# GENG 8010–Part 1: Elements of Differential and Difference Equations

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## Nature of solution(s)– VI

**Example**–Consider

$$\frac{dy}{dx} = 4y$$

verify that $y = e^{4x}$ is a solution.

Now verify that $-5e^{4x}$ and $6e^{4x}$ are also solutions. In fact, $Ke^{4x}$, where $K$ is any constant is a solution.





## Nature of solution(s)– VII

**Remark**

To solve a differential equation try to find the complete family of solutions, expressed in the form of a general solution (e.g. $Ke^{4x}$) above involving one or more arbitrary constants. If you choose a particular value for each arbitrary constant in the general solution of a differential equation, then you obtain a specific solution (e.g.$-5e^{4x}$ and $6e^{4x}$ above) of the differential equation.





## Nature of solution(s)– VIII

based on the above and Theorem 4.1.2, for (4.1), the solution is

$$y_c = K_1 y_1 + K_2 y_2 + \ldots + K_n y_n \tag{4.2}$$

where $K_i$'s are arbitrary constants and $y_c$ denotes the homogeneous or complementary solution of the differential equation.

## Nature of solution(s)– IX

### General solution of differential equation

The **complete solution** of the non-homogeneous differential equation (1.1) is

$$y(t) = y_c + y_p$$

where $y_c$ called **complementary solution** (or sometimes $y_H$ called the **homogeneous solution**) is the solution of the homogeneous equation given by (4.2). $y_p$ is any solution that would satisfy (1.1), and is referred to as the **particular solution** of the differential equation. Any solution for $y_p$ that you can come up with, including guessed solution, if they satisfy (1.1) are allowable.





## Homogeneous equations of $2^{nd}$ order – I

Consider the $2^{nd}$ order homogeneous differential equation

$$(a_2 D^2 + a_1 D + a_0)y(t) = 0 \tag{5.1}$$

Assume a solution of the form

$$y(t) = Ae^{mt}$$

where $A$ & $m$ are constants to be determined.

Substitution gives:

$$Ae^{mt}(a_2 m^2 + a_1 m + a_0) = 0$$





## Homogeneous equations of $2^{nd}$ order – II

$$Ae^{mt}(a_2 m^2 + a_1 m + a_0) = 0$$

- The above has to be satisfied if $y(t) = Ae^{mt}$ is the solution.
- $e^{mt}$ can never be zero
- $A = 0$ implies $y = 0$ which is a trivial solution

So, we conclude that for the above to hold, we must have

$$\boxed{a_2 m^2 + a_1 m + a_0 = 0}$$





## Homogeneous equations of $2^{nd}$ order – III

The above is called **Characteristic or Auxiliary Equation** since its roots *characterize* the nontrivial solutions of diff. eq.

If we let $D$ play the role of $m$, then the characteristic equation for the second order diff. eq. can be written as

$$aD^2 + bD + c = 0$$

which leads to one of the three forms of solutions

## Roots are distinct I

**Case 1–** The roots $m_1$ and $m_2$ are distinct.

In this case, $y_1(t) = e^{m_1 t}$ and $y_2(t) = e^{m_2 t}$, $m_1 \neq m_2$, $\forall t$ and

$$W = \begin{vmatrix} e^{m_1 t} & e^{m_2 t} \\ m_1 e^{m_1 t} & m_2 e^{m_2 t} \end{vmatrix} = (m2 - m_1)e^{(m_1+m_2)t} \neq 0$$

Hence, the complementary solution is of the form

$$\boxed{y(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}}$$





## Roots are repeated I

Case 2– The roots are repeated at $m$.

In this case, one solution is

$$y_1 = e^{mt}$$

How do we get another independent solution?





## Roots are repeated II

If we have two roots at $m$ then the characteristic equation must be

$$(aD^2 + bD + c) = (D - m)^2 = 0$$

meaning that $a = 1$, $b = -2m$ and $c = m^2$.

- Suppose that we know $y_1$ is one solution. Therefore,

$$(aD^2 + bD + c)y_1 = 0$$

- The method of **Reduction of Order** allows us to find another independent solution.





## Roots are repeated III

Let $y(t) = y_2(t) = k(t)y_1(t)$, then

$$Dy = y' = k'y_1 + ky_1'$$

$$D^2y = y'' = k''y_1 + 2k'y_1' + ky_1''$$

Substituting into the diff. eq.

$$\begin{aligned} (aD^2 + bD + c)y &= 0 \\ ak''y_1 + 2ak'y_1' + aky_1'' + bk'y_1 + bky_1' + cky_1 &= 0 \\ \cancelto{0}{(ay_1'' + by_1' + cy_1)}k + y_1ak'' + (2ay_1' + by_1)k' &= 0 \end{aligned}$$

## Roots are repeated IV

Letting $v = k^{'}$ reduces the remaining second order diff. eq. into a first order one given by

$$ay_1v^{'} + (2ay_1^{'} + by_1)v = 0$$

dividing by $(ay_1)$ gives

$$v^{'} + \left(\frac{2}{y_1}y_1^{'} + p\right)v = 0 \qquad where \quad p = \frac{b}{a} = -2m$$





## Roots are repeated V

or by separation of variables

$$\frac{dv}{v} = -\left(\frac{2}{y_1}y_1^{'} + p\right)dt$$

integrating gives

$$\ln|v| = -2ln|y_1| - \int pdt \Longrightarrow \ln|v| + 2ln|y_1| = -\int pdt$$

or

$$v = \frac{1}{y_1^2}e^{-\int pdt}$$





## Roots are repeated VI

Recall $v = k^{'}$, hence,

$$k = \int vdt = \int \frac{1}{y_1^2}e^{-\int pdt}dt$$

$$k = \int \frac{e^{2mt}}{(e^{mt})^2}dt = t$$

again, recall that $y = y_2 = ky_1$, therefore the second solution of the differential equation is

$$y_2 = y = k(t)y_1(t) = te^{mt}$$

Clearly $y_1$ and $y_2$ are linearly independent (check $W \neq 0$) solutions, and hence,

$$\boxed{y(t) = c_1e^{mt} + c_2te^{mt}}$$





## Complex conjugate pair I

Case 3– The roots are a complex conjugate pair at

- $m_1 = p + iq$
- $m_2 = p - iq$

As in the Case 1,

- $y_1 = e^{(p+iq)t}$
- $y_2 = e^{(p-iq)t}$

are solutions, and

$$y = c_1e^{(p+iq)t} + c_2e^{(p-iq)t}$$

# Complex conjugate pair II

**Remark**

- Note that these are complex and not real valued functions. Therefore we would need complex variable theory and need to take a wider view of the solution.
- Also, the form of the solution above involves complex exponentials which are not easy to comprehend.
- We will continue to require all the solutions to be real valued functions, and in forming their linear combinations, we use real-valued parameters.





# Complex conjugate pair III

Recall the **Euler Formula**

$$e^{i\theta} = \cos\theta + i\sin\theta$$

where $\theta$ is any real number. Therefore

$$e^{iqt} = \cos qt + i\sin qt \qquad e^{-iqt} = \cos qt - i\sin qt \tag{5.2}$$

where $\cos(-qt) = \cos qt$ and $\sin(-qt) = -\sin qt$.

Now using (5.2)





# Complex conjugate pair IV

$$y_1 = e^{pt}e^{iqt} = e^{pt}(\cos qt + i\sin qt)$$
$$y_2 = e^{pt}e^{-iqt} = e^{pt}(\cos qt - i\sin qt)$$

multiplying by $\frac{1}{2}$ and adding,

$$y = e^{pt}\cos qt$$

multiplying by $\frac{1}{2i}$ and subtracting,

$$y = e^{pt}\sin qt$$

This and Corollary 4.1.3 (1) indicate that the above are real solutions of (5.1).





# Complex conjugate pair V

To verify, find the characteristic polynomial of with the complex conjugate roots, i.e.,

$$(D - p - iq)(D - p + iq) = D^2 - 2pD + (p^2 + q^2) = 0$$

the above implies that

$$y'' - 2py' + (p^2 + q^2)y = 0 \qquad q \neq 0$$

and verify that $y_1 = e^{pt}\cos qt$ and $y_2 = e^{pt}\sin qt$ are solutions.

# Complex conjugate pair VI

$$
\begin{aligned}
(p^2+q^2)y_1 &= (p^2+q^2)e^{pt}\cos qt\\
-2py_1^{'} &= -2p^2e^{pt}\cos qt + 2pqe^{pt}\sin qt\\
y_1^{''} &= (p^2-q^2)e^{pt}\cos qt - 2pqe^{pt}\sin qt\\
(p^2+q^2)y_2 &= (p^2+q^2)e^{pt}\sin qt\\
-2py_2^{'} &= -2p^2e^{pt}\sin qt - 2pqe^{pt}\cos qt\\
y_2^{''} &= (p^2-q^2)e^{pt}\sin qt + 2pqe^{pt}\cos qt
\end{aligned}
$$

Now if we add the right and left of each three equations involving $y_1$ and $y_2$, respectively, we get





# Complex conjugate pair VII

$$y_1^{''} - 2py_1^{'} + (p^2+q^2)y_1 = 0 \qquad y_2^{''} - 2py_2^{'} + (p^2+q^2)y_2 = 0$$

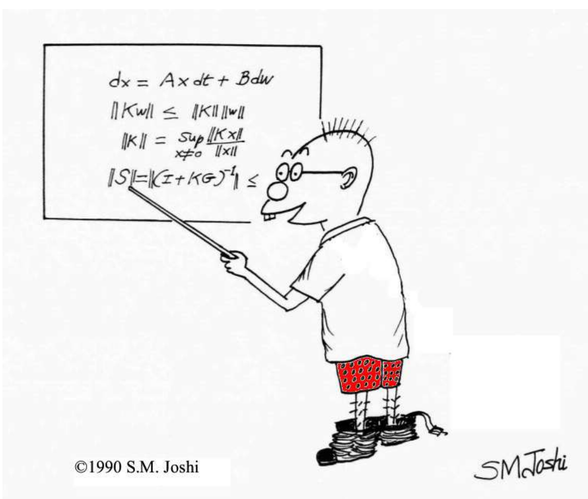
Therefore the general solution is

$$\boxed{y(t) = e^{pt}\left(c_1\cos qt + c_2\sin qt\right)}$$

where $c_1$ and $c_2$ are real parameters.








*"...and so we come to the bottom line!"*





# Summary–solutions of HE of $2^{nd}$ order

| Roots of CE | Coefficients of CE | Complete Solution |
|---|---|---|
| $m_1 \neq m_2$ | $b^2-4ac>0$ | $y = c_1e^{m_1t}+c_2e^{m_2t}$ |
| $m_1 = m_2 = m$ | $b^2-4ac=0$ | $y = c_1e^{mt}+c_2te^{mt}$ |
| $m_{1,2} = p \pm iq$ | $b^2-4ac<0$ | $y = e^{pt}\left(c_1\cos qt + c_2\sin qt\right)$ |

Table 1: Possible solutions for $2nd$ order homogeneous differential equation $(aD^2+bD+c)y(t)=0$

## Examples

**Ex**– Find the solution of $(4D^2 + 16D + 17)y(t) = 0$ for $y(0) = 1$, and $y(\pi) = 0$.

Verify that the roots of the CE are $m1, 2 = -2 \pm \frac{1}{2}i$. Therefore,

$$y(t) = e^{-2t}\left(c_1 \cos\frac{t}{2} + c_2 \sin\frac{t}{2}\right)$$

To find the constants, use the boundary conditions to get $1 = c_1$, and $0 = e^{-2\pi}c_2$ or $c_2 = 0$, and finally

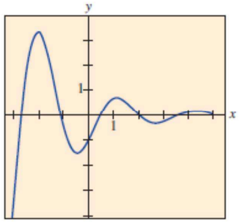
$$\boxed{y(t) = e^{-2t}\cos\frac{t}{2}}$$





## Examples

**Ex**– Find the solution of $(4D^2 + 4D + 17)y(x) = 0$ for $y(0) = -1$, and $y'(0) = 2$.




Verify that the roots of the CE are $m_{1,2} = -\frac{1}{2} \pm 2i$. Therefore,

$$y(x) = e^{-\frac{1}{2}x}\left(c_1 \cos 2x + c_2 \sin 2x\right)$$

for $y(0) = -1$ we get $e^0(c_1 \cos 0 + c_2 \sin 0) = -1$ and $c_1 = -1$.

Differentiating $y(x)$ and then using $y'(0) = 2$ gives $2c_2 + \frac{1}{2} = 2$ or $c_2 = \frac{3}{4}$, and

$$y(x) = e^{-\frac{1}{2}x}\left(-\cos 2x + \frac{3}{4}\sin 2x\right)$$

You can see that the solution is oscillatory but decaying.





## Homogeneous equation

Consider the $n^{th}$ order homogeneous differential equation

$$(a_n D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_0)y(t) = 0$$

Assume the solutions are of the form $y = e^{rt}$ where $r$ is a constant to be determined. Then

$$(a_n r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0)e^{rt} = 0$$

and that implies $\forall t$,

$$a_n r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0 = 0$$

Again the above $n^{th}$ order polynomial with roots $r_1, r_2, \ldots, r_n$ is called the characteristic or auxiliary equation of the system, and the solutions are

$$y_1 = e^{r_1 t}, y_2 = e^{r_2 t}, \ldots, y_n = e^{r_n t}$$





## Homogeneous solution–cont.

As a result of the above, the most general solution of the DE is

$$\boxed{y_c(t) = K_1 e^{r_1 t} + K_2 e^{r_2 t} + \ldots + K_n e^{r_n t}} \qquad \text{(HS)}$$

Again, as in the case of the $2nd$ order equation **three cases** can arise: