Part I–Outline II

Part I–Outline III

The integrating factor method: linear case XIII

integrating (the right side by parts)[†]

$$\begin{aligned} xy^3e^{-y} =& -2\int y^2e^{-y}dy \\ =& 2y^2e^{-y} + 4ye^{-y} + 4e^{-y} + c \end{aligned}$$

and the implicit solution is given by

$$\boxed{xy^3 = 2y^2 + 4y + 4 + ce^y}$$

[†] $\int x^2e^{ax}dx = e^{ax}\left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)$

The integrating factor method: nonlinear case I

Let us start again with

$$M(x, y)dx + N(x, y)dy = 0 \tag{3.3}$$

Suppose $u(x)$ is an integrating factor for (3.3), and $\frac{\partial u}{\partial y} = 0$, and $\frac{\partial u}{\partial x} = \frac{du}{dx}$. Multiplying the (3.3) with $u(x)$ gives

$$uMdx + uNdy = 0$$

If the above is to be exact then $\frac{\partial}{\partial y}(uM) = \frac{\partial}{\partial x}(uN)$, and

$$u\frac{\partial M}{\partial y} + M\frac{\partial u}{\partial y} = u\frac{\partial N}{\partial x} + N\frac{\partial u}{\partial x}$$

The integrating factor method: nonlinear case IV

Example–Solve for the solution of

$$y(x+y+1)dx + x(x+3y+2)dy = 0$$

Calculate

$$\frac{\partial M}{\partial y} = x + 2y + 1 \qquad \frac{\partial N}{\partial x} = 2x + 3y + 2$$

and

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -x - y - 1$$

and

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{x+y+1}{x(x+3y+2)}$$

Note that the above right side is not a function of x only as required in Rule 1.

The integrating factor method: nonlinear case V

So now calculate

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{x+y+1}{y(x+y+1)} = -\frac{1}{y}$$

So now we see that the above right side is a function of y only and therefore Rule 2 applies and $e^{\ln|y|} = |y|$ is the IF. In other words

$$IF = \begin{cases} y & if \quad y > 0 \\ -y & if \quad y < 0 \end{cases}$$

In either case

$$(xy^2 + y^3 + y^2)dx + (x^2y + 3xy^2 + 2xy)dy = 0$$

More on determination of an integrating factor

consider four commonly encountered exact differentials below

$$d(xy) = xdy + ydx \qquad d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$$
$$d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2} \qquad d\left(\tan^{-1}\frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$$

based on the above, group the equation as

$$ydx + (x + x^3y^2)dy = (ydx + xdy) + x^3y^2dy = d(xy) + x^3y^2dy = 0$$

or

$$\frac{1}{(xy)^3}d(xy) + \frac{1}{y}dy = 0$$

which is integrable, so

$$-\frac{1}{2x^2y^2} + \ln|y| + \ln|c| = 0 \Rightarrow \boxed{2x^2y^2\ln|cy| = 1}$$

Another useful trick I

Sometimes a change of variables can result in a transformation of the differential equation in a solvable form. The differential equation itself can be explored for this purpose. Let's illustrate this through an example.

Example– Solve

$$(x + 2y - 1)dx + 3(x + 2y)dy = 0$$

Notice the term $x + 2y$ has occurred twice. So let's define

$$v = x + 2y \Rightarrow dx = dv - 2dy$$

and subsititue to get

$$(v - 1)(dv - 2dy) + 3vdy = 0$$

Bernoulli's Equation I

Bernoulli's Equation is a well know equation that is of the class of equations we have considered, and is given by

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{3.4}$$

- We have dealt already with the case of $n = 1$, in (B) for which the variables are separable.
- So consider the cases when $n \neq 1$. (3.4) can be put into that form.

$$y^{-n}dy + Py^{-n+1}dx = Qdx$$

Bernoulli's Equation II

Let

$$r = y^{-n+1}$$

$$dr = (1-n)y^{-n}dy \Longrightarrow dy = \frac{1}{(1-n)y^{-n}}dr$$

Therefore

$$dr + (1-n)Prdx = (1-n)Qdx$$

Hence, the Bernoulli Equation can be reduced with a change of variable to a standard form that is easily solvable using the techniques already discussed.

Nature of solution(s)– I

We already established that any nth order linear differential equation can be written as

$$(a_n D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0) y(t) = F(t)$$

The above is called a **inhomogeneous differential equation**. Setting the right side equal to 0, we get the **homogeneous differential equation**

$$(a_n D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0) y(t) = 0 \tag{4.1}$$

It turns out that the above differential equation will have at most, n **linearly independent** solutions.

Nature of solution(s)– II

Theorem 4.1.1

A necessary and sufficient condition for the n solutions of (4.1) to be independent is that their Wronskian does not vanish. That is if $y_1, y_2, \ldots, y_n$ *represent the n solutions, the Wronskian* $W(t)$ *is nonzero, where*

$$W(t) = \begin{vmatrix} y_1 & y_2 & \ldots & y_n \\ Dy_1 & Dy_2 & \ldots & Dy_n \\ \ldots & \ldots & \ldots & \ldots \\ D^{n-1}y_1 & D^{n-1}y_2 & \ldots & D^{n-1}y_n \end{vmatrix}$$

Nature of solution(s)– V

Corollary 4.1.3

to Theorem 4.1.2 is

1. *A constant multiple $y = c_1 y_1(x)$ of a solution $y_1(x)$ of a homogeneous linear differential equation is also a solution.*
2. *A homogeneous linear differential equation always possesses the trivial solution $y = 0$.*