GENG 8010–Part 1: Elements of Differential and Difference Equations

Mehdrad Saif ©

University of Windsor

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Part I–Outline I

Part I–Outline II

Part I–Outline III

Higher order non-homogeneous soln.

Consider

$$(a_n D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0) y(t) = F(t)$$

that has a general solution of the form

$$y(t) = y_c + y_p$$

There are two methods for finding y_p:

1. Method of undetermined coefficients
   - Note that this method works when and only when $F(t)$ is itself a particular solution of some homogeneous linear differential equation with constant coefficients.
2. Variation of parameters

Basic idea

The basic procedure here is to assume that the form of y is some linear combination of the terms of $F(t)$ (polynomial function, exponential function, sine or cosine functions or finite sums and products of these functions) and derivatives with each term multiplied by a constant. If however, certain terms in $F(t)$ are similar to those in $y_c(t)$, then certain modifications are necessary.

Example– Find the solution of

$$(D^2 + 2D + 1) y(t) = te^{-t}$$

Note that CE is $m^2 + 2m + 1 = 0$ which gives $m_1 = m_2 = -1$, and so $y_c = c_1 e^{-t} + c_2 te^{-t}$.

Example–cont.

Now notice that if we didn't have similar terms in y_c we would have picked $y_p = Ate^{-t} + Be^{-t}$, as a result, take

$$y_p = At^3 e^{-t} + Bt^2 e^{-t}$$

Now substitute y_p into the diff. eq. and evaluate

$$(D^2 + 2D + 1) y_p(t) = (6At + 2B) e^{-t} = te^{-t} \Longrightarrow A = \frac{1}{6}; B = 0$$

$$\boxed{y(t) = c_1 e^{-t} + c_2 te^{-t} + \frac{1}{6} t^3 e^{-t}}$$

$F(t)$	Educated guess for y_p
k (constant)	A
$pt + q$ (p, q constants)	$At + B$
$pt^2 + qt + k$	$At^2 + Bt + C$
$\sin pt$	$A \sin pt + B \cos pt$
$\cos pt$	$A \sin pt + B \cos pt$
e^{pt}	Ae^{pt}
$(pt + q)e^{kt}$	$(At + B)e^{kt}$
$t^p e^{qt}$	$(A_p t^p + A_{p-1} t^{p-1} + \ldots + A_0) e^{qt}$
$e^{pt} \sin qt$	$e^{pt} (A \sin qt + B \cos qt)$
$e^{pt} \cos qt$	$e^{pt} (A \sin qt + B \cos qt)$
$pt^2 \sin qt$	$(At^2 + Bt + C) \sin qt + (Et^2 + Ft + G) \cos qt$
$te^{pt} \cos qt$	$(At + B)e^{pt} \sin qt + (Ct + E)e^{pt} \cos qt$

More examples

Ex– Fine the general solution of

$$y'' + 5y' + 6y = 4e^{-x} + 5\sin x$$

Characteristic equation is

$$m^2 + 5m + 6 = 0 \Longrightarrow m_1 = -2; m_2 = -3$$

Therefore

$$y_c = c_1 e^{-2x} + c_2 e^{-3x}$$

Now based on the form of the forcing functon $F(x)$, and previous discussion, assume

$$y_p = Ae^{-x} + B\sin x + C\cos x$$

More examples-cont.

substituting into the differential equation and simplifying gives

$$2Ae^{-x} + 5(B - C)\sin x + 5(B + C)\cos x = 4e^{-x} + 5\sin x$$

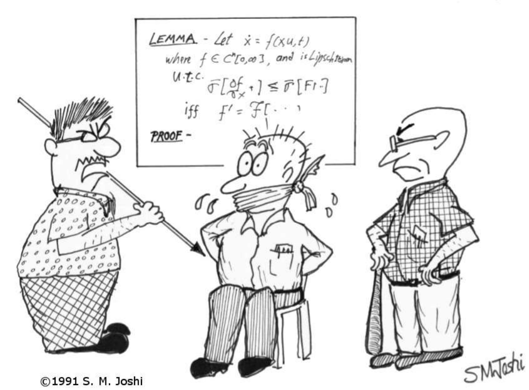
Which gives $2A = 4$ *or* $A = 2$, $5(B - C) = 5$ and $5(B + C) = 0$ which results in $B = \frac{1}{2}$ and $C = -\frac{1}{2}$, and Finally

$$y = y_c + y_p = c_1 e^{-2x} + c_2 e^{-3x} + 2e^{-x} + \frac{1}{2}\sin x - \frac{1}{2}\cos x$$

About VP method

1. Distinct advantage over other methods that it always yields a particular solution y_p if the associated homogeneous equation can be solved.
2. Applicable to linear higher-order equations.
3. Unlike undetermined coefficients, is not limited to cases where the forcing function is a combination of certain functions.
4. No special cases arise due to the nonhomogeneous term being included in the complementary function.
5. It works for (time) varying systems, i.e., $a_i(t)$ or $a_i(x)$.

VP for linear 2^{nd} order ODE I



"After we beat the proof outta him, let's dump him in the theory-practice gap!"

VP for linear 2^{nd} order ODE II

Consider a general linear 2nd order equation

$$(D^2 + a(x)D + b(x))y = f(x)$$

defined over an inteval $\alpha \le x \le \beta$ over which $a(x), b(x)$, and $f(x)$ are defined and continuous. $y_1(x)$ and $y_2(x)$ two known lin. indep. solutions, so

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x)$$

Idea: Replace the constants c_1 and c_2 with two unknown functions u_1, u_2 of x and then

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

differentiating

$$y_p'(x) = u_1(x)y_1'(x) + u_2(x)y_2'(x) + \cancelto{\textit{force } 0}{u_1'(x)y_1(x) + u_2'(x)y_2(x)}$$

VP for linear 2^{nd} order ODE III

In other words,

$$y_p'(x) = u_1(x)y_1'(x) + u_2(x)y_2'(x)$$

subject to requirement that

$$u_1'(x)y_1(x) + u_2'(x)y_2(x) = 0 \qquad \text{(a)}$$

differentiating $y_p'(x)$,

$$y_p''(x) = u_1(x)y_1''(x) + u_2(x)y_2''(x) + u_1'(x)y_1'(x) + u_2'(x)y_2'(x)$$

substituting for y_p, y_p' and y_p'' into the diff. eq.

$$u_1\Big(\cancelto{0}{y_1'' + a(x)y_1' + b(x)y_1}\Big) + u_2\Big(\cancelto{0}{y_2'' + a(x)y_2' + b(x)y_2}\Big) + u_1'y_1' + u_2'y_2' = f(x)$$

VP for linear 2^{nd} order ODE IV

Resulting in

$$u_1'(x)y_1'(x) + u_2'(x)y_2'(x) = f(x) \qquad \text{(b)}$$

(a) and (b) give two eq. in two unknowns u_1, u_2. Using the Cramer Rule to solve for them, we get

$$u_1'(x) = \frac{W_1}{W} = \frac{-y_2(x)f(x)}{W(x)} \qquad u_2'(x) = \frac{W_2}{W} = \frac{y_1(x)f(x)}{W(x)} \qquad \text{(c)}$$

where

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2 \; ; \; W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} \; ; \; W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

You have seen $W(x)$ (Wronskian) before and may recall that for y_1 and y_2 to be independent solutions $W(x)$ cannot be zero.

VP for linear 2^{nd} order ODE V

Integrating the equations in (c) gives u_1 and u_2 and finally the complete solution

$$\boxed{y(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}dx} \qquad (7.1)$$

VP for linear 2^{nd} order ODE VI

Remark

1. We have not included the constants associated with each indefinite integral. Suppose we did and called them $-c_1$ and c_2 , then we would have a addition term like $c_1 y_1(x) + c_2 y_2(x)$ in our solution which is the y_c. When these integration constants are set equal to zero we should get the particular solution $y_p(x)$.

2. **Finally, for an n^{th} order system** (7.1) **can be extended to**

$$\boxed{y_p(x) = \sum_{i=1}^{n} y_i(x) \int \frac{W_i(x)}{W(x)} f(x) dx}$$

VP for linear 2^{nd} order ODE VII

Example–Find the solution of

$$y'' + 2y' + y = e^x$$

Verify that the roots of CE are repeated at $m = -1$, hence

$$y_c(x) = c_1 e^{-x} + c_2 x e^{-x}$$

and

$$y_1(x) = e^{-x} \qquad y_2(x) = xe^{-x}$$

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = e^{-x}\left(e^{-x} - xe^{-x}\right) + xe^{-2x} = e^{-2x}$$

VP for linear 2^{nd} order ODE VIII

substituting in (7.1) gives,

$$\begin{aligned} y_p(x) &= -e^{-x} \int \frac{xe^{-x}e^x}{e^{-2x}} dx + xe^{-x} \int \frac{e^{-x}e^x}{e^{-2x}} dx \\ &= -e^{-x} \int xe^{2x} dx + xe^{-x} \int e^{2x} dx \\ &= -e^{-x}\left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right) + \frac{1}{2}xe^x \\ &= \frac{1}{4}e^x \end{aligned}$$

where we used $\int xe^{ax} dx = e^{ax}\left(\frac{ax-1}{a^2}\right)$. Therefore

$$\boxed{y = c_1 e^x + c_2 x e^x + \frac{1}{4}e^x}$$

VP for linear 2^{nd} order ODE IX

Due to form of $f(x)$ the previous example could have been solved using the method of undetermined coefficients (try solving it using that method). Let us now try an example that cannot be solved using that approach.

Example– Solve the following

$$y'' + y = \csc x$$

Verify that the roots of the CE are at $m_{1,2} = \pm i1$, and hence

$$y_c = c_1 \cos x + c_2 \sin x$$

in which case

$$y_1(x) = \cos x \qquad y_2(x) = \sin x$$

VP for linear 2^{nd} order ODE X

and $W(x) = y_1 y'_2 - y_1' y_2 = \cos^2 x + \sin^2 x = 1$ with $f(x) = \frac{1}{\sin x}$ which using the formula gives

$$y_p(x) = -\cos x \int dx + \sin x \int \cot x dx$$

Using the fact that $\int \cot x dx = \ln|\sin x|$ we get

$$y_p(x) = -x\cos x + \sin x \ln|\sin x|$$

and

$$\boxed{y(x) = c_1 \cos x + c_2 \sin x - x\cos x + \sin x \ln|\sin x|}$$