GENG 8010–Part 1: Elements of Differential and Difference Equations

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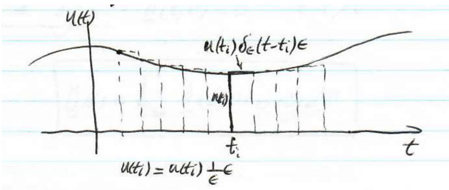
Part I–Outline I

Part I–Outline II

Part I–Outline III

Some system engineering concepts IV



Consider the input $u(t)$ as in the figure on the left. Every piecewise continuous function can be approximated by a series of pulse functions, and

$$u(t) \cong \sum_i u(t_i)\delta_\epsilon(t-t_i)\epsilon$$

$$y(t) = \mathscr{H}u(t) \cong \sum_i \mathscr{H}u(t_i)\delta_\epsilon(t-t_i)\epsilon$$

Some system engineering concepts V

As $\epsilon \longrightarrow 0$, the approximation will tend to exact equality and the summation will become an integration

$$y(t) = \int_{-\infty}^{\infty} \mathscr{H}\delta(t-\tau)u(\tau)d\tau$$

$$h(t,\tau) = \mathscr{H}\delta(t-\tau)$$

Response for time-varying non-causal single variable system

$$y(t) = \int_{-\infty}^{\infty} h(t,\tau)u(\tau)d\tau$$

for a causal system $h(t,\tau) = 0 \ \forall \ \tau > t$, so

Some system engineering concepts VI

Response for time-varying causal single variable system

$$y(t) = \int_{-\infty}^{t} h(t,\tau)u(\tau)d\tau$$

If the system is relaxed at t_0 at which time the input is applied

Response for time-varying relaxed causal single variable system

$$y(t) = \int_{t_0}^{t} h(t,\tau)u(\tau)d\tau$$

Some system engineering concepts VII

Theorem 8.3.1

Relaxed system– *A system is relaxed at t_0 iff $u_{[t_0,\infty)} = 0$ implies* $y_{[t_0,\infty)} = 0$.

For linear time invariant (LTI) systems $h(t,\tau) = h(t-\tau)$, so

Response for LTI relaxed causal single variable system

$$y(t) = \int_{t_0}^{t} h(t-\tau)u(\tau)d\tau$$

Some system engineering concepts VIII

Assume $t_0 = 0$, also let $t - \tau = \lambda \longrightarrow -d\tau = d\lambda$ and at $\tau = 0 \rightarrow t = \lambda$ and $\tau = t \rightarrow \lambda = 0$ which gives.

$$y(t) = \int_{\lambda}^{0} h(\lambda)u(t-\lambda)(-d\lambda) = \int_{0}^{\lambda} h(\lambda)u(t-\lambda)d\lambda$$

Response of LTI single variable system

$$y(t) = \int_{0}^{t} h(t)u(t-\tau)d\tau$$

Compare to previous discussion involving the Green Function

Some system engineering concepts IX

Taking the Laplace transform of the above

I/O relation through transfer function

$$Y(s) = H(s)U(s)$$

and

$$H(s) = \frac{Y(s)}{U(s)} = \mathscr{L}\{h(t)\}$$

First order response I

Consider the following first order system

$$\frac{dx(t)}{dt} + a_0 x(t) = b_0 r(t)$$

Taking Laplace transform

$$sX(s) - x(0) + a_0 X(s) = b_0 R(s) \Longrightarrow (s + a_0)X(s) = b_0 R(s) + x(0)$$

Response of 1st order system

$$X(s) = \underbrace{\frac{b_0}{s+a_0}R(s)}_{\text{zero state response}} + \underbrace{\frac{x(0)}{s+a_0}}_{\text{zero input response}}$$

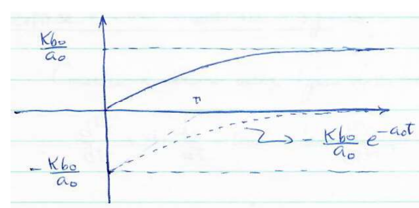
First order response II

$$H(s) = \mathscr{L}\left\{\frac{output}{input}\right\} = \frac{X(s)}{R(s)} = \frac{b_0}{s+a_0}, \quad with \ \ x(0) = 0$$

Considering $r(t) = Ku(t) \Longrightarrow R(s) = \frac{K}{s}$

If $x(0) = 0$

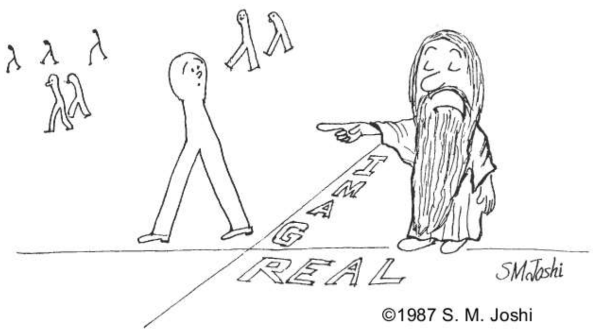
$$X(s) = \frac{Kb_0}{s(s+a_0)} = \frac{Kb_0/a_0}{s} + \frac{-Kb_0/a_0}{s+a_0}$$



$$\boxed{x(t) = \frac{Kb_0}{a_0}\left(1 - e^{-a_0 t}\right)}$$

If $a_0 > 0$ we say that the system is **stable** and $-a_0$ is the **pole** of the system.

First order response III

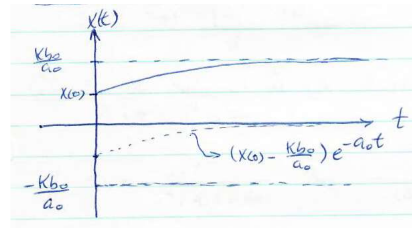


"Go West, young man!"

First order response IV

If $x(0) \neq 0$

$$X(s) = \frac{Kb_0}{s(s+a_0)} + \frac{x(0)}{s+a_0} = \frac{Kb_0}{a_0}\left(\frac{1}{s}\right) + \left(x(0) - \frac{Kb_0}{a_0}\right)\left(\frac{1}{s+a_0}\right)$$



$$\boxed{x(t) = \frac{Kb_0}{a_0} + \left(x(0) - \frac{Kb_0}{a_0}\right) e^{-a_0 t}}$$

Remark– In both cases $x_{ss} = \frac{Kb_0}{a_0}$.

Time Constant–The value of time that makes the exponent of e equal to -1 is called the time constant τ, therefore $\tau = \frac{1}{a_0}$ is the time interval over which the exponential decays by a factor of $\frac{1}{e} = 0.368$.

Second order response I

Consider a second order system described by

$$\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0 x = b_1\frac{dr}{dt} + b_0 r$$

$$X(s) = \underbrace{\frac{b_1 s + b_0}{s^2 + a_1 s + a_0}}_{=\text{H(s)}} R(s) + \frac{\text{1st order polynomial involving IC}}{s^2 + a_1 s + a_0}$$

The characteristic polynomial for this system is

$$\Delta(s) = s^2 + a_1 s + a_0 = 0 \quad \text{with} \quad s_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}$$

Case 1–$a_1^2 > 4a_0 \implies$ distinct and real roots s_1, s_2, in which case,

Second order response II

Overdamped system response

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

Example–

$$H(s) = \frac{12}{s^2 + 4s + 3} \quad \text{with} \quad R(s) = \frac{1}{s} \quad \text{and} \quad I.C. = 0$$

$$x(t) = \mathscr{L}^{-1}\left\{\frac{4}{s} + \frac{2}{s+3} - \frac{6}{s+1}\right\}$$

$$\boxed{x(t) = 4 + 2e^{-3t} - 6e^{-t}}$$

Second order response III

Case 2–$a_1^2 = 4a_0 \Longrightarrow$ repeated roots $s_{1,2} = s_1$, in which case,

$$X(s) = \frac{s+b}{(s-s_1)^2} = \frac{c_1}{s-s_1} + \frac{c_2}{(s-s_1)^2}$$

Critically damped system response

$$x(t) = c_1 e^{s_1 t} + c_2 t e^{s_1 t}$$

Example–

$$H(s) = \frac{9}{s^2+6s+9} \quad \text{with} \quad R(s) = \frac{1}{s} \quad \text{and} \quad I.C. = 0$$

$$X(s) = \frac{1}{s} - \frac{1}{s+3} - \frac{3}{(s+3)^2}$$

Second order response IV

$$\boxed{x(t) = 1 - (1+3t)e^{-3t}}$$

Case 3–$a_1^2 < 4a_0 \Longrightarrow$ complex conjugate pairs $s_{1,2} = \sigma \pm j\omega$, in which case,

$$X(s) = \frac{s+b}{(s+\sigma)^2 + \omega^2}$$

note that if

$$F(s) = \frac{N}{(s+a+jb)(s+a-jb)} = \frac{K_1}{s+a+jb} + \frac{K_1^*}{s+a-jb}$$

and if $K_1 = Ae^{j\theta}$ then $f(t) = 2Ae^{-at}\cos(bt+\theta)$

Underdamped system response

$$x(t) = 2Ae^{-\sigma t}\cos(\omega t + \theta)$$

Second order response V

Example–

$$H(s) = \frac{-3s+17}{s^2+2s+17} \quad \text{with} \quad R(s) = \frac{1}{s} \quad \text{and} \quad I.C. = 0$$

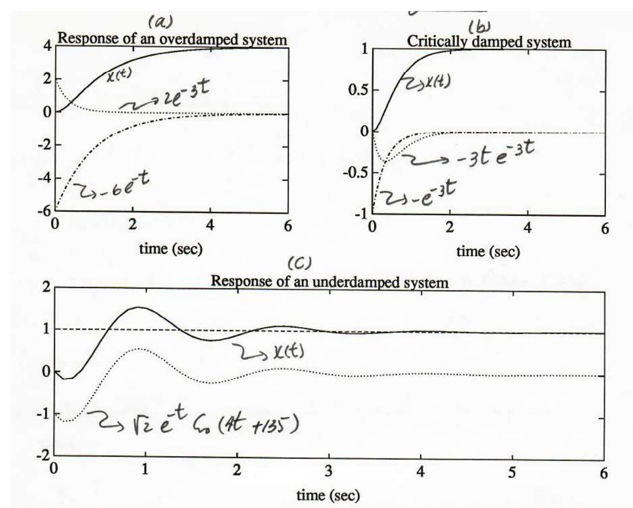
$$X(s) = \frac{-3s+17}{s(s^2+2s+17)} = \frac{1}{s} - \frac{s+5}{(s+1)^2+(4)^2}$$
$$= \frac{1}{s} - \left(\frac{K_1}{s+1+j4} + \frac{K_1^*}{s+1-j4}\right)$$

This gives $K_1 = \sqrt{\frac{1}{2}}\angle 45^\circ$

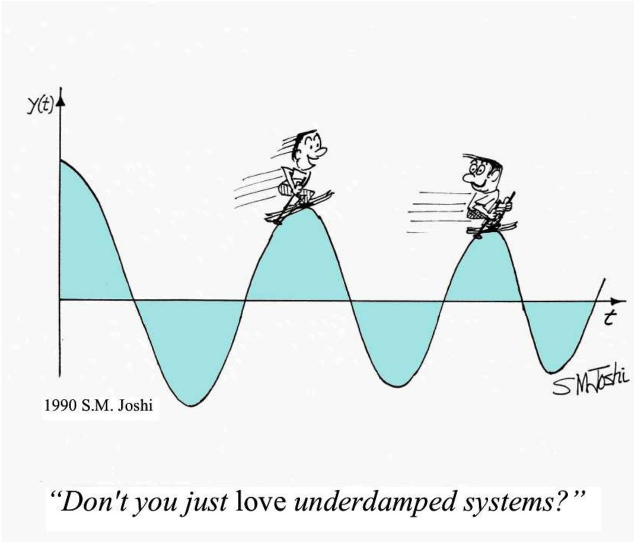
$$\boxed{x(t) = 1 + \sqrt{2}e^{-t}\cos(4t+135)}$$

The response of the three example cases are plotted below

Second order response VI



Second order response VII



Second order response VIII

Undamped natural frequency and damping ratio

Response of 2nd order underdamped system is characterized by the oscillation frequency and the exponential term (ω, σ) given by the CE

$$\Delta(s) = s^2 + a_1 s + a_0 = (s+\sigma)^2 + \omega^2$$

In terms of **natural frequency,** ω_n and **damping ratio** ξ,

$$s^2 + a_1 s + a_0 = s^2 + 2\xi\omega_n s + \omega_n^2 = (s+\sigma)^2 + \omega^2 = s^2 + 2\sigma s + (\omega^2 + \sigma^2)$$

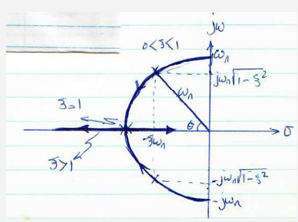
comparing and equating the coefficients of the like terms

Second order response IX

Poles, stability, natural frequency, and damping ratio

$$\xi\omega_n = \sigma \qquad \omega = \omega_n\sqrt{1-\xi^2}$$

$$s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$



$\xi = 0$	poles are on the imaginary axis
$0 < \xi < 1$	poles are on a circle of radius ω_n
$\xi = 1$	repeated roots at $-\omega_n$
$\xi > 1$	distinct real roots

With $\theta = \cos^{-1}\xi$

Remark–The system is **stable** if all the poles are in the left hand plane.

Second order response X

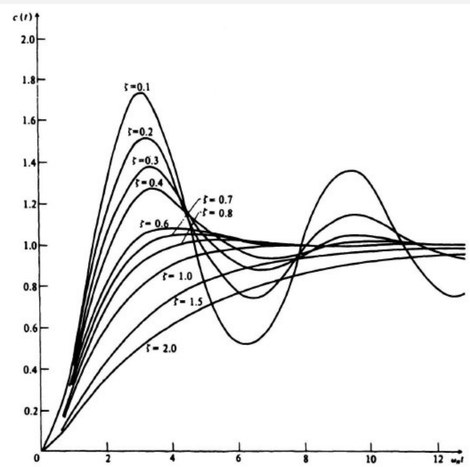
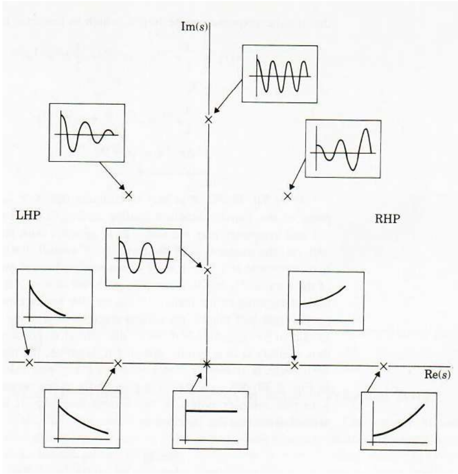
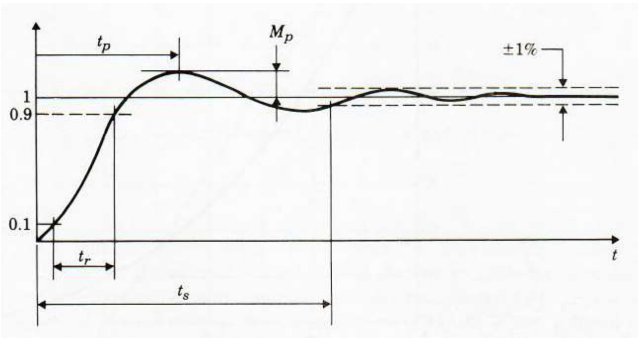


Figure 1: Unit step response of 2nd order system for different values of ξ

Second order response XI



Second order response XII



Pure resonance I

As a final discussion, let us consider the following second order differential equation described by

$$\frac{d^2y}{dt^2} + \omega_0^2 y = F_0 \cos\omega t \qquad y(0) = 0, y'(0) = 0 \tag{8.1}$$

The characteristic polynomial for this system is

$$m^2 + \omega_0^2 = 0 \Longrightarrow m_{1,2} = \pm j\omega_0$$

hence, the form of the homogeneous solution is

$$y_c = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$$

Pure resonance II

Also, given the forcing function on the right, we assume a solution complementary solution of the form

$$y_p = A\sin\omega t + B\cos\omega t$$

pluging into the differential equation gives

$$A(\omega_0^2 - \omega^2)\sin\omega t + B(\omega_0^2 - \omega^2)\cos\omega t = F_0\cos\omega t \longrightarrow A = 0, B = \frac{F_0}{\omega_0^2 - \omega^2}$$

Therefore

$$y(t) = c_1 \sin\omega_0 t + c_2 \cos\omega_0 t + \frac{F_0}{\omega_0^2 - \omega^2}\cos\omega t$$

Pure resonance III

evaluating the constants give

$$c_1\omega_0 = 0$$

$$c_2 + \frac{F_0\omega}{\omega_0^2 - \omega^2} = 0$$

Finally,

$$\boxed{y(t) = \frac{F_0}{\omega_0^2 - \omega^2}\left(\cos\omega t - \cos\omega_0 t\right)}$$

Pure resonance IV

Assume now that $\omega = \omega_0$

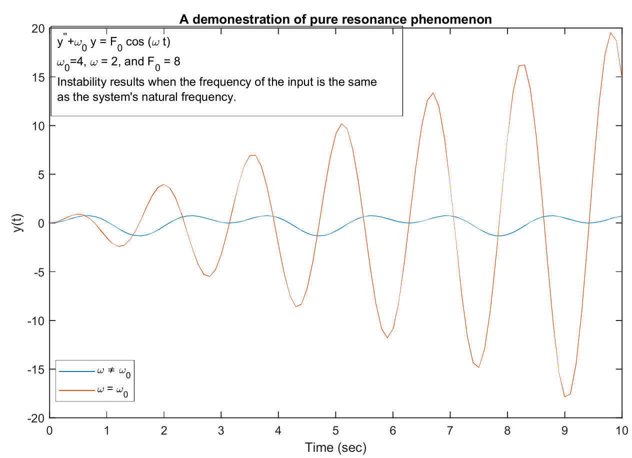
In this case, while the homogeneous solution remains the same, due to the fact that we have similar terms as in the forcing function in x_c, we take

$$y_p = At\sin\omega t + Bt\cos\omega t$$

which leads to $A = \frac{F_0}{2\omega_0}$ and $B = 0$, and from there $c_2 = 0$ and $c_1 = \frac{F_0 t}{2\omega_0}$, and finally

$$\boxed{y(t) = \frac{F_0 t}{\omega_0}\sin\omega_0 t}$$

Pure resonance V



Pure resonance VI

Remark

1. **Pure resonance** occurs exactly when the natural internal frequency ω_0 matches the natural external frequency ω , in which case all solutions of the differential equation are unbounded.
2. Pure resonance was easily demonstrated mathematically by simply taking $\omega = \omega_0$. However, this situation hardly ever happens in real physical engineering systems. Damping is always inherent to physical systems.
3. **Practical Resonance** is said to occur when the external frequency is "tuned" to produce the largest possible solution to the differential equation. We shall illustrate this concept through some analysis and an example.

Practical resonance I

As mentioned before, there is always some damping as a part of the design or perhaps due to frictional forces associated with real physical systems. So rather than (8.1) consider a more realistic system described by

$$my''(t) + by'(t) + ky(t) = F_0 \cos \omega t \tag{8.2}$$

Note that the above is a second order differential equation whose form of homogeneous solution was discussed in Table 1. Also note from Table 1 that as long as the poles of this system have negative real parts. The homogeneous part of the solution will have negative power exponential terms in it and will therefore tend to zero as time goes by. Therefore, let us concentrate on the particular solution.

Practical resonance II

Based on the right hand side of the equation, we take

$$y_p(t) = A \sin \omega t + B \cos \omega t$$

pluggin back into the differential equation gives

$$\begin{cases} (k - m\omega^2) + b\omega B = F_0 \\ -b\omega A + (k - m\omega^2)B = 0 \end{cases} \Longrightarrow \begin{cases} A = \frac{(k-m\omega^2)F_0}{(k-m\omega^2)^2+(b\omega)^2} \\ B = \frac{b\omega F_0}{(k-m\omega^2)^2+(b\omega)^2} \end{cases}$$

Note that

$$A \cos at + B \sin at = \sqrt{A^2 + B^2} \cos\left(at - \tan^{-1}\frac{B}{A}\right) \tag{8.3}$$

Practical resonance III

Hence,

$$y_p(t) = K \cos(\omega t - \alpha)$$

where

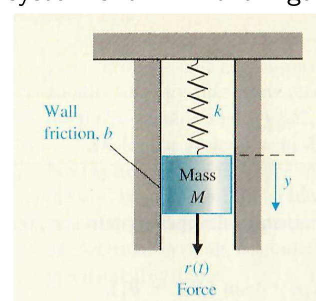
$$K(\omega) = \sqrt{A^2 + B^2} = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (b\omega)^2}} \tag{8.4}$$

Remark

$K(\omega)$ is referred to as *"forced amplitude"* and is always finite. If we had no damping, i.e. $b = 0$ and $\omega = \omega_0 = \sqrt{\frac{k}{m}}$ then $K = \infty$. However, $K(\omega)$ as described above will have a maximum for some value of ω at which **practical resonance** occurs.

Practical resonance IV

Example–Consider the mass, spring, damping system shown in the Figure. Assume that $m = 1, b = 2$, and $k = 26$. Further $r(t) = 82 \cos 4t$, $y(0) = 6$, and $y'(0) = 0$.



The equation of motion for this system is described by

$$y''(t) + 2y'(t) + 26y(t) = 82 \cos 4t$$

Characteristic polynomial for this system is

$$m^2 + 2m + 26 = 0 \Longrightarrow m_{1,2} = -1 \pm j5$$

Therefore,

$$y_c(t) = e^{-t}(c_1 \cos 5t + c_2 \sin 5t)$$

Practical resonance V

Now for the particular solution, take

$$y_p(t) = A\cos 4t + B\sin 4t \implies \begin{cases} 10A + 8B = 82 \\ -8A + 10B = 0 \end{cases} \implies \begin{cases} A = 5 \\ B = 4 \end{cases}$$

Therefore,

$$y(t) = e^{-t}\left(c_1 \cos 5t + c_2 \sin 5t\right) + 5\cos 4t + 4\sin 4t$$

Now using $y(0) = 6$, and $y'(0) = 0$, we obtain $c_1 = 1$ and $c_2 = -3$, so

$$y(t) = \underbrace{e^{-t}\left(\cos 5t - 3\sin 5t\right)}_{\text{transient solution}} + \underbrace{5\cos 4t + 4\sin 4t}_{\text{steady-state solution}}$$

Practical resonance VI

Ignoring the transient part of the solution (since it dies out), consider now the steady-state part of the solution, and the identity (8.3), we get

$$y_{ss} = 5\cos 4t + 4\sin 4t = \sqrt{41}\left(\cos 4t - \alpha\right)$$

where

$$\alpha = tan^{-1}\frac{4}{5} = 0.6747$$

Now based on (8.4) note that the amplitude of the forced response as a function of the frequency is given by

$$K(\omega) = \frac{82}{\sqrt{676 - 48\omega^2 + \omega^4}}$$

Practical resonance VII

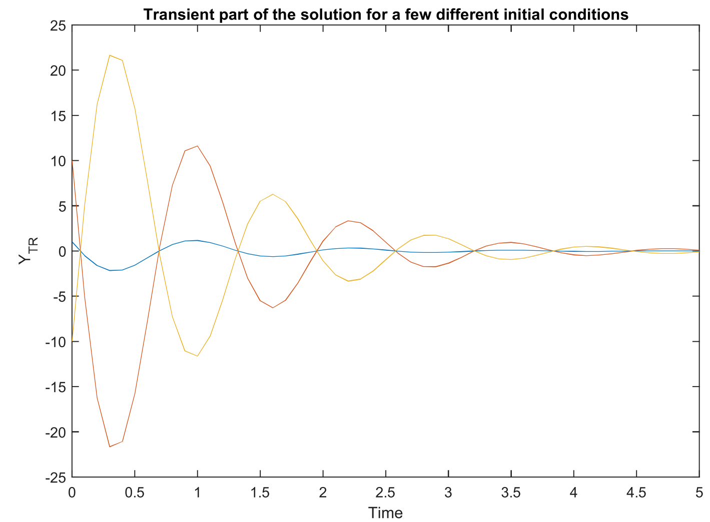
To determine the frequency at which the maximum amplitude occurs differentiate with respect to ω and set equal to zero to solve for ω^*. Hence,

$$K'(\omega) = \frac{-164\omega(\omega^2 - 24)}{(676 - 48\omega^2 + \omega^4)^{\frac{3}{4}}} = 0$$

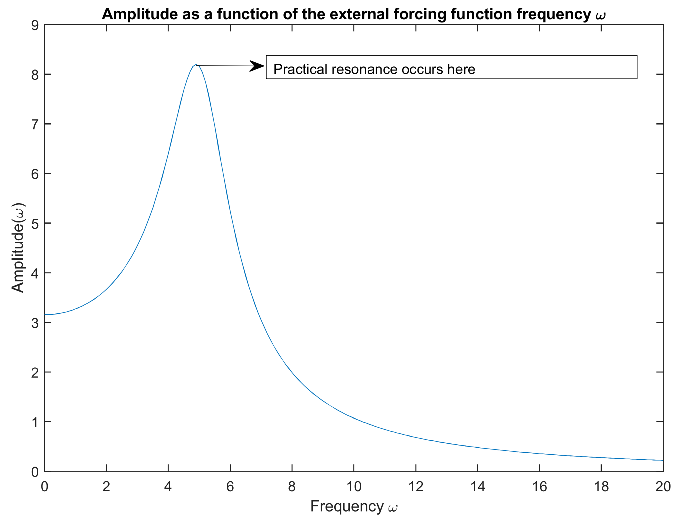
and

$$\omega^* = \sqrt{24} = 4.89$$

Practical resonance VIII



Practical resonance IX



Practical resonance X