Lecture Notes 06th June 2023

- Non-homogeneous ODE with constant coefficients.
- Method of undetermined coefficients.

Theory:

Non-homogeneous ODE with constant coefficients:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x)$$

One way to solve this type of ODE is method of undetermined coefficient.

---

1. We solve the homogeneous ODE ($f(x) = 0$); we call the solution y_h.
2. We then find a particular solution of the non-homogeneous case y_p.
3. The solution of non-homogeneous ODE is $y = y_h + y_p$

To find y_p, we focus on $f(x)$. We have six cases:

1. $f(x)$ is a polynomial.

$f(x)$	y_p
$5x - 7$	$Ax + B$
$6x^2 + 5x - 7$	$Ax^2 + Bx + C$
$x^3 - 1$	$Ax^3 + Bx^2 + Cx + D$

2. $f(x)$ is trigonometric.

$f(x)$	y_p
$\sin 3x$	$A \sin 3x + B \cos 3x$
$\cos 6x$	$A \sin 6x + B \cos 6x$

3. $f(x)$ is exponential.

$f(x)$	y_p
e^{7x}	Ae^{7x}
e^{-3x}	Ae^{-3x}

4. f(x) is multiplication of polynomial and exponential.

$f(x)$	y_p
$(3x-8)e^{4x}$	$(Ax+B)e^{4x}$
$(x^2)e^{3x}$	$(Ax^2+Bx+C)e^{3x}$

5. f(x) is multiplication of trigonometric and exponential.

$f(x)$	y_p
$e^{5x}\sin 7x$	$(A\sin 7x + B\cos 7x)e^{5x}$

6. f(x) is multiplication of trigonometric polynomial.

$f(x)$	y_p
$3x^2\sin 5x$	$(Ax^2+Bx+C)\sin 5x +$ $(Ax^2+Bx+C)\cos 5x$

Solve the following:

Example 1:

$$y'' - 5y' + 6y = x^2 - x + 1$$

$$y'' - 5y' + 6y = 0$$

$$\text{Characteristic equation: } \lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 2)(\lambda - 3) = 0$$

$$\lambda = 2, 3$$

$$y_1 = e^{2x} \text{ and } y_2 = e^{3x};\ y_h = c_1 e^{2x} + c_2 e^{3x}$$

$$\text{Let } y_p = Ax^2 + Bx + C$$

$$y_p{}' = 2Ax + B$$

$$y_p{}'' = 2A$$

$$\text{Substituting values of } yp, yp', yp'' \text{in the given ODE}$$

$$2A - 5(2Ax + B) + 6(Ax^2 + Bx + C) = x^2 - x + 1$$

$$(6A)x^2 - (10A - 6B)x + (2A - 5B + 6C) = x^2 - x + 1$$

$$6A = 1$$

$$10A - 6B = 1$$

$$2A - 5B + 6C = 1$$

$$A = \frac{1}{6}; B = \frac{1}{9}; C = \frac{11}{54}$$

$$y_p = \frac{1}{6}x^2 + \frac{1}{9}x + \frac{11}{54}$$

$$y = y_h + y_p$$

$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{6}x^2 + \frac{1}{9}x + \frac{11}{54}$$

Example 2:

$$y'' + y' - 6y = e^x + \sin 3x$$

$$y'' + y' - 6y = 0$$

Characteristic equation: $\lambda^2 + \lambda - 6 = 0$

$$(\lambda + 3)(\lambda - 2) = 0$$

$$\lambda = 2, -3$$

$y_1 = e^{2x}$ and $y_2 = e^{-3x}$; $y_h = c_1 e^{2x} + c_2 e^{-3x}$

Let $y_p = Ae^x + B \sin 3x + C \cos 3x$

$$y_p{}' = Ae^x + 3B \cos 3x - 3C \sin 3x$$

$$y_p'' = Ae^x - 9B \sin 3x - 9C \cos 3x$$

Substituting values of yp, yp', yp'' in the given ODE

$$Ae^x - 9B \sin 3x - 9C \cos 3x + Ae^x + 3B \cos 3x - 3C \sin 3x - 6(Ae^x + B \sin 3x + C \cos 3x) = e^x + \sin 3x$$

$$(A + A - 6A)e^x + (-9B - 3C - 6B) \sin 3x + (-9C + 3B - 6C) \cos 3x = e^x + \sin 3x + 0. \cos 3x$$

$$-4A = 1$$

$$-15B - 3C -= 1$$

$$-15C + 3B -= 1$$

$$A = -\frac{1}{4}; B = -\frac{5}{78}; C = -\frac{1}{78}$$

$$y_p = -\frac{1}{4} e^x - \frac{5}{78} \sin 3x - \frac{1}{78} \cos 3x$$

$$y = y_h + y_p$$

$$y = c_1 e^{2x} + c_2 e^{-3x} + -\frac{1}{4} e^x - \frac{5}{78} \sin 3x - \frac{1}{78} \cos 3x$$

This method fails for some problems.

If there is any duplication between homogeneous solution y_h and your choice of y_p then this method fails.

To fix this issue instead of y_p we should consider $x.y_p$. Now if $x.y_p$ and y_h have any duplication then we should consider $x^2.y_p$. So $x.y_p, x^2.y_p, \ldots$ continued till there's no duplication.

Solve the following:

Example 1:

$$y'' - 2y' - 8y = e^{-2x}$$

$$y'' - 2y' - 8y = 0$$

$$\text{Characteristic equation: } \lambda^2 - 2\lambda - 6 = 0$$

$$(\lambda - 4)(\lambda + 2) = 0$$

$$\lambda = 4, -2$$

$$y_1 = e^{4x} \text{ and } y_2 = e^{-2x};\ y_h = c_1e^{4x} + c_2e^{-2x}$$

$$\text{Let } y_p = Ae^{-2x}$$

Now we can see that there is a duplication between homogeneous solution y_h and your choice of y_p

$$c_2e^{-2x} \text{ and } Ae^{-2x}$$

$$\therefore y_p = Axe^{-2x}$$

$$y_p' = Ae^{-2x} - 2Axe^{-2x}$$

$$y_p'' = -4Ae^{-2x} + 4Axe^{-2x}$$

Substituting values of yp, yp', yp'' in the given ODE

$$y'' - 2y' - 8y = e^{-2x}$$

$$-4Ae^{-2x} + 4Axe^{-2x} - 2(Ae^{-2x} - 2Axe^{-2x}) - 8(Axe^{-2x}) = e^{-2x}$$

$$(-4A - 2A)e^{-2x} + (4A + 4A - 8A)xe^{-2x} = e^{-2x}$$

$$A = -\frac{1}{6}$$

$$y_p = -\frac{1}{6}xe^{-2x}$$

$$y = y_h + y_p$$

$$y = c_1e^{4x} + c_2e^{-2x} - \frac{1}{6}xe^{-2x}$$

Example 2:

$$y^{(4)} - y'' = 3x^2 - \sin 2x$$

$$y^{(4)} - y'' = 0$$

Characteristic equation: $\lambda^4 - \lambda^2 = 0$

$$\lambda^2(\lambda - 1)(\lambda + 1) = 0$$

$$\lambda = 0, 0, 1, -1$$

$$y_1 = c1, y_2 = c2x, y_3 = e^x \text{ and } y_4 = e^{-x}; y_h = c_1 + c_2x + c_3e^x + c_4e^{-x}$$

$$\text{Let } y_p = Ax^2 + Bx + C + D\sin 2x + E\cos 2x$$

Now we can see that there is a duplication between homogeneous solution $y_h$and your choice of y_p

$$c_1 + c_2x \text{ and } Bx + C$$

$$\therefore y_p = x(Ax^2 + Bx + C) + D\sin 2x + E\cos 2x$$

$$\therefore y_p = (Ax^3 + Bx^2 + Cx) + D\sin 2x + E\cos 2x$$

We can still see that there is a duplication between homogeneous solution $y_h$and your choice of y_p

$$c_2x \text{ and } Cx$$

$$\therefore y_p = x^2(Ax^2 + Bx + C) + D\sin 2x + E\cos 2x$$

$$\therefore y_p = Ax^4 + Bx^3 + Cx^2 + D\sin 2x + E\cos 2x$$

$$y_p' = 4Ax^3 + 3Bx^2 + 2Cx + 2D\cos 2x - 2E\sin 2x$$

$$y_p'' = 12Ax^2 + 6Bx + 2C - 4D\sin 2x - 4E\cos 2x$$

$$y_p''' = 24Ax + 6B - 8D\cos 2x + 8E\sin 2x$$

$$y_p^{(4)} = 24A + 16D\sin 2x + 16E\cos 2x$$

Substituting values of $y_p^{(4)}, y_p''$ in the given ODE

$$y^{(4)} - y'' = 3x^2 - \sin 2x$$

$$24A + 16D\sin 2x + 16E\cos 2x - 12Ax^2 - 6Bx - 2C + 4D\sin 2x + 4E\cos 2x = 3x^2 - \sin 2x$$

$$(-12A)x^2 + (-6B)x + (24A - 2C) - (-16D - 4D)\sin 2x + (16E - 4E)\cos 2x = 3x^2 + 0(x) + 0 - \sin 2x + 0\cos x$$

$$-12A = 3$$

$$-6B = 0$$

$$24A - 2C = 0$$

$$-16D - 4D = 1$$

$$16E - 4E = 0$$

$$A = -\frac{1}{4}; B = 0; C = -3; D = -\frac{1}{20}; E = 0$$

$$\therefore y_p = -\frac{1}{4}x^4 - 3x^2 - \frac{1}{20}\sin 2x$$

$$y = y_h + y_p$$

$$y = c_1 + c_2x + c_3e^x + c_4e^{-x} + -\frac{1}{4}x^4 - 3x^2 - \frac{1}{20}\sin 2x$$