

# GENG 8010–Part 2 - Elements of Applied Linear Algebra

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## Introduction & axiomatic definitions– I

Here we shall specify the collection of objects that form the center of our study.

**Set:** The collection of objects or elements.

Fields that are considered  $\left\{ \begin{array}{l} \text{real numbers} \\ \text{complex numbers} \\ \text{rational functions} \end{array} \right.$

## Introduction & axiomatic definitions– III

**Example–** Is  $\{0, 1\}$  a field?

**Ans.:** No since it violates 1,6.

**But let us define:**

$1 + 1 = 0 + 0 = 0; 0.1 = 0.0 = 0; 1 + 0 = 1; 1.1 = 1$  then  $\{0, 1\}$  is a field.  
It is called the **field of binary numbers**.

**Example–**The set of all real numbers define a field called  $\mathbb{R}$ .

**Example–**The set of all complex numbers define a complex field  $\mathbb{C}$

**Definition (Ring)** A set that satisfies all the axioms in the previous definition except (7) is called a *Commutative Ring* with multiplicative identity.

## Introduction & axiomatic definitions– II

**Definition (Field)–** A field consists of a set  $\mathcal{F}$  of elements called scalars. “+” and “.” are defined and satisfy the following axioms:

- 1  $\forall \alpha, \beta \in \mathcal{F} \exists \alpha + \beta \in \mathcal{F}$  and  $\alpha \cdot \beta \in \mathcal{F}$
- 2  $\forall \alpha, \beta \in \mathcal{F} \alpha + \beta = \beta + \alpha$  and  $\alpha \cdot \beta = \beta \cdot \alpha$   
“+” and “.” are commutative
- 3  $\forall \alpha, \beta, \gamma \in \mathcal{F} (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$  and  $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$   
“+” and “.” are associative.
- 4  $\forall \alpha, \beta, \gamma \in \mathcal{F} \alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$   
“.” is distributive wrt “+”
- 5  $\exists 0 \in \mathcal{F} \ni \alpha + 0 = \alpha$  and  $\exists 1 \in \mathcal{F} \ni 1 \cdot \alpha = \alpha \forall \alpha \in \mathcal{F}$
- 6  $\forall \alpha \in \mathcal{F} \exists \beta \in \mathcal{F} \ni \alpha + \beta = 0$   $\beta$  : additive inverse
- 7  $\forall \alpha \in \mathcal{F}$  other than  $\alpha = 0 \exists \beta \in \mathcal{F} \ni \alpha \cdot \beta = 1$   
 $\beta$  : multiplicative inverse

## Introduction & axiomatic definitions– IV

**Example–**The set of all integers is not a field but is a ring.

**Definition (Vector Space)–**a vector space over a field  $\mathcal{F}$  is defined by  $(\mathcal{X}, \mathcal{F})$  which consists of a set  $\mathcal{X}$  of elements called vectors defined over a field  $\mathcal{F}$  and two operation of vector addition and scalar multiplication such that the following axioms are satisfied:

- 1  $\forall \mathbf{x}_1$ , and  $\mathbf{x}_2 \in \mathcal{X} \exists \mathbf{x}_3 = \mathbf{x}_1 + \mathbf{x}_2 \in \mathcal{X}$ .
- 2  $\forall \mathbf{x}_1$ , and  $\mathbf{x}_2 \in \mathcal{X} \mathbf{x}_1 + \mathbf{x}_2 = \mathbf{x}_2 + \mathbf{x}_1$  “+” is commutative
- 3  $\forall \mathbf{x}_1, \mathbf{x}_2$ , and  $\mathbf{x}_3 \in \mathcal{X} (\mathbf{x}_1 + \mathbf{x}_2) + \mathbf{x}_3 = \mathbf{x}_1 + (\mathbf{x}_2 + \mathbf{x}_3)$  “+” is associative.
- 4  $\exists \mathbf{0} \in \mathcal{X} \ni \mathbf{0} + \mathbf{x} = \mathbf{x} \forall \mathbf{x} \in \mathcal{X}$
- 5  $\forall \mathbf{x} \in \mathcal{X} \exists \hat{\mathbf{x}} \in \mathcal{X} \ni \mathbf{x} + \hat{\mathbf{x}} = \mathbf{0}$
- 6  $\forall \alpha \in \mathcal{F}$  and  $\mathbf{x} \in \mathcal{X} \exists \bar{\mathbf{x}} = \alpha \mathbf{x} \in \mathcal{X}$  called scalar product.
- 7  $\forall \alpha, \beta \in \mathcal{F}$  and  $\mathbf{x} \in \mathcal{X} \alpha \cdot (\beta \cdot \mathbf{x}) = (\alpha \cdot \beta) \cdot \mathbf{x}$  scalar multiplication is associative.

## Introduction & axiomatic definitions– V

- 8  $\forall \alpha \in \mathcal{F}$  and  $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$   $\alpha \cdot (\mathbf{x}_1 + \mathbf{x}_2) = \alpha \cdot \mathbf{x}_1 + \alpha \cdot \mathbf{x}_2$  scalar multiplication is distributive
- 9  $\forall \alpha, \beta \in \mathcal{F}$  and  $\mathbf{x} \in \mathcal{X}$   $(\alpha + \beta)\mathbf{x} = (\alpha \cdot \mathbf{x} + \beta \cdot \mathbf{x})$  scalar multiplication is distributive wrt scalar addition.
- 10  $\forall \mathbf{x} \in \mathcal{X} \exists 1 \in \mathcal{F} \ni 1 \cdot \mathbf{x} = \mathbf{x}$

**Example**—A field can form a vector space over itself, e.g.,  $(\mathbb{R}, \mathbb{R})$ , and  $(\mathbb{C}, \mathbb{C})$  which are the real and complex vector spaces respectively.

**Example**—How about

- 1  $(\mathbb{R}, \mathbb{C})$
- 2  $(\mathbb{C}, \mathbb{R})$