Lecture Notes 27^{th} May 2023

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- Laplace Transformations

Notice:

Midterm – 2

- 1 Question: Undetermined Coefficient - Application
- 1 Question: Taylor Series - Similar to Homework
- 2 Questions: Power Series - Similar to the last question of Homework
- 2 Questions: Laplace - 1 Simple Laplace Transformation, 1 ODE Laplace Application

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Theory:

Let $f(t): [0, \infty) \to \mathbb{R}$

Laplace Transformation of f(t) is:

$$\mathcal{L}\big(f(t)\big) = \int_0^\infty e^{-st}. f(t)\, dt$$

$$f(t) \to F(s)$$

Find Laplace Transform of the following:

Example 1:

$$f(t) = 1$$

$$\mathcal{L}(1) = \int_0^\infty 1. e^{-st} dt$$

$$= -\frac{e^{-st}}{s}\Bigg|_{t=0}^{t=\infty}$$

$$= \left[0 - \left(-\frac{1}{s}\right)\right] \text{ ... (for } s > 0)$$

$$\therefore\ \boldsymbol{\mathcal{L}(1) = \frac{1}{s}}$$

Example 2:

$$f(t) = t$$

$$\mathcal{L}(1) = \int_0^\infty t.e^{-st}dt$$

$$= -\frac{t.e^{-st}}{s}\Bigg|_{t=0}^{t=\infty} + \frac{1}{s}\int_0^\infty e^{-st}\,dt$$

$$\lim_{t\to\infty}\frac{t.e^{-st}}{s} \text{ ... Here } t.e^{-st} \text{ is of the form } \infty \text{ x } 0$$

$$\therefore \text{ We have to change it to } \frac{0}{0} \text{ or } \frac{\infty}{\infty} \text{ ... (L'Hôpital's rule)}$$

$$\lim_{t\to\infty}\frac{t}{e^{st}.s} = \frac{\infty}{\infty}$$

$$\text{L'Hôpital's rule: } \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

$$\therefore \lim_{t\to\infty}\frac{1}{e^{st}.s^2} = \frac{1}{\infty} = 0$$

$$= [0-0] + \frac{1}{s^2} \text{ ... (for } s > 0)$$

$$\therefore \boldsymbol{\mathcal{L}(t) = \frac{1}{s^2}}$$

Example 3:

$$f(t) = e^{-3t}$$

$$\mathcal{L}(1) = \int_0^{\infty} e^{-3t}.e^{-st}dt$$

$$= \int_0^{\infty} e^{-t(s+3)}\,dt$$

$$= -\frac{1}{s+3}.e^{-t(s+3)}\Big|_{t=0}^{t=\infty}$$

$$= \left[0 - \left(-\frac{1}{s+3}\right)\right] \;\ldots(\text{for } s > 0)$$

$$\therefore\ \boldsymbol{\mathcal{L}(e^{-3t}) = \frac{1}{s+3}}$$

Homework:

$$f(t) = \sin 2t$$

Laplace Formulas:

$$\mathcal{L}(1) \qquad \frac{1}{s}$$

$$\mathcal{L}(t^n) \qquad \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(e^{at}) \qquad \frac{1}{s-a}$$

$$\mathcal{L}(\sin at) \qquad \frac{a}{s^2+a^2}$$

$$\mathcal{L}(\cos at) \qquad \frac{s}{s^2+a^2}$$

$$\mathcal{L}(e^{at}.f(t)) \qquad F(s-a)$$

Example 1:

$$\mathcal{L}(t^6 e^{4t}) = \frac{t^6}{(s-4)^7}$$

Example 2:

$$\mathcal{L}(\sin 7t\, e^{5t}) = \frac{7}{(s-5)^2+7^2}$$

Remark:

Laplace is Linear Transformation:

$$\mathcal{L}(a\, f(t) + b\, g(t)) = a\, \mathcal{L}(f(t)) + b\, \mathcal{L}(g(t))$$

Example 1:

$$\mathcal{L}(1+5t) = \mathcal{L}(1) + 5\mathcal{L}(t)$$

$$= \frac{1}{s} + \frac{5}{s^2}$$

Example 2:

$$\mathcal{L}(2e^{-3t} + 5\sin 7t) = 2\mathcal{L}(e^{-3t}) + 5\mathcal{L}(\sin 7t)$$

$$= \frac{2}{s+3} + \frac{5.7}{s^2+7^2}$$

$$= \frac{2}{s+3} + \frac{35}{s^2+49}$$

<u>Laplace Inverse</u> <u>($\mathcal{L}^{-1}$)</u>**:**

$$\mathcal{L}\big(f(t)\big) = F(s)$$

$$\mathcal{L}^{-1}(F(s)) = f(t)$$

Example 1:

$$\mathcal{L}(1) = \frac{1}{s} \rightarrow \mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$$

Example 2:

$$\mathcal{L}(e^{-5t}) = \frac{1}{s+5} \rightarrow \mathcal{L}^{-1}\left(\frac{1}{s+5}\right) = e^{-5t}$$

<u>Remark</u>:

Laplace is Linear Transformation:

$$\mathcal{L}^{-1}(a\,F(s) + b\,G(s)) = a\,\mathcal{L}^{-1}(F(s)) + b\,\mathcal{L}^{-1}(G(s))$$

Example 1:

$$\mathcal{L}^{-1}\left(\frac{1}{s^5}\right) = \frac{t^4}{4!}$$

Example 2:

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+7}\right) = \frac{1}{\sqrt{7}}\sin\sqrt{7}t$$

Example 3:

$$\mathcal{L}^{-1}\left(\frac{-2s+6}{s^2+4}\right) = -2\,\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) + 3\,\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right)$$

$$= -2\cos 2t + 3\sin 2t$$

<u>Application</u>:

ODE by Laplace Transformations:

$$\mathcal{L}\left(y^{(n)}\right) = s^n\,\mathcal{L}(y) - s^{n-1}\,y(0) - s^{n-2}\,y'(0) - s^{n-3}\,y''(0) - \cdots - y^{(n-1)}(0)$$

Example 1:

$$\mathcal{L}(y') = s\,\mathcal{L}(y) - y(0)$$

Example 2:

$$\mathcal{L}(y'') = s^2\,\mathcal{L}(y) - s\,y(0) - y'(0)$$

Example 3:

$$\mathcal{L}(y''') = s^3\,\mathcal{L}(y) - s^2\,y(0) - s\,y'(0) - y''(0)$$

Example 1:

$$y' + 3y = 13\sin 2t\,;\, y(0) = 6$$

$$\mathcal{L}(y') + 3\,\mathcal{L}(y) = 13\,\mathcal{L}(\sin 2t)$$

$$s\,\mathcal{L}(y) - 6 + 3\,\mathcal{L}(y) = \frac{26}{s^2+4}$$

$$(s+3)\mathcal{L}(y) = \frac{26}{s^2+4} + 6$$

$$\mathcal{L}(y) = \frac{1}{s+3}\left(\frac{26}{s^2+4} + 6\right)$$

$$\mathcal{L}(y) = \frac{6s^2+50}{(s^2+4)(s+3)}$$

$$\frac{6s^2+50}{(s^2+4)(s+3)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+4}$$

$$6s^2 + 0s + 50 = (A+B)s^2 + (3B+C)s + (4A+3C)$$

$$(A+B) = 6$$

$$(3B+C) = 0$$

$$4A + 3C = 50$$

$$A = 8;\, B = -2;\, C = 6$$

$$\mathcal{L}(y) = \frac{8}{s+3} + \frac{-2s+6}{s^2+4}$$

$$\mathcal{L}(y) = \frac{8}{s+3} + \frac{-2s}{s^2+4} + \frac{6}{s^2+4}$$

$$y = 8\,\mathcal{L}^{-1}\left(\frac{1}{s+3}\right) - 2\,\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) + 3\,\mathcal{L}^{-1}(\frac{2}{s^2+4})$$

$$\mathbf{y = 8\,e^{-3t} - 2\cos 2t + 3\sin 2t}$$

Example 2:

$$y'' - 3y' + 2y = e^{-4t}; y(0) = 6, y'(0) = 5$$

$$\mathcal{L}(y'') - 3\,\mathcal{L}(y') + 2\,\mathcal{L}(y) = \mathcal{L}(e^{-4t})$$

$$s^2\,\mathcal{L}(y) - s\,y(0) - y'(0) - 3s\,\mathcal{L}(y) + 3\,y(0) + 2\,\mathcal{L}(y) = \frac{1}{s+4}$$

$$\mathcal{L}(y) = \frac{s^2 + 6s + 9}{(s+4)(s-1)(s-2)}$$

$$\frac{s^2 + 6s + 9}{(s+4)(s-1)(s-2)} = \frac{A}{(s+4)} + \frac{B}{(s-1)} + \frac{C}{(s-2)}$$

$$A = \frac{1}{30}; B = -\frac{16}{5}; C = \frac{25}{6}$$

$$\mathcal{L}(y) = \frac{1}{30}\frac{1}{(s+4)} - \frac{16}{5}\frac{1}{(s-1)} + \frac{25}{6}\frac{1}{(s-2)}$$

$$y = \frac{1}{30}\,\mathcal{L}^{-1}\left(\frac{1}{s+4}\right) - \frac{16}{5}\,\mathcal{L}^{-1}\left(\frac{1}{(s-1)}\right) + \frac{25}{6}\,\mathcal{L}^{-1}\left(\frac{1}{(s-2)}\right)$$

$$\mathbf{y = \frac{e^{-4t}}{30} - \frac{16e^{t}}{5} + \frac{25e^{2t}}{6}}$$