GENG 8010–Part 1: Elements of Differential and Difference Equations

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Winter 2023

The integrating factor method: linear case XI

Example–

Solve

$$ydx + (3x - xy + 2)dy = 0$$

Note that we have ydy in the above so this is a nonlinear in y but linear in x.

$$ydx + (3 - y)xdy = -2dy \Rightarrow \frac{dx}{dy} + \left(\frac{3}{y} - 1\right)x = \frac{-2}{y} \qquad y \neq 0$$

$$\frac{dx}{dy} + \left(\frac{3}{y} - 1\right)x = \frac{-2}{y} \qquad y \neq 0$$

The integrating factor method: linear case XII

so the IF is $e^{\int\left(\frac{3}{y}-1\right)dy}$, but

$$\int\left(\frac{3}{y} - 1\right)dy = 3\ln|y| - y$$

so IF is

$$e^{\int\left(\frac{3}{y}-1\right)dy} = e^{(3\ln|y|-y)} = e^{3\ln|y|}e^{-y} = e^{\ln|y|^3}e^{-y} = |y|^3e^{-y}$$

So, for $y > 0$ y^3e^{-y} is an integrating factor for the diff. eq. at the top of the page, and for $y < 0$, $-y^3e^{-y}$ is the integrating factor. In either case

$$IF\left[dx + \left(\frac{3}{y} - 1\right)xdy\right] = IF\left[-\frac{2}{y}dy\right]$$

The integrating factor method: linear case XIII

$$y^3e^{-y}dx + y^2(3 - y)e^{-y}xdy = -2y^2e^{-y}dy$$

$$d\left[xy^3e^{-y}\right] = -2y^2e^{-y}dy$$

integrating the right side by parts or using the following from an integral table

$$\int x^2e^{ax}dx = e^{ax}\left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)$$

we get

$$-2\int y^2e^{-y}dy = -2e^{-y}\left[\frac{y^2}{-1} - \frac{2y}{1} + \frac{2}{-1}\right] + c$$
$$= e^{-y}\left(2y^2 + 4y + 4\right) + c$$

The integrating factor method: linear case XIV

$$xy^3e^{-y} = -2\int y^2e^{-y}dy$$

$$= 2y^2e^{-y} + 4ye^{-y} + 4e^{-y} + c$$

and the implicit solution is given by

$$\boxed{xy^3 = 2y^2 + 4y + 4 + ce^y}$$

The integrating factor method: nonlinear case I

Let us start again with

$$M(x,y)dx + N(x,y)dy = 0 \tag{3.3}$$

Suppose $u(x)$ is an integrating factor for (3.3).

Then $\frac{\partial u}{\partial y} = 0$, and $\frac{\partial u}{\partial x} = \frac{du}{dx}$.

Multiplying the (3.3) with $u(x)$ gives

$$uMdx + uNdy = 0$$

The integrating factor method: nonlinear case II

If the above is to be exact then $\frac{\partial}{\partial y}(uM) = \frac{\partial}{\partial x}(uN)$, and

$$u\frac{\partial M}{\partial y} + M\frac{\partial u}{\partial y} = u\frac{\partial N}{\partial x} + N\frac{\partial u}{\partial x}$$

$$u\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{du}{dx}$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx = \frac{du}{u}$$

If

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$$

Then

$$u(x) = e^{\int f(x)dx}$$

The integrating factor method: nonlinear case III

Similarly, if we go through a similar process as above but this time with $u(y)$, and if we are led to

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y)$$

then

$$u(y) = e^{\int -g(y)dy}$$

The integrating factor method: nonlinear case IV

Remark

Rule:

1. If $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=f(x)$, then $e^{\int f(x)dx}$ is an IF for (3.3).
2. If $\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=g(y)$, then $e^{-\int g(y)dy}$ is an IF for (3.3).

If neither of the above are satisfied, then all we can conclude is that (3.3) does not have an IF that is a function of x or y alone.

The integrating factor method: nonlinear case V

Example–Solve for the solution of

$$y(x+y+1)dx+x(x+3y+2)dy=0$$

Calculate

$$\frac{\partial M}{\partial y}=x+2y+1 \qquad \frac{\partial N}{\partial x}=2x+3y+2$$

and

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=-x-y-1$$

and

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=-\frac{x+y+1}{x(x+3y+2)}$$

Note that the above right side is not a function of x only as required in Rule 1.

The integrating factor method: nonlinear case VI

So now calculate

$$\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=-\frac{x+y+1}{y(x+y+1)}=-\frac{1}{y}$$

So now we see that the above right side is a function of y only and therefore Rule 2 applies and $e^{\ln|y|}=|y|$ is the IF. In other words

$$IF=\begin{cases} y & if \quad y>0 \\ -y & if \quad y<0 \end{cases}$$

In either case

$$(xy^2+y^3+y^2)dx+(x^2y+3xy^2+2xy)dy=0$$

The integrating factor method: nonlinear case VII

Now

$$\frac{\partial F}{\partial x}=M=xy^2+y^3+y^2 \Longrightarrow F=\frac{1}{2}x^2y^2+xy^3+xy^2+R(y)$$

and

$$\frac{\partial F}{\partial y}=N \Longrightarrow x^2y+3xy^2+2xy+R'=x^2y+3xy^2+2xy \Longrightarrow R'=0$$

Hence

$$F=\frac{1}{2}x^2y^2+xy^3+xy^2=\frac{1}{2}c$$

Hence the implicit solution of differential equation is

$$\boxed{xy^2(x+2y+2)=c}$$

More on determination of an integrating factor

Example–Solve for the solution of

$$ydx + (x + x^3y^2)dy = 0$$

Verify that in this case,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{3y^2}{(1+x^2y^2)} \quad and \quad \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = 3x^2y$$

So neither of the two rules are satisfied which tells us that an integrating factor that is a function of x or y alone cannot be found. But lets see if we can get lucky and make an educated guess about the form of an IF.

More on determination of an integrating factor

consider four commonly encountered exact differentials below

$$d(xy) = xdy + ydx \qquad d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$$
$$d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2} \qquad d\left(\tan^{-1}\frac{y}{x}\right) = \frac{xdy - ydx}{x^2+y^2}$$

based on the above, group the equation as

$$ydx + (x + x^3y^2)dy = (ydx + xdy) + x^3y^2dy = d(xy) + x^3y^2dy = 0$$

or

$$\frac{1}{(xy)^3}d(xy) + \frac{1}{y}dy = 0$$

which is integrable, so

$$-\frac{1}{2x^2y^2} + \ln|y| + \ln|c| = 0 \Rightarrow \boxed{2x^2y^2\ln|cy| = 1}$$

Another useful trick I

Sometimes a change of variables can result in a transformation of the differential equation in a solvable form. The differential equation itself can be explored for this purpose. Let's illustrate this through an example.

Example– Solve

$$(x + 2y - 1)dx + 3(x + 2y)dy = 0$$

Notice the term $x + 2y$ has occurred twice. So let's define

$$v = x + 2y \Rightarrow dx = dv - 2dy$$

and subsititue to get

$$(v-1)(dv - 2dy) + 3vdy = 0$$

Another useful trick II

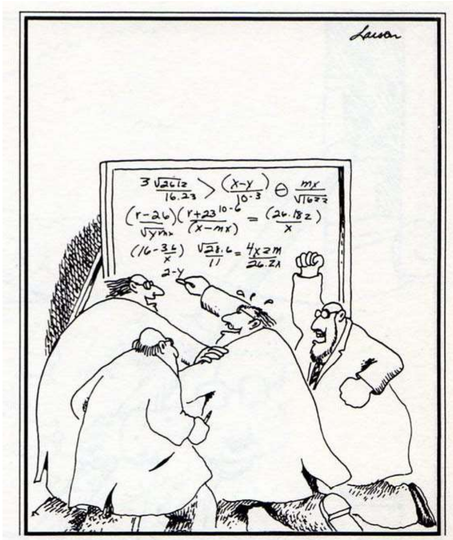
$$(v-1)dv + (v+2)dy = 0$$
$$\frac{v-1}{v+2}dv + dy = 0$$
$$\Rightarrow \left(1 - \frac{3}{v+2}\right)dv + dy = 0 \quad Recalling \quad \int \frac{1}{ax+b} = \frac{1}{a}\ln|ax+b| + c$$

Integrating leads to

$$v - 3\ln|v+2| + y + c = 0$$

Recall that $v = x + 2y$ so,

$$\boxed{x + 3y + c = 3\ln|x + 2y + 2|}$$



"Go for it, Sidney! You've got it! You've got it! Good hands! Don't choke!"

Bernoulli's Equation I

Bernoulli's Equation is a well know equation that is of the class of equations we have considered, and is given by

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{3.4}$$

- We have dealt already with the case of $n = 1$, in (B) for which the variables are separable.
- So consider the cases when $n \neq 1$. (3.4) can be put into that form.

$$y^{-n}dy + Py^{-n+1}dx = Qdx$$

Bernoulli's Equation II

Let

$$r = y^{-n+1}$$

$$dr = (1-n)y^{-n}dy \Longrightarrow dy = \frac{1}{(1-n)y^{-n}}dr$$

Therefore

$$y^{-n}dy + Py^{-n+1}dx = Qdx$$

$$y^{-n}\frac{dr}{(1-n)y^{-n}} + rP(x)dx = Q(x)dx$$

$$\frac{dr}{(1-n)} + rP(x)dx = Q(x)dx$$

Bernoulli's Equation III

$$dr + (1-n)Prdx = (1-n)Qdx$$

or

$$\boxed{\frac{dr(x)}{dx} + (1-n)Pr(x) = (1-n)Q(x)}$$

Hence, the Bernoulli Equation can be reduced with a change of variable to a standard form that is solvable using the techniques already discussed.

Bernoulli's Equation IV

Example– Solve

$$y(6y^2 - x - 1)dx + 2xdy = 0$$

Rearranging to get the equation in the form of (3.4) gives

$$2xdy - y(x+1)dx = -6y^3dx \Longrightarrow \frac{dy}{dx} - \frac{x+1}{2x}y = -\frac{6}{2x}y^3$$

which is in the Bernoulli Equation form with

$$\boxed{n = 3; \quad P(x) = -\frac{x+1}{2x}; \quad Q(x) = -\frac{3}{x}; \quad \textit{and} \;\; r = y^{-3+1} = y^{-2}}$$

so

$$\frac{dr(x)}{dx} + (1-n)Pr(x) = (1-n)Q(x) \Longrightarrow \frac{dr}{dx} + \frac{x+1}{x}r = \frac{6}{x}$$

Bernoulli's Equation V

therefore, $e^{(x+\ln|x|)} = |x|e^x$ is an IF for the above, since

$$\frac{dr}{dx} + r(1+x^{-1}) = 6x^{-1} \qquad IF = e^{\int (1+x^{-1})dx} = |x|e^x$$

$$xe^x\frac{dr}{dx} + re^x(x+1) = 6e^x$$

$$\frac{d}{dx}(rxe^x) = 6e^x$$

which has a solution

$$xre^x = 6e^x + c$$

but $r = y^{-2}$ which if substituted gives,

$$\boxed{y^2(6 + ce^{-x}) = x}$$

Nature of solution(s)– I

We already established that any nth order linear differential equation can be written as

$$(a_nD^n + a_{n-1}D^{n-1} + \ldots + a_1D + a_0)y(t) = F(t)$$

The above is called a **inhomogeneous differential equation**. Setting the right side equal to 0, we get the **homogeneous differential equation**

$$(a_nD^n + a_{n-1}D^{n-1} + \ldots + a_1D + a_0)y(t) = 0 \tag{4.1}$$

It turns out that the above differential equation will have at most, n **linearly independent** solutions.

Nature of solution(s)– II

Theorem 4.1.1

A necessary and sufficient condition for the n solutions of (4.1) to be independent is that their Wronskian does not vanish. That is if $y_1, y_2, \ldots, y_n$ *represent the n solutions, the Wronskian* $W(t)$ *is nonzero, where*

$$W(t) = \begin{vmatrix} y_1 & y_2 & \ldots & y_n \\ Dy_1 & Dy_2 & \ldots & Dy_n \\ \ldots & \ldots & \ldots & \ldots \\ D^{n-1}y_1 & D^{n-1}y_2 & \ldots & D^{n-1}y_n \end{vmatrix}$$

Nature of solution(s)– III

Example–Consider the differential equation

$$\frac{d^2y}{dx^2} - 4y = 0$$

Verify that $y_1 = e^{2x}$ and $y_2 = e^{-2x}$ are both solutions. Therefore,

$$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -4 \neq 0$$

So y_1 and y_2 are independent solutions.