 2D-DFT is by using the formula

$$F = TfT' \qquad ...(1)$$

where T is the N x N DFT matrix.

The DFT matrix is both symmetric as well as separable and hence can be obtained using the relationship

$$F = TfT \qquad ...(2)$$

We shall solve the same example by using this formula

Ex. 3.12.2 : Find the DFT of the given image.

$$\begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 2 \end{bmatrix}$$

Soln. : Since f is a 4×4 matrix, T which is the DFT matrix is given below

$$T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 & 12 & 8 \\ -2 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 \\ -2 & -2 & -2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$F = \begin{bmatrix} 32 & -8 & 0 & -8 \\ -8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -8 & 0 & 0 & 0 \end{bmatrix}$$

This is the same as what we had got using the conventional method.

Summary

This chapter deals with understanding the DFT. The importance of the DFT was discussed. Examples were solved using the conventional method as well as the matrix method. Computation of Inverse DFT was explained. Drawing the Magnitude as well as the Phase spectrum was carried out. We also explained how to identify the frequency components from the x- axis of the DFT spectrum. Properties of the DFT were listed and examples were solved using each of the properties. The chapter also introduced the 2-D DFT.

Review Questions

Q. 1 State and prove differentiation property of Fourier transform.

Q. 2 State and prove circular convolution property of Fourier transform.

Q. 3 State and prove properties of Discrete Time Fourier Transform (DTFT).

Q. 4 Explain the convolution property of Fourier transform of a DT signal.

Q. 5 State and prove following properties of Fourier transform :

(i) Periodicity (ii) Linearity

Q. 6 Explain Overlap-save and Overlap-add algorithms.

Q. 7 Explain Circular convolution.

Q. 8 Prove that linear convolution is obtained using circular convolution property of DFT.

Q. 9 With the help of neat flow diagram explain DIF FFT algorithm consider N = 8.

Q. 10 Explain the relationship between DTFT and DFT.

□□□