GENG 8010–Part 1: Elements of Differential and Difference Equations

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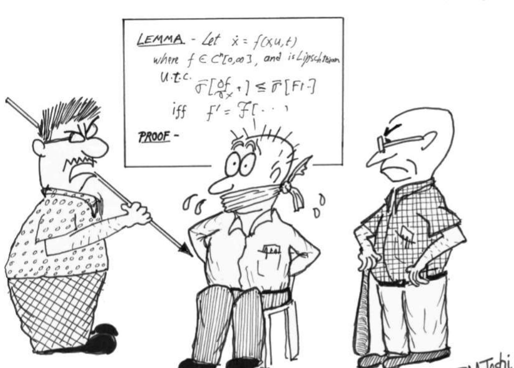
About VP method

1. Distinct advantage over other methods that it always yields a particular solution y_p if the associated homogeneous equation can be solved.
2. Applicable to linear higher-order equations.
3. Unlike undetermined coefficients, is not limited to cases where the forcing function is a combination of certain functions.
4. No special cases arise due to the nonhomogeneous term being included in the complementary function.
5. It works for (time) varying systems, i.e., $a_i(t)$ or $a_i(x)$.

VP for linear 2^{nd} order ODE I



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"After we beat the proof outta him, let's dump him in the theory-practice gap!"

VP for linear 2^{nd} order ODE II

Consider a general linear 2nd order equation

$$(D^2 + a(x)D + b(x))y = f(x)$$

defined over an inteval $\alpha \le x \le \beta$ over which $a(x), b(x)$, and $f(x)$ are defined and continuous. $y_1(x)$ and $y_2(x)$ two known lin. indep. solutions, so

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x)$$

VP for linear 2^{nd} order ODE III

Idea: Replace the constants c_1 and c_2 with two unknown functions u_1, u_2 of x and then

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

differentiating

$$y_p'(x) = u_1(x)y_1'(x) + u_2(x)y_2'(x) + \cancelto{\textit{force}\ \ 0}{u_1'(x)y_1(x) + u_2'(x)y_2(x)}$$

In other words,

$$y_p'(x) = u_1(x)y_1'(x) + u_2(x)y_2'(x)$$

subject to requirement that

$$\boxed{u_1'(x)y_1(x) + u_2'(x)y_2(x) = 0} \qquad \text{(a)}$$

VP for linear 2^{nd} order ODE IV

differentiating $y_p'(x)$,

$$y_p''(x) = u_1(x)y_1''(x) + u_2(x)y_2''(x) + u_1'(x)y_1'(x) + u_2'(x)y_2'(x)$$

substituting for y_p, y_p' and y_p'' into the diff. eq.

$$u_1\cancelto{0}{\left(y_1'' + a(x)y_1' + b(x)y_1\right)} + u_2\cancelto{0}{\left(y_2'' + a(x)y_2' + b(x)y_2\right)} + u_1'y_1' + u_2'y_2' = f(x)$$

Resulting in

$$\boxed{u_1'(x)y_1'(x) + u_2'(x)y_2'(x) = f(x)} \qquad \text{(b)}$$

VP for linear 2^{nd} order ODE V

Consider equations (a, b)

$$\begin{cases} u_1'(x)y_1(x) + u_2'(x)y_2(x) = 0 \\ u_1'(x)y_1'(x) + u_2'(x)y_2'(x) = f(x) \end{cases} \Longrightarrow \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} \begin{bmatrix} u_1'(x) \\ u_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

(a) and (b) give two eq. in two unknowns u_1, u_2. Using the Cramer Rule to solve for them, we get

$$\boxed{u_1'(x) = \frac{W_1}{W} = \frac{-y_2(x)f(x)}{W(x)} \qquad u_2'(x) = \frac{W_2}{W} = \frac{y_1(x)f(x)}{W(x)}} \qquad \text{(c)}$$

where

VP for linear 2^{nd} order ODE VI

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2\ ;\ W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}\ ;\ W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

You have seen $W(x)$ (Wronskian) before and may recall that for y_1 and y_2 to be independent solutions $W(x)$ cannot be zero.

Integrating the equations in (c) gives u_1 and u_2 and finally the complete solution

$$\boxed{y_p(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}dx} \qquad (7.1)$$

VP for linear 2^{nd} order ODE VII

Remark

1. We have not included the constants associated with each indefinite integral. Suppose we did and called them $-c_1$ and c_2 , then we would have a addition term like $c_1 y_1(x) + c_2 y_2(x)$ in our solution which is the y_c. When these integration constants are set equal to zero we should get the particular solution $y_p(x)$.

2. **Finally, for an n^{th} order system** (7.1) **can be extended to**

$$\boxed{y_p(x) = \sum_{i=1}^{n} y_i(x) \int \frac{W_i(x)}{W(x)} f(x) dx}$$

VP for linear 2^{nd} order ODE VIII

Example–Find the solution of

$$y'' + 2y' + y = e^x$$

Verify that the roots of CE are repeated at $m = -1$, hence

$$y_c(x) = c_1 e^{-x} + c_2 x e^{-x}$$

and

$$y_1(x) = e^{-x} \qquad y_2(x) = xe^{-x}$$

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = e^{-x}\left(e^{-x} - xe^{-x}\right) + xe^{-2x} = e^{-2x}$$

VP for linear 2^{nd} order ODE IX

substituting in (7.1) gives,

$$\begin{aligned} y_p(x) &= -y_1(x) \int \frac{y_2(x) f(x)}{W(x)} dx + y_2(x) \int \frac{y_1(x) f(x)}{W(x)} dx \\ &= -e^{-x} \int \frac{xe^{-x}e^x}{e^{-2x}} dx + xe^{-x} \int \frac{e^{-x}e^x}{e^{-2x}} dx \\ &= -e^{-x} \int xe^{2x} dx + xe^{-x} \int e^{2x} dx \\ &= -e^{-x}\left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right) + \frac{1}{2}xe^x = \frac{1}{4}e^x \end{aligned}$$

where we used $\int xe^{ax} dx = e^{ax}\left(\frac{ax-1}{a^2}\right)$. Therefore

$$\boxed{y = c_1 e^x + c_2 x e^x + \frac{1}{4} e^x}$$

VP for linear 2^{nd} order ODE X

Due to form of $f(x)$ the previous example could have been solved using the method of undetermined coefficients (try solving it using that method). Let us now try an example that cannot be solved using that approach.

Example– Solve the following

$$y'' + y = \csc x$$

Verify that the roots of the CE are at $m_{1,2} = \pm i1$, and hence

$$y_c = c_1 \cos x + c_2 \sin x$$

in which case

$$y_1(x) = \cos x \qquad y_2(x) = \sin x$$

VP for linear 2^{nd} order ODE XI

and $W(x) = y_1 y'_2 - y'_1 y_2 = \cos^2 x + \sin^2 x = 1$ with $f(x) = \frac{1}{\sin x}$ which using the formula gives

$$\begin{aligned} y_p(x) &= -y_1(x) \int \frac{y_2(x) f(x)}{W(x)} dx + y_2(x) \int \frac{y_1(x) f(x)}{W(x)} dx \\ &= -\cos x \int dx + \sin x \int \cot x dx \end{aligned}$$

Using the fact that $\int \cot x dx = \ln|\sin x|$ we get

$$y_p(x) = -x \cos x + \sin x \ln|\sin x|$$

and

$$\boxed{y(x) = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \ln|\sin x|}$$

Green Function and 2nd order systems I

Second order systems arise in many engineering domains. The desire is to express the response of a system for different inputs.

Consider again the linear 2nd order equation

$$(D^2 + a(x)D + b(x))y = f(x)$$

defined over an inteval $\alpha \le x \le \beta$ over which $a(x), b(x)$, and $f(x)$ are defined and continuous. Suppose x_0 is a point within the interval for which we know $y(x_0) = y'(x_0) = 0$. Start with (7.1)

$$y_p(x) = -y_1(x) \int \frac{y_2(x) f(x)}{W(x)} dx + y_2(x) \int \frac{y_1(x) f(x)}{W(x)} dx$$

Green Function and 2nd order systems II

Note that the lin. indep. of y_1 and y_2 over the interval, guarantees that $W \neq 0$ for all x within the interval. So IF x and x_0 are within the interval

$$\begin{aligned} y_p(x) &= -y_1(x) \int_{x_0}^{x} \frac{y_2(\tau) f(\tau)}{W(\tau)} d\tau + y_2(x) \int_{x_0}^{x} \frac{y_1(\tau) f(\tau)}{W(\tau)} d\tau \\ &= \int_{x_0}^{x} \frac{-y_1(x) y_2(\tau)}{W(\tau)} f(\tau) d\tau + \int_{x_0}^{x} \frac{y_1(\tau) y_2(x)}{W(\tau)} f(\tau) d\tau \\ &= \int_{x_0}^{x} \frac{y_1(\tau) y_2(x) - y_1(x) y_2(\tau)}{W(\tau)} f(\tau) d\tau \end{aligned}$$

$$\boxed{y_p(x) = \int_{x_0}^{x} G(x,\tau) f(\tau) d\tau}$$

Green Function and 2nd order systems III

Note that with $y_c = 0$, hence, if this was a system at rest, then

$$\boxed{y(x) = \int_{x_0}^{x} G(x,\tau) f(\tau) d\tau}$$

$$\boxed{G(x,\tau) = \frac{y_1(\tau) y_2(x) - y_1(x) y_2(\tau)}{W(\tau)}}$$

where $G(x,\tau)$ is called the **Green's Function**.

Does the above formula look familiar to you?

Green Function and 2nd order systems IV

Example– For $y(0) = \dot{y}(0) = 0$ solve

$$\ddot{y} - y = f(x)$$

Roots of CE are $m_{1,2} = \pm 1$, hence $y_1 = e^x$, and $y_2 = e^{-x}$, and $W(\tau) = -2$, therefore the Green Function is

$$G(x, \tau) = \frac{e^{\tau} e^{-x} - e^{x} e^{-\tau}}{-2} = \frac{e^{x-\tau} - e^{-(x-\tau)}}{2} = \sinh(x - \tau)$$

and hence

$$\boxed{y = y_p = \int_{x_0}^{x} \sinh(x - \tau) f(\tau) d\tau}$$

Introduction, definition and examples I

Remark

1. Differentiation and integrations are transforms in that they *transform* a function into another. They are also linear in that superposition principle holds for them.
2. In the following, we are interested in *integral transforms* where the interval of integration is an unbounded interval $[0, \infty)$

Introduction, definition and examples II

Definition 8.1.1 (**Fundamental definition**)

Assume that $f(t)$ is defined for $t \geq 0$, then the following improper integral is defined as a limit

$$\int_0^{\infty} K(s,t) f(t) dt = \lim_{b \to \infty} \int_0^{b} K(s,t) f(t) dt$$

1. If the above limit exist, the integral is said to be **convergent**, otherwise, it is **divergent** or it fails to exist.
2. The above limit in general exists for certain values of variable s, called **transform variable**.
3. The choice of the **Kernel** of transform, i.e., $K(s,t) = e^{-st}$ results in an important integral transform.

Introduction, definition and examples III

Laplace Transform

The Laplace transform of a function $f(t)$, i.e., $\mathscr{L}\{f(t)\}$ is defined as

$$\boxed{\mathscr{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt}$$

provided that the integral converges.

$$\mathscr{L}\{f(t)\} = F(s) \qquad \mathscr{L}\{y(t)\} = Y(s) \qquad \mathscr{L}\{g(t)\} = G(s)$$

$$f(t) \longleftrightarrow F(s) \qquad y(t) \longleftrightarrow Y(s) \qquad g(t) \longleftrightarrow G(s)$$

- $F(s)$ is **integral transform or Laplace Transform** of $f(t)$.

Introduction, definition and examples IV

Definition–A function is of **exponential order** as $t \to \infty$, $\exists k, M, t_0, \ni$

$$|f(t)| \leq Me^{kt} \quad \forall t > t_0$$

Theorem 8.1.2

If $f(t)$ is piecewise continuous on the interval $[0, \infty)$ and is of exponential order, then $\mathscr{L}\{f(t)\}$ exists for $s > k$.

Note that Theorem 8.1.2 is only a sufficient condition and basically says that $f(t)$ should not grow too fast for its Laplace transform to exist.

Introduction, definition and examples V

Example– Find the Laplace transform of the unit step function $u(t)$.

$$\begin{aligned}\mathscr{L}\{u(t)\} &= \int_0^\infty e^{-st}(1)dt = \lim_{b\to\infty}\int_0^b e^{-st}dt \\ &= \lim_{b\to\infty} \frac{-e^{-st}}{s}\bigg|_0^b = \lim_{b\to\infty}\frac{-e^{-sb}+1}{s} = \frac{1}{s}\end{aligned}$$

so long that $s > 0$ so that $e^{-sb} \to 0$ as $b \to \infty$. Otherwise, the integral would diverge for $s < 0$.

Going forward, we shall drop the limit. Instead

$$\mathscr{L}\{u(t)\} = \int_0^\infty e^{-st}(1)dt = -\frac{e^{-st}}{s}\bigg|_0^\infty = \frac{1}{s}, \quad and \quad s > 0$$

$$1 \longleftrightarrow \frac{1}{s} \quad implies \quad k \longleftrightarrow \frac{k}{s}$$

Introduction, definition and examples VI

Example– Find $\mathscr{L}\{e^{at}\}$

$$\begin{aligned}\mathscr{L}\{e^{at}\} &= \int_0^\infty e^{-st}\left(e^{at}\right)dt = \int_0^\infty e^{-(s-a)t}dt = \frac{e^{-(s-a)t}}{-(s-a)}\bigg|_0^\infty \\ &= \frac{1}{s-a} \quad if \quad (s-a) > 0 \quad or \quad s > a\end{aligned}$$

Introduction, definition and examples VII

Example– Find $\mathscr{L}\{\cos at\}$, and $\mathscr{L}\{\sin at\}$

$$\mathscr{L}\{\cos at\} = \int_0^\infty e^{-st}(\cos at)\,dt$$

$$\mathscr{L}\{\sin at\} = \int_0^\infty e^{-st}(\sin at)\,dt$$

$$dv = e^{-st} \to v = -\frac{1}{s}e^{-st} \quad ; \qquad u = \cos at \to du = -a\sin at$$

$$dv = e^{-st} \to v = -\frac{1}{s}e^{-st} \quad ; \qquad u = \sin at \to du = a\cos at$$

$$\int (.)dx = uv - \int (v)(du)dx$$

where we are setting up to use integration by parts

Introduction, definition and examples VIII

Now integrating by parts will give

$$\mathscr{L}\{\cos at\} = -\frac{1}{s}e^{-st}\cos at\Big|_0^\infty - \frac{a}{s}\int_0^\infty e^{-st}\sin at = \frac{1}{s} - \frac{a}{s}\mathscr{L}\{\sin at\}$$

$$\mathscr{L}\{\sin at\} = -\frac{1}{s}e^{-st}\sin at\Big|_0^\infty + \frac{a}{s}\int_0^\infty e^{-st}\cos at = \frac{a}{s}\mathscr{L}\{\cos at\}$$

$$\mathscr{L}\{\cos at\} = \frac{1}{s} - \frac{a}{s}\left(\frac{a}{s}\mathscr{L}\{\cos at\}\right) \longrightarrow \boxed{\mathscr{L}\{\cos at\} = \frac{s}{s^2+a^2}} \quad \textit{if} \quad s > 0$$

$$\mathscr{L}\{\sin at\} = \frac{a}{s}\frac{s}{s^2+a^2} \longrightarrow \boxed{\mathscr{L}\{\sin at\} = \frac{a}{s^2+a^2}} \quad \textit{if} \quad s > 0$$

Introduction, definition and examples IX

Some common Laplace transforms

	Function $f(t)$	Laplace transforms $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$
(i)	1	$\frac{1}{s}$
(ii)	k	$\frac{k}{s}$
(iii)	e^{at}	$\frac{1}{s-a}$
(iv)	$\sin at$	$\frac{a}{s^2+a^2}$
(v)	$\cos at$	$\frac{s}{s^2+a^2}$
(vi)	t	$\frac{1}{s^2}$
(vii)	t^2	$\frac{2!}{s^3}$
(viii)	t^n $(n = 1, 2, 3, \ldots)$	$\frac{n!}{s^{n+1}}$
(ix)	$\cosh at$	$\frac{s}{s^2-a^2}$
(x)	$\sinh at$	$\frac{a}{s^2-a^2}$

Properties and further definitions I

Transform pairs

Functions $f(t)$ and $F(s)$ are called **Laplace Transform pairs**. Just as $f(t)$ gives a unique $F(s)$, given $F(s)$, $f(t)$ is uniquely defined and is expressed as $\mathscr{L}^{-1}\{F(s)\} = f(t)$.

Linearity property

Linearity property– Let $f_1(t), f_2(t), \ldots, f_n(t)$ have Laplace transforms, and let $\alpha_1, \alpha_2, \ldots \alpha_n$ be some constants. Then

$$\mathscr{L}\{\alpha_1 f_1(t) + \alpha_2 f_2(t) + \ldots + \alpha_n f_n(t)\} = \alpha_1\mathscr{L}\{f_1(t)\} + \alpha_2\mathscr{L}\{f_2(t)\} + \ldots + \alpha_n\mathscr{L}\{f_n(t)\}$$

Properties and further definitions II

Some common inverse transforms

	$F(s) = \mathcal{L}\{f(t)\}$	$\mathcal{L}^{-1}\{F(s)\} = f(t)$
(i)	$\frac{1}{s}$	1
(ii)	$\frac{k}{s}$	k
(iii)	$\frac{1}{s-a}$	e^{at}
(iv)	$\frac{a}{s^2+a^2}$	$\sin at$
(v)	$\frac{s}{s^2+a^2}$	$\cos at$
(vi)	$\frac{1}{s^2}$	t
(vii)	$\frac{2!}{s^3}$	t^2
(viii)	$\frac{n!}{s^{n+1}}$	t^n
(ix)	$\frac{a}{s^2-a^2}$	$\sinh at$
(x)	$\frac{s}{s^2-a^2}$	$\cosh at$
(xi)	$\frac{n!}{(s-a)^{n+1}}$	$e^{at}t^n$
(xii)	$\frac{\omega}{(s-a)^2+\omega^2}$	$e^{at}\sin\omega t$
(xiii)	$\frac{s-a}{(s-a)^2+\omega^2}$	$e^{at}\cos\omega t$
(xiv)	$\frac{\omega}{(s-a)^2-\omega^2}$	$e^{at}\sinh\omega t$
(xv)	$\frac{s-a}{(s-a)^2-\omega^2}$	$e^{at}\cosh\omega t$

Properties and further definitions III

Example–Find $\mathscr{L}\{f(t)\}$ for a piecewise continuous function

$$f(t) = \begin{cases} 0, & 0 \le t < 3 \\ 2, & t \ge 3 \end{cases}$$

$$\begin{aligned} \mathscr{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t)dt \\ &= \int_0^3 e^{-st}(0)dt + \int_3^\infty e^{-st}(2)dt \\ &= -\frac{2e^{-st}}{s}\bigg|_3^\infty \\ &= \frac{2e^{-3s}}{s}, s > 0 \end{aligned}$$

Properties and further definitions IV

Example–Find $\mathscr{L}^{-1}\{F(s)\}$ where

$$F(s) = \frac{4s-5}{s^2-s-2}$$

$$\frac{4s-5}{s^2-s-2} = \frac{4s-5}{(s-2)(s+1)} = \frac{A}{(s-2)} + \frac{B}{(s+1)} = \frac{A(s+1)+B(s-2)}{(s-2)(s+1)}$$

at $s = 2 \longrightarrow A = 1$ and at $s = -1 \longrightarrow B = 3$. Therefore

$$\boxed{f(t) = \mathscr{L}^{-1}\{F(s)\} = \mathscr{L}^{-1}\left\{\frac{1}{s-2} + \frac{3}{s+1}\right\} = e^{2t} + 3e^{-t}}$$

Properties and further definitions V

Example–Find $\mathscr{L}^{-1}\{F(s)\}$ where

$$F(s) = \frac{3s^3+s^2+12s+2}{(s-3)(s+1)^3}$$

$$\frac{3s^3+s^2+12s+2}{(s-3)(s+1)^3} = \frac{A}{s-3} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3}$$

Find $A = 2, B = 1, C = -4$, and $D = 3$ to get

$$\boxed{f(t) = \mathscr{L}^{-1}\{F(s)\} = 2e^{3t} + e^{-t} - 4te^{-t} + \frac{3}{2}t^2e^{-t}}$$

Properties and further definitions VI

Theorem 8.2.1

If $f(t)$ be is continuous on $0 \le t < \infty$, and let $f'(t)$ be piecewise continuous on every finite interval contained in $t \ge 0$. Then if $\mathscr{L}\{f(t)\} = F(s)$,

$$\boxed{\mathscr{L}\{f'(t)\} = sF(s) - f(0)}$$

Proof.

$$\mathscr{L}\{f'(t)\} = \int_0^\infty e^{-st}f'(t)dt \qquad ; \int uv'dx = uv - \int u'vdx$$

$$\mathscr{L}\{f'(t)\} = e^{-st}f(t)\bigg|_0^\infty + s\int_0^\infty e^{-st}f(t)dt = sF(s) - f(0)$$

□

Properties and further definitions VII

The above can be **extended to higher derivatives** if the similar conditions hold, then it can be shown that

$$\boxed{\mathscr{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)}$$

Theorem 8.2.2

First translation or s-Shift Theorem $\mathscr{L}\{f(t)\} = F(s)$ *and* a *is a real number, then*

$$\mathscr{L}\{e^{at}f(t)\} = F(s-a)$$

Properties and further definitions VIII

Example– Find $\mathscr{L}\{f(t)\}$ if $f(t) = e^{-2t}\sin 3t$

$$\mathscr{L}\{e^{-2t}\sin 3t\} = \frac{3}{(s-(-2))^2+3^2} = \frac{3}{(s+2)^2+9} = \frac{3}{s^2+4s+13}$$

Define the t-shifted unit step function

$$u(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

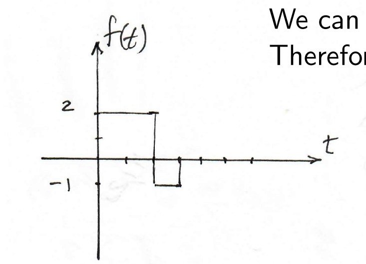
Theorem 8.2.3

Second translation or t-Shift Theorem *If* $\mathscr{L}\{f(t)\} = F(s)$ *and* $a > 0$ *is a real number, then*

$$\mathscr{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

Properties and further definitions IX

Example– Find $\mathscr{L}\{f(t)\}$ for the function graphed below



We can write $f(t) = 2u(t) - 3u(t-2) + u(t-3)$. Therefore

$$\boxed{\mathscr{L}\{f(t)\} = \frac{2}{s} - \frac{3e^{-2s}}{s} + \frac{e^{-3s}}{s}}$$

Properties and further definitions X

Theorem 8.2.4

Multiplication by t^n-differentiation of transform–*Let* $\mathscr{L}\{f(t)\} = F(s)$, *then*

$$\mathscr{L}\{t^nf(t)\} = (-1)^n\frac{d^nF(s)}{ds^n}$$

Theorem 8.2.5

Scaling theorem–*Let* $\mathscr{L}\{f(t)\} = F(s)$, *then if* $k > 0$, *then*

$$\mathscr{L}\{f(kt)\} = \frac{1}{k}F\left(\frac{s}{k}\right)$$

Definition (Convolution)–Convolution of two functions $f(t)$ and $g(t)$ is defined by $(f * g)(t)$ and is given by

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

Properties and further definitions XI

Theorem 8.2.6

Convolution theorem–*Let* $\mathscr{L}\{f(t)\} = F(s)$, *and* $\mathscr{L}\{g(t)\} = G(s)$, *then*

$$\mathscr{L}\{(f * g)(t)\} = F(s)G(s)$$

Definition (Volterra Integral Equation)–The following *integral equation* (often arising in initial value problems) is called Volterra integral equation where λ is a parameter and $K(t,\tau)$ is the *Kernel* of the integral equation.

$$y(t) = f(t) + \lambda \int_0^t K(t,\tau)y(\tau)d\tau$$

Special case of Volterra equation is when the kernel $K(t,\tau)$ is a function of $t-\tau$, i.e. $K(t,\tau) = K(t-\tau)$, in which case the integral part in the above becomes a convolution integral.

Properties and further definitions XII

Example– Solve the integral equation

$$y(t) = 2e^{-t} + \int_0^t \sin(t-\tau)y(\tau)d\tau$$

Taking $\mathscr{L}\{y(t)\}$ and using Theorem 8.2.6

$$Y(s) = \frac{2}{s+1} + \frac{Y(s)}{s^2+1} \longrightarrow Y(s) = \frac{2(s^2+1)}{s^2(s+1)}$$
$$= -\frac{2}{s} + \frac{2}{s^2} + \frac{4}{s+1}$$

$$\boxed{y(t) = -2 + 2t + 4e^{-t}}$$