Lecture Notes 23rd May 2023

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Theory:

ODE with constant coefficients

General form: $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x)$

$$a_n, a_{n-1}, \ldots, a_1, a_0 = \text{constants}$$

First, let us consider the homogeneous case; $f(x) = 0;\ \text{for } n = 1$

$$a_1 y' + a_0 y = 0$$

$$\frac{1}{y} dy = -\frac{a_0}{a_1} dx$$

$$\int \frac{1}{y} dy = \int -\frac{a_0}{a_1} dx$$

$$\ln y = -\frac{a_0}{a_1} x \ldots \text{(note: no + c)}$$

$$y = e^{-\frac{a_0}{a_1} x} = e^{\lambda x} \text{ (Only solution)}$$

Discussion:

$$\ln y = -\frac{a_0}{a_1} x + c \ldots \text{(note: +c)}$$

$$y = e^{-\frac{a_0}{a_1} x + c} = e^{\lambda x + c} = Ae^{\lambda x} \ldots \text{(Scalar multiple of the only solution)}$$

Remark:

Any homogeneous ODE of order n has exactly n good/basis solutions such as $y_1, y_2, \ldots, y_n$ and the general solution can be written as;

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

$$c_1, c_2, c_n = \text{any scalar}$$

Now, let us consider n = 2;

$$a_2y'' + a_1y' + a_0y = 0$$

Using the remark, this ODE has two basis/good solutions (say y_1 and y_2)

Maybe one of them looks like $y = e^{\lambda x}$

Let us check to see if we can find such a ‘λ’

Substitute $y = e^{\lambda x}$ in the ODE $a_2y'' + a_1y' + a_0y = 0$

$$y = e^{\lambda x};\ y' = \lambda e^{\lambda x};\ y'' = \lambda^2 e^{\lambda x}$$

$$a_2\lambda^2 e^{\lambda x} + a_1\lambda e^{\lambda x} + a_0 e^{\lambda x} = 0$$

$$e^{\lambda x}(a_2\lambda^2 + a_1\lambda + a_0) = 0$$

$$(a_2\lambda^2 + a_1\lambda + a_0) = 0 \ldots (e^{\lambda x} \text{ can never be } 0)$$

$$(a_2\lambda^2 + a_1\lambda + a_0) = 0 \ldots (\text{Characteristic equation})$$

$$\lambda = \frac{-a_1 \pm \sqrt{{a_1}^2 - 4a_2a_0}}{2a_2}$$

Now, there are 3 cases;

1. ${a_1}^2 - 4a_2a_0 > 0$
2. ${a_1}^2 - 4a_2a_0 = 0$
3. ${a_1}^2 - 4a_2a_0 < 0$

<u>Case 1</u>: $y_1 = e^{\lambda 1x}$ and $y_2 = e^{\lambda 2x}$; $y = c_1e^{\lambda 1x} + c_2e^{\lambda 2x}$

<u>Case 2</u>: $y_1 = e^{\lambda x}$ and $y_2 = xe^{\lambda x}$; $y = c_1e^{\lambda x} + c_2xe^{\lambda x}$

<u>Case 3</u>: $y_1 = e^{(\alpha+i\beta)x} = e^{\alpha x}\cos\beta x$ and $y_2 = e^{(\alpha-i\beta)x} = e^{\alpha x}\sin\beta x$;

$$y = c_1e^{\alpha x}\cos\beta x + c_2e^{\alpha x}\sin\beta x$$

<u>Solve the following</u>:

Example 1:

$$y'' + 3y' + 2y = 0$$

Characteristic equation: $\lambda^2 + 3\lambda + 2 = 0$

$$(\lambda + 1)(\lambda + 2) = 0$$

$$\lambda_1 = -1;\ \lambda_2 = -2$$

$y_1 = e^{-x}$ and $y_2 = e^{-2x}$; $y = c_1 e^{-x} + c_2 e^{-2x}$

Example 2:

$$y'' - 4y' + 4y = 0$$

Characteristic equation: $\lambda^2 - 4\lambda + 4 = 0$

$$(\lambda - 2)^2 = 0$$

$$\lambda = 2, 2$$

$y_1 = e^{2x}$ and $y_2 = xe^{2x}$; $y = c_1 e^{2x} + c_2 x e^{2x}$

Example 3:

$$y'' + y' + y = 0$$

Characteristic equation: $\lambda^2 + \lambda + 1 = 0$

$$\lambda = \frac{-1 \pm \sqrt{1 - 4}}{2}$$

$$\lambda = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$y_1 = e^{-\frac{1}{2}x} \cos\frac{\sqrt{3}}{2}x$ and $y_2 = e^{-\frac{1}{2}x} \sin\frac{\sqrt{3}}{2}x$;

$$y = c_1 e^{-\frac{1}{2}x} \cos\frac{\sqrt{3}}{2}x + c_2 e^{-\frac{1}{2}x} \sin\frac{\sqrt{3}}{2}x$$

Example 4:

$$y'' + y = 0$$

Characteristic equation: $\lambda^2 + 1 = 0$

$$\lambda = \pm i$$

$$y_1 = \cos x \text{ and } y_2 = \sin x;$$

$$y = c_1 \cos x + c_2 \sin x$$

Example 5:

$$y''' - 4y'' + 5y' - 2y = 0$$

Characteristic equation: $\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$

$$(\lambda - 1)^2(\lambda - 2) = 0$$

$$\lambda = 1, 1, 2$$

$$y = c_1 e^x + c_2 x e^x + c_3 e^{2x}$$

Example 6:

$$y''' - y = 0$$

Characteristic equation: $\lambda^3 - 1 = 0$

$$(\lambda - 1)(\lambda^2 + \lambda + 1) = 0$$

$$\lambda = 1$$

$$\lambda = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$y = c_1 e^x + c_1 e^{-\frac{1}{2}x} \cos\frac{\sqrt{3}}{2}x + c_2 e^{-\frac{1}{2}x} \sin\frac{\sqrt{3}}{2}x$$

Example 7:

$$(\lambda - 2)^3(\lambda + 5)(\lambda^2 + 1) = 0$$

$$y = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x} + c_4 e^{-5x} + c_5 \cos x + c_6 \sin x$$