GENG 8010–Part 1: Elements of Differential and Difference Equations

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Winter 2023

Homogeneous equation I

Consider the n^{th} order homogeneous differential equation

$$(a_n D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0) y(t) = 0$$

Assume the solutions are of the form $y = e^{rt}$ where r is a constant to be determined. Then

$$(a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0) e^{rt} = 0$$

and that implies $\forall t$,

$$a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0 = 0$$

Homogeneous equation II

Again the above n^{th} order polynomial with roots $r_1, r_2, \ldots, r_n$ is called the characteristic or auxiliary equation of the system, and the solutions are

$$y_1 = e^{r_1 t}, y_2 = e^{r_2 t}, \ldots, y_n = e^{r_n t}$$

As a result of the above, the most general solution of the DE is

$$\boxed{y_c(t) = K_1 e^{r_1 t} + K_2 e^{r_2 t} + \ldots + K_n e^{r_n t}} \qquad \text{(HS)}$$

Again, as in the case of the $2nd$ order equation **three cases** can arise:

Case 1–All the roots are real and distinct

There are n solutions: $y_i = e^{r_i t}, i = 1, 2, \ldots n$ and recall the Wronskian

$$W(t) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} & \ldots & e^{r_n t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} & \ldots & r_n e^{r_n t} \\ \ldots & \ldots & \ldots & \ldots \\ r_1^{n-1} e^{r_1 t} & r_2^{n-1} e^{r_2 t} & \ldots & r_n^{n-1} e^{r_n t} \end{vmatrix} \neq 0$$

Hence, these solutions are linearly independent and (HS) is the most general solution.

Homogeneous solution–cont.

Case 2–All the roots are real but some are repeated

Suppose r_1 is repeated twice. Then, clearly the first two column of $W(t)$ are the same and

$$W(t)=\begin{vmatrix} e^{r_1 t} & e^{r_1 t} & \dots & e^{r_n t} \\ r_1 e^{r_1 t} & r_1 e^{r_1 t} & \dots & r_n e^{r_n t} \\ \dots & \dots & \dots & \dots \\ r_1^{n-1} e^{r_1 t} & r_1^{n-1} e^{r_1 t} & \dots & r_n^{n-1} e^{r_n t} \end{vmatrix} = 0$$

However, it can be shown that $y_1(t)=e^{r_1 t}$, and $y_2(t)=te^{r_1 t}$ satisfy the DE and are independent solutions. Generalizing, if $r_1=r_2=\ldots=r_k$, then

$$\boxed{y_c(t)=K_1e^{r_1 t}+K_2te^{r_1 t}+\ldots+K_kt^{k-1}e^{r_1 t}+K_{k+1}e^{r_{k+1} t}+\ldots+K_ne^{r_n t}}$$

Homogeneous solution–cont.

Case 3–Some roots appear in complex conjugate pairs

Assume that $r_{1,2}=a\pm jb$. Then

$$\begin{aligned} K_1e^{r_1 t}+K_2e^{r_2 t} =& e^{at}\left(K_1e^{jbt}+K_2e^{-jbt}\right) \\ =& e^{at}\left[(K_1+K_2)\cos bt+j(K_1-K_2)\sin bt\right] \\ =& e^{at}\left(A\cos bt+B\sin bt\right) \end{aligned}$$

Two trignometric functions with the same frequency can always be written as a single term with a phase angle. So,

$$K_1e^{r_1 t}+K_2e^{r_2 t}=Ke^{at}\cos(bt+\phi)$$

Homogeneous solution–Example

Example–Suppose a homogeneous differential equation's characteristic polynomial has the following roots

$$m_1=m_2=m_3=-2;\quad m_4=m_5=3;\quad m_6=-4;\quad m_{7,8,9,10}=-3\pm i2$$

Then based on our discussion

$$\begin{aligned} y(t)=&(c_1+c_2t+c_3t^2)e^{-2t}+(c_4+c_5t)e^{3t}+c_6e^{-4t} \\ &+e^{-3t}(c_7\cos(2t)+c_8\sin(2t))+te^{-3t}(c_9\cos(2t)+c_{10}\sin(2t)) \end{aligned}$$

Higher order non-homogeneous soln.

Consider

$$(a_nD^n+a_{n-1}D^{n-1}+\ldots+a_1D+a_0)y(t)=F(t)$$

that has a general solution of the form

$$y(t)=y_c+y_p$$

There are two methods for finding y_p:

1. Method of Undetermined Coefficients (MUC)
2. Variation of Parameters (VP)

About MUC I

- Note that this method works when and only when $F(t)$ is itself a particular solution of some homogeneous linear differential equation with constant coefficients.
- The basic procedure here is to assume that the form of y is some linear combination of the terms of $F(t)$ (polynomial function, exponential function, sine or cosine functions or finite sums and products of these functions) and derivatives with each term multiplied by a constant. If however, certain terms in $F(t)$ are similar to those in $y_c(t)$, then certain modifications are necessary.

Examples I

Example–Find the general solution of

$$y'' + 5y' + 6y = 4e^{-x} + 5\sin x$$

Characteristic equation is

$$D^2 + 5D + 6 = 0 \Longrightarrow m_1 = -2;\ m_2 = -3$$

Therefore

$$y_c = c_1 e^{-2x} + c_2 e^{-3x}$$

Now based on the form of the forcing functon $F(x)$, and previous discussion, assume

$$y_p = Ae^{-x} + B\sin x + C\cos x$$

Examples II

substituting into the differential equation and simplifying gives

$$2Ae^{-x} + 5(B - C)\sin x + 5(B + C)\cos x = 4e^{-x} + 5\sin x$$

Which gives $2A = 4$ *or* $A = 2$, $5(B - C) = 5$ and $5(B + C) = 0$ which results in $B = \frac{1}{2}$ and $C = -\frac{1}{2}$, and finally

$$\boxed{y = y_c + y_p = c_1 e^{-2x} + c_2 e^{-3x} + 2e^{-x} + \frac{1}{2}\sin x - \frac{1}{2}\cos x}$$

Examples III

Example–Find the solution of

$$(D^2 + 2D + 1)y(t) = te^{-t}$$

Note that CE is

$$D^2 + 2D + 1 = 0$$

which gives $m_1 = m_2 = -1$, and so

$$y_c = c_1 e^{-t} + c_2 te^{-t}$$

Examples IV

Now notice that if we didn't have similar terms in y_c we would have picked $y_p = Ate^{-t} + Be^{-t}$, as a result, take

$$y_p = At^3e^{-t} + Bt^2e^{-t}$$

Now substitute y_p into the diff. eq. and evaluate

$$(D^2 + 2D + 1)y_p(t) = (6At + 2B)e^{-t} = te^{-t} \Longrightarrow A = \frac{1}{6}; B = 0$$

$$\boxed{y(t) = c_1e^{-t} + c_2te^{-t} + \frac{1}{6}t^3e^{-t}}$$

Selection of y_p

$F(t)$	Educated guess for y_p
k (constant)	A
$pt + q$ $(p, q$ constants)	$At + B$
$pt^2 + qt + k$	$At^2 + Bt + C$
$\sin pt$	$A\sin pt + B\cos pt$
$\cos pt$	$A\sin pt + B\cos pt$
e^{pt}	Ae^{pt}
$(pt + q)e^{kt}$	$(At + B)e^{kt}$
t^pe^{qt}	$(A_pt^p + A_{p-1}t^{p-1} + \ldots + A_0)e^{qt}$
$e^{pt}\sin qt$	$e^{pt}(A\sin qt + B\cos qt)$
$e^{pt}\cos qt$	$e^{pt}(A\sin qt + B\cos qt)$
$pt^2\sin qt$	$(At^2 + Bt + C)\sin qt + (Et^2 + Ft + G)\cos qt$
$te^{pt}\cos qt$	$(At + B)e^{pt}\sin qt + (Ct + E)e^{pt}\cos qt$

More examples I

Example–Consider

$$(D^4 + D^3)y(t) = 1 - t^2e^{-t}$$

Note for the above $m_1 = m_2 = m_3 = 0$ and $m_4 = -1$. So

$$y_c = c_1 + c_2t + c_3t^2 + c_4e^{-t}$$

Now ordinarily we would pick

$$y_p = A + Be^{-t} + Cte^{-t} + Dt^2e^{-t}$$

But notice that many terms in y_p are in y_c. Multiply each component of the solution with t^n where n is the minimum integer that would get rid of the duplication. Therefore,

$$y_p = At^3 + Bte^{-t} + Ct^2e^{-t} + Dt^3e^{-t}$$

About VP method

1. Distinct advantage over other methods that it always yields a particular solution y_p if the associated homogeneous equation can be solved.
2. Applicable to linear higher-order equations.
3. Unlike undetermined coefficients, is not limited to cases where the forcing function is a combination of certain functions.
4. No special cases arise due to the nonhomogeneous term being included in the complementary function.
5. It works for (time) varying systems, i.e., $a_i(t)$ or $a_i(x)$.