GENG 8010–Part 1: Elements of Differential and Difference Equations

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Second order response VIII

Undamped natural frequency and damping ratio

Response of 2nd order underdamped system is characterized by the oscillation frequency and the exponential term (ω, σ) given by the CE

$$\Delta(s) = s^2 + a_1 s + a_0 = (s+\sigma)^2 + \omega^2$$

In terms of **natural frequency,** ω_n and **damping ratio** ξ,

$$s^2 + a_1 s + a_0 = s^2 + 2\xi\omega_n s + \omega_n^2 = (s+\sigma)^2 + \omega^2 = s^2 + 2\sigma s + (\omega^2 + \sigma^2)$$

comparing and equating the coefficients of the like terms

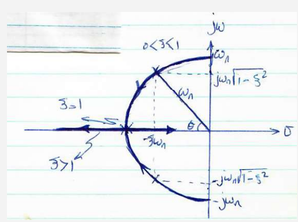
Second order response IX

Poles, stability, natural frequency, and damping ratio

$$\xi\omega_n = \sigma \qquad \omega = \omega_n\sqrt{1-\xi^2}$$

$$s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

$\xi = 0$	poles are on the imaginary axis
$0 < \xi < 1$	poles are on a circle of radius ω_n
$\xi = 1$	repeated roots at $-\omega_n$
$\xi > 1$	distinct real roots



With $\theta = \cos^{-1}\xi$

Remark–The system is **stable** if all the poles are in the left hand plane.

Second order response X

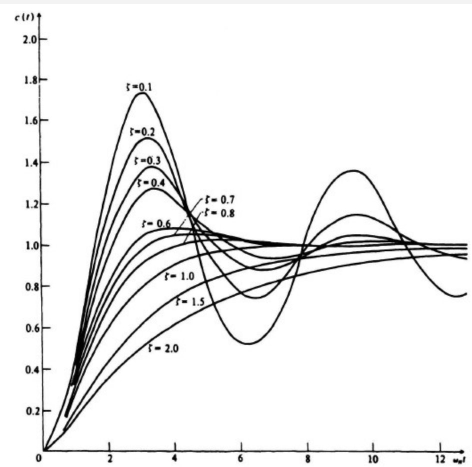
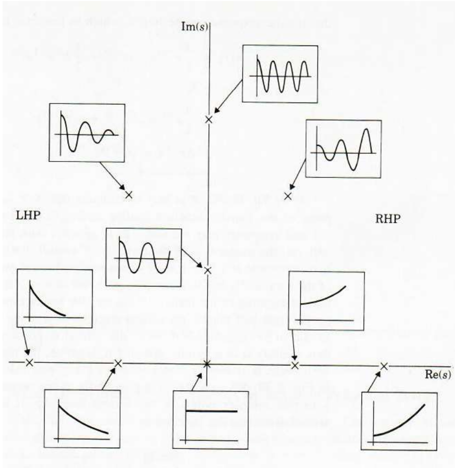
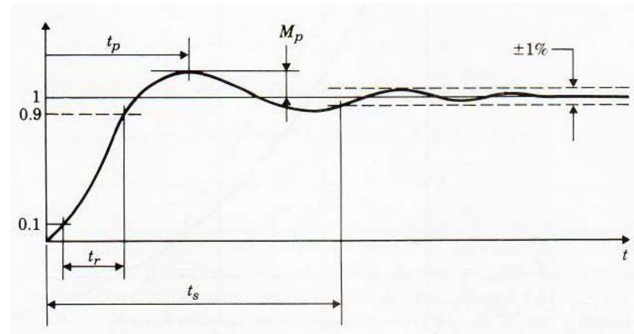


Figure 1: Unit step response of 2nd order system for different values of ξ

Second order response XI



Second order response XII



Pure resonance I

As a final discussion, let us consider the following second order differential equation described by

$$\frac{d^2y}{dt^2} + \omega_0^2 y = F_0 \cos \omega t \qquad y(0) = 0, y'(0) = 0 \tag{8.1}$$

The characteristic polynomial for this system is

$$m^2 + \omega_0^2 = 0 \Longrightarrow m_{1,2} = \pm j\omega_0$$

hence, the form of the homogeneous solution is

$$y_c = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

Pure resonance II

Also, given the forcing function on the right, we assume a solution complementary solution of the form

$$y_p = A \sin \omega t + B \cos \omega t$$

pluging into the differential equation gives

$$A(\omega_0^2 - \omega^2) \sin \omega t + B(\omega_0^2 - \omega^2) \cos \omega t = F_0 \cos \omega t \longrightarrow A = 0, B = \frac{F_0}{\omega_0^2 - \omega^2}$$

Therefore

$$y(t) = c_1 \sin \omega_0 t + c_2 \cos \omega_0 t + \frac{F_0}{\omega_0^2 - \omega^2} \cos \omega t$$

Pure resonance III

evaluating the constants give

$$c_1\omega_0 = 0$$

$$c_2 + \frac{F_0\omega}{\omega_0^2 - \omega^2} = 0$$

Finally,

$$\boxed{y(t) = \frac{F_0}{\omega_0^2 - \omega^2}\left(\cos\omega t - \cos\omega_0 t\right)}$$

Pure resonance IV

Assume now that $\omega = \omega_0$

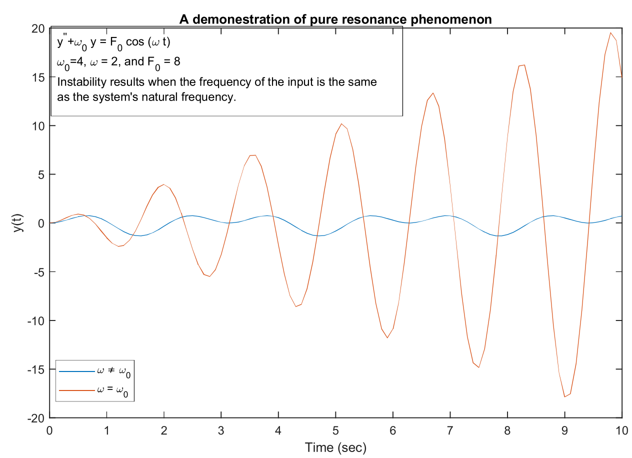
In this case, while the homogeneous solution remains the same, due to the fact that we have similar terms as in the forcing function in x_c, we take

$$y_p = At\sin\omega t + Bt\cos\omega t$$

which leads to $A = \frac{F_0}{2\omega_0}$ and $B = 0$, and from there $c_2 = 0$ and $c_1 = \frac{F_0 t}{2\omega_0}$, and finally

$$\boxed{y(t) = \frac{F_0 t}{\omega_0}\sin\omega_0 t}$$

Pure resonance V



Pure resonance VI

Remark

1. **Pure resonance** occurs exactly when the natural internal frequency ω_0 matches the natural external frequency ω , in which case all solutions of the differential equation are unbounded.
2. Pure resonance was easily demonstrated mathematically by simply taking $\omega = \omega_0$. However, this situation hardly ever happens in real physical engineering systems. Damping is always inherent to physical systems.
3. **Practical Resonance** is said to occur when the external frequency is "tuned" to produce the largest possible solution to the differential equation. We shall illustrate this concept through some analysis and an example.

Practical resonance I

As mentioned before, there is always some damping as a part of the design or perhaps due to frictional forces associated with real physical systems. So rather than (8.1) consider a more realistic system described by

$$my''(t) + by'(t) + ky(t) = F_0 \cos \omega t \tag{8.2}$$

Note that the above is a second order differential equation whose form of homogeneous solution was discussed in Table 1. Also note from Table 1 that as long as the poles of this system have negative real parts. The homogeneous part of the solution will have negative power exponential terms in it and will therefore tend to zero as time goes by. Therefore, let us concentrate on the particular solution.

Practical resonance II

Based on the right hand side of the equation, we take

$$y_p(t) = A \sin \omega t + B \cos \omega t$$

pluggin back into the differential equation gives

$$\begin{cases} (k - m\omega^2)A + b\omega B = F_0 \\ -b\omega A + (k - m\omega^2)B = 0 \end{cases} \implies \begin{cases} A = \frac{(k-m\omega^2)F_0}{(k-m\omega^2)^2+(b\omega)^2} \\ B = \frac{b\omega F_0}{(k-m\omega^2)^2+(b\omega)^2} \end{cases}$$

Note that

$$A \cos at + B \sin at = \sqrt{A^2 + B^2} \cos\left(at - \tan^{-1}\frac{B}{A}\right) \tag{8.3}$$

Practical resonance III

Hence,

$$y_p(t) = K \cos(\omega t - \alpha)$$

where

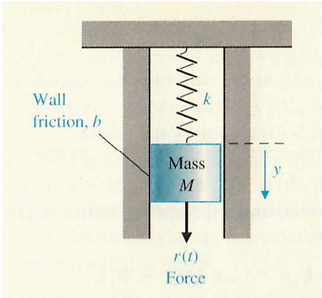
$$K(\omega) = \sqrt{A^2 + B^2} = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (b\omega)^2}} \tag{8.4}$$

Remark

$K(\omega)$ is referred to as *"forced amplitude"* and is always finite. If we had no damping, i.e. $b = 0$ and $\omega = \omega_0 = \sqrt{\frac{k}{m}}$ then $K = \infty$. However, $K(\omega)$ as described above will have a maximum for some value of ω at which **practical resonance** occurs.

Practical resonance IV

Example–Consider the mass, spring, damping system shown in the Figure. Assume that $m = 1, b = 2$, and $k = 26$. Further $r(t) = 82 \cos 4t$, $y(0) = 6$, and $y'(0) = 0$.



The equation of motion for this system is described by

$$y''(t) + 2y'(t) + 26y(t) = 82 \cos 4t$$

Characteristic polynomial for this system is

$$m^2 + 2m + 26 = 0 \implies m_{1,2} = -1 \pm j5$$

Therefore,

$$y_c(t) = e^{-t}(c_1 \cos 5t + c_2 \sin 5t)$$

Practical resonance V

Now for the particular solution, take

$$y_p(t) = A\cos 4t + B\sin 4t \implies \begin{cases} 10A + 8B = 82 \\ -8A + 10B = 0 \end{cases} \implies \begin{cases} A = 5 \\ B = 4 \end{cases}$$

Therefore,

$$y(t) = e^{-t}\left(c_1 \cos 5t + c_2 \sin 5t\right) + 5\cos 4t + 4\sin 4t$$

Now using $y(0) = 6$, and $y'(0) = 0$, we obtain $c_1 = 1$ and $c_2 = -3$, so

$$y(t) = \underbrace{e^{-t}\left(\cos 5t - 3\sin 5t\right)}_{\text{transient solution}} + \underbrace{5\cos 4t + 4\sin 4t}_{\text{steady-state solution}}$$

Practical resonance VI

Ignoring the transient part of the solution (since it dies out), consider now the steady-state part of the solution, and the identity (8.3), we get

$$y_{ss} = 5\cos 4t + 4\sin 4t = \sqrt{41}\left(\cos 4t - \alpha\right)$$

where

$$\alpha = tan^{-1}\frac{4}{5} = 0.6747$$

Now based on (8.4) note that the amplitude of the forced response as a function of the frequency is given by

$$K(\omega) = \frac{82}{\sqrt{676 - 48\omega^2 + \omega^4}}$$

Practical resonance VII

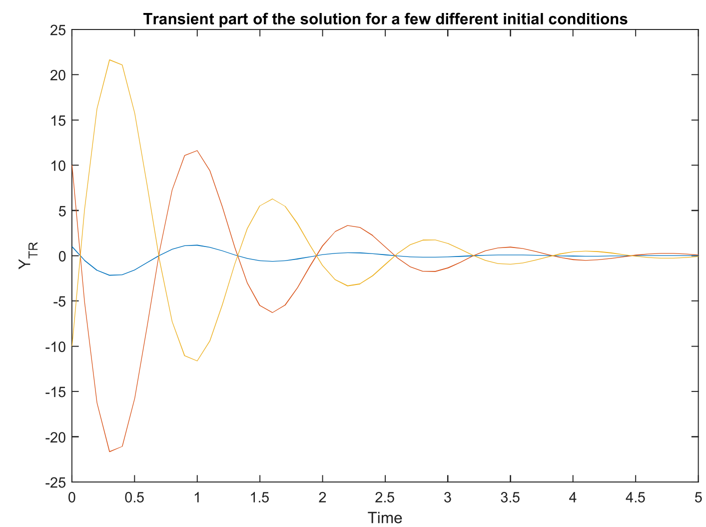
To determine the frequency at which the maximum amplitude occurs differentiate with respect to ω and set equal to zero to solve for ω^*. Hence,

$$K'(\omega) = \frac{-164\omega(\omega^2 - 24)}{(676 - 48\omega^2 + \omega^4)^{\frac{3}{2}}} = 0$$

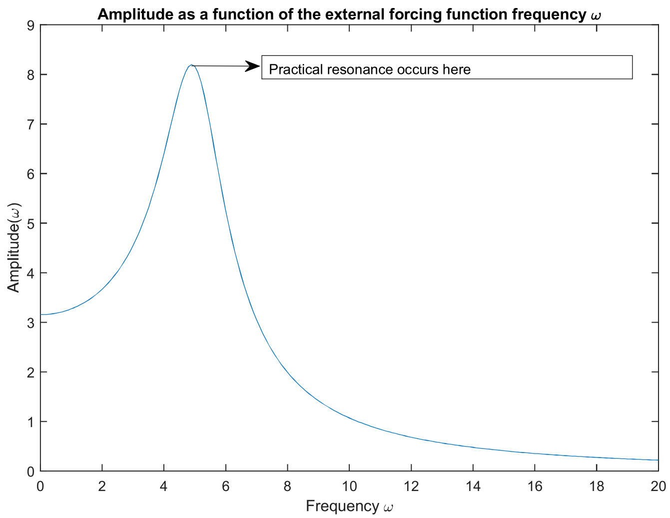
and

$$\omega^* = \sqrt{24} = 4.89$$

Practical resonance VIII



Practical resonance IX



Practical resonance X

Practical resonance XI

Remark

1. Note that in the ideal case, pure resonance for the above system would occur if there was no damping, i.e. $b = 0$ and that would happen at

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{26} = 5.1$$

   Compare ω_0 for pure resonance to ω^* for practical resonance. They are indeed, very close!

2. Consider some interesting examples:
   - Resonance explained
   - Resonance in mass-spring system
   - Breaking a glass
   - Resonance in an RLC circuit
   - Tacoma Bridge, November 1940. Was not resonance