GENG 8010–Part 2 - Elements of Applied Linear Algebra

Mehrdad Saif ©

University of Windsor

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Outline of part 2– I

Outline of part 2– II

Outline of part 2– III

Introduction & axiomatic definitions– VI

Example– Given $\mathscr{F}$, let

$$\boldsymbol{x}_i = \begin{bmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{bmatrix}$$

If we define the usual operations

$$\boldsymbol{x}_i + \boldsymbol{x}_j = \begin{bmatrix} x_{1i} + x_{1j} \\ x_{2i} + x_{2j} \\ \vdots \\ x_{ni} + x_{nj} \end{bmatrix} \quad \text{and} \quad \alpha.\boldsymbol{x}_i = \begin{bmatrix} \alpha.x_{1i} \\ \alpha.x_{2i} \\ \vdots \\ \alpha.x_{ni} \end{bmatrix}$$

Introduction & axiomatic definitions– VII

Then $\mathscr{F}^n, \mathscr{F}$ is an n-dimensional vector space. $(\Re^n, \Re)$ and $(\mathbb{C}^n, \mathbb{C})$ are the n-dimensional real and complex vector spaces respectively.

Definition-Subspace of a vector space

Let $(\mathscr{X}, \mathscr{F})$ be a vector space and $\mathscr{Y} \subset \mathscr{X}$. Then $(\mathscr{Y}, \mathscr{F})$ is a **subspace** of $(\mathscr{X}, \mathscr{F})$ if under the same operations of $(\mathscr{X}, \mathscr{F})$, $\mathscr{Y}$ itself form a vector space over $\mathscr{F}$.

Introduction & axiomatic definitions– VIII

Example– $(\Re^n, \Re)$ is a subspace of $(\mathbb{C}^n, \Re)$.

Remark

Since "+", and "." of vectors are defined for $(\mathscr{X}, \mathscr{F})$, they satisfy 2,3,7-10. To check 1,4-6 and have them satisfied, we will require the following instead.

Alternate subspace definition

A set $\mathscr{Y} \subset \mathscr{X}$ is a subspace of $(\mathscr{X}, \mathscr{F})$ if

$$\forall \boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathscr{Y} \text{ and } \alpha_1, \alpha_2 \in \mathscr{F}, \quad \alpha_1 \boldsymbol{y}_1 + \alpha_2 \boldsymbol{y}_2 \in \mathscr{Y}$$

Introduction & axiomatic definitions– IX

Example– Consider $(\Re^2, \Re)$.

Any line that passes through the origin is a subspace of $(\Re^2, \Re)$, e.g. $\begin{bmatrix} x_1 \\ \alpha x_1 \end{bmatrix}$.

Linear independence– I

Q. What is the reason for us needing a coordinate system?

Q. What is the coordinate system equivalent for a vector space?

Linear independence

A set of vector $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ in a linear space $(\mathscr{X}, \mathscr{F})$ is said to be **linearly dependent** iff $\exists$ scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathscr{F}$ not all zeros, such that

$$\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2 + \ldots + \alpha_n \boldsymbol{x}_n = \mathbf{0}$$

If the only values of α_i for which the above is true is for $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$ then $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ are **linearly independent**.

Linear independence– II

Example–

$$\boldsymbol{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}; \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}; \boldsymbol{x}_3 = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$

$$\alpha_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{cases} 2\alpha_1 + \alpha_2 + 4\alpha_3 = 0 \\ \alpha_1 + 2\alpha_3 = 0 \\ \alpha_1 + \alpha_2 + 2\alpha_3 = 0 \end{cases}$$

$$\Longrightarrow \alpha_2 = 0 \quad \text{and} \quad \alpha_1 = -2\alpha_3$$

and there is infinitely many α_i's that satisfy these equations. So the vectors are linearly dependent.

Linear independence– III

Alternative definition of linear independence

If $\boldsymbol{\alpha} = [\alpha_1 \ \alpha_2 \ \ldots \ \alpha_n]^T$, then the vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ are linearly independent if

$$[\boldsymbol{x}_1 \ \boldsymbol{x}_2 \ \ldots\ldots \ \boldsymbol{x}_n]\boldsymbol{\alpha} = \mathbf{0} \ \text{ implies } \ \boldsymbol{\alpha} = \mathbf{0} \quad \boldsymbol{\alpha} \in \mathscr{F}^n$$

Linear independence– IV

"Of course I am an experienced space scientist- I know a lot about Banach spaces, Hilbert spaces, Lebesgue spaces, Sobolev spaces,...."

Linear independence– V

Dimension of a vector space

The maximum number of linearly independent vectors in a linear space is called the **dimension** of the linear space.

Example– In $(\Re^2, \Re)$ we cannot find three linearly independent vectors.

Example– Consider the set of all real valued piecewise continuous functions over $(-\infty, \infty)$. It forms a linear space over the field of real numbers (called a *function space*). The zeros vector is one which is zeros over $(-\infty, \infty)$. The following functions belong to this space:
$t, t^2, t^3, \ldots \qquad -\infty < t < \infty.$

Linear independence– VI

Example– In the above example the set of functions $\{t^n, n = 1, 2, \ldots\}$ is linearly independent, because there are no real constants α_i's not all zero for such that

$$\sum_i \alpha_i t^i = 0$$

and since there are infinitely many of these functions, the vector space is a infinite dimensional one.

Definition of a basis

A set of linearly independent vectors in $(\mathscr{X}, \mathscr{F})$ is called a **basis** if every vector $\boldsymbol{x} \in \mathscr{X}$ can be written as a unique linear combination of the basis vectors.

Linear independence– VII

Theorem 2.2.1

in an n-dimensional linear vector space any n-linearly independent vectors, $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ *will qualify as basis.*

Proof.
Since $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ are linearly independent, then

$$\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2 + \ldots\ldots + \alpha_n \boldsymbol{x}_n + \alpha_{n+1} \boldsymbol{x}_{n+1} = 0$$

and not all α_i's are zero. Clearly $\alpha_{n+1} \neq 0$ because if it was, the implication of that would be that $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ would be linearly dependent which is not the case. Thus,

$$\boldsymbol{x}_{n+1} = -\frac{1}{\alpha_{n+1}} (\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2 + \ldots\ldots + \alpha_n \boldsymbol{x}_n)$$

Linear independence– VIII

Thus, $\boldsymbol{x}_{n+1}$ can be expressed as a linear combination of $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$. Now all that we need to show is that this representation is unique. Thus, suppose otherwise,

$$\boldsymbol{x}_{n+1} = \beta_1 \boldsymbol{x}_1 + \beta_2 \boldsymbol{x}_2 + \ldots\ldots + \beta_n \boldsymbol{x}_n$$
$$\boldsymbol{x}_{n+1} = \hat{\beta}_1 \boldsymbol{x}_1 + \hat{\beta}_2 \boldsymbol{x}_2 + \ldots\ldots + \hat{\beta}_n \boldsymbol{x}_n \quad \text{and subtract from above}$$

$$\boldsymbol{0} = (\beta_1 - \hat{\beta}_1)\boldsymbol{x}_1 + (\beta_2 - \hat{\beta}_2)\boldsymbol{x}_2 + \ldots\ldots + (\beta_n - \hat{\beta}_n)\boldsymbol{x}_n$$

Now β_i must be equal to $\hat{\beta}_i, \forall\, i$ since if it was otherwise, the implication would be that the set of $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ is dependent which would be a contradiction. ■

Linear independence– IX

Remark

As a result of above, if $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n\}$ is given as a basis, then for every $\boldsymbol{x}$

$$\boldsymbol{x} = [\boldsymbol{x}_1 \ \boldsymbol{x}_2 \ \ldots\ldots \ \boldsymbol{x}_n]\beta \qquad \beta \in (\mathscr{F}^n, \mathscr{F})$$

Theorem 2.2.2

In an n-dimensional linear space an orthogonal set of vectors will also be linearly independent. The converse is not necessarily true.

Proof.

Consider two vectors $\boldsymbol{x}_1$, and $\boldsymbol{x}_2$ and recall that orthogonality means

$$< \boldsymbol{x}_1, \boldsymbol{x}_2 >= 0$$

Linear independence– X

If $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ were linearly dependent, then

$$\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2 = 0 \ \text{ iff } \ \alpha_1 \neq 0, \alpha_2 \neq 0$$

$$< \boldsymbol{x}_1, \alpha_1\boldsymbol{x}_1 + \alpha_2\boldsymbol{x}_2 >= 0 = \alpha_1 < \boldsymbol{x}_1, \boldsymbol{x}_1 > + \alpha_2 < \boldsymbol{x}_1, \boldsymbol{x}_2 >= \alpha_1 < \boldsymbol{x}_1, \boldsymbol{x}_1 >$$

Therefore, $\alpha_1 = 0$. Similarly we can prove $\alpha_2 = 0$. So we have shown that if $\boldsymbol{x}_1, \boldsymbol{x}_2$ are orthogonal, the only way $\alpha_1\boldsymbol{x}_1 + \alpha_2\boldsymbol{x}_2$ could be zero is if $\alpha_1 = \alpha_2 = 0$ which implies the vectors are linearly independent. This can be extended to n vectors.

Orthonormal basis and Gram-Schmidt process– I

Orthonormal vectors form a convenient basis. So we are interested to find such a basis given n linearly independent vectors. This can be done by using the Gram-Schmidt orthogonalization procedure.

- Applications such as representing a signal in terms of some elementary time functions called *basis functions*.

Gram-Schmidt Process– Given a set of linearly independent vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$, we wish to generate a set of orthonormal vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n$

1. Select $\hat{\boldsymbol{e}}_1 = \boldsymbol{v}_1$.
2. Find $< \hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_1 >$, and normalize $\hat{\boldsymbol{e}}_1$ to get $\boldsymbol{e}_1 = \frac{\hat{\boldsymbol{e}}_1}{\sqrt{<\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_1>}}$.

Orthonormal basis and Gram-Schmidt process– II

3. Let $\hat{\boldsymbol{e}}_2 = \alpha_1 \boldsymbol{e}_1 + \boldsymbol{v}_2$

$$< \boldsymbol{e}_1, \hat{\boldsymbol{e}}_2 > = \alpha_1 < \boldsymbol{e}_1, \boldsymbol{e}_1 > + < \boldsymbol{e}_1, \boldsymbol{v}_2 >$$
$$0 = \alpha_1 + < \boldsymbol{e}_1, \boldsymbol{v}_2 > \Longrightarrow \alpha_1 = - < \boldsymbol{e}_1, \boldsymbol{v}_2 >$$

4. Find $< \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_2 >$, and normalize $\hat{\boldsymbol{e}}_2$ to get $\boldsymbol{e}_2 = \frac{\hat{\boldsymbol{e}}_2}{\sqrt{<\hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_2>}}$.
5. Let $\hat{\boldsymbol{e}}_3 = \alpha_2 \boldsymbol{e}_1 + \alpha_3 \boldsymbol{e}_2 + \boldsymbol{v}_3$ and get

$$\alpha_2 = - < \boldsymbol{e}_1, \boldsymbol{v}_3 >$$
$$\alpha_3 = - < \boldsymbol{e}_2, \boldsymbol{v}_3 >$$

6. Find $< \hat{\boldsymbol{e}}_3, \hat{\boldsymbol{e}}_3 >$, and normalize $\hat{\boldsymbol{e}}_3$ to get $\boldsymbol{e}_3 = \frac{\hat{\boldsymbol{e}}_3}{\sqrt{<\hat{\boldsymbol{e}}_3, \hat{\boldsymbol{e}}_3>}}$.
7. Continue in this manner.

Orthonormal basis and Gram-Schmidt process– III

Example–Find an orthonormal set by using

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Verify that the above two vectors are linearly independent, and then

1. Let $\hat{\boldsymbol{e}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
2. $< \hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_1 >= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2$ and $\boldsymbol{e}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Orthonormal basis and Gram-Schmidt process– IV

3. $\hat{\boldsymbol{e}}_2 = \alpha_1 \boldsymbol{e}_1 + \boldsymbol{x}_2$

   $\alpha_1 = - < \boldsymbol{e}_1, \boldsymbol{x}_2 >= -\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}$

4. $\hat{\boldsymbol{e}}_2 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \frac{-1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

   $\boldsymbol{e}_2 = \frac{-\sqrt{2}}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Representation of a vector with respect to a basis– I

Vector representation

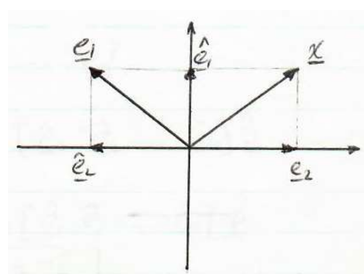
In an n-dimensional vector space $(\mathscr{F}^n, \mathscr{F})$ if $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n\}$ is chosen as a basis $(\mathscr{B})$, then every vector can be written as

$$\boldsymbol{x} = [\boldsymbol{x}_1 \ \boldsymbol{x}_2 \ \ldots\ldots \ \boldsymbol{x}_n]\beta \qquad \beta \in (\mathscr{F}^n, \mathscr{F})$$

β is called the **representation** of $\boldsymbol{x}$ with respect to $\mathscr{B}$.

Representation of a vector with respect to a basis– II

Example–Consider $(\Re^2, \Re)$.



$$\boldsymbol{e}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}; \quad \boldsymbol{e}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boldsymbol{x} = \boldsymbol{e}_1 + 2\boldsymbol{e}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Longrightarrow \boxed{\beta = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

β is the representation of $\boldsymbol{x}$ wrt $\mathscr{B} = \{\boldsymbol{e}_1 \boldsymbol{e}_2\}$.

Representation of a vector with respect to a basis– III

$$\hat{\mathbf{e}}_1 = \begin{bmatrix}0\\1\end{bmatrix}, \ \hat{\mathbf{e}}_2 = \begin{bmatrix}-1\\0\end{bmatrix} \Longrightarrow \mathbf{x} = \hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 = \begin{bmatrix}0\\1\end{bmatrix} - \begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

$$= [\hat{\mathbf{e}}_1 \ \ \hat{\mathbf{e}}_2]\begin{bmatrix}1\\-1\end{bmatrix} \Longrightarrow \boxed{\hat{\beta} = \begin{bmatrix}1\\-1\end{bmatrix}}$$

where $\hat{\beta}$ is representation of $\mathbf{x}$ wrt to $\hat{\mathscr{B}} = \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2\}$.

Representation of a vector with respect to a basis– IV

Alternatively:

$$\mathbf{x} = \begin{bmatrix}1\\1\end{bmatrix} = [\mathbf{e}_1 \ \ \mathbf{e}_2]\beta \Longrightarrow \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-1 & 1\\1 & 0\end{bmatrix}\beta \Longrightarrow \boxed{\beta = \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}}$$

$$\mathbf{x} = [\hat{\mathbf{e}}_1 \ \ \hat{\mathbf{e}}_2]\hat{\beta} \Longrightarrow \hat{\beta} = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}^{-1}\begin{bmatrix}1\\1\end{bmatrix} \Longrightarrow \boxed{\hat{\beta} = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}}$$

Representation of a vector with respect to a basis– V

Theorem 2.2.3

In an n-dimensional vector space $(\mathscr{X}, \mathscr{F})$ any $x \in (\mathscr{X}, \mathscr{F})$ can be represented with respect to an orthonormal basis $\mathscr{B} = \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ by the representation $\boldsymbol{\beta}$ where $\beta_i =< \mathbf{e}_i, \mathbf{x} >$.

Representation of a vector with respect to a basis– VI

Example–Represent $\mathbf{x} = \begin{bmatrix}3\\-1\end{bmatrix}$ with respect to the orthonormal basis obtained in the last example.

$$\beta_1 =< \mathbf{e}_1, \mathbf{x} >= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}3\\-1\end{bmatrix} = \frac{4}{\sqrt{2}}$$

$$\beta_2 =< \mathbf{e}_2, \mathbf{x} >= \frac{-\sqrt{2}}{2}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}3\\-1\end{bmatrix} = -\sqrt{2}$$

Now check that

$$\mathbf{x} = [\mathbf{e}_1 \ \ \mathbf{e}_2]\begin{bmatrix}\beta_1\\\beta_2\end{bmatrix} = \begin{bmatrix}3\\-1\end{bmatrix}$$

Representation of x in $\mathscr{B}$ vs. $\hat{\mathscr{B}}$– I

Question– In an n-dimensional space is there any way of finding the representation of a vector in one basis from its representation in another basis?

Consider

$$[\boldsymbol{e}_1 \ \ \boldsymbol{e}_2 \ \ \ldots \ \ \boldsymbol{e}_n] = [\hat{\boldsymbol{e}}_1 \ \ \hat{\boldsymbol{e}}_2 \ \ \ldots \ \ \hat{\boldsymbol{e}}_n]\boldsymbol{T}$$

$$\therefore \ \ \boldsymbol{T} = [\hat{\boldsymbol{e}}_1 \ \ \hat{\boldsymbol{e}}_2 \ \ \ldots \ \ \hat{\boldsymbol{e}}_n]^{-1}[\boldsymbol{e}_1 \ \ \boldsymbol{e}_2 \ \ \ldots \ \ \boldsymbol{e}_n]$$

Also,

$$\boldsymbol{x} = [\boldsymbol{e}_1 \ \ \boldsymbol{e}_2 \ \ \ldots \ \ \boldsymbol{e}_n]\beta = [\hat{\boldsymbol{e}}_1 \ \ \hat{\boldsymbol{e}}_2 \ \ \ldots \ \ \hat{\boldsymbol{e}}_n]\hat{\beta}$$

Then

$$[\hat{\boldsymbol{e}}_1 \ \ \hat{\boldsymbol{e}}_2 \ \ \ldots \ \ \hat{\boldsymbol{e}}_n]\boldsymbol{T}\beta = [\hat{\boldsymbol{e}}_1 \ \ \hat{\boldsymbol{e}}_2 \ \ \ldots \ \ \hat{\boldsymbol{e}}_n]\hat{\beta}$$

$$\Longrightarrow \boxed{\hat{\beta} = \boldsymbol{T}\beta} \quad \text{or} \quad \boxed{\beta = \boldsymbol{T}^{-1}\hat{\beta}}$$

Representation of x in $\mathscr{B}$ vs. $\hat{\mathscr{B}}$– II

Example–Consider the last example.

$$[\boldsymbol{e}_1 \ \ \boldsymbol{e}_2] = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \qquad [\hat{\boldsymbol{e}}_1 \ \ \hat{\boldsymbol{e}}_2] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

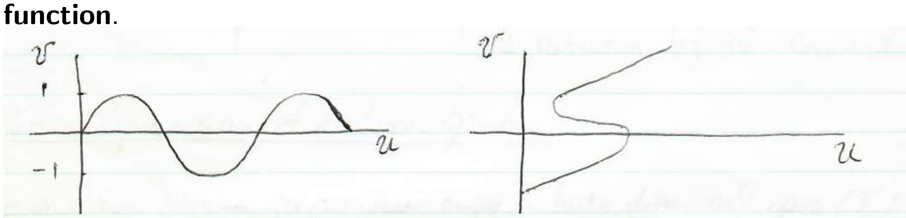
$$\boldsymbol{T} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

Now given $\beta = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, get

$$\hat{\beta} = \boldsymbol{T}\beta = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Preliminaries and definitions– I

Consider two sets $\mathcal{U}$ and $\mathcal{V}$ and suppose that we assign to each element of $\mathcal{U}$ one and only one element of $\mathcal{V}$. Then the rule of assignment is called a **function**.



The curve on the left represents a function $f : \mathcal{U} \longrightarrow \mathcal{V}$ with $v = f(u)$ and $v \in \mathcal{V}$ and $u \in \mathcal{U}$.

Furthermore,

- The set $\mathcal{U}$ on which a function is defined is called a **domain** of the function.

Preliminaries and definitions– II

- The subset of $\mathcal{V}$ that is assigned to some elements of $\mathcal{U}$ is called the **range** of the function (in which case f maps $\mathcal{U}$ <u>into</u> $\mathcal{V}$.
- If the range of the function is all of $\mathcal{V}$, then the function maps $\mathcal{U}$ <u>onto</u> $\mathcal{V}$.

The functions that we are interested in are called *linear functions or linear operators or linear mapping or transformation*. The sets associated with these linear operators are required to be vector spaces over the same field.

$$\mathscr{L} : (\mathscr{X}, \mathscr{F}) \longrightarrow (\mathscr{Y}, \mathscr{F})$$

Preliminaries and definitions– III

Linear operator

A function that maps $(\mathscr{X}, \mathscr{F})$ into $(\mathscr{Y}, \mathscr{F})$ is said to be a linear operator iff

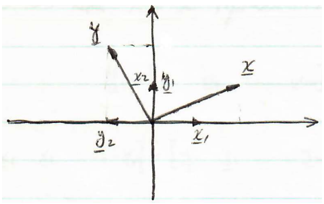
$$\mathscr{L}(\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2) = \alpha_1 \mathscr{L}(\boldsymbol{x}_1) + \alpha_2 \mathscr{L}(\boldsymbol{x}_2)$$

$$\forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathscr{X} \text{ and } \forall \alpha_1, \alpha_2 \in \mathscr{F}$$

Preliminaries and definitions– IV

Example–Let $\mathscr{L}$ be a 90 degrees CCW rotation.

Note that if we have $\{\boldsymbol{x}_1, \boldsymbol{x}_2\}$; and $\alpha_1 = 2$ and $\alpha_2 = 1$, then $\mathscr{L}(\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2) = \alpha_1 \mathscr{L}(\boldsymbol{x}_1) + \alpha_2 \mathscr{L}(\boldsymbol{x}_2) = \boldsymbol{y}$ and in both cases the 90 degrees CCW rotation will result in $\boldsymbol{y}$ and hence is a linear operator.



Matrix representation of $\mathscr{L}$– I

"There I was, walking on the bridge between theory and practice, and suddenly, out of nowhere, comes this gigantic, strongly positive, self-adjoint operator with a dense domain, and smashes the bridge all to pieces! What a nightmare!"