Engineering Mathematics – Midterm 1

1. Solve the ODE by any method. 10

$$y' = \frac{y^2 - 2y + 2}{\sqrt{x}(x+1)}$$

Solution:

$$\frac{dy}{dx} = \frac{(y-1)^2 + 1}{\sqrt{x}(x+1)}$$

$$\frac{1}{(y-1)^2 + 1} dy = \frac{1}{\sqrt{x}(x+1)} dx$$

$$\int \frac{1}{(y-1)^2 + 1} dy = \int \frac{1}{\sqrt{x}(x+1)} dx$$

$$\tan^{-1}(y-1) = \int \frac{1}{\sqrt{x}(x+1)} dx$$

$$Let\ \sqrt{x} = u$$

$$\frac{1}{2\sqrt{x}} = \frac{du}{dx}$$

$$\frac{1}{\sqrt{x}} dx = 2\ du$$

$$\tan^{-1}(y-1) = 2\int \frac{1}{(u^2+1)} du$$

$$\tan^{-1}(y-1) = 2\tan^{-1} u + c$$

$$\tan^{-1}(y-1) = 2\tan^{-1} \sqrt{x} + c$$

2. Solve the ODE by method of integration factor. 10

$$x^2y' + 13xy = \frac{x^2+3}{x^{11}(x+2)}$$

Solution:

$$y' + \frac{13}{x}y = \frac{x^2+3}{x^{13}(x+2)}$$

Comparing with the standard equation $y' + p(x)y = q(x)$ we get,

$$p(x) = \frac{13}{x};\ q(x) = \frac{x^2+3}{x^{13}(x+2)}$$

$$e^{\int p(x)dx} = e^{\int \frac{13}{x}dx}$$

$$e^{\int p(x)dx} = x^{13}$$

Multiplying throughout by the integration factor we get,

$$x^{13}.y' + 13x^{12}.y = \frac{x^2+3}{(x+2)}$$

$$\frac{d(y.x^{13})}{dx} = \frac{x^2+3}{(x+2)}$$

$$\int \frac{d(y.x^{13})}{dx}dx = \int \frac{x^2+3}{(x+2)}dx$$

$$y.x^{13} = \int \frac{(x+2)(x-2)+7}{(x+2)}dx$$

$$y.x^{13} = \int x - 2 + \frac{7}{(x+2)}dx$$

$$y.x^{13} = \frac{x^2}{2} - 2x + 7\ln(x+2) + c$$

$$y = \frac{1}{x^{13}}(\frac{x^2}{2} - 2x + 7\ln(x+2) + c)$$

3. Solve the ODE by method of variation of parameters. 10

$$y' + \frac{1}{x}y = \cos(5x)$$

Solution: First, we solve for the homogeneous case

$$y' + \frac{1}{x}y = 0$$

$$\frac{1}{y}dy = -\frac{1}{x}dx$$

$$\int \frac{1}{y}dy = \int -\frac{1}{x}dx$$

$$\ln(y) = -\ln(x) + c$$

$$y = e^{-\ln(x)+c} = \frac{1}{x}.e^{c}$$

$$y_h = Ax^{-1}$$

Varying A to A(x); $y = A(x).x^{-1}$

$$y' = A'(x).x^{-1} - A(x).x^{-2}$$

Substituting y and y' in the given ODE we get,

$$A'(x).x^{-1} - A(x).x^{-2} + \frac{1}{x}A(x).x^{-1} = \cos(5x)$$

$$A'(x).x^{-1} = \cos(5x)$$

$$A(x) = \int x.\cos(5x)\,dx$$

$$A(x) = x\frac{\sin(5x)}{5} + \frac{\cos(5x)}{25} + c$$

$$y = \left(x\frac{\sin(5x)}{5} + \frac{\cos(5x)}{25} + c\right).x^{-1}$$

4. Solve the ODE by any method that you prefer. 10

$$y' + 2\cot(x)\, y = \cot(x)\, y^6$$

Solution: Comparing with Bernoulli's equation $y' + p(x)y = \mathrm{q}(x)y^n$

$$p(x) = 2\cot(x)\,;\, q(x) = \cot(x)\,;\, n = 6$$

$$y' + p(x)y = \mathrm{q}(x)y^n$$

$$y^{-n}y' + p(x)y^{1-n} = q(x)$$

$$Let\ y^{1-n} = v$$

$$1 - n.\, y^{-n} = \frac{dv}{dy}$$

$$y^{-n} = \frac{1}{1-n}\frac{dv}{dy}$$

$$\frac{1}{1-n}\frac{dv}{dy}.\frac{dy}{dx} + p(x).\, v = q(x)$$

$$v' + (1-n).\, p(x).\, v = (1-n).\, q(x)$$

$$Let\ (1-n).\, p(x) = P(x);\ (1-n).\, q(x) = Q(x)$$

$$v' + P(x).\, v = Q(x)$$

$$Integration\ Factor = e^{\int P(x)dx}$$

$$e^{\int P(x)dx}.\, v' + e^{\int P(x)dx}.\, P(x).\, v = e^{\int P(x)dx}.\, Q(x)$$

$$\frac{d(v.\, e^{\int P(x)dx})}{dx} = e^{\int P(x)dx}.\, Q(x)$$

$$\int \frac{d(v.\, e^{\int P(x)dx})}{dx}\, dx = \int e^{\int P(x)dx}.\, Q(x)\, dx$$

$$v.e^{\int P(x)dx} = \int e^{\int P(x)dx}.Q(x)\,dx$$

$$y^{1-n}.e^{(1-n)\int p(x)dx} = (1-n)\int e^{(1-n)\int p(x)dx}.q(x)\,dx$$

Resubstituting $p(x) = 2\cot(x)\,;q(x) = \cot(x)\,;n = 6$

$$y^{1-6}.e^{(1-6)\int 2\cot(x)dx} = (1-6)\int e^{(1-6)\int 2\cot(x)dx}.\cot(x)\,dx$$

$$y^{-5}.e^{-10\int \cot(x)dx} = -5\int e^{-10\int \cot(x)dx}.\cot(x)\,dx$$

$$y^{-5}.e^{-10.\ln(\sin x)} = -5\int e^{-10.\ln(\sin x)}.\cot(x)\,dx$$

$$y^{-5}.\sin^{-10}x = -5\int \frac{\cot(x)}{\sin^{10}x}dx$$

$$y^{-5}.\sin^{-10}x = -5\int \frac{\cos x}{\sin^{11}x}dx$$

$$y^{-5}.\sin^{-10}x = -5\int \frac{1}{u^{11}}du$$

$$y^{-5}.\sin^{-10}x = -5(\frac{1}{-10.u^{10}})$$

$$y^{-5}.\sin^{-10}x = \left(\frac{1}{2.\sin^{10}x}\right) + c$$

$$y^{-5} = \frac{1}{2} + c\sin^{10}x$$

5. Solve the ODE by method of exact equations. 10

$$\left(\frac{y}{1+x^2y^2}+ye^{xy}\right)dx+\left(\frac{x}{1+x^2y^2}+xe^{xy}\right)dy=0$$

Solution: Here, $M=\left(\frac{y}{1+x^2y^2}+ye^{xy}\right); N=\left(\frac{x}{1+x^2y^2}+xe^{xy}\right)$

$$\frac{\partial M}{\partial y}=\frac{1+x^2y^2-y(2x^2y)}{(1+x^2y^2)^2}+e^{xy}+y.(e^{xy}).x$$

$$\frac{\partial N}{\partial x}=\frac{1+x^2y^2-x(2xy^2)}{(1+x^2y^2)^2}+e^{xy}+x.(e^{xy}).y$$

Therefore, the given ODE is exact

Solution is $f(x,y)=c$ where, $\frac{\partial f}{\partial x}=M \; and \; \frac{\partial f}{\partial y}=N$

$$\frac{\partial f}{\partial x}=M$$

$$f=\int M\,dx+g(y)$$

$$f=\int\left(\frac{y}{1+x^2y^2}+ye^{xy}\right)\,dx+g(y)$$

$$f=y\int\left(\frac{1}{1+x^2y^2}+e^{xy}\right)\,dx+g(y)$$

$$f=y\left(\frac{1}{y}\tan^{-1}xy+\frac{1}{y}e^{xy}\right)+g(y)$$

$$f=\tan^{-1}xy+e^{xy}+g(y)$$

$$\frac{\partial f}{\partial y}=N$$

$$\frac{\partial f}{\partial y}=\frac{x}{1+x^2y^2}+xe^{xy}+g'(y)$$

$$N = \frac{x}{1 + x^2 y^2} + x e^{xy}$$

$$\frac{x}{1 + x^2 y^2} + y e^{xy} + g'(y) = \frac{x}{1 + x^2 y^2} + x e^{xy}$$

$$g'(y) = 0$$
$$g(y) = k$$

$$f(x, y) = c$$

$$f = \tan^{-1} xy + e^{xy} + g(y)$$

$$f = \tan^{-1} xy + e^{xy} + k$$

$$\tan^{-1} xy + e^{xy} + k = c$$

$$\tan^{-1} xy + e^{xy} = C$$

6. Suppose that a 9-th order homogeneous linear differential equation with constant coefficients has characteristic roots: 10

2, 2, 3, 4+2i, 4-2i, 4+2i, 4-2i, 7-3i, 7+3i

What are the general solutions of the differential equation?

Solution:

$$y_1 = e^{2x}$$
$$y_2 = x.e^{2x}$$
$$y_3 = e^{3x}$$
$$y_4 = e^{4x}\cos 2x$$
$$y_5 = e^{4x}\sin 2x$$
$$y_6 = x.e^{4x}\cos 2x$$
$$y_7 = x.e^{4x}\sin 2x$$
$$y_8 = e^{7x}\cos 3x$$
$$y_9 = e^{7x}\sin 3x$$

$$y = c_1e^{2x} + c_2xe^{2x} + c_3e^{3x} +$$
$$c_4e^{4x}\cos 2x + c_5e^{4x}\sin 2x +$$
$$c_6x.e^{4x}\cos 2x + c_7x.e^{4x}\sin 2x +$$
$$c_8e^{7x}\cos 3x + c_9e^{7x}\sin 3x$$