GENG 8010–Part 1: Elements of Differential and Difference Equations

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Part I–Outline I

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Part I–Outline II

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Part I–Outline III

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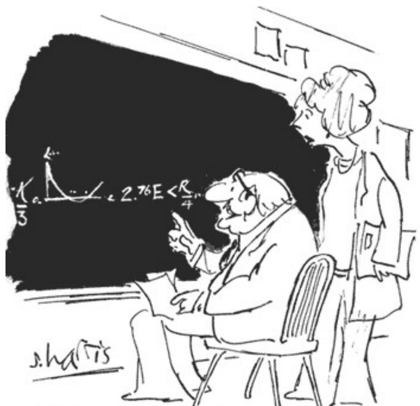
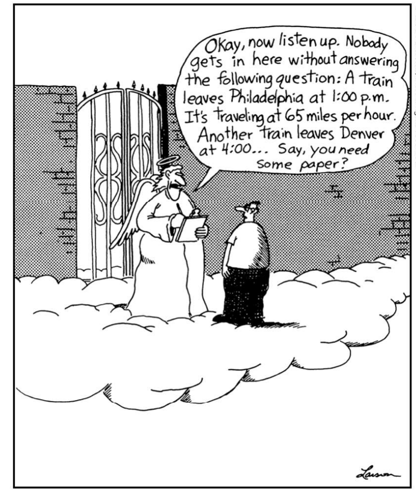
Introduction & definitions I



"The beauty of this is that it is only of theoretical importance, and there is no way it can be of any practical use whatsoever."

Introduction & definitions II



Math phobic's nightmare

Introduction & definitions III

Many physical systems' behaviors in science and engineering are described with differential or partial differential equations.

Differential equations

- An equation relating an unknown dependent function and one or more of its derivatives with respect to an independent variable is called a **differential equation**.
- If the DE contains only ordinary derivatives of one or more functions with respect to a single independent variable, then the DE is called to be an **ordinary differential equation**.
- If the DE involves partial derivatives of one or more functions of two or more independent variables, then it is called a **partial differential equation**.

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Ordinary vs. Partial DE

- Ordinary: dy, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\dot{y}$, $\ddot{y}$, dx
- Partial: $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial x \partial y}$, u_{xx}, u_{xy}

Order of DE

Order of a differential equation is defined by the order of the highest derivative in the equation.

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Types of variables

- Variables that denote values of a function are often called the **dependent variables**.
- An **independent variable** is one that may take on any values in the domain of the function which the dependent variables stands for.

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Example–Consider the following differential equations

$$\frac{dy}{dx} = e^{2x} + \cos x \tag{a}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t} \tag{b}$$

$$y^{''} - 2y^{'} + y = \sin x \tag{c}$$

$$4x^3 dx - 3y dy = 0 \tag{d}$$

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Example–In the previous example

- In (a,c), y is the dependent variable and x is the independent variable.
- In (b), u is the dependent variable and, x, y and t are the independent variables.
- in (d) either x or y can be thought of the dependent variable and then the other would be the independent variable.

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Example–

$$\frac{dy}{dx} = 3y \tag{e}$$

or

$$\frac{d^2y}{dx^2} - 6x\frac{dy}{dx} + 3xy = cos(x) \tag{f}$$

In the above $y(x)$ is a function of x. Hence

- y is dependent variable
- x is independent variable

The order of the differential equation is the order of the highest derivative that appears in the equation. So

- Equation (e) is first order
- Equation (f) is a second order differential equation

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Differential Form

A first order differential equation in **differential form**

$$M(x,y)dx + N(x,y)dy = 0$$

Example–Consider

$$(2y+3x)dx + 2dy = 0$$

by assuming that y is the dependent variable and the fact that differential dy is defined as $dy = y^{'}dx$, we get

$$\frac{dy}{dx} + y + \frac{3}{2}x = 0$$

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General vs. Normal Form

- **General Form** of an *nth* order ordinary equation in one dependent variable
$$F(x,y,y^{'},y^{''},\cdots,y^{(n)}) = 0$$
- Assuming that it is possible to solve for the highest derivative. Then the **Normal Form** of the differential equation is

$$\frac{d^n y}{dx^n} = f(x,y,y^{'},\cdots,y^{(n-1)})$$

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Linear vs. Non-Linear DE

Linearity is a property of differential equations which relates to the relationship of the function to its derivatives. For our purposes, **linearity is not affected by anything happening to the independent variable**

- Linear terms: $t\dot{y}, t^3y, t^2\ddot{y}, cos(t)\dot{y}, e^{-2t}\ddot{y}$.
- Nonlinear terms: $y^3, y\dot{y}, sin(y)y, uu_y, u_tu_y$.

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Example–Consider the following equations

$$y^{''} + 5xy^{'} - 8y = \sin x \qquad \text{Linear despite the term } xy^{'}$$

$$y^{''} + 4yy^{'} - 10y = \cos x \qquad \text{nonlinear because of } yy^{'}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial t} + u + v = \sin u$$

This last equation is linear in v but nonlinear in u because of $\sin u$ so the equation is nonlinear.

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“ ... and according to the legend, there are many, many applications at the other end of the rainbow!”

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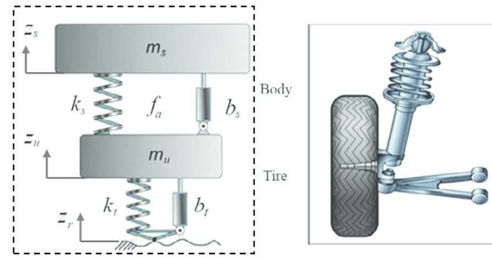
Quarter car active suspension system



The equations of motion for this system are

$$m_s\ddot{z}_s = -b_s\left(\dot{z}_s - \dot{z}_u\right) - k_s\left(z_s - z_u\right) + f_a$$
$$m_u\ddot{z}_u = b_s\left(\dot{z}_s - \dot{z}_u\right) + k_s\left(z_s - z_u\right) - f_a + b_t\left(\dot{z}_r - \dot{z}_u\right) + k_t\left(z_r - z_u\right)$$

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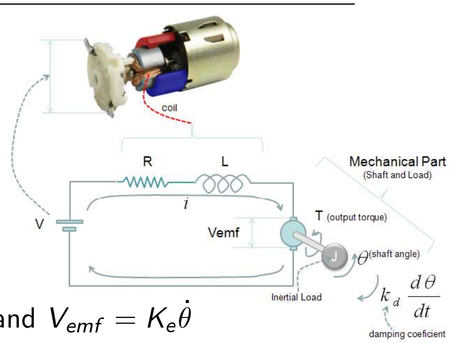
Electric (DC) motor driving an inertial load



Note $T = K_i i(t)$ and $V_{emf} = K_e\dot{\theta}$

$$V = Ri + L\frac{di}{dt}V_{emf} \Longrightarrow \frac{di}{dt} = -\frac{R}{L}i(t) - \frac{K_e}{L}\omega(t) + \frac{1}{L}V$$
$$J\frac{d^2\theta}{dt^2} = K_i i - K_d\frac{d\theta}{dt} \Longrightarrow \frac{d\omega}{dt} = -\frac{1}{J}K_d\omega(t) + \frac{1}{J}K_i i(t)$$

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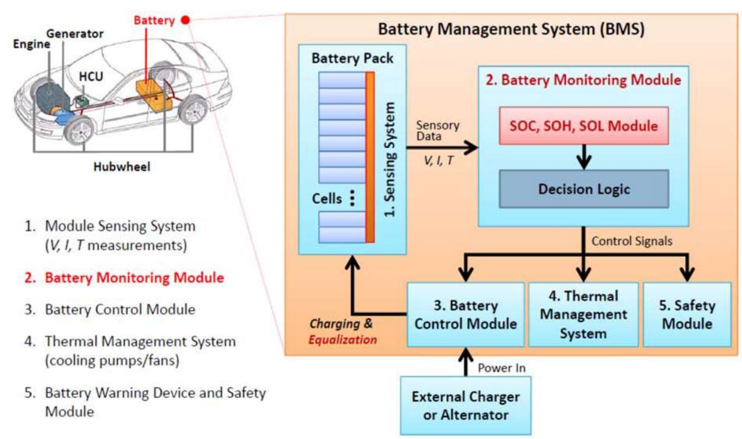
Li-ion battery in electrified vehicles



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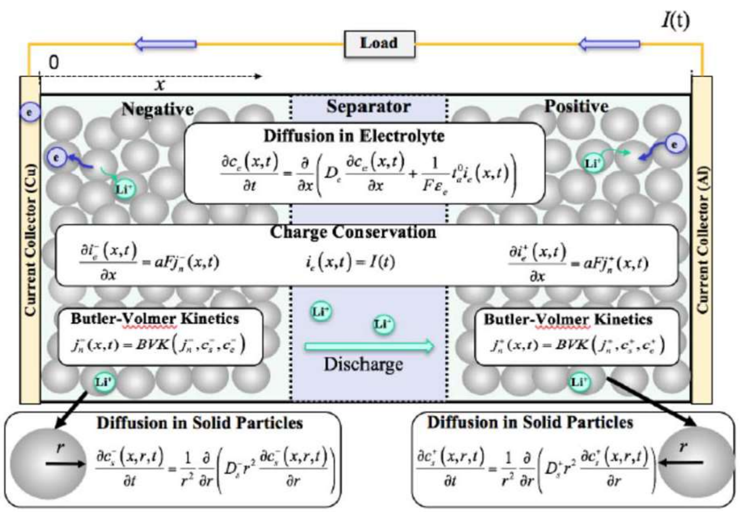

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Simplified multi-story shear building model

A multi-degrees of freedom model of a building where only the x and y horizontal sway motions and the z vertical motion of each floor is considered.

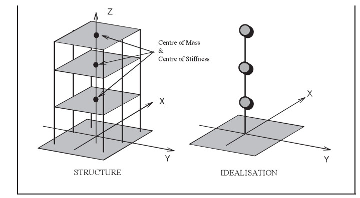

Figure 1: Multi-storey shear building

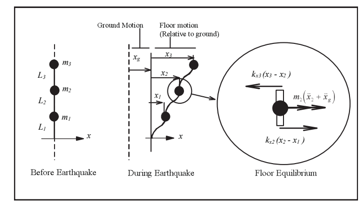

Figure 2: Horizontal motion

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Horizontal Equations of Motion

The forces on each floor must balance. Hence the system of equations for the overall three stories would be

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}_x\boldsymbol{x} = -\boldsymbol{M}\boldsymbol{1}\ddot{x}_g$$

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} k_{x_1} + k_{x_2} & -k_{x_2} & 0 \\ -k_{x_2} & k_{x_2} + k_{x_3} & -k_{x_3} \\ 0 & -k_{x_3} & k_{x_3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -\boldsymbol{M} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \ddot{x}_g$$

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Vertical Equations of Motion

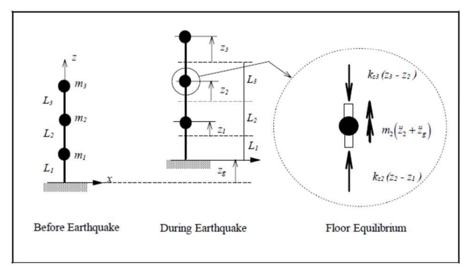

Figure 4: Vertical Motion

$$\boldsymbol{M}\ddot{\boldsymbol{z}} + \boldsymbol{K}_z\boldsymbol{z} = -\boldsymbol{M}\boldsymbol{1}\ddot{z}_g$$