GENG 8010–Part 2 - Elements of Applied Linear Algebra

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Outline of part 2– I

Outline of part 2– II

Outline of part 2– III

Basics of linear algebra– I

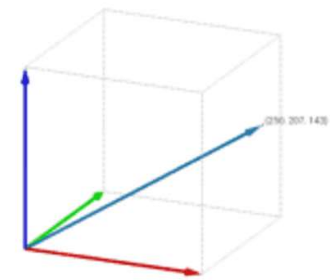
- Single numbers such as $a, x, b, u, v, \ldots$ are **scalars**.
- Vectors are 1D arrays that have scalar elements that could be real, complex, integers, etc.

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \text{in the case where } x_i \text{ are real we denote } \boldsymbol{x} \in \Re^n$$

The above is a column vector but we may have a row vector as well

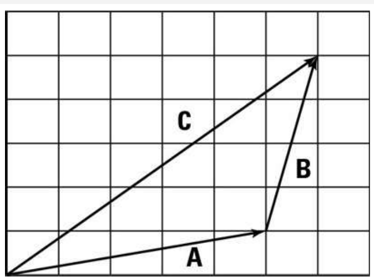
$$\boldsymbol{r} = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix}$$

Basics of linear algebra– II



Vectors can represent any quantity of interest such as color, displacement, time, daily temperature, hourly load demand on a power system, speech, etc.

Basics of linear algebra– III



Where

$$\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

Basics of linear algebra– IV

- An n vector with all of its entries being 1 is represented as $\mathbf{1}_n$ or just $\mathbf{1}$.
- Unit vectors have one entry of 1 and rest are 0, e.g.,

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix};\ \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix};\ \boldsymbol{e}_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Note that $\boldsymbol{e}_i^T \boldsymbol{x}$ picks the *ith* entry of $\boldsymbol{x}$.

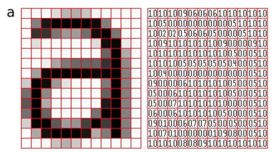
Similarly, $\mathbf{1}^T \boldsymbol{x} = x_1 + x_2 + \cdots + x_n$ gives sum of the entries.

Basics of linear algebra– V

- A **matrix** is a 2D array of numbers/scalars with m rows and n columns

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{if } a_{ij} \text{ are real } \boldsymbol{x} \in \Re^{mxn}$$

where a_{ij} is the row ith and column jth element of the matrix. Matrices can represent variety of things, e.g. an image



Basics of linear algebra– VI

A matrix $\boldsymbol{A} \in \Re^{mxn}$ is

- **Tall** if $m > n$.
- **Long** if $m < n$.
- **Square** if $m = n$.

- If two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same dimension, then

$$\boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

Further $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$ and $(\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C} = (\boldsymbol{A} + \boldsymbol{C}) + \boldsymbol{B} = \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C})$.

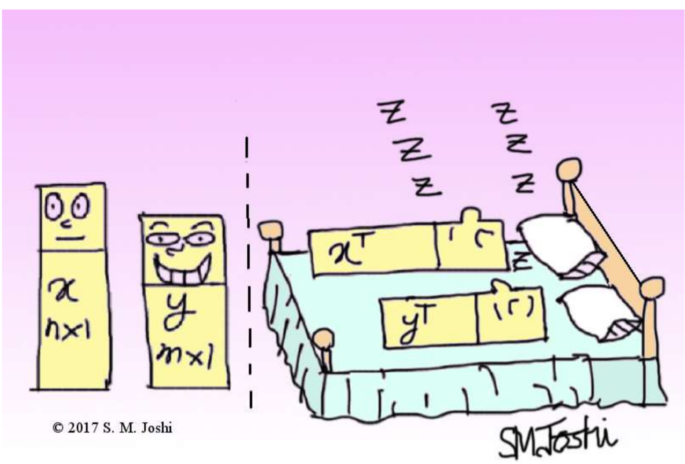
Basics of linear algebra– VII

- For a matrix $\boldsymbol{A} \in \Re^{mxn}$ **transpose** of $\boldsymbol{A}$ is another matrix denoted by $\boldsymbol{A}^T \in \Re^{nxm}$ and is defined according to the following

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\boldsymbol{A}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

Basics of linear algebra– VIII



x and y transpose themselves to get some z's

Basics of linear algebra– IX

- Consider the following

$$\boldsymbol{Pa} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \\ a_{51} \end{bmatrix} = \begin{bmatrix} a_{21} \\ a_{41} \\ a_{11} \\ a_{51} \\ a_{31} \end{bmatrix}$$

The matrix $\boldsymbol{P}$ is a **Permutation Matrix**. It permutes the rows of another matrix. The row 1 is replaced by row 2, row 2 by row 1, row 3 by row 4, row 4 by row 5, and row 5 by row 3.

Basics of linear algebra– X

On the other hand

$$\boldsymbol{aP} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} a_{13} & a_{11} & a_{15} & a_{12} & a_{14} \end{bmatrix}$$

In this case $\boldsymbol{P}$ permutes the column of another matrix where the column 1 is replaced by column 3, and so on.

For non-square matrices permutations $\boldsymbol{P}$ and $\boldsymbol{Q}$ can be found that would do the same.

Basics of linear algebra– XI

Example–consider

$$\boldsymbol{A} = \begin{bmatrix} 1 & 8 & -2 \\ -1 & 1 & 4 \\ 3 & 9 & 6 \\ -2 & 1 & 0 \end{bmatrix};\ \boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix};\ \boldsymbol{Q}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Basics of linear algebra– XII

$$\boldsymbol{PAQ}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 8 & -2 \\ -1 & 1 & 4 \\ 3 & 9 & 6 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 4 \\ 1 & 8 & -2 \\ -2 & 1 & 0 \\ 3 & 9 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 1 & -1 \\ -2 & 8 & 1 \\ 0 & 1 & -2 \\ 6 & 9 & 3 \end{bmatrix}$$

Basics of linear algebra– XIII

Suppose now that we wish to further change the 2nd column of the resulting matrix with its third column, so

$$\begin{bmatrix} 4 & 1 & -1 \\ -2 & 8 & 1 \\ 0 & 1 & -2 \\ 6 & 9 & 3 \end{bmatrix} \boldsymbol{Q}_2 = \begin{bmatrix} 4 & 1 & -1 \\ -2 & 8 & 1 \\ 0 & 1 & -2 \\ 6 & 9 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 1 \\ -2 & 1 & 8 \\ 0 & -2 & 1 \\ 6 & 3 & 9 \end{bmatrix}$$

$$\boldsymbol{Q} = \boldsymbol{Q}_1 \boldsymbol{Q}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Basics of linear algebra– XIV

Now if we go back and compute

$$\boldsymbol{PAQ} = \begin{bmatrix} -1 & 1 & 4 \\ 1 & 8 & -2 \\ -2 & 1 & 0 \\ 3 & 9 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 1 \\ -2 & 1 & 8 \\ 0 & -2 & 1 \\ 6 & 3 & 9 \end{bmatrix}$$

which is the same result as before.

Basics of linear algebra– XV

- A matrix $\boldsymbol{A} \in \Re^{nxn}$ is **symmetric** if $\boldsymbol{A} = \boldsymbol{A}^T$.
  If $\boldsymbol{A} \in \Re^{mxn}$ and $\boldsymbol{B} \in \Re^{nxp}$, and α is a constant scalar, then
  - $(\boldsymbol{A}^T)^T = \boldsymbol{A}$.
  - $(\alpha \boldsymbol{A})^T = \alpha \boldsymbol{A}^T$.
  - $(\boldsymbol{AB})^T = \boldsymbol{B}^T \boldsymbol{A}^T$.
  - $(\boldsymbol{A}+\boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$
- To get $\alpha \boldsymbol{A}$ where α is a scalar, all elements of $\boldsymbol{A}$ are multiplied by the scalar α.
- If $\boldsymbol{A}$ is nxm and $\boldsymbol{B}$ is mxp, then

$$\boldsymbol{AB} = \boldsymbol{C} \quad \text{where } c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

Basics of linear algebra– XVI

Example–consider

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\boldsymbol{AB} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} \end{bmatrix}$$

Note that when matrices commute, $\boldsymbol{AB} \neq \boldsymbol{BA}$. However, $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$.

Basics of linear algebra– XVII

- The **scalar product, or inner product** of two vectors $\boldsymbol{x}$, and $\boldsymbol{y} \in (\mathbb{C}^n, \mathbb{C})$, is a complex number denoted

$$< \boldsymbol{x}, \boldsymbol{y} >= \boldsymbol{x}^* \boldsymbol{y} = \bar{x}_1 y_1 + \bar{x}_2 y_2 + \cdots + \bar{x}_n y_n$$

where $\boldsymbol{x}^*$ is the complex conjugate transpose of $\boldsymbol{x}$ and $\bar{x}_i$ refers to complex conjugate of the ith element of $\boldsymbol{x}$. The above have the following properties:

1. $< \overline{\boldsymbol{x}, \boldsymbol{y}} >=< \boldsymbol{y}, \boldsymbol{x} >$
2. $< \alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2, \boldsymbol{y} >= \bar{\alpha}_1 < \boldsymbol{x}_1, \boldsymbol{y} > + \bar{\alpha}_2 < \boldsymbol{x}_2, \boldsymbol{y} >$
3. $< \boldsymbol{x}, \boldsymbol{x} > > 0 \ \forall \ \boldsymbol{x} \neq 0$

Basics of linear algebra– XVIII

Note that (1) and (2) above imply

$$< \boldsymbol{x}, \alpha \boldsymbol{y} >=< \overline{\alpha \boldsymbol{y}, \boldsymbol{x}} >= \bar{\bar{\alpha}} < \overline{\boldsymbol{y}, \boldsymbol{x}} >= \alpha < \boldsymbol{x}, \boldsymbol{y} >$$

For any square matrix $\boldsymbol{A}$

$$< \boldsymbol{x}, \boldsymbol{A}\boldsymbol{y} >= \boldsymbol{x}^* \boldsymbol{A}\boldsymbol{y}$$

$$< \boldsymbol{A}^* \boldsymbol{x}, \boldsymbol{y} >= (\boldsymbol{A}^* \boldsymbol{x})^* \boldsymbol{y} = \boldsymbol{x}^* \boldsymbol{A}\boldsymbol{y}$$

Therefore,

$$\boxed{< \boldsymbol{x}, \boldsymbol{A}\boldsymbol{y} >=< \boldsymbol{A}^* \boldsymbol{x}, \boldsymbol{y} >}$$

Basics of linear algebra– XIX

Orthogonality

- Two vectors $\boldsymbol{x}$, and $\boldsymbol{y}$ are **orthogonal** iff $< \boldsymbol{x}, \boldsymbol{y} >= 0$.
- A set of vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n$ are **orthonormal** iff

$$< \boldsymbol{x}_i, \boldsymbol{x}_j >= \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq \text{j} \end{cases}$$

Basics of linear algebra– XX

Example–consider

$$\boldsymbol{a} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$$

$$< \boldsymbol{a}, \boldsymbol{b} >= \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} = 0 \quad \text{vectors are orthogonal}$$

$$< \boldsymbol{a}, \boldsymbol{a} >= \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 2 \quad \text{they are not orthonormal}$$

Basics of linear algebra– XXI

Example–

In ML or statistical applications, $\boldsymbol{w}$ often is a weight vector and $\boldsymbol{x}$ is a feature vector whose elements are called **regressors**, then $<\boldsymbol{w}, \boldsymbol{x}>= \boldsymbol{w}^T\boldsymbol{x}$ is the score. A regression model can have a form (an affine form) of

$$\hat{\boldsymbol{x}} = \boldsymbol{x}^T\boldsymbol{w} + \boldsymbol{v}$$

where $\boldsymbol{v}$ is called an *offset*.

Basics of linear algebra– XXII

Norm ($\|.\|$) is a measure of size of a vector.

- L_p norm is

$$\| \boldsymbol{x} \|_p = \left(\sum_i |x_i|^p\right)^{\frac{1}{p}}$$

- L_2 norm is a common norm defined $\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x}^T\boldsymbol{x}} = \sqrt{\sum_{i=1}^n x_i^2}$
- L_1 norm is simply $\|\boldsymbol{x}\|_1 = \sum_i |x_i|$
- Max norm or infinity norm is $\|\boldsymbol{x}\|_\infty = \max_i |x_i|$

Basics of linear algebra– XXIII

Some properties of norm

- $\|\boldsymbol{x}\| > 0$ and $\|\boldsymbol{x}\| = 0 \implies \boldsymbol{x} = 0$ (positivity).
- $\|\alpha\boldsymbol{x}\| = |\alpha|\|\boldsymbol{x}\|$ (scaling)
- *Triangle inequality–* $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$
- *mean square value* of a vector is

$$\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n} = \frac{\|\boldsymbol{x}\|^2}{n}$$

- *Root-mean square (RMS)* (useful for comparing size of vectors of different length) of a vector is

$$\boldsymbol{x}_{rms} = \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}} = \frac{\|\boldsymbol{x}\|}{\sqrt{n}}$$

Basics of linear algebra– XXIV

- $\boldsymbol{A} \in \Re^{nxn}$ is **invertible** iff $\exists \boldsymbol{B} \ni \boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$. In that case $\boldsymbol{A}^{-1} = \boldsymbol{B}$. additionally
    1. $(\alpha\boldsymbol{A})^{-1} = \frac{1}{\alpha}\boldsymbol{A}^{-1}$.
    2. $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.
    3. $(\boldsymbol{A}^T)^{-1} = (\boldsymbol{A}^{-1})^T$
    4. A 2x2 matrix is invertible iff $a_{11}a_{22} \neq a_{12}a_{21}$ in which case $\boldsymbol{A}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$
    5. $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$ (assuming that the inverse exist)
    6. $(\boldsymbol{A}^T)^{-1} = (\boldsymbol{A}^T)^{-1}$
    7. $\boldsymbol{A}^0 = \boldsymbol{I}$, and $\boldsymbol{A}^m\boldsymbol{A}^n = \boldsymbol{A}^{m+n}$.
    8. $(\boldsymbol{A}^{-1})^k = (\boldsymbol{A}^k)^{-1}$ to see this note $\boldsymbol{A}^{-2} = \boldsymbol{A}^{-1}\boldsymbol{A}^{-1} = (\boldsymbol{AA})^{-1}$ and so on it can be showed for higher powers.

Basics of linear algebra– XXV

- **Trace** of a matrix is defined as

$$tr(\boldsymbol{A}) = \sum_i a_{ii} \qquad tr(\boldsymbol{AB}) = tr(\boldsymbol{BA})$$

- Determinant is a means to determine if a matrix is invertible, or non-singular.
    - $\det(\alpha \boldsymbol{A}) = \alpha^n \det \boldsymbol{A}$
    - $\det \boldsymbol{A}^T = \det \boldsymbol{A}$
    - $\det(\boldsymbol{AB}) = \det \boldsymbol{A} \det \boldsymbol{B}$
- A matrix whose inverse and transpose are, i.e. $\boldsymbol{A}^{-1} = \boldsymbol{A}^T$, the same is called an **orthogonal** matrix. For such matrices $\boldsymbol{AA}^T = \boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}$.

Basics of linear algebra– XXVI

Operations involving differentiation

- Let $\boldsymbol{y} = \boldsymbol{Ax}$ where $\boldsymbol{A} \in \Re^{m,n}$, and $\boldsymbol{A}$ does not depend on $\boldsymbol{x}$, then

$$\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} = \boldsymbol{A}$$

- Let $\boldsymbol{y} = \boldsymbol{Ax}$ where $\boldsymbol{A} \in \Re^{m,n}$, and $\boldsymbol{A}$ does not depend on $\boldsymbol{x}$ or $\boldsymbol{z}$, however, $\boldsymbol{x}$ is a function of $\boldsymbol{z}$, then

$$\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{z}} = \boldsymbol{A}\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{z}}$$

Basics of linear algebra– XXVII

- Let $\alpha = \boldsymbol{y}^T\boldsymbol{Ax}$ where $\boldsymbol{y}$ is $mx1$, $\boldsymbol{x}$ is $nx1$ and $\boldsymbol{A}$ is mxn with $\boldsymbol{A}$ independent of $\boldsymbol{x}, \boldsymbol{y}$, then

$$\frac{\partial \alpha}{\partial \boldsymbol{x}} = \boldsymbol{y}^T\boldsymbol{A} \qquad \text{and} \quad \frac{\partial \alpha}{\partial \boldsymbol{y}} = \boldsymbol{x}^T\boldsymbol{A}^T$$

- In case of a quadratic $\alpha = \boldsymbol{x}^T\boldsymbol{Ax}$, we have

$$\frac{\partial \alpha}{\partial \boldsymbol{x}} = \boldsymbol{x}^T(\boldsymbol{A} + \boldsymbol{A}^T)$$

- In we consider a further special case of a quadratic $\alpha = \boldsymbol{x}^T\boldsymbol{Ax}$, with a symmetric matrix $\boldsymbol{A}$, we have

$$\frac{\partial \alpha}{\partial \boldsymbol{x}} = 2\boldsymbol{x}^T\boldsymbol{A}$$

Basics of linear algebra– XXVIII

- If $\alpha = \boldsymbol{y}^T\boldsymbol{x}$ where $\boldsymbol{y}, \boldsymbol{x}$ are $nx1$ and a function of $\boldsymbol{z}$,

$$\frac{\partial \alpha}{\partial \boldsymbol{z}} = \boldsymbol{x}^T\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{z}} + \boldsymbol{y}^T\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{z}}$$

- If $\alpha = \boldsymbol{x}^T\boldsymbol{x}$

$$\frac{\partial \alpha}{\partial \boldsymbol{z}} = 2\boldsymbol{x}^T\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{z}}$$

- Let $\alpha = \boldsymbol{y}^T\boldsymbol{Ax}$, then

$$\frac{\partial \alpha}{\partial \boldsymbol{z}} = \boldsymbol{x}^T\boldsymbol{A}^T\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{z}} + \boldsymbol{y}^T\boldsymbol{A}\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{z}}$$

- Let $\alpha = \boldsymbol{x}^T\boldsymbol{Ax}$, then

$$\frac{\partial \alpha}{\partial \boldsymbol{z}} = \boldsymbol{x}^T(\boldsymbol{A} + \boldsymbol{A}^T)\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{z}} \qquad \text{and if } \boldsymbol{A} = \boldsymbol{A}^T \text{ then } \frac{\partial \alpha}{\partial \boldsymbol{z}} = 2\boldsymbol{x}^T\boldsymbol{A}\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{z}}$$

Basics of linear algebra– XXIX

- If $\boldsymbol{A}(\alpha)$, then

$$\frac{\partial \boldsymbol{A}^{-1}}{\partial \alpha} = -\boldsymbol{A}^{-1}\frac{\partial \boldsymbol{A}}{\partial \alpha}\boldsymbol{A}^{-1}$$

Basics of linear algebra– XXX

- **Complexity** of vector computation is determined in terms of basic operations such as addition, multiplication, etc. and are called floating point operations or **flops** (since numbers in computers are stored in floating point format, i.e. block of 64 bits or 8 bytes which is a group of 8 bits).
- Therefore the complexity of an algorithm is the total number of flops needed which is based on the input. For instance an estimate of time required to perform an inner product (depends on the computer speed as well) can be found by finding the required number of flops.
- As an example $\boldsymbol{x} + \boldsymbol{y}$ requires n flops for n additions.
- $\boldsymbol{x}^T\boldsymbol{y}$ requires n multiplication and $n-1$ addition, so $2n-1$ or simply $2n$ flops.
- Typical computers can perform $10^9 flops/sec$ or even more.