GENG 8010–Part 2 - Elements of Applied Linear Algebra

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Winter 2023

Introduction & axiomatic definitions– I

Here we shall specify the collection of objects that form the center of our study.

Set: The collection of objects or elements.

$$\text{Fields that are considered} \begin{cases} \text{real numbers} \\ \text{complex numbers} \\ \text{rational functions} \end{cases}$$

Introduction & axiomatic definitions– II

Definition (Field)– A field consists of a set $\mathscr{F}$ of elements called scalars. "+" and "." are defined and satisfy the following axioms:

1. $\forall \alpha, \beta \in \mathscr{F} \;\; \exists\; \alpha + \beta \in \mathscr{F}$ and $\alpha.\beta \in \mathscr{F}$
2. $\forall \alpha, \beta \in \mathscr{F} \;\; \alpha + \beta = \beta + \alpha$ and $\alpha.\beta = \beta.\alpha$
   "+" and "." are commutative
3. $\forall \alpha, \beta, \gamma \in \mathscr{F} \;\; (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ and $(\alpha.\beta).\gamma = \alpha.(\beta.\gamma)$
   "+" and "." are associative.
4. $\forall \alpha, \beta, \gamma \in \mathscr{F} \;\; \alpha.(\beta + \gamma) = \alpha.\beta + \alpha.\gamma$
   '.' is distributive wrt "+"
5. $\exists 0 \in \mathscr{F} \ni \alpha + 0 = \alpha$ and $\exists 1 \in \mathscr{F} \ni 1.\alpha = \alpha \forall \alpha \in \mathscr{F}$
6. $\forall \alpha \in \mathscr{F} \;\; \exists \;\; \beta \in \mathscr{F} \ni \alpha + \beta = 0 \qquad \beta$: additive inverse
7. $\forall \alpha \in \mathscr{F}$ other than $\alpha = 0 \;\; \exists \beta \in \mathscr{F} \ni \alpha.\beta = 1$
   β : multiplicative inverse

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Example– Is $\{0, 1\}$ a field?
Ans.: No since it violates 1,6.

But let us define:
$1 + 1 = 0 + 0 = 0; 0.1 = 0.0 = 0; 1 + 0 = 1; 1.1 = 1$ then $\{0, 1\}$ is a field.
It is called the **field of binary numbers.**

Example–The set of all real numbers define a field called $\Re$.

Example–The set of all complex numbers define a complex field $\mathbb{C}$

Definition (Ring) A set that satisfies all the axioms in the previous definition except (7) is called a *Commutative Ring* with multiplicative identity.

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Example–The set of all integers is not a field but is a ring.

Definition (Vector Space)-a vector space over a field $\mathscr{F}$ is defined by $(\mathscr{X}, \mathscr{F})$ which consists of a set $\mathscr{X}$ of elements called vectors defined over a field $\mathscr{F}$ and two operation of vector addition and scalar multiplication such that the following axioms are satisfied:

1. $\forall \boldsymbol{x}_1$, and $\boldsymbol{x}_2 \in \mathscr{X}$ $\exists\, \boldsymbol{x}_3 = \boldsymbol{x}_1 + \boldsymbol{x}_2 \in \mathscr{X}$.
2. $\forall \boldsymbol{x}_1$, and $\boldsymbol{x}_2 \in \mathscr{X}$ $\boldsymbol{x}_1 + \boldsymbol{x}_2 = \boldsymbol{x}_2 + \boldsymbol{x}_1$ "+" is commutative
3. $\forall \boldsymbol{x}_1, \boldsymbol{x}_2$, and $\boldsymbol{x}_3 \in \mathscr{X}$ $(\boldsymbol{x}_1 + \boldsymbol{x}_2) + \boldsymbol{x}_3 = \boldsymbol{x}_1 + (\boldsymbol{x}_2 + \boldsymbol{x}_3)$ "+" is associative.
4. $\exists \boldsymbol{0} \in \mathscr{X} \ni \boldsymbol{0} + \boldsymbol{x} = \boldsymbol{x}$ $\forall \boldsymbol{x} \in \mathscr{X}$
5. $\forall \boldsymbol{x} \in \mathscr{X} \exists\, \hat{\boldsymbol{x}} \in \mathscr{X} \ni \boldsymbol{x} + \hat{\boldsymbol{x}} = \boldsymbol{0}$
6. $\forall \alpha \in \mathscr{F}$ and $\boldsymbol{x} \in \mathscr{X}$ $\exists\, \bar{\boldsymbol{x}} = \alpha \boldsymbol{x} \in \mathscr{X}$ called scalar product.
7. $\forall \alpha, \beta \in \mathscr{F}$ and $\boldsymbol{x} \in \mathscr{X}$ $\alpha.(\beta.\boldsymbol{x}) = (\alpha.\beta).\boldsymbol{x}$ scalar multiplication is associative.

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8. $\forall \alpha \in \mathscr{F}$ and $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathscr{X}$ $\alpha.(\boldsymbol{x}_1 + \boldsymbol{x}_2) = \alpha.\boldsymbol{x}_1 + \alpha.\boldsymbol{x}_2$ scalar multiplication is distributive
9. $\forall \alpha, \beta \in \mathscr{F}$ and $\boldsymbol{x} \in \mathscr{X}$ $(\alpha + \beta)\boldsymbol{x} = (\alpha.\boldsymbol{x} + \beta.\boldsymbol{x}$ scalar multiplication is distributive wrt scalar addition.
10. $\forall \boldsymbol{x} \in \mathscr{X}$ $\exists 1 \in \mathscr{F} \ni 1.\boldsymbol{x} = \boldsymbol{x}$

Example–A field can form a vector space over itself, e.g., $(\Re, \Re)$, and $(\mathbb{C}, \mathbb{C})$ which are the real and complex vector spaces respectively.

Example–How about

1. $(\Re, \mathbb{C})$
2. $(\mathbb{C}, \Re)$

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Example– Given $\mathscr{F}$, let

$$\boldsymbol{x}_i = \begin{bmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{bmatrix}$$

If we define the usual operations

$$\boldsymbol{x}_i + \boldsymbol{x}_j = \begin{bmatrix} x_{1i} + x_{1j} \\ x_{2i} + x_{2j} \\ \vdots \\ x_{ni} + x_{nj} \end{bmatrix} \quad \text{and} \quad \alpha.\boldsymbol{x}_i = \begin{bmatrix} \alpha.x_{1i} \\ \alpha.x_{2i} \\ \vdots \\ \alpha.x_{ni} \end{bmatrix}$$

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Then $\mathscr{F}^n, \mathscr{F}$ is an n-dimensional vector space. $(\Re^n, \Re)$ and $(\mathbb{C}^n, \mathbb{C})$ are the n-dimensional real and complex vector spaces respectively.

Definition-Subspace of a vector space

Let $(\mathscr{X}, \mathscr{F})$ be a vector space and $\mathscr{Y} \subset \mathscr{X}$. Then $(\mathscr{Y}, \mathscr{F})$ is a **subspace** of $(\mathscr{X}, \mathscr{F})$ if under the same operations of $(\mathscr{X}, \mathscr{F})$, $\mathscr{Y}$ itself form a vector space over $\mathscr{F}$.

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Example– $(\Re^n, \Re)$ is a subspace of $(\mathbb{C}^n, \Re)$.

Remark

Since "$+$", and "$\cdot$" of vectors are defined for $(\mathscr{X}, \mathscr{F})$, they satisfy 2,3,7-10. To check 1,4-6 and have them satisfied, we will require the following instead.

Alternate subspace definition

A set $\mathscr{Y} \subset \mathscr{X}$ is a subspace of $(\mathscr{X}, \mathscr{F})$ if

$$\forall \boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathscr{Y} \text{ and } \alpha_1, \alpha_2 \in \mathscr{F}, \quad \alpha_1 \boldsymbol{y}_1 + \alpha_2 \boldsymbol{y}_2 \in \mathscr{Y}$$

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Example– Consider $(\Re^2, \Re)$.

Any line that passes through the origin is a subspace of $(\Re^2, \Re)$, e.g. $\begin{bmatrix} x_1 \\ \alpha x_1 \end{bmatrix}$.