Lecture Notes 9th May 2023

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	Subjects to Cover	
1	ODE of the first order	First Midterm
2	ODE of higher orders	
3	Power Series Solutions	Second Midterm
4	Laplace Transforms	
5	Quantitative Methods: Numerical Solution	If time permits
6	Fourier Series, Fourier Integral, Fourier Transform	Everything for Finals
7	Sturm-Liouville problem	
8	Partial Differential Equations (PDE)	
9	Diffusion (heat) PDE	
10	Wave PDE	
11	Laplace PDE	

<u>Ordinary Differential Equations (ODE)</u>:

A differential equation is an equation containing a function like y(t) and its derivatives y', y'', y''', …

It is called ordinary if the dependent function y has only 1 independent variable.

Example 1:

$$2t^2y'' - e^ty' + \cos t \; y = \sin t$$

Highest degree = 2, therefore Second Order ODE

Example 2: Electrical Engineering (RLC Circuit)

$$L.\, I'' + R.\, I' + \frac{1}{C} I = V'$$

L, R, C = constants

I, V are functions of Time t

I(t) = current at time t

V(t) = voltage at time t

Second Order, homogenous if V’ = 0, also linear

Differential equation = f(t)

If f(t) = 0 ==> homogeneous

If f(t) ≠ 0 ==> non-homogeneous

Example 3: Mechanical Engineering (Mechanical Vibrations)

$$mx'' + cx' + kx = f(t)$$

m,c,k = constants

Second order,

If f(t) = 0 ==> homogeneous

If f(t) ≠ 0 ==> non-homogeneous

Linear

Example 4: Civil Engineering (Elastic beam)

$$EV^{(4)} + KV = 0$$

(4) ==> Fourth Derivative

Fourth Order

Linear

Homogeneous

Example 5: Industrial Engineering (Speed control of a DC motor)

$$aw' + bw = 0$$

a, b = constant

First order

Linear

Homogeneous

First Order Differential Equation:

- Separation of variables
- Integration Factor
- Variation of parameters
- Exact Equations

Separation of Variables:

$$\frac{dy}{dx} = f(x)g(x)$$

$$\frac{1}{g(x)} dy = f(x)dx$$

$$\int \frac{1}{g(x)} dy = \int f(x)dx$$

Solve the following ODE using separation of variables:

Example 1:

$$\frac{dy}{dx} = -\frac{x}{y}, \text{when } y(4) = -3$$

$$y\, dy = -x\, dx$$

$$\int y\ dy = -\int x\ dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$

$$\frac{(-3)^2}{2} = -\frac{(4)^2}{2} + c$$

$$\frac{9}{2} = -\frac{16}{2} + c$$

$$c = \frac{25}{2}$$

$$\therefore\ \frac{y^2}{2} = -\frac{x^2}{2} + \frac{25}{2}$$

Example 2:

$$y' = \frac{4x}{1+2e^y}$$

$$\frac{dy}{dx} = \frac{4x}{1+2e^y}$$

$$(1+2e^y)\,dy = 4x\,dx$$

$$\int 1 + 2e^y\,dy \;=\; 4\int x\;dx$$

$$\int 1\;dy + 2\int e^y\;dy \;=\; 4\left(\frac{x^2}{2}\right) + c$$

$$y + 2e^y = 2x^2 + c$$

Example 3:

$$\frac{dy}{dx} = \frac{y(y-2)}{x(y-1)}$$

$$\frac{y-1}{y(y-2)}\,dy = \frac{1}{x}\,dx$$

$$\int \frac{y-1}{y(y-2)}\,dy = \int \frac{1}{x}\,dx$$

$$\frac{1}{2}\int \left(\frac{1}{y}+\frac{1}{y-2}\right)dy = \int \frac{1}{x}\,dx \;\ldots\; \text{(By Partial Fraction)}$$

$$\frac{1}{2}(\ln|y| + \ln|y-2|) = \ln|x| + c$$

$$\ln\left|\frac{y(y-2)}{x^2}\right| = 2c$$

$$\frac{y(y-2)}{x^2} = e^{2c}$$

Integration factor:

$$y' + p(x)y = q(x) \ldots (1)$$

$$\text{Integration Factor} = e^{\int p(x)dx}$$

Multiplying equation (1) by the Integration Factor we get,

$$y'e^{\int p(x)dx} + p(x)ye^{\int p(x)dx} = q(x)e^{\int p(x)dx}$$

By Product Rule of Differentiation, the LHS becomes

$$\frac{d\left(ye^{\int p(x)dx}\right)}{dx} = q(x)e^{\int p(x)dx}$$

Integrating both sides

$$\int \frac{d\left(ye^{\int p(x)dx}\right)}{dx} = \int q(x)e^{\int p(x)dx}$$

$$ye^{\int p(x)dx} = \int q(x)e^{\int p(x)dx}$$

$$y = \frac{1}{e^{\int p(x)dx}}\left[\int q(x)e^{\int p(x)dx} + c\right]$$

Solve the following ODE using Integration Factor:

Example 1:

$$\frac{dy}{dx} + 5y = 3$$

$$y' + p(x) = q(x)$$

$$p(x) = 5, q(x) = 3$$

$$\text{Integration Factor} = e^{\int p(x)dx}$$

$$= e^{\int 5\, dx}$$

$$= e^{5x}$$

Multiplying both sides of the given differential equation by the integration factor we get,

$$e^{5x}.\frac{dy}{dx} + e^{5x}.5y = e^{5x}.3$$

By Product rule we can simplify the LHS as,

$$\frac{d(y.e^{5x})}{dx} = 3e^{5x}$$

Integrating both sides we get,

$$\int \frac{d(y.e^{5x})}{dx}dx = \int 3e^{5x}dx$$

$$ye^{5x} = \frac{3}{5}e^{5x} + c$$

$$y = \frac{3}{5} + ce^{-5x}$$

Example 2:

$$y' + 5y = x$$

$$y' + p(x) = q(x)$$

$$p(x) = 5, q(x) = x$$

$$\text{Integration Factor} = e^{\int p(x)dx}$$

$$= e^{\int 5\,dx}$$

$$= e^{5x}$$

Multiplying both sides of the given differential equation by the integration factor we get,

$$e^{5x}.y' + e^{5x}.5y = e^{5x}.x$$

By Product rule we can simplify the LHS as,

$$\frac{d(y.e^{5x})}{dx} = xe^{5x}$$

Integrating both sides we get,

$$\int \frac{d(y.e^{5x})}{dx}dx = \int xe^{5x}dx$$

$$ye^{5x} = \frac{xe^{5x}}{5} - \frac{1}{5}\int e^{5x}dx$$

$$ye^{5x} = \frac{xe^{5x}}{5} - \frac{e^{5x}}{25} + c$$

$$y = 5x - 1 + 25ce^{-5x}$$

Example 3:

$$xy' + 3y = xe^{x^4}$$

$$y' + \frac{3}{x}y = e^{x^4}$$

$$y' + p(x) = q(x)$$

$$p(x) = \frac{3}{x}, q(x) = e^{x^4}$$

$$\text{Integration Factor} = e^{\int p(x)dx}$$

$$= e^{\int \frac{3}{x} dx}$$

$$= x^3$$

Multiplying both sides of the given differential equation by the integration factor we get,

$$x^3 . y' + x^3 . \frac{3}{x}y = x^3 . e^{x^4}$$

$$x^3 . y' + 3x^2 . y = x^3 . e^{x^4}$$

By Product rule we can simplify the LHS as,

$$\frac{d(y.x^3)}{dx} = x^3 e^{x^4}$$

Integrating both sides we get,

$$\int \frac{d(y.x^3)}{dx} dx = \int x^3 e^{x^4} dx \ldots \text{(i)}$$

Let $x^4 = u$

$$\therefore x^3 dx = \frac{1}{4} du$$

Substituting in (i) we get,

$$\therefore yx^3 = \frac{1}{4}\int e^u du$$

$$yx^3 = \frac{1}{4} e^u + c \ldots \text{(ii)}$$

Resubstituting $x^4 = u$ in (ii)

$$yx^3 = \frac{1}{4} e^{x^4} + c$$