Engineering Mathematics

Lecture Notebook

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Engineering Mathematics GENG 8010-31-R-2021S

Textbook:

Advanced Engineering Mathematics
2nd Edition
by Michael D. Greenberg

Subjects:

1st mid
(i) ODE of first order
(ii) ODE of higher orders
(iii) Power Series / Taylor Series

2nd mid
(iv) Laplace Transformation
(v) Fourier Analysis

After 2nd mid toward final
(vi) Sturm-Liouville
(vii) PDE
(viii) Heat PDE
(ix) wave PDE
(x) Laplace PDE

ODE = Ordinary Differential Equations

PDE = Partial " "

First Order:

O D E

Ordinary Differential Equation

* ordinary means the function has only one variable.

$\boxed{\text{Ex}-1}$

$$L I''(t) + R I'(t) + \frac{1}{C} I(t) = V'(c)$$

(RLC circuit, Elec. Engg.)

$I(t)$ = function

L, R, C = constant

degree/order = highest derivative

ODE of order 2, linear

$V' = 0 \longrightarrow$ homogeneous

$V' \neq 0 \longrightarrow$ non-homo

$$y'' + 2y' = x$$

non homo

$$y'' + 2y' = 0$$

homogeneous

Practice

$$2xy'' + 3x^2y' - 2y = 10$$ linear

$$2xy'' + 3xy'^2 - 5 = 0$$ non-linear

* $\sqrt{y}$, $\sqrt{y'}$, y'^2, $\log y$ + $\frac{1}{y''}$ non-linear terms.

$$2xy'' - 5y' - 5x = 0$$

$$2xy'' - 5y' = 5x$$ non-homo

LHS = RHS

(y'', y', y) = (other Term)

$\boxed{\text{Ex-2}}$

constant

$$\frac{dy}{dx} = ky$$ (exp. growth,)

$$y' - ky = 0$$

first order

homogeneous, linear

$\boxed{\text{Ex-3}}$ $$mx''(t) + cx'(t) + kx(t) = f(t)$$

[mechanical vibration]

Second order ODE, linear.

$f(t) = 0 \longrightarrow$ homogeneous

$f(t) \neq 0 \longrightarrow$ non-homo

$\boxed{\text{Ex}-4}$

$$EV^{(4)}(x) + KV(x) = q$$

elastic beam, civil Engg.

ODE of order 4, linear.

$q = 0 \longrightarrow$ homo

$q \neq 0 \longrightarrow$ non-homo

$\boxed{\text{Ex}-5}$

speed control of a DC motor Industrial Engg.

$$aw'(t) + bw(t) = v(t)$$

$a, b \longrightarrow$ constant/coefficient

First order ODE, linear

$v(t) = 0 \longrightarrow$ homo

$v(t) \neq 0 \longrightarrow$ non-homo

$2x^2y' - 5xy = 7$ linear

(i) $\boxed{\text{First order ODE}}$

today
① Separation of Variables
② Integration factor

next class
③ Variation of parameters
④ Exact Equations

① Separation of Variables

$$\frac{dy}{dx} = f(x)\,g(y)$$

$$\int \frac{dy}{dx} = \int f(x)\,dx \longrightarrow \text{do integration}$$

Exercise-1

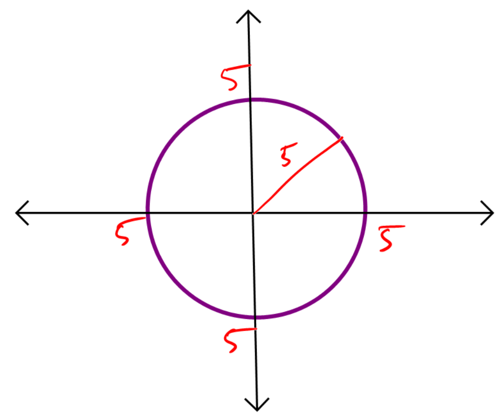
Solve $\frac{dy}{dx} = -\frac{x}{y}$ when $y(4) = -3$ Boundary condition (BC)

$$\int y\,dy = \int -x\,dx$$

$$\Rightarrow \frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$$

$$\frac{1}{2}(-3)^2 = -\frac{1}{2}(4)^2 + c \qquad [y(4)=3]$$

$$\Rightarrow c = \frac{25}{2}$$



$$x^2 + y^2 = 25$$

$$\Rightarrow y^2 = 25 - x^2$$

$$\Rightarrow y = \pm\sqrt{25-x^2}$$

$$\frac{1}{2}x^2 + \frac{1}{2}y^2 = \frac{25}{2}$$

$x^2 + y^2 = 25 \longrightarrow$ this is OK

Exercise-2

Solve $y' = \dfrac{4x}{1+2e^y}$

$$\Rightarrow \frac{dy}{dx} = \frac{4x}{1+2e^y}$$

$$\Rightarrow \int (1+2e^y)\,dy = \int 4x\,dx$$

$$\Rightarrow y + 2e^y = \frac{4}{2}x^2 + c$$

$$\Rightarrow y + 2e^y = 2x^2 + c$$

Exercise-3

Solve $y' = \dfrac{y(y-2)}{x(y-1)}$

$$\Rightarrow \frac{dy}{dx} = \frac{y(y-2)}{x(y-1)}$$

$$\Rightarrow \int \frac{y-1}{y(y-2)}\,dy = \int \frac{1}{x}\,dx$$

$$= \ln x + C$$

Partial Fraction (PF)

$$\int \frac{P(x)}{Q(x)}\,dx$$

$P(x)$: Poly^n

$Q(x)$: Poly^n

if

$\deg P < \deg Q$

* If $\deg P \geq \deg Q$ then we have to first do long division and the PF.

$$\frac{y-1}{y(y-2)} = \frac{A}{y} + \frac{B}{y-2}$$

$$\Rightarrow \frac{y-1}{y(y-2)} = \frac{A(y-2) + By}{y(y-2)}$$

$A = \frac{1}{2}$, $B = \frac{1}{2}$

please practice!

$$\int \frac{y-1}{y(y-2)}\, dy = \int \frac{\frac{1}{2}}{y}\, dy + \int \frac{\frac{1}{2}}{y-2}\, dy$$

$$= \frac{1}{2}\ln y + \frac{1}{2}\ln(y-2)$$

$$\frac{1}{2}\ln y + \frac{1}{2}\ln(y-2) = \ln x + C$$

② Integration Factor:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

case ①: $Q(x) = 0$

$$\frac{dy}{dx} + P(x)\cdot y = 0$$

$$\Rightarrow \frac{dy}{dx} = -P(x)y$$

$$\Rightarrow \int \frac{dy}{y} = \int -P(x)\, dx$$

$$\Rightarrow \ln y = \int -P(x)\, dx$$

$$\Rightarrow y = e^{-\int P(x)\, dx}$$

easy

Case ②: $Q(x) \neq 0$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$e^{\int P(x)dx}$ ← Integration Factor (IF)

Multiplying both side of $\frac{dy}{dx} + P(x)y = Q(x)$ by IF.

$$e^{\int P(x)dx} \cdot \frac{dy}{dx} + e^{\int P(x)dx} \cdot P(x)y = e^{\int P(x)dx} Q(x)$$

$$\frac{d\left(ye^{\int P(x)dx}\right)}{dx} = e^{\int P(x)dx} Q(x)$$

magic

$(e^u)' = u'e^u$

$\left(e^{\int P(x)dx}\right)' = P(x)\, e^{\int P(x)dx}$

$(uv)' = u'v + uv'$

$$ye^{\int P(x)dx} = \int e^{\int P(x)dx} Q(x)\, dx$$

$$\Rightarrow y = \frac{RHS}{e^{\int P(x)dx}}$$

Exercise

Solve $\frac{dy}{dx} + \overset{p(x)}{5}y = 3$

multiply

Integration factor

$$IF = e^{\int p(x)dx} = e^{\int 5dx} = e^{5x}$$

$$e^{5x}\frac{dy}{dx} + 5e^{5x}y = 3e^{5x}$$

magic

$$\int \frac{d(ye^{5x})}{dx}dx = \int 3e^{5x}dx$$

$$\Rightarrow ye^{5x} = \frac{3}{5}e^{5x} + C$$ ← this is ok in exam

$$\Rightarrow y = \frac{3}{5} + ce^{-5x}$$

Exercise

Solve $\frac{dy}{dx} + \overset{p(x)}{5}y = x$

$$IF = e^{\int p(x)dx} = e^{\int 5dx} = e^{5x}$$

$$e^{5x}\frac{dy}{dx} + 5e^{5x}y = xe^{5x}$$

(magic)

$$\Rightarrow \frac{d(ye^{5x})}{dx} = xe^{5x}$$

Integration by part

$$\int u\,dv = uv - \int v\,du$$

$$\int \underbrace{x}_{u}\,\underbrace{e^{5x}\,dx}_{dv}$$

$u = x$	$dv = e^{5x}\,dx$
$du = dx$	$v = \frac{1}{5}e^{5x}$

$$uv - \int v\,du = (x)\cdot\left(\frac{1}{5}e^{5x}\right) - \int \frac{1}{5}e^{5x}\,dx$$

$$= \frac{1}{5}xe^{5x} - \frac{1}{25}e^{5x} + c$$

So,

$$ye^{5x} = \frac{1}{5}xe^{5x} - \frac{1}{25}e^{5x} + c$$

$$y = \frac{1}{5}x - \frac{1}{25} + ce^{-5x}$$

fancy simplification

Exercise

Solve $x\dfrac{dy}{dx} + 3y = xe^{x^4}$ not standard

$$\frac{dy}{dx} + \overset{P(x)}{\frac{3}{x}}\,y = e^{x^4}$$ standard

$$IF = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$$

$$x^3 \frac{dy}{dx} + 3x^2 y = x^3 e^{x^4}$$

$$\Rightarrow \int \frac{d(yx^3)}{dx}\,dx = \int x^3 e^{x^4}\,dx$$

$$\Rightarrow yx^3 = \int x^3 e^{x^4}\,dx$$

Substitution

$\int x^3 e^{x^4}\,dx$ let, $x^4 = u$

$4x^3\,dx = du$

$$\int \frac{1}{4} e^u\,du = \frac{1}{4} e^u + C$$

So, $yx^3 = \frac{1}{4} e^{x^4} + C$

- * Partial Fraction
- * Integration by Parts → $x\sin x$, $x\cos x$, xe^x, $x\ln x$, $e^x \cdot \sin x$
- * Integration by Substitution
- * Basic Formulas

practice these

Homeworks will be posted

21-May-2021

(3) Variation of parameters:

- discovered by Italian mathematician Lagrange.

mechanics

$$y' + p(x)y = q(x)$$

$$q(x) \downarrow 0$$

$$y' + p(x)y = 0$$

Lagrange first solved this for homogeneous case, $q(x) = 0$

$$y' + p(x)y = 0$$

$$\Rightarrow y' = -p(x)y$$

$$\Rightarrow \frac{y'}{y} = -p(x)$$

$$\Rightarrow \int \frac{y'}{y}\,dx = \int -p(x)\,dx$$

$$\Rightarrow \ln y = \int -p(x)\,dx + c$$

$$\Rightarrow y = e^{\int -p(x)\,dx + c}$$

$$= e^{\int -p(x)\,dx}\, \underbrace{e^{c}}_{A}$$

constant/parameter

$$y_h = A e^{\int -p(x)\,dx}$$

y_h special case of general solution

Now, we vary A to a function $A(x)$, and get general solution

$$y = A(x)\, e^{\int -P(x)\,dx}$$

$$y' + P(x)\,y = q(x)$$ Original ODE

magic

$$\underbrace{A'(x)\, e^{\int -P(x)\,dx} + A(x)\left[\cancel{-P(x)\, e^{\int -P(x)\,dx}}\right]}_{y'} + \cancel{P(x)\,A(x)\,e^{\int -P(x)\,dx}}$$

$$= q(x)$$

$$\Rightarrow A'(x)\, e^{\int -P(x)\,dx} = q(x)$$

$$\Rightarrow A'(x) = q(x)\, e^{\int P(x)\,dx}$$

$$\Rightarrow A(x) = \int q(x)\, e^{\int P(x)\,dx}$$

So, $$y = \int q(x)\, e^{\int P(x)\,dx} \cdot e^{\int -P(x)\,dx}$$

Ex-1 Solve $y' + 5y = 3$

First we solve $y' + 5y = 0$

$$\Rightarrow y' = -5y$$

$$\Rightarrow \frac{y'}{y} = -5$$

$$\Rightarrow \ln y = -5x + c$$

$$\Rightarrow y = e^{-5x+c}$$

$$\Rightarrow y = e^{-5x} \cdot e^{c}$$

$$\Rightarrow y_h = Ae^{-5x}$$

Now, we do variation of A to $A(x)$

$$y = A(x)e^{-5x}$$

Now substitute this to the equation,

magic

$$A'(x)e^{-5x} + \cancel{A(x)\left[-5e^{-5x}\right]} + \cancel{5A(x)e^{-5x}} = 3$$

$$\Rightarrow A'(x)e^{-5x} = 3$$

$$\Rightarrow A'(x) = 3e^{5x}$$

$$\Rightarrow A(x) = \frac{3}{5}e^{5x} + c$$

So, $y = \left[\frac{3}{5}e^{5x} + c\right]e^{-5x}$

$$= \frac{3}{5} + ce^{-5x}$$

Ex-2 Solve $x\dfrac{dy}{dx} + 3y = xe^{x^4}$ not standard

$$\frac{dy}{dx} + \frac{3}{x}y = e^{x^4}$$ Standard

First we solve, $y' + \frac{3}{x}y = 0$

$$\Rightarrow \frac{y'}{y} = -\frac{3}{x}$$

$$\Rightarrow \ln y = -3\ln x + c$$

$$\Rightarrow y = e^{-3\ln x + c}$$

$$\Rightarrow y = e^{-3\ln x} \cdot e^{c}$$

$$\Rightarrow y = Ae^{-3\ln x}$$

$$\Rightarrow y_h = Ax^{-3}$$

Now, we do variation of A to $A(x)$

$$y = A(x)x^{-3}$$

$$y' + \frac{3}{x}y = e^{x^4}$$

$$\Rightarrow A'(x)x^{-3} + \cancel{A(x)(-3)x^{-4}} + \cancel{\frac{3}{x}A(x)x^{-3}} = e^{x^4}$$

$$\Rightarrow A'(x) = e^{x^4} \cdot x^3$$

$$\Rightarrow A(x) = \int x^3 e^{x^4}\, dx$$

$$= \int \frac{1}{4}\, du\, e^u$$

$$= \frac{1}{4} e^u + C$$

$$= \frac{1}{4} e^{x^4} + C$$

Let, $x^4 = u$

$4x^3\, dx = du$

$x^3\, dx = \frac{1}{4}\, du$

So, $y = \left[\frac{1}{4} e^{x^4} + c\right] x^{-3}$ ✓

4. Exact Equations:

→ { y is a function of x, looking for y

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

we can solve for y if ↑ is exact (this ODE)

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{exactness}$$

→ Solution is something like $f(x,y) = C$

such that, $\frac{\partial f}{\partial x} = M$, $\frac{\partial f}{\partial y} = N$

$M(x,y)\,\partial x + N(x,y)\,\partial y = 0$, solution $f(x,y) = C$

* First check exactness.

* Then,

Start $\frac{\partial f}{\partial x} = M \longrightarrow f(x,y) = \int M(x,y)\,dx + g(y)$

$$\frac{\partial f}{\partial y} = N \longrightarrow \frac{df}{\partial y} = \frac{\partial\left(\int M dx\right)}{dy} + g'(y)$$

$$\text{So, } g'(y) = N - \frac{\partial\left(\int M dx\right)}{dy}$$

* See LHS only y, but RHS (x,y). So??

$\rightarrow$ $g(y) = $ find $\xrightarrow{\text{sub}}$ $f(x,y) = \ldots\ldots$ equation.

Ex-1

Solve $\underbrace{2xy^2}_{M}dx + \underbrace{2(x^2y-1)}_{N}dy = 0$

Exactness

$$\frac{dM}{dy} = 4xy$$

$$\frac{dN}{dx} = 4xy$$

So, this is exact.

Solution is $f(x,y) = C$

$$\frac{\partial f}{\partial x} = M$$

$$\Rightarrow f(x,y) = \int M \, \partial x + g(y)$$

$$\Rightarrow f(x,y) = \int 2xy^2 \, \partial x + g(y)$$

$$= x^2y^2 + g(y)$$

Now, $\frac{\partial f}{\partial y} = N$

$$\Rightarrow \cancel{2x^2y} + g'(y) = \cancel{2x^2y} - 2$$

$$\Rightarrow g'(y) = -2$$

$$\Rightarrow g(y) = -2y$$

So, $f(x,y) = x^2y^2 - 2y$

Solution. $f(x,y) = c$

$$x^2y^2 - 2y = c$$

Answer.

$$2xy^2 dx + 2(x^2y - 1) = 0$$

$$2xy^2 + x^2(2yy') - 2y' = 0$$

$$\Rightarrow 2xy^2 + y'[2x^2y - 2] = 0$$

$$\Rightarrow 2xy^2 + \frac{dy}{dx} 2[x^2y - 1] = 0$$

Implisidu derivative

Ex-2 Solve $\left(\frac{1}{1+y^2} + \cos x - 2xy\right)\frac{dy}{dx} = y(y+\sin x)$

$$\Rightarrow \underbrace{y(y+\sin x)}_{M}\,dx - \underbrace{\left[\frac{1}{1+y^2} + \cos x - 2xy\right]}_{N}dy$$

Exactness
$$\frac{\partial M}{\partial y} = 2y + \sin x$$
$$\frac{\partial N}{\partial x} = \sin x + 2y$$

So, this is exact.

$$\frac{\partial f}{\partial x} = M \longrightarrow f(x,y) = \int M\,dx + g(y)$$
$$f(x,y) = y^2x - y\cos x + g(y)$$

magic

Now, $\frac{\partial f}{\partial y} = N \longrightarrow \cancel{2xy} - \cancel{\cos x} + g'(y) = \frac{-1}{1+y^2} + \cancel{\cos x} + \cancel{2xy}$

$$\Rightarrow g'(y) = -\frac{1}{1+y^2}$$
$$\Rightarrow g(y) = -\tan^{-1}y$$

solution $f(x,y) = C$

$$y^2x - y\cos x - \tan^{-1}y = C$$

Answer.

28-May-2021

ODE with constant coefficients:

homogeneous

$$a_n y^{(n)} + a_{n-1} y^{n-1} + \cdots + a_1 y' + a_0 y = 0$$

Start with $n=1$,

$$a_1 y' + a_0 y = 0$$

$$\Rightarrow a_1 y' = -a_0 y$$

$$\Rightarrow \int \frac{y'}{y}\,dx = \int -\frac{a_0}{a_1}\,dx$$

$$\Rightarrow \ln y = -\frac{a_0}{a_1} x$$

$$\Rightarrow y = e^{-\frac{a_0}{a_1} x}$$

$-\frac{a_0}{a_1} = \lambda$, so $y = e^{\lambda x}$

basis/general solution

fundamental sol^n.

$\boxed{\text{Remark}}$

Any homogeneous ODE of order n has exactly n basis solutions, like $y_1, y_2 \cdots y_n$ and the general solution of ODE is all linear combination/superposition of them.

$$y = c_1 y_1 + c_2 y_2 + \cdots\cdots + c_n y_n$$

$3y'' - 6y' + y = 0$

$$y = c_1 y_1 + c_2 y_2$$

Degree 2

$$a_2 y'' + a_1 y' + a_0 y = 0$$

for this ODE

We hope $y = e^{\lambda x}$ is a solution

Sub in and find λ,

$$y = e^{\lambda x}, \quad y' = \lambda e^{\lambda x}, \quad y'' = \lambda^2 e^{\lambda x}$$

$$a_2 \lambda^2 e^{\lambda x} + a_1 \lambda e^{\lambda x} + a_0 e^{\lambda x} = 0$$

$$\Rightarrow e^{\lambda x}\left[a_2 \lambda^2 + a_1 \lambda + a_0\right] = 0$$

$$e^{\lambda x} \neq 0$$

So, $a_2 \lambda^2 + a_1 \lambda + a_0 = 0$

characteristic equation/
characteristic polynomial

$$\lambda = \frac{-a \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Real
- Identical
- complex

3 cases

Case ①

$$a_1^2 - 4a_0a_2 > 0$$

two real solutions λ_1 & λ_2 then we get two basis solution of ODE

$$y_1 = e^{\lambda_1 x} \quad \text{and} \quad y_2 = e^{\lambda_2 x}$$

So, $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

Case ②,

$$a_1^2 - 4a_0a_2 = 0$$

λ

$\lambda_1 = \lambda_2$

Two identical solution

$$Y_1 = e^{\lambda x} \quad \text{and} \quad Y_2 = x e^{\lambda x}$$

So, $Y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$

Case ③,

$$a_1 - 4a_0a_2 < 0$$

two complex conjugate solution.

$\alpha \pm \beta \qquad i = \sqrt{-1}$

$\rightarrow \quad a_2 \lambda^2 - a_1 \lambda + a_0 = 0$

$$\lambda = \frac{-a_1 \pm \sqrt{neg}}{2a_2}$$

$$= \underset{\alpha}{\frac{a_1}{2a_2}} \pm \underset{i \quad \beta}{\frac{\sqrt{}}{2a_2}}$$

$y_1 = c_1 e^{(\alpha + i\beta)x}$ and $y_2 = c_2 e^{(\alpha - i\beta)x}$

$$y = y_1 + y_2$$
$$= c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$

Euler, $\quad e^{i\theta} = \cos\theta + i\sin\theta$

Real solutions, $\quad y_1 = e^{\alpha x}\cos(\beta x) \quad , y_2 = e^{\alpha x}\sin(\beta x)$

$$y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$$

Ex-1 $\quad$ Solve $\quad y'' + 3y' + 2y = 0$

$a_2 y'' + a_1 y' + a_0 y = 0$
$a_2 \lambda^2 + a_1 \lambda + a_0 = 0$

Characteristic equation, $\quad \lambda + 3\lambda + 2 = 0$

$\Rightarrow (\lambda + 2)(\lambda + 1) = 0$

$\lambda = -1, -2$

$$y = c_1 e^{-x} + c_2 e^{-2x}.$$

Ex-2

Solve $y'' - 4y' + 4y = 0$

characteristic equation, $\lambda^2 - 4\lambda + 4 = 0$

$$\Rightarrow (\lambda - 2)^2 = 0$$

$$\Rightarrow \lambda = 2, 2$$

$$y = c_1 e^{2x} + c_2 x e^{2x}$$

Ex-3

Solve $y'' + y' + y = 0$

$\sqrt{-3} = i\sqrt{3}$

characteristic equation, $\lambda^2 + \lambda + 1 = 0$

$\lambda = \dfrac{-1 \pm \sqrt{-3}}{2}$

$$\lambda = \underbrace{-\tfrac{1}{2}}_{\alpha} \pm i \underbrace{\tfrac{\sqrt{3}}{2}}_{\beta}$$

$e^{\alpha x} \cos\beta x \qquad e^{\alpha x} \sin\beta x$

$$y = c_1 e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right)$$

Ex-4

Solve $y'' + y = 0$

characteristic equation, $\lambda^2 + 1 = 0$

$$\lambda^2 = -1$$

$$\lambda = \pm i$$

$$\alpha = 0, \quad \beta = 1$$

$$y_1 = e^{0 \cdot x} \cos(1 \cdot x) \qquad y_2 = e^{0 \cdot x} \sin(1 \cdot x)$$

$$y = c_1 \cos x + c_2 \sin x$$

General case

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

char. eq^n: $a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0$

$\boxed{\text{Remark} - 1}$

if $\lambda_1, \lambda_2 \cdots \lambda_k$ are k distincts real solutions $(k \le n)$ then

$e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots\ldots e^{\lambda_k x}$ are basis solutions.

$\boxed{\text{Remark} - 2}$

Recall multiplicity

$$(\lambda - 3)^5 (\lambda - 7)^2 = 0$$

$\lambda = 3,3,3,3,3 \longrightarrow$ multi 5

$\lambda = 7,7 \longrightarrow$ multi 2

If α is a root of multiplicity k,

$$(\lambda-\alpha)^k \; p(\lambda)$$

then there are k basis solutions as follows

$$e^{\alpha x}, x e^{\alpha x}, x^2 e^{\alpha x}, \ldots\ldots, x^{k-1} e^{\alpha x}$$

$\boxed{\text{Remark-3}}$

If $\lambda = \alpha \pm i\beta$ is a complex conjugate solution of multiplicity k then there are $2k$ basis soln.

$$\begin{array}{ll} e^{\alpha x}\cos\beta x & e^{\alpha x}\sin\beta x \\ x e^{\alpha x}\cos\beta x & x e^{\alpha x}\sin\beta x \\ x^2 e^{\alpha x}\cos\beta x & x^2 e^{\alpha x}\sin\beta x \\ \vdots & \vdots \\ x^{k-1} e^{\alpha x}\cos\beta x & x^{k-1} e^{\alpha x}\sin\beta x \end{array}$$

Ex-1 Solve $y''' - 4y'' + 5y' - 2y = 0$

$$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$$

$$(\lambda-1)^2(\lambda-2) = 0$$

for such kind of problem characteristic eqn will be given in exam

$\frac{-2}{1} = -2$ $\pm 1, \pm 2$ chinese theorem

rational roots of a polynomial

$\lambda = 1, \lambda = 2$

multi = 2 multi = 1

$$y = c_1 e^{x} + c_2 x e^{x} + c_3 e^{2x}$$

Ex-2

$y''' - y = 0$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$\lambda^3 - 1 = 0$

$(\lambda - 1)(\lambda^2 + \lambda + 1) = 0$

1

$$\lambda = \frac{-1 \pm \sqrt{-3}}{2} = \underbrace{-\tfrac{1}{2}}_{\alpha} \pm \underbrace{\frac{\sqrt{3}}{2}}_{\beta}$$

$$y = c_1 e^{x} + c_2 e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}\right) + c_3 e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}\right)$$

Ex-3

$$\underbrace{(\lambda - 2)^3}_{\text{multi } 3} (\lambda - 5)$$

$$y = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x} + c_4 e^{5x}.$$

next class

non-homogeneous ODE

with constant coefficient.

$$a_n y^{(n)} + \dots\dots + a_1 y' + a_0 y = f(x)$$

To solve, first we solve the homogeneous

Suppose, solution of homogeneous is y_h

$\Rightarrow$ then we need to find a solution for non-homogeneous (particular) (Undetermined coeff.) y_p.

General solution $y = y_h + y_p$

today next class

1st midterm, June 18, Friday

~~9-11 am~~ bb

10-12

till next class

* 6 Question
* 6 different uploads.

04-June-2021

Non-homogeneous ODE with constant coefficient:

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = f(x)$$

① First solve $a_1 y^{(n)} + \cdots + a_0 y = 0$, y_h

today ② Find a particular solution of non-homogeneous ODE, y_p

③ $y = y_h + y_p$

Undetermined co-efficients y_p

→ y_p depends on $f(x)$.

Six cases

(i) $f(x)$ is polynomial.

$f(x)$	y_p
$8x-6$	$Ax+B$
$2x^2-5x+10$	Ax^2+Bx+C
x^3-2	Ax^3+Bx^2+Cx+D

(ii) $f(x)$ is Trigonometric function.

$\sin^2 x$
$\sin^2 5x$

$f(x)$	y_p
$\sin(3x)$	$A\sin(3x) + B\cos(3x)$
$\cos(5x)$	$A\sin(5x) + B\cos(5x)$

explain fun math idea!

$RHS = f(x)$ $\quad\quad$ $ODE = f(x)$

$y_p \rightsquigarrow f(x), f'(x), f''(x), f'''(x) \ldots\ldots$

Suppose $f(x) = \sin(3x)$ $\rightsquigarrow y_p = A\sin 3x + B\cos 3x$

$f(x) = x^2$

$\rightarrow y_p = Ax^2 + Bx + C$

$f(x) = \frac{1}{x}$

$f'(x) = x^{-2}$ $\quad f''(x) = 2x^{-3} \ldots\ldots$ infinite

$f(x) = \tan x = \frac{\sin x}{\cos x}$ $\quad f' \quad f'' \quad f'''$

derivation will go forever

$\sec x = \frac{1}{\cos x}$ $\quad$ same

(iii) $f(x)$ is exponential function.

$f(x)$	y_p
e^{ax}	Ae^{ax}
e^{5x}	Ae^{5x}

(iv) Mix of Polynomial and exponential.

multiplication

$f(x)$	y_p
$(2x-5)\,e^{4x}$	$(Ax+B)\,e^{4x}$
$x^2 e^{3x}$	$(Ax^2+Bx+C)\,e^{3x}$

(v) Mix of trigonometric and exponential function.

multiplication

$f(x)$	y_p
$e^{8x}\sin(5x)$	$e^{8x}\left[A\sin(5x)+B\cos(5x)\right]$ or $Ae^{8x}\sin(5x)+Be^{8x}\cos(5x)$

(vi) Mix of trigonometric and polynomials

$f(x)$	y_p
$3x^2\sin(5x)$	$(Ax^2+Bx+C)\sin 5x$ $+(Dx^2+Ex+F)\cos 5x$

$\longrightarrow (Ax^2+Bx+C)(D\sin 5x+E\cos 5x) = (Ax^2+Bx+C)\sin x + \ldots$

Now:

Ex: $f(x) = e^{5x} + 3\cos 7x$

$$y_p = \underbrace{Ae^{5x}}_{y_{p_1}} + \underbrace{B\sin 7x + C\cos 7x}_{y_{p_2}}$$

So, $f(x) + g(x) + \cdots\cdots$

$\longrightarrow y_{p_1} + y_{p_2} + \cdots\cdots$

Ex-1 Solve $y'' - 5y' + 6y = x^2 - x + 1$ using method of Undetermined coefficient.

First we solve, $y'' - 5y' + 6y = 0$

characteristic equation, $\lambda^2 - 5\lambda + 6 = 0$

$\Rightarrow (\lambda-3)(\lambda-2) = 0$

So, $\lambda = 2, 3$

6
-3 -2

$$y_h = c_1 e^{2x} + c_2 e^{3x}$$ homogeneous

So, $y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B$

$y_p'' = 2A$

Now, substituting these to the main equation,

$$[2A] - 5[2Ax + B] + 6[Ax^2 + Bx + C] = x^2 - x + 1$$

$$\Rightarrow \quad x^2[6A] + x[-10A + 6B] + [2A - 5B + 6C] = x^2 - x + 1$$

Equating coefficient,

$$\begin{cases} 6A = 1 & A = \frac{1}{6} \\ -10A + 6B = -1 & B = \frac{1}{9} \\ 2A - 5B + 6C = 1 & C = \frac{11}{54} \end{cases}$$

$$y_p = \frac{1}{6}x^2 + \frac{1}{9}x + \frac{11}{54}$$

So, $y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{6}x^2 + \frac{1}{9}x + \frac{11}{54}$.

1st HMW. solve $y'' + y' - 6y = e^x + \sin(3x)$

Hint: First $y'' + y' - 6y = 0$

$$\lambda^2 + \lambda - 6 = 0$$

$$(\lambda + 3)(\lambda - 2) = 0$$

$$\lambda = 2, -3$$

$6 \to 3, -2$

$$y_h = c_1 e^{2x} + c_2 e^{-3x}$$

$y_p = Ae^x + B\sin 3x + C\cos 3x.$

HMW $\longrightarrow$ find y', y'' then substitute

$\nearrow e^x[A + A - 6A] + \sin(3x)[-9B - 3C - 6B]$

$+ \cos(3x)[-9C + 9B - 6C]$

$= e^x + \sin(3x)$

$$\begin{cases} -4A = 1 \\ -15B - 3C = 1 \\ -15C + 3B = 0 \end{cases} \qquad \begin{aligned} A &= -1/4 \\ B &= -5/78 \\ C &= -1/78 \end{aligned}$$

$y = y_h + y_p = c_1 e^{2x} + c_2 e^{-3x} - \frac{1}{4}e^x - \frac{5}{78}\sin 3x - \frac{1}{78}\cos 3x$

You can write y_{p_1} $\underline{y_{p_2}}$

$y'' + y' - 6y = e^x \longrightarrow y_{p_1}$

Done !!

This method has a problem

!!

Remark:

* This method fails sometimes.

* If there is any duplication between y_h and your choice of y_p (table) then this method fails.

* You should modify y_p

* Consider xy_p instead of y_p

new particular solution

* If there is still duplication between xy_p and y_h then Consider $x^2 y_p$ ...... (and continue)

y_h

y_p xy_p x^2y_p, x^3y_p . . .

continue till you do not see any duplication.

Ex-1 Solve $y'' - 2y' - 8y = e^{-2x}$

First $y'' - 2y' - 8y = 0$

$$\lambda^2 - 2\lambda - 8 = 0$$

$$\Rightarrow (\lambda+2)(\lambda-4) = 0$$

$$\Rightarrow \lambda = -2, 4$$

$\therefore y_h = c_1 e^{4x} + c_2 e^{-2x}$

$\rightarrow y_p = Ae^{-2x}$ duplication

*If you continue with this you would get $0 = 1$

So, $y_p = Axe^{-2x}$ (modified)

$y_p' = Ae^{-2x} + Ax\left[-2e^{-2x}\right]$ $\rightarrow -2Axe^{-2x}$

$y_p'' = -2Ae^{-2x} - 2Ae^{-2x} + 4Axe^{-2x}$

Sub in main,

$$\left[-4Ae^{-2x} + 4Axe^{-2x}\right] - 2\left[Ae^{-2x} - 2Axe^{-2x}\right] - 8\left[Axe^{-2x}\right] = e^{-2x}$$

$$\Rightarrow e^{-2x}(-4A - 2A) + xe^{-2x}(4A + 4A - 8A) = e^{-2x}$$

$$\Rightarrow e^{-2x}(-6A) + xe^{-2x}(0) = e^{-2x}$$

$-6A = 1$

$A = -1/6$

$y = y_h + y_p$

$= c_1 e^{4x} + c_2 e^{-2x} - \frac{1}{6} x e^{-2x}$

$\underline{2^{nd} \text{ HMW:}}$ Solve $y^{(4)} - y'' = 3x^2 - \sin(2x)$

First solve $y^{(4)} - y'' = 0$

$$\Rightarrow \lambda^4 - \lambda^2 = 0$$

$$\Rightarrow \lambda^2(\lambda^2 - 1) = 0$$

$\lambda^2 = 0$ $\quad\quad\quad$ $\lambda^2 - 1 = 0$

$\lambda = 0, 0$ $\quad\quad\quad$ $(\lambda+1)(\lambda-1) = 0$

$\lambda = 1, -1$

$$y_h = c_1 + c_2 x + c_3 e^x + c_4 e^{-x}$$

$$y_p = Ax^2 + Bx + C + D\sin(2x) + E\cos(2x)$$

y_{p_1} has duplication, not y_{p_2}

* multiply y_{p_1} with x $\rightarrow$ again we get duplication.

So, $y_p = Ax^4 + Bx^3 + Cx^2 + D\sin(2x) + E\cos(2x)$

$\underline{\text{HMW}}$ $\quad y', y'', y''', y^{(4)}$ $\quad$ sub in

$A = -\frac{1}{4}$, $B = 0$, $C = -3$, $D = -\frac{1}{20}$, $E = 0$

$$y = y_h + y_p$$

So, $y = c_1 + c_2 x + c_3 e^x + c_4 e^{-x} - \frac{1}{4}x^4 - 3x^2 - \frac{1}{20}\sin(2x)$

Midterm - June 18, (10 am to 12) EDT, till this class, A, B, C version

11 - June - 2021

Power Series :

Numerical Series

$$\text{finite} \leftarrow \sum_{n=0}^{K} a_n = a_0 + a_1 + \cdots\cdots + a_K$$

$$\text{Infinite} \leftarrow \sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots\cdots$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \underbrace{1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\cdots}_{\text{Geometric series}} = \frac{1}{1-1/2}$$

$$= 2$$

(convergent)

$$\sum_{n=1}^{\infty} n = 1+2+3+\cdots\cdots = \text{(divergent)}$$

\# Power series in one variable is an infinite series of the form.

$$\text{function} = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0^2) + \cdots\cdots$$

center of the series

Coefficient
(constant coeff.)

* It looks like ploynomials
* Polynomials are nice !

Polynomials are continuous and differentiable
*very easy to integrate.

Idea: can we approximate a function by a power Series!!

differentiable of any order (smooth) / Infinite differentiable

Some functions (nice!) can be approximated by power series [Taylor Series]

$$f(x) \cong \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$= f(x_0) + \frac{f(x_0)}{\lfloor 1}(x-x_0) + \frac{f''(x_0)}{\lfloor 2}(x-x_0)^2 + \cdots\cdots$$

Ex-1 Find Taylor Series of $f(x) = e^x$ at $x_0 = 0$

$$f(x) = e^x = \sum \frac{f^n(0)}{\lfloor n} x^n.$$

$f(x) = e^x$

$f(0) = 1$

$f'(x) = e^x \longrightarrow f'(0) = 1$

$f''(x) = e^x \longrightarrow f''(0) = 1$

$$e^x \cong 1 + \frac{1}{\lfloor 1}x + \frac{1}{\lfloor 2}x^2 + \frac{1}{\lfloor 3}x^3 + \cdots\cdots \quad \sum \frac{x^n}{\lfloor n}$$

$$= 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots\cdots$$

convergent
$-\infty < x < \infty$
everywhere

** Taylor Series at $x_0 = 0$ is called Maclaurin Series.

Application:

Find $\underbrace{\int e^{x^2} dx}_{\text{terrible Integral}}$

* so we can use Taylor series

$$\int e^{x^2} \cong \int 1 + x^2 + \frac{x^4}{\underline{|2}} + \frac{x^6}{\underline{|3}} + \cdots\cdots$$

Integration
$$= x + \frac{1}{3}x^3 + \frac{1}{10}x^5 + \cdots\cdots$$

HMW: Find Taylor series at $x_0 = 0$

$f(x)$	Answer	3 terms	Extra (convergent)
① $\ln(1+x)$	$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$-1 \le x \le 1$
② $\sin x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{\underline{|(2n+1)}}$ * $n=1 \longrightarrow (2n-1)$	$x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \cdots$	$-\infty < x < \infty$
③ $\cos x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{\underline{|2n}}$	$1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \cdots$	$-\infty < x < \infty$

+ Also check email Homeworks!

Power series Applications

① Approximation of function
Ex: Integration

② ODE

Idea of power series method of ODE.

1a → Taylor
2a → power (mid-2)

Ex-1: consider $y'' + P(x)y' + q(x)y = 0$

$P(x), q(x)$ = functions (new ODE)

★ **Definitions:** A function is called 'Analytic' if it can be approximated by power series

$e^x, \sin x, \cos x$

★ **Remark:** If $P(x)$ and $q(x)$ are analytic the the solutions of ODE are analytic.

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

analytic

$$y'' + \frac{1}{x-1} y' + y = 0$$

not analytic

★ In exam everything will be analytic

$x_0 = 1$

$x_0 = 0 \longrightarrow$ OK! then it's $\boxed{\text{analytic}}$

derivation at any point exists!

$$y'' + x^2 y' + 5y = 0$$

$p(x) = x^2$ $\qquad$ $q(x) = 5$

Analytic = $\underbrace{\text{Taylor series}}_{\text{at any point}}$ (have the derivative $[f^{(n)}(x_0)]$)

x_0 = given

$f^n(x_0) \longrightarrow$ Taylor series

$f(x)$ = analytic

Ex-1 Solve $y'' + y = 0$ at $x_0 = 0$

$$y'' + p(x)y' + q(x)y = 0$$

$p(x) = 0$ $\qquad$ $q(x) = 1$

$\wr\wr$ $\qquad$ $\wr\wr$

Taylor = 0 $\qquad$ Taylor = 1

analytic

* Taylor series of 10 is 10.

$f(x) = x^2$

$f'(x) = 2x \longrightarrow \frac{2x}{1}$

$f''(x) = 2$

$f(x) = x, \; x^2, \; !!$

taylor series $\rightarrow 2$

$$y'' + y = 0 \;;$$

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots\cdots$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$\xrightarrow{\text{sub in}}$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow \sum_{n=0}^{\infty} x^n \underbrace{\left[(n+2)(n+1) a_{n+2} + a_n \right]}_{\text{zero}} = 0$$

$$\sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} (a_n + b_n)$$

$$\sum_{n=2}^{\infty} a_n = a_2 + a_3 + \cdots = \sum_{n=0}^{\infty} a_{n+2}$$

So, $(n+2)(n+1) a_{n+2} + a_n = 0$ for $n = 0, 1, 2, 3 \ldots$

$n=0 \longrightarrow 2a_2 + a_0 = 0 \longrightarrow a_2 = -\frac{1}{2} a_0$

$n=1 \longrightarrow 3 \times 2 a_3 + a_1 = 0 \longrightarrow a_3 = -\frac{1}{3 \times 2} a_1$

$n=2 \longrightarrow 4 \times 3 a_4 + a_2 = 0 \longrightarrow a_4 = \frac{-1}{4 \times 3} a_2 = \frac{1}{4 \times 3 \times 2} a_0$

$n=3 \longrightarrow 5 \times 4 a_5 + a_3 = 0 \longrightarrow a_5 = \frac{-1}{5 \times 4} a_3 = \frac{1}{5 \times 4 \times 3 \times 2} a_1$

*** no need to find pattern in exam.

So,

$$a_2 = -\frac{1}{\underline{|2}} a_0$$

$$a_3 = \frac{-1}{\underline{|3}} a_1$$

$$a_4 = \frac{1}{\underline{|4}} a_0$$

$$a_5 = \frac{a_1}{\underline{|5}}$$

not needed in exam

$$a_{2n} = \frac{(-1)^n}{(2n)!} a_0 \qquad a_{2n+1} = \frac{(-1)^n}{(2n+1)!} a_1$$

$$y = \sum_{n=0}^{\infty} a_n x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$= a_0 + a_1 x - \frac{1}{\lfloor 2} a_0 x^2 - \frac{1}{\lfloor 3} a_1 x^3 + \frac{1}{\lfloor 4} a_0 x^4 + \frac{1}{\lfloor 5} a_1 x^5 + \cdots$$

$$= a_0 \Big(\underbrace{1 - \frac{1}{\lfloor 2} x^2 + \frac{1}{\lfloor 4} x^4 + \cdots}_{\text{basis}}\Big) + a_1 \Big(\underbrace{x - \frac{1}{\lfloor 3} x^3 + \frac{1}{\lfloor 5} x^5 + \cdots}_{\text{basis}}\Big)$$

$$= a_0 \cos x + a_1 \sin x$$

Question was $y'' + y = 0$ $\qquad \lambda^2 + 1 = 0$, $\lambda = \pm i$ $\qquad \alpha = 0$, $\beta = 1$

$e^{0x} \cos(1x) \qquad e^{0x} \sin(1x)$

$y = c_1 \cos x + c_2 \sin x$ $\qquad$ verified

$y'' + xy = 0$ $\qquad$ this type will come in the exam.

Exam = June 18

$10^{am} \longrightarrow 12^{noon}$ (Windsor time)

15 + 5
- 15: solve
- 5: uploading

* 2 hours
* 6 Questions

solve $x^2 + 1 = 0$

attatch file $\quad$ click and attatch

one single file (PDF)

Check (resources) page.

*** cannot go to next Question until you attatch!

02 - July - 2021

Power series: (continued):

① $\sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} (a_n + b_n)$

② shift formula: $\sum_{n=0}^{\infty} a_n = \sum_{n=1}^{\infty} a_{n-1}$

③ $\underbrace{\sum_{n=0}^{\infty} a_n}_{a_0+a_1+a_2+\cdots} = \sum_{n=1}^{\infty} a_{n+1}$

Shift up-down

$= a_{-1} + a_0 + a_1 + a_2 + \cdots$

False-created

$a_{-1} = 0$ assume

LHS = RHS

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_{n-2}$$

assume: $\boxed{a_{-2} = 0 = a_{-1}}$

extra condition

Ex-1 Solve $(x-1)y'' + y' + 2(x-1)y = 0$ where $y(4)=5$, $y'(4)=0$

Boundary condition

In exam

center $x_0 = 4$

$$y = \sum_{n=0}^{\infty} a_n (x-4)^n$$

solution

$a_0, a_1, a_2 \cdots$

$$y' = \sum_{n=1}^{\infty} n a_n (x-4)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n (x-4)^{n-2}$$

$$x-1 = 3+(x-4)$$

sub in →

$$\left[3+(x-4)\right] \sum_{n=2}^{\infty} n(n-1)\, a_n (x-4)^{n-2} + \sum_{n=1}^{\infty} n\, a_n (x-4)^{n-1}$$

$$+ 2\left[3+(x-4)\right] \sum_{n=0}^{\infty} a_n (x-4)^{n} = 0$$

$$\Rightarrow 3\sum_{n=2}^{\infty} n(n-1)\, a_n (x-4)^{n-2} + \sum_{n=2}^{\infty} n(n-1)\, a_n (x-4)^{n-1} + \sum_{n=1}^{\infty} n\, a_n (x-4)^{n-1}$$

$$+ 6\sum_{n=0}^{\infty} a_n (x-4)^{n} + 2\sum_{n=0}^{\infty} a_n (x-4)^{n+1} = 0$$

$$\Rightarrow 3\sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2} (x-4)^{n} + \sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2} (x-4)^{n+1}$$

$$+ \sum_{n=0}^{\infty} (n+1)\, a_{n+1} (x-4)^{n} + 6\sum_{n=0}^{\infty} a_n (x-4)^{n} + 2\sum_{n=0}^{\infty} a_n (x-4)^{n+1} = 0$$

$$\Rightarrow 3\sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2} (x-4)^{n} + \sum_{n=0}^{\infty} (n+1)\, n\, a_{n+1} (x-4)^{n}$$

$$+ \sum_{n=0}^{\infty} (n+1)\, a_{n+1} (x-4)^{n} + 6\sum_{n=0}^{\infty} a_n (x-4)^{n} + 2\sum_{n=0}^{\infty} a_{n-1} (x-4)^{n} = 0$$

[assume $a_{-1} = 0$]

$$\Rightarrow \sum_{n=0}^{\infty} (x-4)^{n} \Big[3(n+2)(n+1)\, a_{n+2} + (n+1)\, n\, a_{n+1} + (n+1)\, a_{n+1}$$

$$+ 6a_n + 2a_{n-1} \Big] = 0$$

Now, $3(n+2)(n+1)a_{n+2} + (n+1)^2 a_{n+1} + 6a_n + 2a_{n-1} = 0$

for $n = 0, 1, 2, \ldots$

$$a_{n+2} = -\frac{(n+1)}{3(n+2)} a_{n+1} - \frac{2a_n}{(n+2)(n+1)} - \frac{2a_{n-1}}{3(n+2)(n+1)}$$

$$n=0 \longrightarrow a_2 = -\frac{1}{3\times 2} a_1 - \frac{2a_0}{2\times 1} - \frac{2a_{-1}}{3\times 2\times 1}$$

$$= -\frac{1}{6} a_1 - a_0$$

$$n=1 \longrightarrow a_3 = -\frac{2}{3\times 3} a_2 - \frac{2a_1}{3\times 2} - \frac{2a_0}{3\times 3\times 2}$$

$$= -\frac{2}{9} a_2 - \frac{a_1}{3} - \frac{a_0}{9}$$

$$= -\frac{2}{9}\left(-\frac{1}{6} a_1 - a_0\right) - \frac{a_1}{3} - \frac{a_0}{9}$$

$$= \frac{-8}{27} a_1 + \frac{1}{9} a_0$$

$$n=2 \longrightarrow a_4 = \frac{5}{108} a_1 + \frac{5}{36} a_0$$

$$y = a_0 + a_1(x-4) + a_2(x-4)^2 + a_3(x-4)^3 + \cdots$$

$$= a_0 + a_1(x-4) + \left(-\frac{1}{6}a_1 - a_0\right)(x-4)^2 + \left(-\frac{8}{27}a_1 + \frac{1}{9}a_0\right)(x-4)^3$$

$$+ \left(\frac{5}{108}a_1 + \frac{5}{36}a_0\right)(x-4)^4 + \cdots$$

$$= a_0\left[1-(x-4)^2+\frac{1}{9}(x-4)^3+\frac{5}{36}(x-4)^4+\ldots\ldots\right]$$

$$+a_1\left[(x-4)-\frac{1}{6}(x-4)^2-\frac{8}{27}(x-4)^3+\frac{5}{108}(x-4)^4+\ldots\ldots\right]$$

*** In exam two non-zero or three non-zero terms will be asked to include.

extra condition,

$y(4)=5$ $\qquad\qquad$ $y'(4)=0$

$\Rightarrow a_0[1-0]=5$

$\Rightarrow a_0=5$

again, $y'(4)=0$

Now, $(x-4)+a_1=0$

$\Rightarrow a_1=0$ $\qquad\qquad$ $[y'(4)=0,\ x=4]$

Laplace Transformation :

$$L : \underset{\text{time}}{\overset{\text{function}}{t}} \longrightarrow \underset{\text{frequency}}{\overset{\text{function}}{s}}$$

$$f : [0, \infty] \longrightarrow \mathbb{R}$$

$$\mathcal{L}\{f(t)\} = \int_0^t e^{-st} f(t)\, dt$$

(Improper Integration)

Ex-1 Find Laplace of $f(t) = 1$

$$\mathcal{L}(1) = \int_0^\infty e^{-st}\, dt$$

$$= -\frac{1}{s} e^{-st} \Big|_0^\infty$$

$$= 0 - \left(-\frac{1}{s}\right) = \frac{1}{s}$$

$s > 0$
convergent
(we get an answer)

Ex-2 Find Laplace of $f(t) = t$

$$\mathcal{L}(t) = \int_0^\infty \overbrace{e^{-st}}^{dv} \underbrace{t}_{u}\, dt$$

Integration by parts
$uv - \int v\, du$

$$= \underbrace{\frac{-te^{-st}}{s}\Bigg|_0^{\infty}}_{\text{!! Zero}} + \underbrace{\frac{1}{s}\int_0^{\infty} e^{-st}\,dt}_{\frac{1}{s}\left(\frac{1}{s}\right)=\frac{1}{s^2}} \qquad s>0$$

L'Hospitals

$$\frac{-te^{-st}}{s}\Bigg|_{t=0}^{t=\infty} \qquad -\infty \times 0 \quad \text{undefined}$$

$$\lim_{t\to\infty} \frac{-t}{e^{st}} = \frac{\infty}{\infty} = \text{Hopital}$$

$$= \lim_{t\to\infty} \frac{-1}{se^{st}} = \frac{-1}{\infty} = 0$$

Hopital

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty} \quad \text{or} \quad \frac{0}{0}$$

$$= \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)}$$

so, finally $\mathcal{L}\{t\} = \frac{1}{s^2}$

Ex-3 Find $\mathcal{L}(e^{-3t}) = ?$ $\qquad f(t) = e^{-3t}$.

$$\mathcal{L}(e^{-3t}) = \int_0^{\infty} e^{-st} e^{-3t}\,dt$$

$$= \int_0^{\infty} e^{-t(s+3)}\,dt \qquad s+3>0$$

$$= \frac{1}{s+3} e^{-t(s+3)} \Big|_0^\infty$$

$$= \frac{1}{s+3}$$

HMW: Find $\mathcal{L}(\sin 2t)$

Hint: $\mathcal{L}(\sin 2t) = \int_0^\infty \overbrace{e^{-st} \overbrace{\sin(2t)}^{u}\, dt}^{dv}$

twice integration by parts

$$= \frac{2}{s^2+4}$$

Remark:

linear

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

$\mathcal{L}(2e^{-3t} - 7\sin 2t)$

$$= 2\mathcal{L}(e^{-3t}) - 7\mathcal{L}(\sin 2t)$$

$$= \frac{2}{s+3} - \frac{14}{s^2+4}$$

Laplace Table:

① $\mathcal{L}(1) = \frac{1}{s}$

② $\mathcal{L}(t^n) = \frac{\lfloor n}{s^{n+1}}$

③ $\mathcal{L}(e^{at}) = \frac{1}{s-a}$

④ $\mathcal{L}(\sin at) = \frac{a}{s^2+a^2}$

⑤ $\mathcal{L}(\cos at) = \frac{s}{s^2+a^2}$

⑥ $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

$\mathcal{L}\{f(t)\} = F(s)$

shift formula
shift of
the variables

\# $\mathcal{L}(t^4 e^{3t}) = \frac{\lfloor 4}{(s-3)^5}$

\# $\mathcal{L}(t^5) = \frac{\lfloor 5}{s^6}$

\# $\mathcal{L}(e^{-2t} \sin 5t) = \frac{5}{(s+2)^2+25}$

$\mathcal{L}$: function $t \longrightarrow$ function s

$$\mathcal{L}\{f(t)\} = F(s)$$

Laplace is 1-1 function

not bijection

$$(*) \quad \mathcal{L}^{-1}\{F(s)\} = f(t)$$

$\mathcal{L}(t^2) = \frac{\lfloor 2}{s^3}$

$$\mathcal{L}^{-1}\left(\frac{\lfloor 2}{s^3}\right) = t^2$$

(*) $\mathcal{L}^{-1}$ is also linear

$$\mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\}$$

$\mathcal{L}^{-1}\left(\frac{1}{s^5}\right) = \mathcal{L}^{-1}\left(\frac{\frac{1}{\lfloor 4}\cdot\lfloor 4}{s^5}\right)$

$$= \frac{1}{\lfloor 4}\mathcal{L}^{-1}\left(\frac{\lfloor 4}{s^5}\right)$$

$$= \frac{1}{\lfloor 4} t^4$$

$\frac{1}{\sqrt{7}}\mathcal{L}^{-1}\left(\frac{\sqrt{7}}{s^2+7}\right) = \frac{1}{\sqrt{7}}\sin(\sqrt{7}t)$

\# $\mathcal{L}^{-1}\left(\frac{-2s+6}{s^2+4}\right)$

$$= -2\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) + \frac{6}{2}\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right)$$

$$= -2\cos(2t) + 3\sin 2t$$

Application:

next week

① Solve ODE by Laplace

② Convolution Theorem

included in 2nd mid

2^{nd} midterm

$$\underbrace{\text{Taylor}}_{1} + \underbrace{\text{Power}}_{2} + \underbrace{\text{Laplace}}_{2}$$

* 5 or 6 questions.

9-July-2021

Laplace Transformation:

$\begin{cases} \text{ODE} \\ \text{convolution} \end{cases}$

Remark:

☆ $y(t)$

$$\mathcal{L}\{y^{(n)}\} = s^n \mathcal{L}(y) - s^{n-1}y(0) - s^{n-2}y'(0) - s^{n-3}y''(0)$$
$$- s^{n-4}y'''(0) - \cdots\cdots - y^{n-1}(0).$$

$n=1$ $\quad \mathcal{L}(y') = s\mathcal{L}(y) - y(0)$

$n=2$ $\quad \mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0)$

$n=3$ $\quad \mathcal{L}(y''') = s^3\mathcal{L}(y) - s^2y(0) - sy'(0) - y''(0)$

ODE

Solve $y' + 3y = 13\sin 2t$ with $y(0) = 6$

Take Laplace of both side,

$$\mathcal{L}(y' + 3y) = \mathcal{L}(13\sin 2t)$$

$$\Rightarrow \mathcal{L}(y') + 3\mathcal{L}(y) = 13\mathcal{L}(\sin 2t) \qquad \text{[linearity]}$$

$$\Rightarrow [s\mathcal{L}(y) - \underbrace{y(0)}_{6}] + 3\mathcal{L}(y) = \frac{26}{s^2+4}$$

$$\Rightarrow s\mathcal{L}(y) - 6 + 3\mathcal{L}(y) = \frac{26}{s^2+4}$$

$$\Rightarrow \mathcal{L}(y)\,[s+3] = \frac{26}{s^2+4} + 6$$

$$\Rightarrow \mathcal{L}(y) = \frac{26}{(s^2+4)(s+3)} + \frac{6}{(s+3)}$$

$$= \frac{26 + 6(s^2+4)}{(s^2+4)(s+3)}$$

$$= \frac{6s^2+50}{(s^2+4)(s+3)}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{ \frac{6s^2+50}{(s^2+4)(s+3)} \right\}$$

$$\frac{6s^2+50}{(s^2+4)(s+3)} = \frac{As+B}{(s^2+4)} + \frac{A}{s+3}$$

$A = 8$
$B = -2$
$C = 6$

$$y(t) = \mathcal{L}^{-1}\left(\frac{-2s+6}{s^2+4}\right) + \mathcal{L}^{-1}\left(\frac{8}{s+3}\right)$$

$$= -2\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) + 6\mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right) + 8\mathcal{L}^{-1}\left(\frac{1}{s+3}\right)$$

$$= -2\cos 2t + 3\sin 2t + 8e^{-3t}$$

Solve $y'' - 3y' + 2y = e^{-4t}$, $y(0)=1$, $y'(0)=5$

Laplace of both side and linearity,

$$\left[s^2\mathcal{L}(y) - sy(0) - y'(0)\right] - 3\left[s\mathcal{L}(y) - y(0)\right] + 2\mathcal{L}(y) = \frac{1}{s+4}$$

$$\Rightarrow s^2\mathcal{L}(y) - s - 5 - 3s\mathcal{L}(y) + 3 + 2\mathcal{L}(y) = \frac{1}{s+4}$$

$$\Rightarrow \mathcal{L}(y)\left[s^2 - 3s + 2\right] = \frac{1}{s+4} + s + 2$$

$$\begin{array}{ccc} & 2 & \\ -2 & & -1 \end{array}$$

$$\Rightarrow \mathcal{L}(y) = \frac{1}{(s+4)(s-2)(s-1)} + \frac{s+2}{(s-1)(s-2)}$$

$$= \frac{1 + (s+2)(s+4)}{(s+4)(s-2)(s-1)}$$

common denominator

$s^2 + 2s + 4s + 8$

$$= \frac{s^2 + 6s + 9}{(s+4)(s-1)(s-2)}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s+4)(s-1)(s-2)}\right\}$$

$$\frac{s^2+6s+9}{(s+4)(s-1)(s-2)} = \frac{A}{(s+4)} + \frac{B}{(s-1)} + \frac{C}{(s-2)}$$

$$A = \frac{1}{30}, \quad B = -\frac{16}{5}, \quad C = \frac{25}{6}.$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{1/30}{s+4}\right) + \mathcal{L}^{-1}\left(\frac{-16/5}{s-1}\right) + \mathcal{L}^{-1}\left(\frac{25/6}{s-2}\right)$$

$$= \frac{1}{30} e^{-4t} - \frac{16}{5} e^{t} + \frac{25}{6} e^{2t}$$

\# Solve $y'' - 6y' + 9y = t^2 e^{3t}$, $y(0) = 2$, $y'(0) = 17$

Taking Laplace of both side,

$$\mathcal{L}(y'') - 6\mathcal{L}(y') + 9\mathcal{L}(y) = \mathcal{L}(t^2 e^{3t})$$

$$\Rightarrow [s^2\mathcal{L}(y) - sy(0) - y'(0)] - 6[s\mathcal{L}(y) - y(0)] + 9\mathcal{L}(y)$$

$$= \frac{2}{(s-3)^3}$$

$$\Rightarrow s^2\mathcal{L}(y) - 2s - 17 - 6s\mathcal{L}(y) + 12 + 9\mathcal{L}(y) = \frac{2}{(s-3)^3}$$

$$\Rightarrow \mathcal{L}(y)[s^2 - 6s + 9] = \frac{2}{(s-3)^3} + 2s + 5$$

$$\Rightarrow \mathcal{L}(y) = \frac{2}{(s-3)^5} + \frac{2s+5}{(s-3)^2}$$

$$= \frac{2}{(s-3)^5} + \frac{2(s-3)+11}{(s-3)^2}$$

$$= \frac{2}{(s-3)^5} + \frac{2}{(s-3)} + \frac{11}{(s-3)^2}$$

$$y(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^5}\right\} + 2\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) + 11\left\{\frac{1}{(s-3)^2}\right\}$$

$$= \frac{2}{14} e^{3t} + 2e^{3t} + 11e^{-3t} t$$

t	s
p	d
x	z

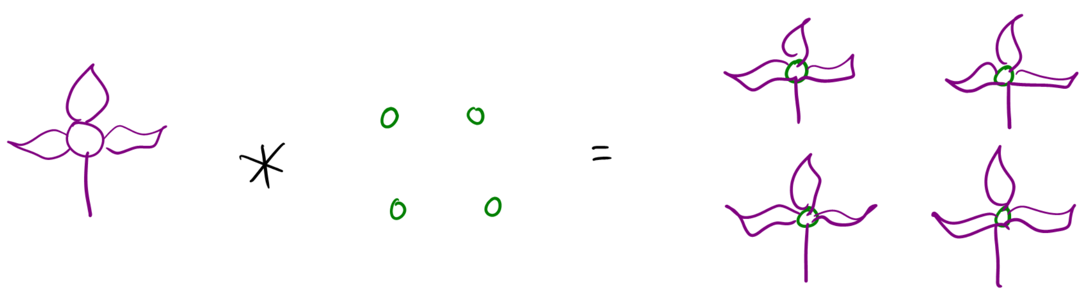
$$\mathcal{L}(0) = 0$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty 0 = 0$$

Convolution Theorem:

$$f(t),\ g(t) \xrightarrow{\text{New}} (f * g)(t)$$

$$(f * g)(t) = \int_0^t f(x)\, g(t-x)\, dx$$



Good Resolution

Image Processing

microscope

obj $*$ f $=$  zooming

Properties:

$$(f*g)(t) = \int_0^t f(x)\,g(t-x)\,dx$$

(i) $f * g = g * f$

(ii) $f * (g * h) = (f * g) * h$

(rule of associativity)

(iii) $f * (g + h) = f * g + f * h$

(iv) $f * 0 = 0$

Think

(v) $f * \delta = f$

unit
impulse

ramp

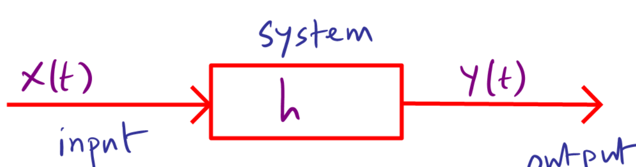
delta

Convolution Theorem

$$\mathcal{L}(f*g) = \mathcal{L}(f)\cdot\mathcal{L}(g)$$

$$f*g = \mathcal{L}^{-1}\left\{\mathcal{L}(f)\cdot\mathcal{L}(g)\right\}$$

Predict system output



$h(t) * x(t)$

* feedback
* feedback control
* industry

convolution idea

$$\mathcal{L}^{-1}\left\{\mathcal{L}(h(t))\,\mathcal{L}(y(t))\right\}$$

Compute $\mathcal{L}^{-1}\left\{\frac{2s}{(s^2+1)^2}\right\}$

convolution

$\mathcal{L}(f*g)=\mathcal{L}(f)\,\mathcal{L}(g)$

$f*g=\mathcal{L}^{-1}(FG)$

$$=\mathcal{L}^{-1}\left(2\cdot\frac{1}{s^2+1}\cdot\frac{s}{s^2+1}\right)$$

$$=2\mathcal{L}^{-1}\left\{\mathcal{L}(\sin t)\cdot\mathcal{L}(\cos t)\right\}$$

$$=2(\sin t*\cos t)$$

$$=2\int_0^t \sin x\cos(t-x)\,dx$$

$$=2\int_0^t \frac{1}{2}\left[\sin(x+t-x)+\sin(x-t+x)\right]dx$$

$$=\int_0^t\left\{\sin t+\sin(2x-t)\right\}dx$$

$$=x\sin t-\frac{1}{2}\cos(2x-t)\Big|_0^t$$

$$=t\sin t-\cancel{\frac{1}{2}\cos t}+\cancel{\frac{1}{2}\cos t}$$

$$=t\sin t$$

$$\mathcal{L}^{-1}\left(\frac{2s}{(s^2+1)^2}\right)=t\sin t$$

convolution

$t\sin t$

$\cos(-t) = \cos t$ (even function)

$\sin(-t) = -\sin t$ (odd function)

$$\sin a \cdot \cos b = \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right]$$

HMW will be posted.

Test-2 (Friday, July 16, 10 – 12 am EDT)

Taylor, power, Laplace $\begin{cases} \to \text{ODE} \\ \to \text{convolution} \end{cases}$

Taylor $\to$ (1), power $\to$ (2), Laplace $\to$ (2 or 3)

(total 5 or 6)

2 hours

23-July-2021

Fourier Series :

* Joseph Fourier (French)
  Heat Equation in a metal plate.

* Taylor Series : $f \cong$ Polynomial (approximate)
  Fourier Series : $f \cong$ sine/cos

TS : f is differentiable

FS : f not even continuous.

more f

Fourier Analysis

- F series (in our class)
- F transform

Question: which functions have FS ?

Answer: Piecewise continuous.

A function f on $[a,b]$ (interval) is called piecewise continuous if

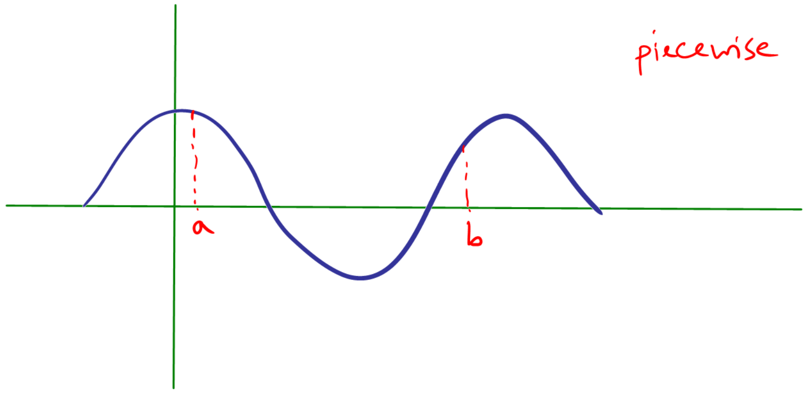
a) f is continuous on $[a,b]$ except perhaps at finitely many points.

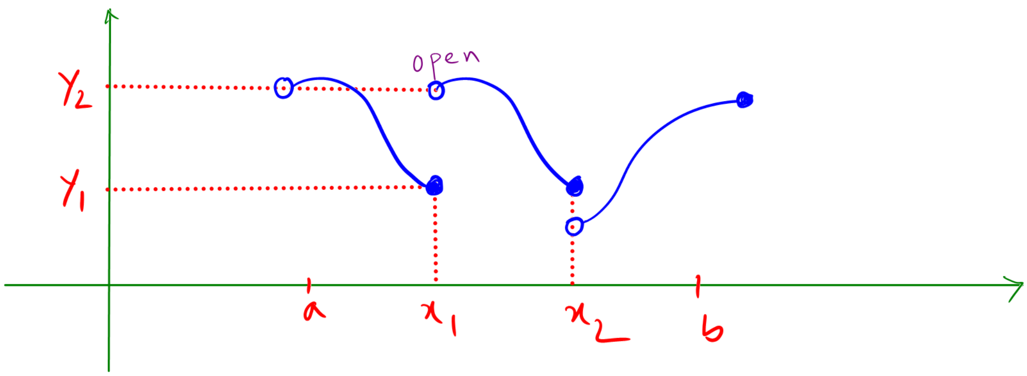
(finite # of discont.)

(b) f at those discontinuity points has left and right limit.

Example:

piecewise continuous





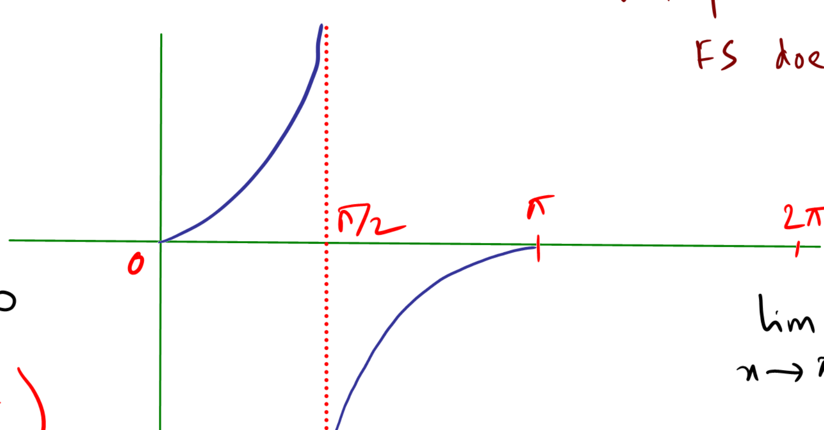
f is not continuous at x_1 & x_2

$\lim_{x \to x_1} f(x) = Y_1$ exist $\qquad \lim_{x \to x_1} f(x) = Y_2$

$f(x) = \tan x$ on $(0, 2\pi)$

not piecewise continuous

FS does not exist.



$\lim_{x \to \pi/2^-} \tan x = \infty$

(Also does not exist)

$\lim_{x \to \pi/2^+} \tan x = \infty$

(does not exist)

* Let $f(x)$ be a piecewise continuous function on $[-\pi, \pi)$,

Then the FS

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

a_0, a_n, b_n called Fourier coefficients.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

* n is natural numbers

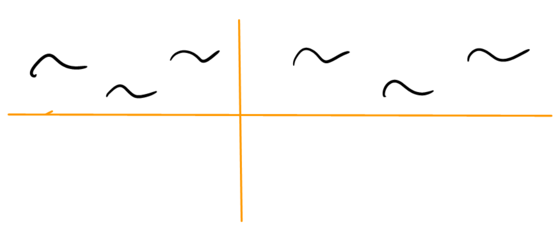
$n = 1, 2, 3, \ldots\ldots$

HMW: Think

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

How we find a_0, a_n, b_n

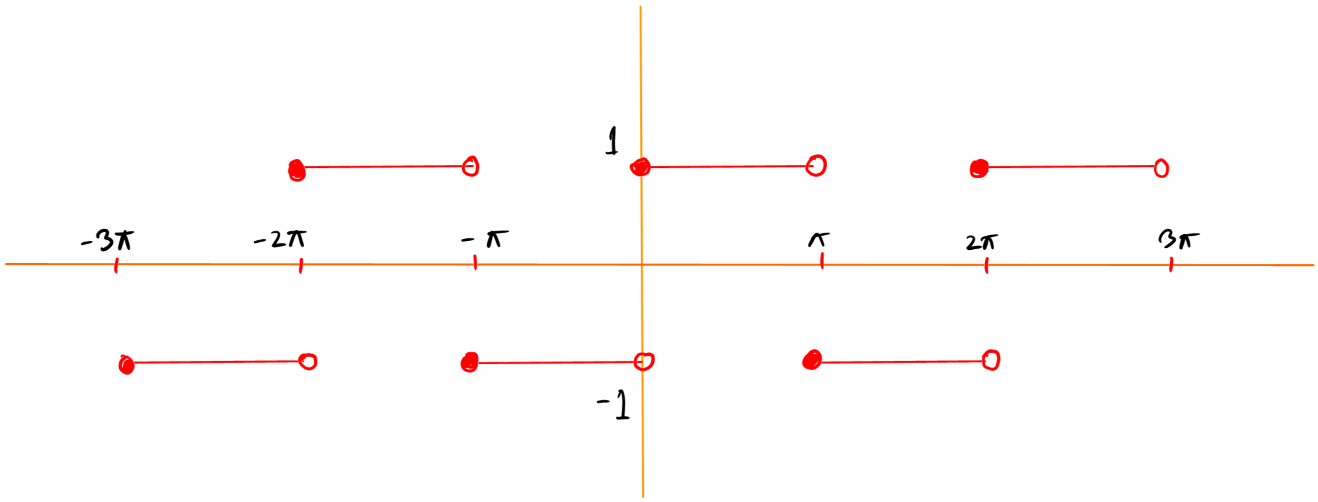
Hint: integration on both side



periodic

Find FS of $f(x) = \begin{cases} -1 & -\pi \le x < 0 \\ 1 & 0 \le x < \pi \end{cases}$

(2π periodic)



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -1\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1 \cdot dx$$

$$= -\frac{1}{\pi}\left[x\right]_{-\pi}^{0} + \frac{1}{\pi}\left[x\right]_{0}^{\pi}$$

$$= -\frac{1}{\pi}(0+\pi) + \frac{1}{\pi}(\pi - 0)$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -\cos(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi}\cos(nx)\,dx$$

$$= -\frac{1}{n\pi}\left[\sin(nx)\right]_{-\pi}^{0} + \frac{1}{n\pi}\left[\sin(nx)\right]_{0}^{\pi}$$

$$= -\frac{1}{n\pi}(0-0) + \frac{1}{n\pi}(\cancel{\sin n\pi}^{\,0} - 0)$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -\sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} \sin(nx)\,dx$$

$$= \frac{1}{n\pi}\Big[\cos(nx)\Big]_{-\pi}^{0} + \frac{-1}{n\pi}\Big[\cos(n\pi)\Big]_{0}^{\pi}$$

$$= \frac{1}{n\pi}\Big[1-\cos(n\pi)\Big] - \frac{1}{n\pi}\Big[\cos(n\pi - 1)\Big]$$

$$= \frac{2}{n\pi}\Big[1-\cos(n\pi)\Big]$$

$$= \begin{cases} 0 & n = \text{even} \\ \dfrac{4}{n\pi} & n = \text{odd} \end{cases}$$

$$f(x) = \frac{1}{2}\cancel{a_0}^{\,0} + \sum_{n=1}^{\infty} \cancel{a_n}^{\,0}\cos(nx) + b_n\sin(nx)$$

$$= \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$= \frac{4}{\pi}\sin(x) + \frac{4}{3\pi}\sin(3x) + \frac{4}{5\pi}\sin(5x)$$

$$= \sum_{n=\text{odd}} \frac{4}{n\pi}\sin(nx)$$

$$= \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin\{(2n-1)x\}$$

← professor likes this one

* all of them correct.

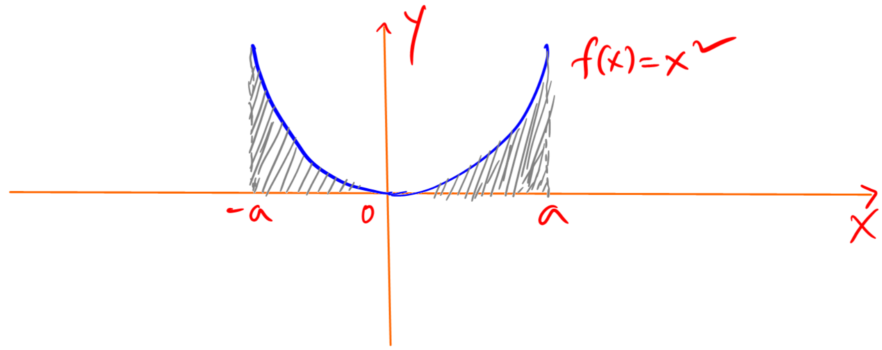
HMW: Find FS on $(-\pi, \pi)$

① $f(x) = x$	$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx)$ $a_0 = a_n = 0$, $b_n = \frac{2(-1)^{n+1}}{n}$
② $f(x) = \|\sin x\|$ $a_n = \begin{cases} 0 & n(\text{odd}) \\ \dots & n(\text{even}) \end{cases}$	$\frac{2}{\pi} - \sum \frac{4}{\pi(4n^2-1)} \cos(2nx)$ $a_0 = \frac{4}{\pi}$, $a_n = \frac{4}{\pi(4n^2-1)}$, $b_n = 0$
③ $f(x) = \|x\|$ $a_n = \begin{cases} 0 & n = \text{even} \\ \dots & n = \text{odd} \end{cases}$	$\frac{\pi}{2} - \sum \frac{4}{\pi} \frac{\cos\{(2n-1)x\}}{(2n-1)^2}$ $b_n = 0$, $a_0 = \pi$, $a_n = \frac{4}{\pi(2n-1)^2}$
④ $f(x) = \sin^2 x$	$\frac{1}{2} - \frac{1}{2}\cos(2x)$ $a_0 = 1$, $b_n = 0$, a_n all zero except a_2

Even function:

$$f(-x) = f(x).$$

$x^2, x^4, \cos x$ are even function.



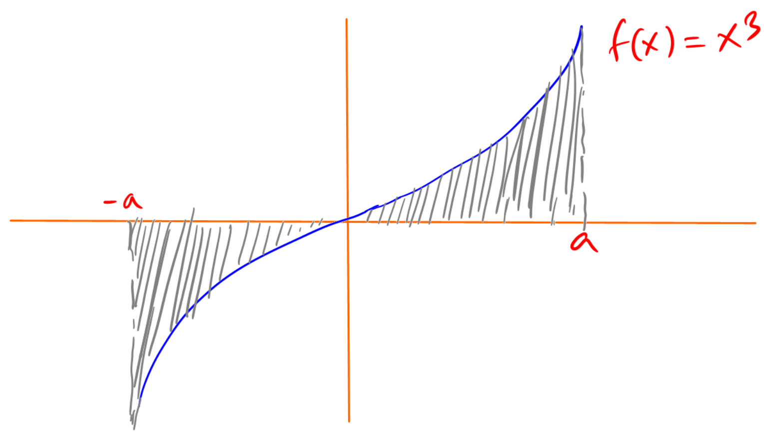
* This is symmetric with y axis.

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

odd function:

$$f(-x) = -f(x)$$

$\sin x$, x^3, x^5 are odd functions.



* Symmetric with respect to origin.

$$\int_{-a}^{a} f(x)\,dx = 0$$

FS of odd functions:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{o}\,\underbrace{\cos(nx)}_{e}\,dx = 0$$

odd × even = odd

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{o}\,\underbrace{\sin(nx)}_{o}\,dx$$

odd × odd = even

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\,dx$$

** In the exam, we can say since this is an odd function a_n & a_0 are zero.

$$f(x) \cong \sum b_n \sin(nx).$$

* Also apply for even function.

FS of an even function:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{e}\,\underbrace{\sin(nx)}_{o}\,dx$$

even × odd = odd

$$= 0$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{e}\,\underbrace{\cos(nx)}_{e}\,dx$$

even × even = even

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(nx)\,dx$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos(nx)$$

HMW: (same as before)

① $f(x)=x$	o	$a_0=a_n=0$
② $f(x)=\lvert\sin x\rvert$	e	$b_n=0$
③ $f(x)=\lvert x\rvert$	e	$b_n=0$
④ $f(x)=\sin^2 x$	e	$b_n=0$

Engine energy

$f(x) =$ → positive

↳ time , heat function

$f(x)$ on $[0,\pi]$.

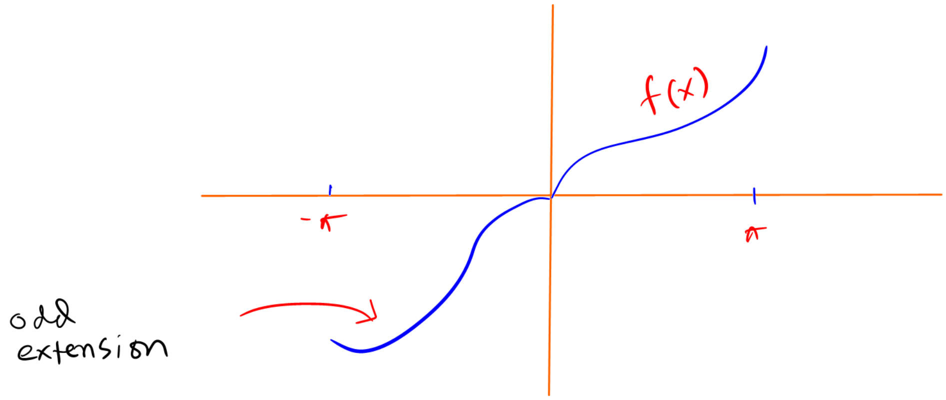
FS = ?

$a_0, a_n, b_n \longrightarrow \cancel{\int_{-\pi}^{\pi}}$ does not work.

Idea of Extension

Suppose $f(x)$ is defined only on $[0,\pi]$ (rather than $[-\pi,\pi]$) Then how we calculate FS ?

we manually extend the function.



* Fourier Sine Series:

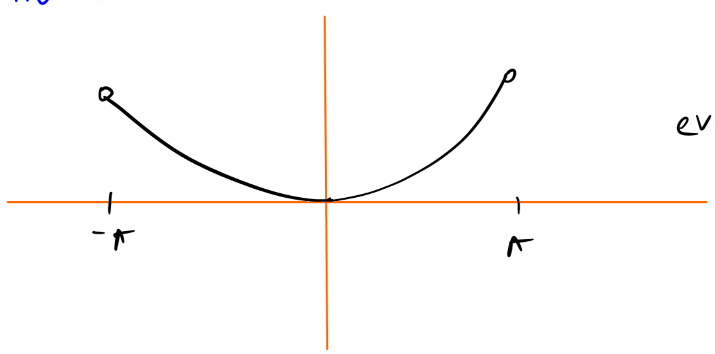
odd extension.

* there will be no a_0, a_n.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx.$$

* Fourier Cosine Series:



even extension

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

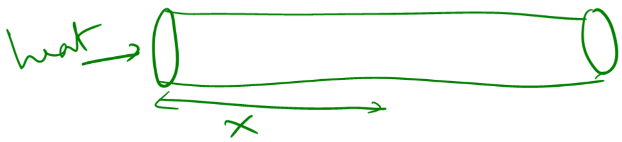
$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)$$

Practical scenario

temp
= { $\longrightarrow$ FS

60% course

$h(x,t)$ ↳ position



** Force you to use sine/cosine series.

* either one of them.

$f(x)$ on $[0,\pi]$.

even $\frac{1}{2}a_0 + \sum a_n \cos nx = \sum b_n \sin(nx)$

$x < [0,\pi]$.

exactly the same

Amazing!!

will happen!!

<u>HMW:</u> consider $f(x) = x$ on $[0,\pi]$.

① Find Fourier sine series.

$$\text{Ans: } \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx)$$

⑪ Find Fourier cosine series.

$$\text{Ans: } \pi/2 - \sum \frac{4}{\pi} \frac{\cos\{(2n-1)x\}}{(2n-1)^2}$$

Hint: $f(x) = \sum b_n \sin(nx)$

$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \ldots$ $\quad |x|$

$[-\pi, \pi]$

$[-L, L]$

Fourier series can be similarly defined on $[-L, L)$ for 2L-periodic function.

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

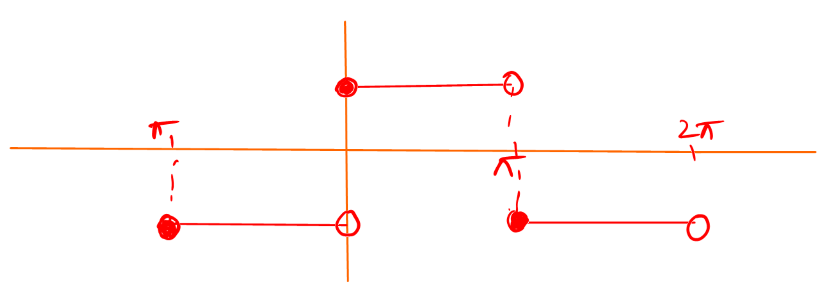
$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

close-open dibacle.

$$f(x) = \begin{cases} -1 & -\pi \leq x < 0 \\ 1 & 0 \leq x < \pi \end{cases}$$



* both of them can't be close and both of them can't be open.

Book: Fourier Analysis and its Application
— by Folland

HMW: Find FS of $f(x) = \begin{cases} 0 & -2 \le x < 0 \\ (2-x) & 0 \le x < 2 \end{cases}$

(L=2, 4-periodic)

Answer:

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx$$

$$= \frac{1}{2}\int_{-2}^{0} 0 + \frac{1}{2}\int_{0}^{2}(2-x)\,dx$$

$$a_n = \frac{1}{2}\int_{-2}^{0} 0\,dx + \frac{1}{2}\int_{0}^{2}(2-x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$\vdots$

$$= \begin{cases} 0 & n = \text{even} \\ \frac{4}{n^2\pi^2} & n = \text{odd} \end{cases}$$

$$b_n = \frac{1}{2}\int_{-2}^{0} 0 \cdot dx + \frac{1}{2}\int_{0}^{2}(2-x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{2}{n\pi}$$

Other topics

$$y'' + 7xy' + 8y = 4x$$

$\sum$ .... Line this short.

$n=0,1,2,3\ldots\ldots$

$n=1$

30-July-2021

Sturm - Liouville (SL) problem:

* Let $X = X(x)$ be a function on $[a,b]$.

* An Elementary SL problem is :

$$X'' = \lambda X$$

ODE

λ = constant

* ODE of order 2, homogeneous, with constant coefficient.

* we need to find both function X and scalar λ.

eigen function (eigen vector/value) — eigenvalue

$\Rightarrow$ SL: $X'' = \lambda X$ with different BC

Boundary condition

✓ (I) Dirichlet BC:

X on $[0, l]$

$$X(0) = X(l) = 0$$

✓ (II) Neumann BC

X on $[0, l]$

$$X'(0) = X'(l) = 0$$

HMW (III) Periodic BC

X on $[-l, l]$

$$\begin{cases} X(-l) = X(l) \\ X'(-l) = X'(l) \end{cases}$$

HMW ④ Mixed BC $\quad X(0) = X'(\ell) = 0$

X on $[0,\ell]$ $\quad$ OR

$X'(0) = X(\ell) = 0$

<u>Drichlet</u> $\quad$ SL: $X'' = \lambda X \quad , \quad X(0) = X(\ell) = 0$

ℓ is a numerical number

$$X'' - \lambda X = 0$$

characteristic equation,

$$m^2 - \lambda = 0$$

$$\Rightarrow m = \pm\sqrt{\lambda}.$$

We have 3 cases, $\underset{①}{\underline{\lambda > 0}}, \underset{②}{\underline{\lambda = 0}}, \underset{③}{\underline{\lambda < 0}}$

<u>Case ①</u>, $\lambda > 0$

$m = \pm\sqrt{\lambda}$, two real solution.

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$$

BC: $X(0) = 0 \longrightarrow 0 = c_1 + c_2 \longrightarrow c_1 = -c_2$

BC: $X(\ell) = 0 \longrightarrow c_1 e^{\sqrt{\lambda}\ell} + c_2 e^{-\sqrt{\lambda}\ell} = 0$ $\quad c_1 = -c_2$

$\Rightarrow -c_2 e^{\sqrt{\lambda}\ell} + c_2 e^{-\sqrt{\lambda}\ell} = 0$

$$\Rightarrow c_2\big[\underbrace{-e^{\sqrt{\lambda}\,l} + e^{-\sqrt{\lambda}\,l}}_{\text{never zero}}\big] = 0$$

So, $c_2 = 0$

$c_1 = 0 \qquad X(x) = 0$ Trivial solution for ODE.

Prove: $-e^{-\sqrt{\lambda}\,l} + e^{-\sqrt{\lambda}\,l}$ never be zero

We argue by contradiction.

Assume, $-e^{-\sqrt{\lambda}\,l} + e^{\sqrt{\lambda}\,l} = 0$

$$\Rightarrow e^{-\sqrt{\lambda}\,l} = e^{\sqrt{\lambda}\,l}$$

$$\Rightarrow \frac{1}{e^{\sqrt{\lambda}\,l}} = e^{\sqrt{\lambda}\,l}$$

$$\Rightarrow e^{2\sqrt{\lambda}\,l} = 1$$

$$\Rightarrow 2\sqrt{\lambda}\,l = 0$$

$\lambda > 0, \; l > 0$

$\boxed{\text{Contradiction}}$

So, we show $-e^{-\sqrt{\lambda}\,l} + e^{-\sqrt{\lambda}\,l} \neq 0$

We don't have to do this step in the exam.

Case 2, $\lambda = 0$

$$X'' = \lambda X$$

$$\Rightarrow X'' = 0 \qquad [\lambda = 0]$$

$$\Rightarrow (X')' = 0$$

$$\Rightarrow X' = \text{constant}$$

$$\Rightarrow X' = B \qquad [\text{let constant} = B]$$

$$\Rightarrow X = Bx + A \qquad [A \text{ is a constant}]$$

BC: $X(0) = 0 \longrightarrow A = 0 \longrightarrow X(x) = Bx$

BC: $X(l) = 0 \longrightarrow Bl = 0, \quad l > 0$

So, $B = 0$

Hence, $X(x) = 0$ Trivial solution.

Case 3, $\lambda < 0$

$$m = \pm\sqrt{\lambda}$$

$$= \pm\sqrt{(-1)(-\lambda)}$$

$$= \pm\sqrt{(-1)}\sqrt{-\lambda}$$

$$= \pm i\sqrt{-\lambda}$$

two complex conjugate solution.

$$\alpha = 0, \quad \beta = \sqrt{-\lambda}$$

$e^{\alpha x}\cos\beta x$
$e^{\alpha x}\sin\beta x$

$$X(x) = c_1\cos\left(\sqrt{-\lambda}\,x\right) + c_2\sin\left(\sqrt{-\lambda}\,x\right)$$

BC: $X(0)=0 \longrightarrow c_1=0 \longrightarrow X(x) = c_2\sin\left(\sqrt{-\lambda}\,x\right)$

BC: $X(l)=0 \longrightarrow c_2\sin\left(\sqrt{-\lambda}\,l\right)=0$ $\nearrow c_2=0$ $\searrow \sin\left(\sqrt{-\lambda}\,l\right)=0$

when $c_2=0$

then $X(x)=0$ Trivial solution.

when. $\sin\left(\sqrt{-\lambda}\,l\right)=0$

$\Rightarrow \sqrt{-\lambda}\,l = n\pi$ $\quad n=0,\pm1,\pm2,\pm3,\ldots\ldots$

$$\underbrace{\sqrt{-\lambda}}_{+}\,\underbrace{l}_{+} = n\underbrace{\pi}_{+}$$

n → has to be positive

$n=1,2,3,4,\ldots.$

$$\sqrt{-\lambda} = \frac{n\pi}{l}$$

$$\Rightarrow \lambda_n = -\left(\frac{n\pi}{l}\right)^2$$

$$X_n(x) = c_2\sin\left(\sqrt{-\lambda}\,x\right)$$

$$= c_2\sin\left(\frac{n\pi x}{l}\right)$$

Since c_2 is any scaler multiplication, we can remove it.

So,
$$\begin{cases} \lambda_n = -\left(\frac{n\pi}{l}\right)^2 \\ X_n = \sin\left(\frac{n\pi x}{l}\right) \\ n = 1, 2, 3, \ldots \end{cases}$$

infinity many solution.

$$X = \sum_{n=1}^{\infty} c_n X_n$$

$X'' = \lambda X$.

Fourier Sine Series

we'll see later.

Neumann BC:

$$X'' = \lambda X \quad , X'(0) = X'(l) = 0$$

$$X'' - \lambda X = 0$$

Characteristic polynomial,

$$\Rightarrow m^2 - \lambda = 0$$

$$\Rightarrow m = \pm\sqrt{\lambda}$$

3 cases, $\underset{1}{\underline{\lambda > 0}}$, $\underset{2}{\underline{\lambda = 0}}$, $\underset{3}{\underline{\lambda < 0}}$

Case ①, $\lambda > 0$

$m = \pm\sqrt{\lambda}$ two real solution.

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}.$$

$$X'(x) = c_1\sqrt{\lambda}e^{\sqrt{\lambda}x} - c_2\sqrt{\lambda}e^{-\sqrt{\lambda}x}$$

BC: $X'(0) = 0 \longrightarrow c_1 - c_2 = 0 \longrightarrow c_1 = c_2$

BC: $X'(l) = 0 \longrightarrow c_1\sqrt{\lambda}e^{\sqrt{\lambda}l} - c_2\sqrt{\lambda}e^{-\sqrt{\lambda}l} = 0$ $\quad c_1 = c_2$

$c_1 = c_2$

so, $\underbrace{c_2\sqrt{\lambda}}_{\text{not zero}}\ \big[\underbrace{e^{\sqrt{\lambda}l} - e^{-\sqrt{\lambda}l}}_{\text{never zero}}\big] = 0$

So, $c_2 = 0$

$X(x) = 0$ Trivial solution.

Case ②, $\lambda = 0$

$$X'' = \lambda X$$

$$\Rightarrow X'' = 0$$

$\Rightarrow X' = B$ [B is a constant]

$\Rightarrow X = Bx + A$ [A is a constant]

BC: $X'(0)=0 \longrightarrow 0=B \longrightarrow X(x) = A$ constant

BC: $X'(l)=0 \longrightarrow B=0$

$X(x) = \text{constant}.$

Case 3, $\lambda<0$,

$m = \pm\sqrt{\lambda}$ two complex

$\Rightarrow m = \pm\sqrt{(-1)(-\lambda)}$

$= \pm i\sqrt{-\lambda}$

$\alpha = 0, \quad \beta = \sqrt{-\lambda}.$

$X(x) = c_1\cos(\sqrt{-\lambda}\,x) + c_2\sin(\sqrt{-\lambda}\,x)$

$X'(x) = -c_1\sqrt{-\lambda}\sin(\sqrt{-\lambda}\,x) + c_2\sqrt{-\lambda}\cos(\sqrt{-\lambda}\,x)$

BC: $X'(0)=0 \longrightarrow c_2\sqrt{-\lambda}=0 \xrightarrow{\lambda<0} c_2=0$

BC: $X'(l)=0 \longrightarrow -c_2\sqrt{-\lambda}\sin(\sqrt{-\lambda}\,l)=0$ → $c_1=0$ or $\sin(\sqrt{-\lambda}\,x)=0$

when $c_1=0$

$X(x)=0$ Trivial solution.

when, $\sin(\sqrt{-\lambda}\,l)=0$

$\Rightarrow \sqrt{-\lambda}\,x = n\pi$

$n = 1, 2, 3, 4, \ldots\ldots$

$$\lambda_n = -\left(\frac{n\pi}{\ell}\right)$$

$$X(x) = c_1 \cos\left(\sqrt{-\lambda}\, x\right)$$

$$\Rightarrow X(x) = c_1 \cos\left(\frac{n\pi x}{\ell}\right)$$

$$X_n = \cos\left(\frac{n\pi x}{\ell}\right)$$

All the non-trivial solutions $(\lambda < 0, \lambda = 0)$,

$$\lambda_n = -\left(\frac{n\pi}{\ell}\right)^2$$

$$X_n = \cos\left(\frac{n\pi x}{\ell}\right)$$

$$n = \underbrace{0}_{\lambda=0}, \underbrace{1, 2, 3, \ldots\ldots\ldots}_{\lambda<0}$$

HMW:

(III) periodic $X(-\ell) = X(\ell)$

$X'(-\ell) = X'(\ell)$

Ans: $\lambda_n = -\left(\frac{n\pi}{\ell}\right)^2$

X_n combination, $1, \cos\left(\frac{n\pi x}{\ell}\right), \sin\left(\frac{n\pi x}{\ell}\right)$

$n = 0, 1, 2, 3, \ldots\ldots$

(IV) Mixed BC

$$x(0) = x'(\ell) = 0$$

or

$$x'(0) = x(1) = 0$$

Ans: $$\lambda_n = -\left\{ \frac{(2n-1)\pi}{2\ell} \right\}^2$$

$$X_n = \sin\left\{ \frac{(2n+1)\pi x}{2\ell} \right\}$$

$$n = 0, 1, 2, 3, \ldots\ldots$$

//

Introduction to Partial Differential Equation (PDE):

A PDE is a DE which the function involved has more than one variable.

Ex: $f(x,t)$

$$2 \cdot \frac{\partial f}{\partial x} - 5t^2 \frac{\partial^2 f}{\partial x \partial t} = 2x$$

* Most important example of PDE $u(x,t)$.

1. $c^2 U_{xx} = U_{tt}$ wave equation
2. $c^2 U_{xx} = U_t$ Heat equation

Homogeneous $6U_{xx} - U_{tt} = 0$

(3) $U_{xx} + U_{tt} = 0$ Laplace equation

Wave equation, $6U_{xx} = U_{tt}$, $4U_{xx} = U_{tt}$.

Final Exam:

2 PDE
2 FS
1 SL

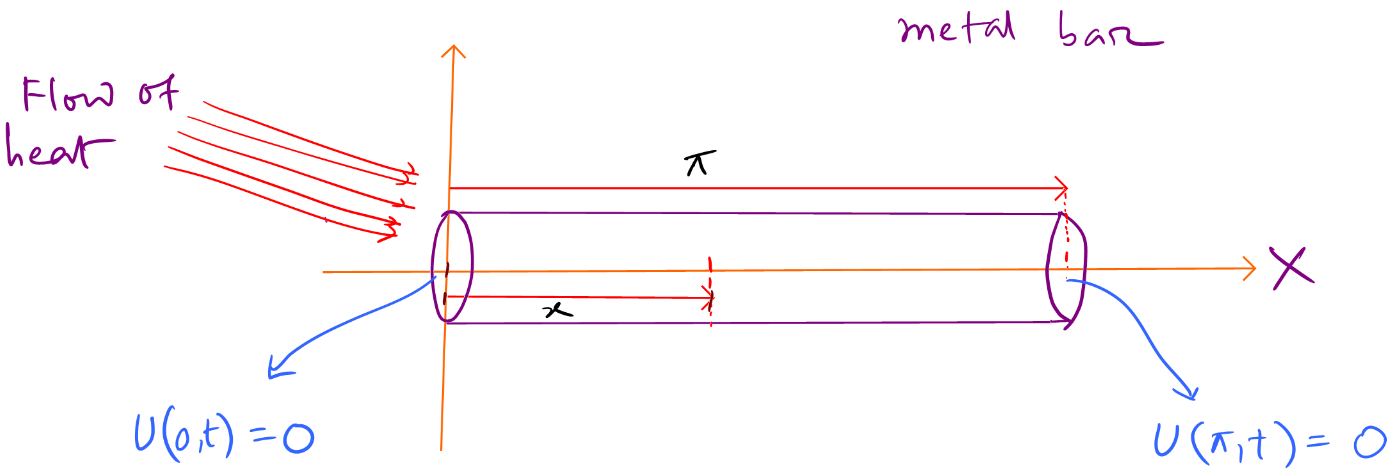
8-9 Q#

* Power Series, 1 #
* Laplace, 1 #
* From 1st mid, 1 #.

$35\text{ min} < Q < 5\text{ min}$

06-Aug-2021

PDE



$U(x,t)$ = temp. of bar at position x at time t.

Solve the following PDE,

$$4U_{xx} = U_t \qquad 0 \leq x \leq \pi, \; t > 0$$

① $U(0,t) = 0$ $\quad t > 0$ $\quad$ (BC 1)

② $U(\pi,t) = 0$ $\quad t > 0$ $\quad$ (BC 2)

③ $U(x,t) = 0$ $\quad 0 < x < \pi$ $\quad$ IV (Initial value)

We use the method of separation of variables.

Question is to find $U(x,t) = ?$

Assume $U(x,t) = X(x) \cdot T(t)$ completely separable.

$$U = XT.$$

So,
$$U_x = X'T \qquad U_t = XT'$$
$$U_{xx} = X''T$$

Substitute to the Heat Equation,

$$4X''T = XT'$$ Put 4 in the place of first derivative

$$\Rightarrow \frac{X''}{X} = \frac{T'}{4T} = \text{constant} = \lambda$$

$$\left\{ \begin{array}{l} \text{(i)} \ \dfrac{X''}{X} = \lambda \longrightarrow \text{ODE} \\ \text{(ii)} \ \dfrac{T'}{4T} = \lambda \longrightarrow \text{ODE of 1st order} \end{array} \right]$$ homogeneous with const. coeff.

First let's solve the first order,

$$\frac{T'}{4T} = \lambda$$
$$\Rightarrow \frac{T'}{T} = 4\lambda$$
$$\Rightarrow \int \frac{T'}{T} dt = \int 4\lambda \, dt$$
$$\Rightarrow \ln T = 4\lambda t$$
$$\Rightarrow T(t) = e^{4\lambda t}$$

Again, $$\frac{X''}{X} = \lambda$$

$$\Rightarrow X'' = \lambda X$$

$$\Rightarrow X'' - \lambda X = 0$$

$U(x,t) = X(x) \cdot T(t)$.

BC: $U(0,t) = X(0)\,T(t) = 0 \longrightarrow X(0) = 0$

BC: $U(\pi,t) = X(\pi) \cdot T(t) = 0 \longrightarrow X(\pi) = 0$

$T(t) \neq 0$

It satisfy the Drichlet BC. $X(0) = X(L) = 0$.

where $L = \pi$

* In exam solution will be given for SL. BC..

solution. $$\lambda_n = -\left(\frac{n\pi}{\pi}\right)^2 = -n^2$$

$$X_n = \sin\left(\frac{n\pi x}{\pi}\right) = \sin(nx)$$

$n = 1, 2, 3 \ldots\ldots$

Now, $T(t) = e^{4\lambda t}$

So, $T_n(t) = e^{-4n^2 t}$

$$U_n = X_n T_n = \sin(nx)\, e^{-4n^2 t}$$

$U_1, U_2, U_3 \ldots\ldots$

So, $U = \underbrace{a_1 U_1 + a_2 U_2 + a_3 U_3 + \ldots\ldots}_{\text{general solution}}$

General solution,

$$U(x,t) = \sum_{n=1}^{\infty} a_n \sin(nx) e^{-4n^2 t}$$

$a_1, a_2, a_3 \ldots\ldots$

$a_n = ?$

IV: $U(x,0) = x$

$t=0$

So, $$U(x,0) = \sum_{n=1}^{\infty} a_n \sin(nx)$$

Hence, we have Fourier Sine Series of $f(x)=x$ on $0<x<\pi$ or $[0,\pi]$.

* The Fourier Series will be provided in the exam.

So,

Fourier Sine Series of $f(x)=x$ is given.

$$f(x)=x = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin(nx)$$

compare

IV: $$x = \sum_{n=1}^{\infty} a_n \sin(nx)$$

So, After comparing we get,

$$a_n = \frac{2}{n}(-1)^{n+1}$$

Therefore,

$$U(x,t) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin(nx)\, e^{-4n^2 t}$$

Ans.

solve $9U_{xx} = U_{tt}$ $\quad 0 \le x \le \pi \quad t \ge 0$

BC ① $U(0,t) = 0$ $\quad t > 0$

BC ② $U(\pi,t) = 0$ $\quad t > 0$

IV FS ③ $U_t(x,0) = 0$

IV FS ④ $U(x,0) = f(x) = \begin{cases} x & 0 \le x \le \pi/2 \\ \pi - x & \pi/2 \le x \le \pi \end{cases}$

Here in the question X contains SL.

Which variable gives sturm Liouville "DO not" put constant that side.

Assume $U(x,t) = X(x) \cdot T(t)$

$U = XT$

so, $U_x = X'T$ $\quad\quad U_t = XT'$

$U_{xx} = X''T$ $\quad\quad U_{tt} = XT''$

Substituting these to the wave PDE,

$$9X''T = XT''$$

$$\Rightarrow \frac{X''}{X} = \frac{T''}{9T} = \lambda \qquad [\lambda \text{ is a constant}]$$

Now, $\left.\begin{aligned}\frac{X''}{X} &= \lambda \\ \frac{T''}{9T} &= \lambda\end{aligned}\right\}$ Homogeneous ODE   2nd order

Now, $\frac{X''}{X} = \lambda$ $U = XT$

$\Rightarrow X'' = \lambda X$

BC: $U(0,t) = 0 \longrightarrow X(0)T(t) = 0 \longrightarrow X(0) = 0$

BC: $U(\pi,t) = 0 \longrightarrow X(\pi)T(t) = 0 \longrightarrow X(\pi) = 0$

$T(t) \neq 0$

So, $X(0) = X(\pi) = 0$

Dirichlet BC, $X'' = \lambda X$. $X(0) = X(L) = 0$, where $L = \pi$

Hence, $\lambda_n = -\left(\frac{n\cancel{\pi}}{\cancel{\pi}}\right)^2 = -n^2$

$$X_n = \sin\left(\frac{n\pi x}{\pi}\right) = \sin(nx)$$

$$n = 1, 2, 3, \ldots\ldots$$

Again, $\frac{T''}{9T} = \lambda$

$$\Rightarrow T'' - 9T\lambda = 0$$

characteristic polynomial, $m^2 - 9\lambda = 0$

$$\Rightarrow m = \pm 3\sqrt{\lambda}$$

$$= \pm 3\sqrt{-n^2} \qquad [\lambda = -n^2]$$

$$= \pm 3in$$

$$\alpha = 0, \quad \beta = 3n$$

So, $T_n(t) = A_n \cos(3nt) + B_n \sin(3nt)$

$$U_n = X_n T_n = \sin(nx)\left[A_n \cos(3nt) + B_n \sin(3nt)\right]$$

$$U(x,t) = \sum_{n=1}^{\infty} \sin(nx)\left[A_n \cos(3nt) + B_n \sin(3nt)\right]$$

We don't need to put any constant C_n, because A_n, B_n two constant are already there, and const. $\times$ const. $=$ const.

IV: $U_t(x,0) = 0$

So, $U_t(x,t) = \sum_{n=1}^{\infty} \sin(nx)\left[-3nA\sin(3nt) + 3nB_n\cos(3nt)\right]$

Hence, plug in $t=0$ to get $U_t(x,0)$

So, $U_t(x,0) = \sum_{n=1}^{\infty} 3nB_n \sin(nx) = 0$

Now, Fourier Sine Series of $f(x)=0$ on $[0,\pi]$ is 0.

Hence, By comparing we get,

$$3nB_n = 0$$

$$\Rightarrow B_n = 0$$

If we calculate

$b_n = \frac{1}{\pi}\int 0 \ldots$

$[\text{const}]_0^{\pi} = 0$

* we will get $b_n = 0$

Therefore, $U(x,t) = \sum_{n=1}^{\infty} A_n \sin(nx)\cos(3nt)$

$A_n = ?$

IV: $U(x,0) = f(x) = \begin{cases} x & 0 \le x \le \pi/2 \\ \pi - x & \pi/2 \le x \le 0 \end{cases}$

So, $U(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin(nx)$ —— (I)

Now, Fourier Sine series for $f(x)$ is given,

$$f(x) = \sum_{n=1}^{\infty} \frac{4(-1)^{n-1}}{\pi(2n-1)^2} \sin\{(2n-1)x\}$$ —— (II)

Comparing (I) & (II),

$$A_{2n-1} = \frac{4(-1)^{n-1}}{\pi(2n-1)^2}$$

$$A_{2n} = 0$$

Therefore,

$$U(x,t) = \sum_{n=1}^{\infty} \frac{4(-1)^{n-1}}{\pi(2n-1)^2} \sin\{(2n-1)x\} \cdot \cos\{3(2n-1)t\}$$

$5 < a < 35$ min

for final exam.

Mohammad Abidul Basir

16-Aug-2021.