Lecture Notes 16th May 2023

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Variation of parameters:

$$y' + p(x)y = q(x)$$

$$y' + p(x)y = 0$$

$$\frac{dy}{dx} = -p(x)y$$

$$\frac{1}{y}dy = -p(x)dx$$

$$\int \frac{1}{y}dy = \int -p(x)dx$$

$$\ln y = \int -p(x)dx + c$$

$$e^{\ln y} = e^{\int -p(x)dx+c}$$

$$y_h = e^{\int -p(x)dx}.e^{c}$$

$$y_h = A.e^{\int -p(x)dx}$$

$$y = A(x).e^{\int -p(x)dx}$$

$$y' = A'(x).e^{\int -p(x)dx} - p(x)A(x)e^{\int -p(x)dx}$$

$$y' + p(x)y = q(x)$$

$$A'(x).e^{\int -p(x)dx} - p(x)A(x).e^{\int -p(x)dx} + p(x)A(x).e^{\int -p(x)dx} = q(x)$$

$$A'(x) = q(x).e^{\int p(x)dx}$$

$$A(x) = \int q(x).e^{\int p(x)dx}$$

$$y = A(x).e^{\int -p(x)dx}$$

Solve the following ODE using variation of parameter:

Example 1:

$$y' + 5y = 3$$

$$\frac{dy}{dx} + 5y = 3$$

First, we solve for the homogeneous case

$$\frac{dy}{dx} + 5y = 0$$

$$\frac{dy}{dx} = -5y$$

$$\frac{1}{y}dy = -5\ dx$$

$$\int \frac{1}{y}dy = \int -5\ dx$$

$$\ln y = -5x + c$$

$$y = e^{-5x+c}$$

$$y = e^{-5x}.e^{c} = A.e^{-5x}$$

Varying A to A(x)

$$y = A(x).e^{-5x}$$

$$y' = A'(x).e^{-5x} - 5A(x)e^{-5x}$$

Substituting y and y' in the give ODE we get,

$$A'(x).e^{-5x} - 5A(x).e^{-5x} + 5A(x).e^{-5x} = 3$$

$$A'(x) = 3e^{5x}$$

$$A(x) = \frac{3}{5}e^{5x} + c$$

$$y = (\frac{3}{5}e^{5x} + c).e^{-5x}$$

Example 2:

$$x\frac{dy}{dx} + 3y = xe^{x^4}$$

$$\frac{dy}{dx} + \frac{3}{x}y = e^{x^4}$$

First, we solve for the homogeneous case

$$\frac{dy}{dx} + \frac{3}{x}y = 0$$

$$\frac{dy}{dx} = -\frac{3}{x}y$$

$$\frac{1}{y}dy = -\frac{3}{x}\ dx$$

$$\int \frac{1}{y}dy = \int -\frac{3}{x}\ dx$$

$$\ln y = -3\ln x + c$$

$$y = e^{-3\ln x + c}$$

$$y = x^{-3}.e^{c} = A.x^{-3}$$

Varying A to A(x)

$$y = A(x).x^{-3}$$

$$y' = A'(x).x^{-3} - 3A(x)x^{-4}$$

Substituting y and y' in the give ODE we get,

$$A'(x).x^{-3} - 3A(x)x^{-4} + \frac{3}{x}A(x).x^{-3} = e^{x^4}$$

$$A'(x) = x^3e^{x^4}$$

$$A(x) = \int x^3e^{x^4}dx$$

$$\text{Let } x^4 = u; x^3dx = \tfrac{1}{4}du$$

$$A(x) = \frac{1}{4}\int e^{u}du$$

$$y = (\frac{1}{4}e^{x^4} + c).x^{-3}$$

Method of exact equations:

$$M(x,y)dx + N(x,y)dy = 0$$

Condition for exactness ==> $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Solution ==> $f(x,y) = c$

Where $\frac{\partial f}{\partial x} = M(x,y)$, $\frac{\partial f}{\partial y} = N(x,y)$

$$f(x,y) = \int M(x,y)dx + g(y)$$

$$\frac{\partial f(x,y)}{\partial y} = N(x,y)$$

$$N(x,y) = \frac{\partial\left(\int M(x,y)dx\right)}{\partial y} + g'(y)$$

$$g'(y) = N(x,y) - \frac{\partial\left(\int M(x,y)dx\right)}{\partial y}$$

It can be proved that since the ODE is exact then all terms of x will be cancelled out in the LHS

Resubstitute g(y) in $f(x,y) = \int M(x,y)dx + g(y)$

Solution is $f(x,y) = c$

$$\therefore \int M(x,y)dx + g(y) = c$$

Solve the following ODE using method of exact equations:

Example 1:

$$2xy^2dx + 2(x^2y - 1)dy = 0$$

First, we check for exactness i.e., $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Here, $\frac{\partial M}{\partial y} = 4xy$ and $\frac{\partial N}{\partial x} = 4xy$

Therefore, the give ODE is exact

Solution is $f(x, y) = c$

Where $\frac{\partial f}{\partial x} = M(x, y)$, $\frac{\partial f}{\partial y} = N(x, y)$

$$f(x, y) = \int M(x, y)dx + g(y)$$

$$f(x, y) = \int 2xy^2dx + g(y)$$

$$f(x, y) = x^2y^2 + g(y)$$

$$\frac{\partial f}{\partial y} = N$$

$$\frac{\partial(x^2y^2 + g(y))}{\partial y} = 2(x^2y - 1)$$

$$2x^2y + g'(y) = 2x^2y - 2$$

$$g'(y) = -2$$

$$g(y) = -2y$$

$$f(x, y) = x^2y^2 - 2y$$

$$f(x, y) = c$$

$$x^2y^2 - 2y = c$$

Example 2:

$$\left(\frac{1}{1+y^2}+\cos x-2xy\right)\frac{dy}{dx}=y(y+\sin x)$$

$$y(y+\sin x)dx-\left(\frac{1}{1+y^2}+\cos x-2xy\right)dy=0$$

First, we check for exactness i.e., $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

Here, $\frac{\partial M}{\partial y}=2y+\sin x$ and $\frac{\partial N}{\partial x}=2y+\sin x$

Therefore, the give ODE is exact

Solution is $f(x,y)=c$

Where $\frac{\partial f}{\partial x}=M(x,y),\ \frac{\partial f}{\partial y}=N(x,y)$

$$f(x,y)=\int M(x,y)dx+g(y)$$

$$f(x,y)=\int y(y+\sin x)dx+g(y)$$

$$f(x,y)=y^2x-y.\cos x+g(y)$$

$$\frac{\partial f}{\partial y}=N$$

$$\frac{\partial(y^2x-y.\cos x+g(y))}{\partial y}=-\left(\frac{1}{1+y^2}+\cos x-2xy\right)$$

$$2xy-\cos x+g'(y)=-\left(\frac{1}{1+y^2}+\cos x-2xy\right)$$

$$g'(y)=-\frac{1}{1+y^2}$$

$$g(y)=-\tan^{-1}y$$

$$f(x,y)=y^2x-y.\cos x-\tan^{-1}y$$

$$f(x,y)=c$$

$$y^2x-y.\cos x-\tan^{-1}y=c$$