GENG 8010–Part 1: Elements of Differential and Difference Equations

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Part I–Outline I

Part I–Outline II

Part I–Outline III

Nonhomogeneous solution of difference equations I

As in the case of differential equations, There are two methods for finding $y_p(k)$:

1. Method of undetermined coefficients
   - Note that this method works when repeated application of E to forcing function $F(k)$ produces a finite number of linearly independent terms. Examples of possible $F(k)$ are $k^m + \ldots + b_1 k + b_0$, $\sin\theta k$, $\cosh\theta k$, etc.
2. Variation of parameters (parallel process to finding a solution for differential equations but will not be pursued).

Nonhomogeneous solution of difference equations II

Example–Solve the following by the method of undetermined coefficients:

$$(E^2 + 2E + 1)y(k) = k(-1)^k$$

CE is

$$\beta^2 + 2\beta + 1 = 0$$

with $\beta_{1,2} = -1$, $y_c(k) = \alpha_1(-1)^k + \alpha_2 k(-1)^k$. Since $F(k)$ contains $(-1)^k$ which is also in the homogeneous solution, rather than choosing a form as such $(Ak + B)(-1)^k$, we choose

Nonhomogeneous solution of difference equations III

$$y_p(k) = (Ak^3 + Bk^2)(-1)^k$$

and

$$Ey_p = -\left[A(k+1)^3 + B(k+1)^2\right](-1)^k$$
$$E^2 y_p = \left[A(k+2)^3 + B(k+2)^2\right](-1)^k$$

substituting into the equation and equating results in $A = \frac{1}{6}$, and $B = -\frac{1}{2}$, and complete solution is then

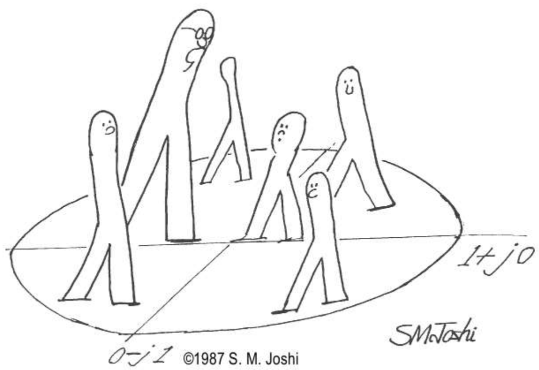
$$\boxed{y(k) = \left(\alpha_1 + \alpha_2 k - \frac{1}{2}k^2 + \frac{1}{6}k^3\right)(-1)^k}$$

System concepts I

Stability of linear discrete-time systems

1. Roots of CE are the poles of the system. Poles that lie within a unit circle produce terms in the homogeneous equation that converge to zero as $k \longrightarrow \infty$ (stable).
2. Roots outside the unit circle produce terms of growing magnitude without bound as k increases (unstable).
3. Simple roots on the unit circle produce terms of constant magnitude (stable or marginally stable), but repeated roots on the unit circle will produce terms that increase without limit as k increases (unstable).

System concepts II



"And then in the seventies came the digital revolution, and they confined us to this silly circle!"

System concepts III

Much in the same way as in continuous time systems, the input-output relation between in a discrete-time system can be represented as

$$y(k) = \mathscr{H}_D u(k)$$

consider now the discrete-Delta function defined as

$$\delta(k) = \begin{cases} 1, & \text{if } k = 0. \\ 0, & \text{if } k \neq 0. \end{cases}$$

Now represent the input to the system as

$$u(k) = \sum_{m=-\infty}^{\infty} u(m)\delta(k-m)$$

$$y(k) = \mathscr{H}_D u(k) = \mathscr{H}_D \left(\sum_{m=-\infty}^{\infty} u(m)\delta(k-m) \right) = \sum_{m=-\infty}^{\infty} u(m) g(k,m)$$

System concepts IV

Weighting sequence and response

$g(k,m)$ is the **impulse response** or **weighting sequence**.

$$\boxed{y(k) = \sum_{m=-\infty}^{\infty} u(m) g(k,m)}$$

Response for noncausal time varying systems

$$\boxed{y(k) = \sum_{m=k_0}^{k} u(m) g(k,m)}$$

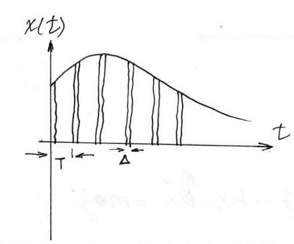
$k > m$, Response for causal time varying systems with input at k_0

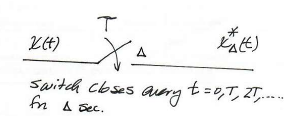
$$\boxed{y(k) = \sum_{m=k_0}^{k} u(m) g(k-m)}$$

$k > m$, for time invariant systems

Introduction & practical sampling I

Consider a sampled data system where the sampling duration Δ is much smaller than the sampling period and the largest time constant of the input signal $x(t)$, and as such can be approximated by a series of flat topped pulses as





$$x^*_\Delta(t) = \begin{cases} x(kT), & \text{if } kT \le t \le kT + \Delta \\ 0, & \text{if } kT + \Delta \le t \le (k+1)T \end{cases}$$

$$x^*_\Delta(t) = \sum_{k=0}^{\infty} x(kT)\left(u(t-kT) - u(t-kT-\Delta)\right)$$

Introduction & practical sampling II

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$$
\begin{aligned}
X^*_\Delta(s) &= \mathscr{L}\left\{\sum_{k=0}^{\infty} x(kT)\left(u(t-kT)-u(t-kT-\Delta)\right)\right\} \\
&= \sum_{k=0}^{\infty} x(kT)\left(\frac{e^{-kTs}}{s}-\frac{e^{-(kT+\Delta)s}}{s}\right) \\
&= \sum_{k=0}^{\infty} x(kT)\left(\frac{(1-e^{-\Delta s})e^{-kTs}}{s}\right)
\end{aligned}
$$

If $\Delta \longrightarrow 0$, then

Introduction & practical sampling III

$$
1-e^{-\Delta s} \approx 1-\left(1-\Delta s+\frac{(\Delta s)^2}{2}-\dots\right) = \Delta s \qquad e^a = \sum_{n=0}^{\infty}\frac{a^n}{n!}
$$

$$
X^*_\Delta(s) = \sum_{k=0}^{\infty} x(kT)\Delta e^{-kTs}
$$

but $\mathscr{L}^{-1}\left(e^{-kTs}\right) = \delta(t-kT)$, therefore

$$
\boxed{x^*_\Delta(t) \approx \Delta \sum_{k=0}^{\infty} x(kT)\delta(t-kT) \quad \text{train of impulses with strength of } \Delta x(kT)}
$$

Introduction & practical sampling IV

Remark

The above means that the finite pulse width sampler can be replaced with an ideal sampler ($\Delta = 0$) cascaded by a gain term with an attenuation of Δ. The output of an ideal sampler is given by

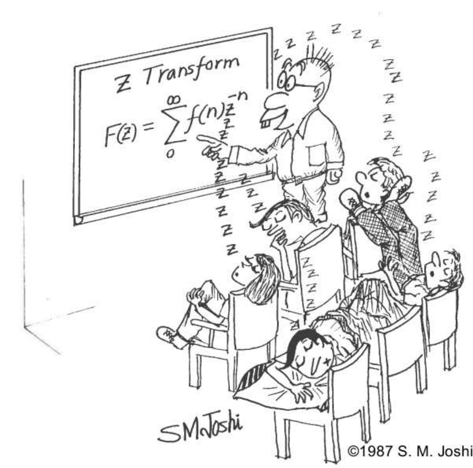
$$
x^*(t) \triangleq x(t)\delta_T(t) \ \text{ where } \ \delta_T(t) \triangleq \sum_{0}^{\infty}\delta(t-kT)
$$

$$
x^*(t) = \sum_{k=0}^{\infty} x(kT)\delta(t-kT)
$$

$$
\mathscr{L}[x^*(t)] = X^*(s) = \sum_{k=0}^{\infty} x(kT)e^{-(kTs)}
$$

Infinite series involves factors of e^{sT} and its powers make $\mathscr{L}^{-1}$ difficult.

Introduction & practical sampling V



Introduction & practical sampling VI

$\mathscr{Z}$ transform definition

Define

$$z \triangleq e^{Ts} \longrightarrow s = \frac{1}{T}\ln z \quad \text{and then}$$

$$X^*\left(s = \frac{1}{T}\ln z\right) \triangleq X(z) = \sum_{k=0}^{\infty} x(kT)z^{-k} \qquad \text{One-sided } \mathscr{Z} \text{ transform}$$

The above converges absolutely for $|z| > c$, where c is the radius of convergence.

Summary steps

1. Sample $x(t)$ to get $x^*(t)$
2. Obtain $\mathscr{L}(x^*(t)) = \sum_{k=0}^{\infty} x(kT)e^{-(kTs)}$
3. Find $X(z) = \sum_{k=0}^{\infty} x(kT)z^{-k}$

Introduction & practical sampling VII

Theorem 10.1.1

Ratio Test–*For the series $f_1(z) + f_2(z) + \ldots + f_n(z) + \ldots$, let*

$$\lim_{n\to\infty}\left|\frac{f_{n+1}(z)}{f_n(z)}\right| = |r(z)|$$

Then the given series converges absolutely for those values of z for which $0 \le |r(z)| < 1$ and diverges for those values of z for which $|r(z)| > 1$. The values of z for which $|r(z)| = 1$ form the boundary of the region of convergence of the series, and at those points the ratio test provides no information about the convergence or divergence of the series.

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Theorem 10.1.2

Two functions of time have the same $\mathscr{Z}$ transformation if and only if they are identical $\forall t = nT(n = 0, 1, 2, \ldots)$.

Remark

This theorem tells us that $\mathscr{Z}$ transform is not unique, and that $F(z)$ contains no information about $f(t)$ except at sampling instants.

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Example–Find $\mathscr{Z}(u(t))$

$$u^*(t) = \sum_{k=0}^{\infty} u(kT)\delta(t - kT) = \sum_{k=0}^{\infty}\delta(t - kT)$$

$$U^*(s) = \sum_{k=0}^{\infty} e^{-(kTs)} \Longrightarrow U(z) = \sum_{k=0}^{\infty} z^{-k}$$

$$U(z) = 1 + z^{-1} + z^{-2} + \ldots \qquad \text{multiply by } z - 1$$

$$(z-1)U(z) = z + 1 + z^{-1} + z^{-2} + \ldots - 1 - z^{-1} - z^{-2} - \ldots$$

$$(z-1)U(z) = z$$

$$\boxed{U(z) = \frac{z}{z-1}} \qquad \text{Series converges for } |z| > 1$$

Introduction & practical sampling X

Consider the following series:

$$S_k = a + ar + ar^2 + ar^3 + \cdots + ar^{k-1}$$
$$rS_k = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^k$$
$$S_k - rS_k = a - ar^k \Longrightarrow S_k = \frac{a(1-r^k)}{1-r} \qquad r \neq 1$$

if $k \to \infty$ and $|r| < 1$, then

$$S_k = \frac{a}{1-r}$$

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Example–Find $\mathscr{Z}\left(e^{-at}u(t)\right)$

$$x^*(t) = \sum_{k=0}^{\infty} e^{-akT}\delta(t-kT) \Longrightarrow X^*(s) = \sum_{k=0}^{\infty} e^{-akT}e^{-kTs}$$
$$X^*(s) = \sum_{k=0}^{\infty} e^{-(s+a)kT} = \frac{1}{1-e^{-(s+a)T}} \qquad |e^{-(s+a)T}| < 1$$

replacing e^{sT} by z

$$\boxed{X(z) = \frac{z}{z-e^{-aT}}}$$

Introduction & practical sampling XII

Example–Find $\mathscr{Z}\left(\beta^t\right)$

Let $\beta = e^{-a}$ in the last example to get

$$\beta^t \longleftrightarrow \frac{z}{z-\beta^T} \qquad \text{for } T=1 \quad \boxed{\beta^t \longleftrightarrow \frac{z}{z-\beta}}$$

Example–Note that

$$\mathscr{Z}\left(e^{j\omega t}\right) = \mathscr{Z}\left(\cos\omega t + j\sin\omega t\right) = \mathscr{Z}(\cos\omega t) + j\mathscr{Z}(\sin\omega t)$$

$$e^{j\omega t} \longleftrightarrow \frac{z}{z-e^{j\omega T}} = \frac{z}{z-(\cos\omega T + j\sin\omega T)}$$
$$= \frac{z^2 - z\cos\omega T + jz\sin\omega T}{z^2 + \sin^2\omega T + \cos^2\omega T - 2z\cos\omega T}$$

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$$\mathscr{Z}\left(e^{j\omega t}\right) = \mathscr{Z}\left(\cos\omega t + j\sin\omega t\right) = \mathscr{Z}(\cos\omega t) + j\mathscr{Z}(\sin\omega t)$$

$$e^{j\omega t} \longleftrightarrow \frac{z}{z-e^{j\omega T}} = \frac{z^2 - z\cos\omega T + jz\sin\omega T}{z^2 - 2z\cos\omega T + 1}$$

$$\therefore \quad \boxed{\cos\omega t \longleftrightarrow \frac{z(z-\cos\omega T)}{z^2 - 2z\cos\omega T + 1}} \qquad \boxed{\sin\omega t \longleftrightarrow \frac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1}}$$

Some properties of $\mathscr{Z}$: I

1. Linearity property–

$$\begin{aligned}\mathscr{Z}[a_1 f_1(t) + a_2 f_2(t)] &= a_1 \mathscr{Z}[f_1(t)] + a_2 \mathscr{Z}[f_2(t)] \\ &= \sum_{k=0}^{\infty} [a_1 f_1(kT) + a_2 f_2(kT)] z^{-k} \\ &= a_1 \sum_{k=0}^{\infty} f_1(kT) z^{-k} + a_2 \sum_{k=0}^{\infty} f_2(kT) z^{-k} \\ &= a_1 F_1(z) + a_2 F_2(z)\end{aligned}$$

Some properties of $\mathscr{Z}$: II

2. Shifting property– As an example let's find $\mathscr{Z}[f(t+mT)]$

$$\begin{aligned}\mathscr{Z}[f(t+mT)] &= \sum_{k=0}^{\infty} f(kT+mT) z^{-k} \qquad \text{let } i = k+m \\ &= \sum_{i=m}^{\infty} f(iT) z^{-i+m} = z^m \sum_{i=m}^{\infty} f(iT) z^{-i} \\ &= z^m \sum_{i=0}^{\infty} f(iT) z^{-i} - z^m \sum_{i=0}^{m-1} f(iT) z^{-i} \\ &= z^m F(z) - \sum_{i=0}^{m-1} f(iT) z^{m-i}\end{aligned}$$

$$\therefore \boxed{f(t+mT) \longleftrightarrow z^m F(z) - z^m f(0) - z^{m-1} f(T) - \ldots - z f(mT-T)}$$

Some properties of $\mathscr{Z}$: III

3. $\sum_{k=0}^{\infty} f_1(t-kT) f_2(kT) \longleftrightarrow F_1(z) F_2(z)$
4. $t^m f(t) \longleftrightarrow (-Tz)^m \frac{d^m}{dz} F(z)$
5. $\frac{f(t)}{t} \longleftrightarrow -\int \frac{1}{T} \frac{F(z)}{z} dz$
6. $e^{-at} f(t) \longleftrightarrow F\left(z e^{aT}\right)$ where $|e^{aT} z| < R^-$
7. $\frac{\partial}{\partial \alpha} f(t, \alpha) \longleftrightarrow \frac{\partial}{\partial \alpha} F(z, \alpha)$
8. $f(0) = \lim_{z \to \infty} F(z)$
   Initial Value Theorem: provided that the limit exist.
9. $f(\infty) = \lim_{z \to 1} (1 - z^{-1}) F(z)$
   Final Value Theorem: if $(1 - z^{-1}) F(z)$ is analytic for $|z| > 1$.

Solution of difference equations I

$\mathscr{Z}$ transform

$\mathscr{Z}$ transform is a useful tool for solution of constant coefficient difference (discrete) equations that arise in many fields, including signal processing, digital control, manufacturing systems, operations research, etc., where a phenomenon physically and naturally is described by a difference equation or is transformed into a discrete-time process.

Solution of difference equations II

Example– Solve $(E+1)y(k) = (-1)^k$ with $y(0) = 1$.
Taking $\mathscr{Z}$ transform

$$
\begin{aligned}
zY(z) - zY(0) + Y(z) &= \frac{z}{z+1} \\
(z+1)Y(z) &= z + \frac{z}{z+1} = \frac{z^2+2z}{z+1} \\
\frac{Y(z)}{z} &= \frac{K_1}{z+1} + \frac{K_2}{(z+1)^2} \qquad K_1 = 1, K_2 = 1 \\
Y(z) &= \frac{z}{z+1} + \frac{z}{(z+1)^2} \\
y(k) &= (-1)^k - k(-1)^k
\end{aligned}
$$

where $k(-1)^k \leftrightarrow \frac{-z}{(z+1)^2}$