GENG 8010–Part 1: Elements of Differential and Difference Equations

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Part I–Outline I

Part I–Outline II

Part I–Outline III

Green Function and 2nd order systems I

Second order systems arise in many engineering domains. The desire is to express the response of a system for different inputs.

Consider again the linear 2nd order equation

$$(D^2 + a(x)D + b(x))y = f(x)$$

defined over an inteval $\alpha \leq x \leq \beta$ over which $a(x), b(x)$, and $f(x)$ are defined and continuous. Suppose x_0 is a point within the interval for which we know $y(x_0) = y'(x_0) = 0$. Start with (7.1)

$$y_p(x) = -y_1(x) \int \frac{y_2(x)f(x)}{W(x)} dx + y_2(x) \int \frac{y_1(x)f(x)}{W(x)} dx$$

Green Function and 2nd order systems II

Note that the lin. indep. of y_1 and y_2 over the interval, guarantees that $W \neq 0$ for all x within the interval. So IF x and x_0 are within the interval

$$\begin{aligned} y_p(x) &= -y_1(x) \int_{x_0}^{x} \frac{y_2(t)f(t)}{W(t)} dt + y_2(x) \int_{x_0}^{x} \frac{y_1(t)f(t)}{W(t)} dt \\ &= \int_{x_0}^{x} \frac{-y_1(x)y_2(t)}{W(t)} f(t) dt + \int_{x_0}^{x} \frac{y_1(t)y_2(x)}{W(t)} f(t) dt \\ &= \int_{x_0}^{x} \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} f(t) dt \end{aligned}$$

$$\boxed{y_p(x) = \int_{x_0}^{x} G(x,t) f(t) dt}$$

Green Function and 2nd order systems III

Note that with $y_c = 0$, hence, if this was a system at rest, then

$$\boxed{y(x) = \int_{x_0}^{x} G(x,t) f(t) dt}$$

$$\boxed{G(x,t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}}$$

where $G(x,t)$ is called the **Green's Function**.

Does the above formula look familiar to you?

Green Function and 2nd order systems IV

Example– For $y(0) = \dot{y}(0) = 0$ solve

$$\ddot{y} - y = f(x)$$

Roots of CE are $m_{1,2} = \pm 1$, hence $y_1 = e^x$, and $y_2 = e^{-x}$, and $W(t) = -2$, therefore the Green Function is

$$G(x,t) = \frac{e^t e^{-x} - e^x e^{-t}}{-2} = \frac{e^{x-t} - e^{-(x-t)}}{2} = \sinh(x-t)$$

and hence

$$y = y_p = \int_{x_0}^{x} \sinh(x-t) f(t) dt$$

Introduction, definition and examples I

Differentiation and integrations are transforms in that they *transform* a function into another. They are also linear in that superposition principle holds for them.
In the following, we are interested in *integral transforms* where the interval of integration is an unbounded interval $[0, \infty)$
Fundamental definition– Assume that $f(t)$ is defined for $t \geq 0$, then the following improper integral is defined as a limit

$$\int_0^\infty K(s,t)f(t)dt = \lim_{b\to\infty}\int_0^b K(s,t)f(t)dt$$

If the above limit exist, the integral is said to be **convergent**, otherwise, it is **divergent** or it fails to exist. The above limit in general exists for certain values of variable s. The choice of $K(s,t) = e^{-st}$ results in an important integral transform.

Introduction, definition and examples II

Laplace Transform

The Laplace transform of a function $f(t)$, i.e., $\mathscr{L}\{f(t)\}$ is defined as

$$\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}f(t)dt$$

provided that the integral converges.

$$\mathscr{L}\{f(t)\} = F(s) \qquad \mathscr{L}\{y(t)\} = Y(s) \qquad \mathscr{L}\{g(t)\} = G(s)$$

$$f(t) \longleftrightarrow F(s) \qquad y(t) \longleftrightarrow Y(s) \qquad g(t) \longleftrightarrow G(s)$$

and $F(s)$ is **integral transform** of $f(t)$, $K(t,s)$ is the **Kernel** of transform, and s is **transform variable**.

Introduction, definition and examples III

Definition–A function is of **exponential order** as $t \to \infty$, $\exists k, M, t_0, \ni$

$$|f(t)| \leq Me^{kt} \quad \forall t > t_0$$

Theorem 8.1.1

If $f(t)$ is piecewise continuous on the interval $[0, \infty)$ and is of exponential order, then $\mathscr{L}\{f(t)\}$ exists for $s > k$.

Note that Theorem 8.1.1 is only a sufficient condition and basically says that $f(t)$ should not grow too fast for its Laplace transform to exist.

Introduction, definition and examples IV

Example– Find the Laplace transform of the unit step function $u(t)$.

$$\mathscr{L}\{u(t)\} = \int_0^\infty e^{-st}(1)dt = \lim_{b\to\infty}\int_0^b e^{-st}dt$$

$$= \lim_{b\to\infty} \left.\frac{-e^{-st}}{s}\right|_0^b = \lim_{b\to\infty}\frac{-e^{-sb}+1}{s} = \frac{1}{s}$$

so long that $s > 0$ so that $e^{-sb} \to 0$ as $b \to \infty$. Otherwise, the integral would diverge for $s < 0$.

Going forward, we shall drop the limit. Instead

$$\mathscr{L}\{u(t)\} = \int_0^\infty e^{-st}(1)dt = -\left.\frac{e^{-st}}{s}\right|_0^\infty = \frac{1}{s}, \quad \textit{and} \quad s > 0$$

$$1 \longleftrightarrow \frac{1}{s} \quad \textit{implies} \quad k \longleftrightarrow \frac{k}{s}$$

Introduction, definition and examples V

Example– Find $\mathscr{L}\{e^{at}\}$

$$\mathscr{L}\{e^{at}\} = \int_0^\infty e^{-st}\left(e^{at}\right)dt = \int_0^\infty e^{-(s-a)t}dt = \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty$$
$$= \frac{1}{s-a} \quad \textit{if} \quad (s-a) > 0 \ \ \textit{or} \ \ s > a$$

Introduction, definition and examples VI

Example– Find $\mathscr{L}\{\cos at\}$, and $\mathscr{L}\{\sin at\}$

$$\mathscr{L}\{\cos at\} = \int_0^\infty e^{-st}\left(\cos at\right)dt$$
$$\mathscr{L}\{\sin at\} = \int_0^\infty e^{-st}\left(\sin at\right)dt$$
$$v' = e^{-st} \rightarrow v = -\frac{1}{s}e^{-st} \quad ; \qquad u = \cos at \rightarrow u' = -a\sin at$$
$$v' = e^{-st} \rightarrow v = -\frac{1}{s}e^{-st} \quad ; \qquad u = \sin at \rightarrow u' = a\cos at$$
$$\int uv'\,dx = uv - \int u'\,vdx$$

where we are setting up to use integration by parts

Introduction, definition and examples VII

Now integrating by parts will give

$$\mathscr{L}\{\cos at\} = -\frac{1}{s}e^{-st}\cos at\Big|_0^\infty - \frac{a}{s}\int_0^\infty e^{-st}\sin at = \frac{1}{s} - \frac{a}{s}\mathscr{L}\{\sin at\}$$
$$\mathscr{L}\{\sin at\} = -\frac{1}{s}e^{-st}\sin at\Big|_0^\infty + \frac{a}{s}\int_0^\infty e^{-st}\cos at = \frac{a}{s}\mathscr{L}\{\cos at\}$$
$$\mathscr{L}\{\cos at\} = \frac{1}{s} - \frac{a}{s}\left(\frac{a}{s}\mathscr{L}\{\cos at\}\right) \longrightarrow \boxed{\mathscr{L}\{\cos at\} = \frac{s}{s^2+a^2}} \quad \textit{if} \quad s > 0$$
$$\mathscr{L}\{\sin at\} = \frac{a}{s}\left(\frac{a}{s} - \frac{a}{s}\mathscr{L}\{\sin at\}\right) \longrightarrow \boxed{\mathscr{L}\{\sin at\} = \frac{a}{s^2+a^2}} \quad \textit{if} \quad s > 0$$

Introduction, definition and examples VIII

Some common Laplace transforms

	Function $f(t)$	Laplace transforms $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$
(i)	1	$\frac{1}{s}$
(ii)	k	$\frac{k}{s}$
(iii)	e^{at}	$\frac{1}{s-a}$
(iv)	$\sin at$	$\frac{a}{s^2+a^2}$
(v)	$\cos at$	$\frac{s}{s^2+a^2}$
(vi)	t	$\frac{1}{s^2}$
(vii)	t^2	$\frac{2!}{s^3}$
(viii)	t^n $(n = 1, 2, 3, \ldots)$	$\frac{n!}{s^{n+1}}$
(ix)	$\cosh at$	$\frac{s}{s^2-a^2}$
(x)	$\sinh at$	$\frac{a}{s^2-a^2}$

Properties and further definitions I

Transform pairs

Functions $f(t)$ and $F(s)$ are called **Laplace Transform pairs**. Just as $f(t)$ gives a unique $F(s)$, given $F(s)$, $f(t)$ is uniquely defined and is expressed as $\mathscr{L}^{-1}\{F(s)\} = f(t)$.

Linearity property

Linearity property– Let $f_1(t), f_2(t), \ldots, f_n(t)$ have Laplace transforms, and let $\alpha_1, \alpha_2, \ldots \alpha_n$ be some constants. Then

$$\mathscr{L}\{\alpha_1 f_1(t) + \alpha_2 f_2(t) + \ldots + \alpha_n f_n(t)\} = \alpha_1 \mathscr{L}\{f_1(t)\} + \alpha_2 \mathscr{L}\{f_2(t)\} + \ldots + \alpha_n \mathscr{L}\{f_n(t)\}$$

Properties and further definitions II

Some common inverse transforms

	$F(s)=\mathcal{L}\{f(t)\}$	$\mathcal{L}^{-1}\{F(s)\}=f(t)$
(i)	$\frac{1}{s}$	1
(ii)	$\frac{k}{s}$	k
(iii)	$\frac{1}{s-a}$	e^{at}
(iv)	$\frac{a}{s^2+a^2}$	$\sin at$
(v)	$\frac{s}{s^2+a^2}$	$\cos at$
(vi)	$\frac{1}{s^2}$	t
(vii)	$\frac{2!}{s^3}$	t^2
(viii)	$\frac{n!}{s^{n+1}}$	t^n
(ix)	$\frac{a}{s^2-a^2}$	$\sinh at$
(x)	$\frac{s}{s^2-a^2}$	$\cosh at$
(xi)	$\frac{n!}{(s-a)^{n+1}}$	$e^{at}t^n$
(xii)	$\frac{\omega}{(s-a)^2+\omega^2}$	$e^{at}\sin\omega t$
(xiii)	$\frac{s-a}{(s-a)^2+\omega^2}$	$e^{at}\cos\omega t$
(xiv)	$\frac{\omega}{(s-a)^2-\omega^2}$	$e^{at}\sinh\omega t$
(xv)	$\frac{s-a}{(s-a)^2-\omega^2}$	$e^{at}\cosh\omega t$

Properties and further definitions III

Example–Find $\mathscr{L}\{f(t)\}$ for a piecewise continuous function

$$f(t) = \begin{cases} 0, & 0 \le t < 3 \\ 2, & t \ge 3 \end{cases}$$

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t) dt \\
&= \int_0^3 e^{-st}(0)dt + \int_3^\infty e^{-st}(2)dt \\
&= -\frac{2e^{-st}}{s}\Big|_3^\infty \\
&= \frac{2e^{-3s}}{s}, s > 0
\end{aligned}$$

Properties and further definitions IV

Example–Find $\mathscr{L}^{-1}\{F(s)\}$ where

$$F(s) = \frac{4s-5}{s^2-s-2}$$

$$\frac{4s-5}{s^2-s-2} = \frac{4s-5}{(s-2)(s+1)} = \frac{A}{(s-2)} + \frac{B}{(s+1)} = \frac{A(s+1)+B(s-2)}{(s-2)(s+1)}$$

at $s = 2 \longrightarrow A = 1$ and at $s = -1 \longrightarrow B = 3$. Therefore

$$\boxed{f(t) = \mathscr{L}^{-1}\{F(s)\} = \mathscr{L}^{-1}\left\{\frac{1}{s-2} + \frac{3}{s+1}\right\} = e^{2t} + 3e^{-t}}$$

Properties and further definitions V

Example–Find $\mathscr{L}^{-1}\{F(s)\}$ where

$$F(s) = \frac{3s^3 + s^2 + 12s + 2}{(s-3)(s+1)^3}$$

$$\frac{3s^3 + s^2 + 12s + 2}{(s-3)(s+1)^3} = \frac{A}{s-3} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3}$$

Find $A = 2, B = 1, C = -4$, and $D = 3$ to get

$$\boxed{f(t) = \mathscr{L}^{-1}\{F(s)\} = 2e^{3t} + e^{-t} - 4te^{-t} + \frac{3}{2}t^2e^{-t}}$$

Properties and further definitions VI

Theorem 8.2.1

If $f(t)$ be is continuous on $0 \le t < \infty$, and let $f'(t)$ be piecewise continuous on every finite interval contained in $t \ge 0$. Then if $\mathscr{L}\{f(t)\} = F(s)$,

$$\boxed{\mathscr{L}\{f'(t)\} = sF(s) - f(0)}$$

Proof.

$$\mathscr{L}\{f'(t)\} = \int_0^\infty e^{-st}f'(t)dt \qquad ; \int uv'dx = uv - \int u'vdx$$

$$\mathscr{L}\{f'(t)\} = e^{-st}f(t)\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)dt = sF(s) - f(0)$$

□

Properties and further definitions VII

The above can be **extended to higher derivatives** if the similar conditions hold, then it can be shown that

$$\boxed{\mathscr{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)}$$

Theorem 8.2.2

First translation or s-Shift Theorem *$\mathscr{L}\{f(t)\} = F(s)$ and a is a real number, then*

$$\mathscr{L}\{e^{at}f(t)\} = F(s-a)$$

Properties and further definitions VIII

Example– Find $\mathscr{L}\{f(t)\}$ if $f(t) = e^{-2t}\sin 3t$

$$\mathscr{L}\{e^{-2t}\sin 3t\} = \frac{3}{(s-(-2))^2 + 3^2} = \frac{3}{(s+2)^2 + 9} = \frac{3}{s^2 + 4s + 13}$$

Define the t-shifted unit step function

$$u(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

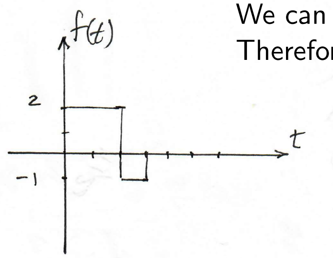
Theorem 8.2.3

Second translation or t-Shift Theorem *If $\mathscr{L}\{f(t)\} = F(s)$ and $a > 0$ is a real number, then*

$$\mathscr{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

Properties and further definitions IX

Example– Find $\mathscr{L}\{f(t)\}$ for the function graphed below



We can write $f(t) = 2u(t) - 3u(t-2) + u(t-3)$. Therefore

$$\boxed{\mathscr{L}\{f(t)\} = \frac{2}{s} - \frac{3e^{-2s}}{s} + \frac{e^{-3s}}{s}}$$

Properties and further definitions X

Theorem 8.2.4

Multiplication by t^n-differentiation of transform–*Let $\mathscr{L}\{f(t)\} = F(s)$, then*

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}$$

Theorem 8.2.5

Scaling theorem–*Let $\mathscr{L}\{f(t)\} = F(s)$, then if $k > 0$, then*

$$\mathscr{L}\{f(kt)\} = \frac{1}{k} F\left(\frac{s}{k}\right)$$

Definition (Convolution)–Convolution of two functions $f(t)$ and $g(t)$ is defined by $(f * g)(t)$ and is given by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

Properties and further definitions XI

Theorem 8.2.6

Convolution theorem–*Let $\mathscr{L}\{f(t)\} = F(s)$, and $\mathscr{L}\{g(t)\} = G(s)$, then*

$$\mathscr{L}\{(f * g)(t)\} = F(s)G(s)$$

Definition (Volterra Integral Equation)–The following *integral equation* (often arising in initial value problems) is called Volterra integral equation where λ is a parameter and $K(t,\tau)$ is the *Kernel* of the integral equation.

$$y(t) = f(t) + \lambda \int_0^t K(t,\tau)y(\tau)d\tau$$

Special case of Volterra equation is when the kernel $K(t,\tau)$ is a function of $t-\tau$, i.e. $K(t,\tau) = K(t-\tau)$, in which case the integral part in the above becomes a convolution integral.

Properties and further definitions XII

Example– Solve the integral equation

$$y(t) = 2e^{-t} + \int_0^t \sin(t-\tau)y(\tau)d\tau$$

Taking $\mathscr{L}\{y(t)\}$ and using Theorem 8.2.6

$$Y(s) = \frac{2}{s+1} + \frac{Y(s)}{s^2+1} \longrightarrow Y(s) = \frac{2(s^2+1)}{s^2(s+1)}$$

$$= -\frac{2}{s} + \frac{2}{s^2} + \frac{4}{s+1}$$

$$\boxed{y(t) = -2 + 2t + 4e^{-t}}$$

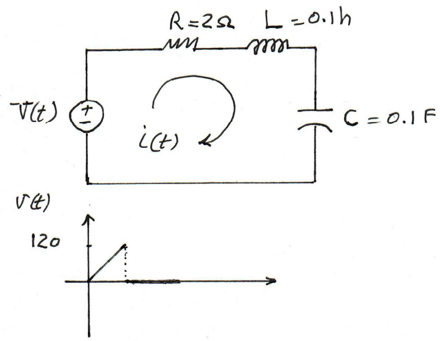
Properties and further definitions XIII

Integro–differential equations– Let us illustrate this class of problems with the following simple RLC circuit

The circuit is described by

$$v(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt$$

The above is an integro-differential equation with $v(t) = 120t - 120tu(t-1)$



Properties and further definitions XIV

Before taking Laplace Transform, we shall write $v(t)$ as:

$$v(t) = 120t - 120(t-1+1)u(t-1) = 120t - 120(t-1)u(t-1) - 120u(t-1)$$

giving

$$V(s) = 120\left(\frac{1}{s^2} - e^{-s}\left[\frac{1}{s^2} + \frac{1}{s}\right]\right)$$

Taking the Laplace transform of the integro-differential equation gives (with $i(0) = 0$)

$$2I(s) + 0.1sI(s) + 10\frac{I(s)}{s} = 120\left[\frac{1}{s^2} - \left(\frac{e^{-s}}{s^2} + \frac{e^{-s}}{s}\right)\right]$$

Properties and further definitions XV

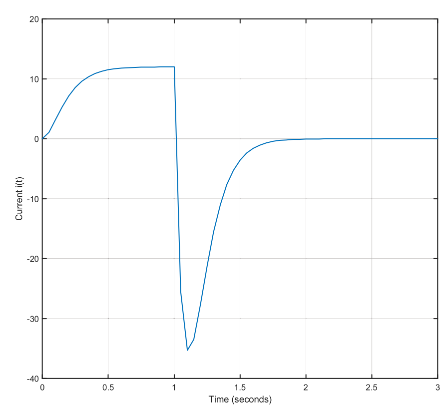
$$I(s) = 1200\left[\frac{0.01}{s} - \frac{0.01}{s+10} - \frac{0.1}{(s+10)^2} - \frac{0.01}{s}e^{-s} + \frac{0.01}{s+10}e^{-s} \right.$$
$$\left. + \frac{0.1}{(s+10)^2}e^{-s} - \frac{1}{(s+10)^2}e^{-s}\right]$$

leading to

$$i(t) = \begin{cases} 12 - 12e^{-10t} - 120te^{-10t}, & 0 \le t < 1 \\ -12e^{-10t} + 12e^{-10(t-1)} - 120te^{-10t} - 1080(t-1)e^{-10(t-1)}, & t \ge 1 \end{cases}$$

Properties and further definitions XVI

The MATLAB response of the current in response to the applied voltage (solution of the differential equation for the given forcing function) is given below.



Properties and further definitions XVII

Theorem 8.2.7

Transform of an integral– *Let $f(t)$ be of exponential order and $\mathscr{L}\{f(t)\} = F(s)$, then*

$$\mathscr{L}\{\int_0^t f(\tau)d\tau\} = \frac{F(s)}{s} \qquad s > k$$

Theorem 8.2.8

Integral of a transform– *Let $\frac{f(t)}{t}$ be of exponential order and $\mathscr{L}\{\frac{f(t)}{t}\} = G(s)$, for $s > k$ and $\mathscr{L}\{f(t)\} = F(s)$, then*

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(x)dx$$

Properties and further definitions XVIII

Theorem 8.2.9

Property of Dirac Delta Function– *Assuming $f(t)$ is defined and integrable and is continuous in the vicinity of a, then*

$$\int_0^\infty f(t)\delta(t-a)dt = f(a)$$

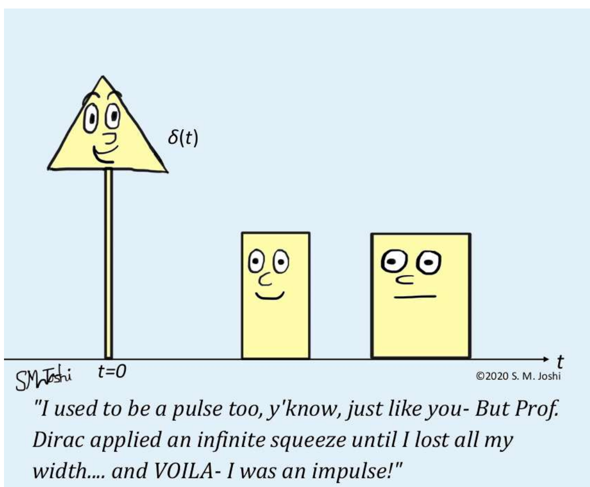
$$\mathscr{L}\{\delta(t-a)\} = e^{-as}$$

$$\mathscr{L}\{\delta(t)\} = 1$$

Summary: advantages of Laplace transform

1) Solution is routine an progresses systematically; 2) it gives the total solution; and 3) initial conditions are taken care of in the process.

Properties and further definitions XIX



Some system engineering concepts I

Some defentions

[System] *A system is a collection of things which are related in such a way which make sense to think of them as whole.*

[Single vs. multivariable] *A system is single variable iff $\exists$ one input and one output. Otherwise, it is multivariable.*

[Relaxedness] *A system is relaxed at time t_0 iff the output $y_{[t_0,\infty]}$ is solely and uniquely defined by input $u_{[t_0,\infty]}$.*

[Memoryless] *A system is zero memory if $y(t_1)$ depends only on $u(t_1)$.*

[Memory] *A system has "memory" if $y(t_1)$ depends on $u(t_1)$ as well as $u(t)$, $t < t_1$.*

Some system engineering concepts II

Remark

If a system has memory, it is impossible to determine $y(t)$ without assuming the system has been relaxed or at rest. In that case,

$$y(t) = \mathscr{H} u(t)$$

where $\mathscr{H}$ is some operator mapping the input space into output space. $\mathscr{H} : \{u(t)\} \longrightarrow \{y(t)\}$, or $\mathscr{H}$ is the **mathematical model of the system**.

Some system engineering concepts III

[Linearity] *A system is linear iff*

$$\mathscr{H}\left[(a_1 u_1(t) + a_2 u_2(t)\right] = a_1 \mathscr{H} u_1(t) + a_2 \mathscr{H} u_2(t)$$

[Causality] *A system is causal or non-anticipatory if the output of the system at time t does not depend on inputs applied after t.*

[Time invariance] *If the system's characteristic remains the same with time, then its is time invariant or stationary*

$$\mathscr{H}\mathscr{S}_\tau u(t) = \mathscr{S}_\tau \mathscr{H} u(t)$$

where $\mathscr{S}_\tau$ is the shifting operator, i.e., $\mathscr{S}_\tau u(t) = u(t+\tau)$. Otherwise, the system is time varying.