GENG 8010–Part 2 - Elements of Applied Linear Algebra

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Winter 2022

Outline of part 2– I

Outline of part 2– II

Outline of part 2– III

Matrix representation of $\mathscr{L}$– II

Claim –Every linear operator that maps a finite dimensional space $(\mathscr{X}, \mathscr{F})$ into a finite dimensional space $(\mathscr{Y}, \mathscr{F})$ has a matrix representation with coefficients in $\mathscr{F}$.

Theorem 2.5.1

Let $(\mathscr{X}, \mathscr{F})$ and $(\mathscr{Y}, \mathscr{F})$ be n and m dimensional linear spaces. Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ be a set of linearly independent vectors in $\mathscr{X}$. Then the linear operator $\mathscr{L} : (\mathscr{X}, \mathscr{F}) \longrightarrow (\mathscr{Y}, \mathscr{F})$ is uniquely determined by the n−pairs of mapping $\boldsymbol{y}_i = \mathscr{L}\{\boldsymbol{x}_i\}$, for $i = 1, 2, \ldots, n$. Furthermore, with respect to the basis $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n\}$ of $\mathscr{X}$ and $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\}$ of $\mathscr{Y}$, $\mathscr{L}$ can be represented by an mxn matrix $\boldsymbol{A}$ with coefficients in $\mathscr{F}$. The ith column of $\boldsymbol{A}$ is the representation of $\boldsymbol{y}_i$ with respect to the basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\}$.

Matrix representation of $\mathscr{L}$– III

Matrix representation of $\mathscr{L}$– IV

Proof.
a) Let $\boldsymbol{x} \in \mathscr{X}$, then

$$\boldsymbol{x} = \alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2 + \ldots + \alpha_n \boldsymbol{x}_n$$

Since $\mathscr{L}$ is a linear operator

$$\begin{aligned} \mathscr{L}\{\boldsymbol{x}\} &= \alpha_1 \mathscr{L}\{\boldsymbol{x}_1\} + \alpha_2 \mathscr{L}\{\boldsymbol{x}_2\} + \ldots + \alpha_n \mathscr{L}\{\boldsymbol{x}_n\} \\ &= \alpha_1 \boldsymbol{y}_1 + \alpha_2 \boldsymbol{y}_2 + \ldots + \alpha_n \boldsymbol{y}_n \end{aligned}$$

which implies that for an $\boldsymbol{x} \in \mathscr{X}$, $\mathscr{L}\{\boldsymbol{x}\}$ is uniquely determined by $\boldsymbol{y}_i = \mathscr{L}\{\boldsymbol{x}_i\}$, $i = 1, 2, \ldots, n$.

Matrix representation of $\mathscr{L}$– V

b) Let us now find representation of $\boldsymbol{y}_i$ wrt basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\}$.

$$\boldsymbol{y}_i = [\boldsymbol{u}_1 \;\; \boldsymbol{u}_2 \;\; \ldots \;\; \boldsymbol{u}_m] \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix}, \quad i = 1, \ldots, n \;; \quad a_{ij} \in \mathscr{F}$$

Matrix representation of $\mathscr{L}$– VI

$$\begin{aligned}
\mathscr{L}\{[\boldsymbol{x}_1 \;\; \boldsymbol{x}_2 \;\; \dots \;\; \boldsymbol{x}_n]\} &= [\boldsymbol{y}_1 \;\; \boldsymbol{y}_2 \;\; \dots \;\; \boldsymbol{y}_n] \\
&= [\boldsymbol{u}_1 \;\; \boldsymbol{u}_2 \;\; \dots \;\; \boldsymbol{u}_m] \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \\
&= [\boldsymbol{u}_1 \;\; \boldsymbol{u}_2 \;\; \dots \;\; \boldsymbol{u}_m]\boldsymbol{A}, \quad \boldsymbol{A} \in \Re^{mxn}
\end{aligned}$$

Matrix representation of $\mathscr{L}$– VII

With respect to their respective bases, the linear operator $\boldsymbol{y} = \mathscr{L}\{\boldsymbol{x}\}$ can be written as

$$\begin{aligned}
\boldsymbol{y} &= \mathscr{L}\{\boldsymbol{x}\} \\
[\boldsymbol{u}_1 \;\; \boldsymbol{u}_2 \;\; \dots \;\; \boldsymbol{u}_m]\beta &= \mathscr{L}\{[\boldsymbol{x}_1 \;\; \boldsymbol{x}_2 \;\; \dots \;\; \boldsymbol{x}_n]\alpha\} \\
&= [\boldsymbol{y}_1 \;\; \boldsymbol{y}_2 \;\; \dots \;\; \boldsymbol{y}_n]\alpha \\
&= [\boldsymbol{u}_1 \;\; \boldsymbol{u}_2 \;\; \dots \;\; \boldsymbol{u}_m]\boldsymbol{A}\alpha
\end{aligned}$$

where α and β are the representations of $\boldsymbol{x}$ and $\boldsymbol{y}$ with respect to their own basis.

Matrix representation of $\mathscr{L}$– VIII

After the basis is chosen, there is no difference between specifying $\boldsymbol{x}$ and $\boldsymbol{y}$ and α and β. Hence, in studying $\boldsymbol{y} = \mathscr{L}\{\boldsymbol{x}\}$, we may just study the relationship between their representations. So,

$$[\boldsymbol{u}_1 \;\; \boldsymbol{u}_2 \;\; \dots \;\; \boldsymbol{u}_m]\beta = [\boldsymbol{u}_1 \;\; \boldsymbol{u}_2 \;\; \dots \;\; \boldsymbol{u}_m]\boldsymbol{A}\alpha \Longrightarrow \boxed{\beta = \boldsymbol{A}\alpha} \qquad \blacksquare$$

Remark

It is important to note that $\boldsymbol{A}$ is completely dependent on the choice of bases chosen for $(\mathscr{X}, \mathscr{F})$ and $(\mathscr{Y}, \mathscr{F})$.

Matrix representation of $\mathscr{L}$– IX

A subclass of operators are those that map a linear space $(\mathscr{X}, \mathscr{F})$ into itself

$$\mathscr{L} : (\mathscr{X}, \mathscr{F}) \longrightarrow (\mathscr{X}, \mathscr{F})$$

If we choose $\mathscr{B} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n\}$, then a matrix representation for $\mathscr{L}$ wrt $\mathscr{B}$ is $\boldsymbol{A} \in \Re^{nxn}$
If we choose $\hat{\mathscr{B}}$, then the matrix representation of the operator wrt to this basis is $\hat{\boldsymbol{A}} \in \Re^{nxn}$

Question–What is the relationship between $\boldsymbol{A}$ and $\hat{\boldsymbol{A}}$?

Matrix representation of $\mathscr{L}$– X

Matrix representation of $\mathscr{L}$– XI

Let $\boldsymbol{x} \in \mathscr{X}$ with two representation $\boldsymbol{\alpha}$ and $\hat{\boldsymbol{\alpha}}$ wrt to $\mathscr{B}$ and $\hat{\mathscr{B}}$. Also,

$$\boldsymbol{y} = \mathscr{L}\{\boldsymbol{x}\}$$

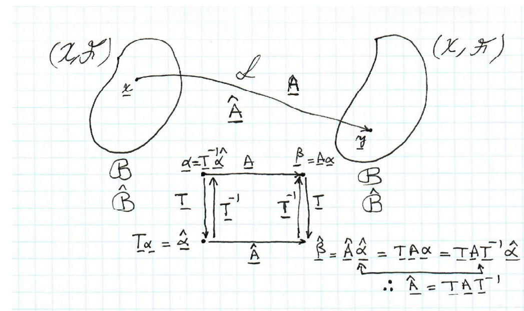


Figure 1: An operator mapping a space into itself.

Matrix representation of $\mathscr{L}$– XII

So find its representation β and $\hat{\beta}$ wrt to $\mathscr{B}$ and $\hat{\mathscr{B}}$. We know

$$\hat{\alpha} = \boldsymbol{T}\alpha \qquad \hat{\beta} = \boldsymbol{T}\beta \qquad \text{where } \boldsymbol{T} \text{ is non-singular}$$

$$\hat{\beta} = \hat{\boldsymbol{A}}\hat{\alpha} \qquad \beta = \boldsymbol{A}\alpha$$

$$\text{thus, } \quad \hat{\boldsymbol{A}} = \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1} \tag{1}$$

$$\boldsymbol{A} = \boldsymbol{T}^{-1}\hat{\boldsymbol{A}}\boldsymbol{T} \tag{2}$$

Similar matrices

Two matrices $\boldsymbol{A}$ and $\hat{\boldsymbol{A}}$ are **similar** if there exist a nonsingular matrix $\boldsymbol{T}$ satisfying (1) and (2). The transformation is called a **similarity transformation**.

Matrix representation of $\mathscr{L}$– XIII

Remark (Interpretation of a matrix)

Given an *nxn* matrix $\boldsymbol{A}$ with coefficients in $\mathscr{F}$, if it is not specified to be the representation of some operator, we shall consider it as a linear operator that maps $(\mathscr{F}^n, \mathscr{F})$ into itself.

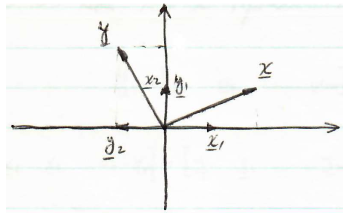
Example–Consider $(\Re^2, \Re)$ and our example before where we chose $\mathscr{B} = \{\boldsymbol{x}_1, \boldsymbol{x}_2\}$,

Recall that $\mathscr{L}$ was a 90 degrees CCW rotation. We are now interested to find the representation of this linear operator with respect to the natural basis, i.e. $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$.

Matrix representation of $\mathscr{L}$– XIV

$$y_1 = \mathscr{L}\{x_1\} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$y_2 = \mathscr{L}\{x_2\} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$



$$\text{thus, } \boldsymbol{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{\alpha} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{rep. } \boldsymbol{x} \text{ wrt } \mathscr{B}$$

$$\beta = \boldsymbol{A\alpha} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Solutions of $Ax = y$– I

Consider

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= y_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= y_2 \\ \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= y_m \end{aligned}$$

where $a_{ij} \in \mathscr{F}$ and $y_i \in \mathscr{F}$, also $x_i \in \mathscr{F}$. The above can be written as

$$\boldsymbol{Ax} = \boldsymbol{y} \qquad \begin{matrix} \boldsymbol{A} : mxn \\ \boldsymbol{x} : nx1 \\ \boldsymbol{y} : mx1 \end{matrix}$$

Solutions of $Ax = y$– II

Questions:

1. **Existence**– Condition on $\boldsymbol{A}$ and $\boldsymbol{y}, \ni \boldsymbol{x}$ exits.
2. **Number of solutions**– Number of linearly independent $\boldsymbol{x}$'s $\ni \boldsymbol{Ax} = \boldsymbol{y}$.

Range

The **range** of a linear operator $\boldsymbol{A}$ is the set $\mathscr{R}(\boldsymbol{A})$ defined as:

$$\mathscr{R}(\boldsymbol{A}) = \{\text{all } \boldsymbol{y} \in (\mathscr{F}^m, \mathscr{F}) \text{ for which } \exists \text{ at least one } \boldsymbol{x} \in (\mathscr{F}^n, \mathscr{F}) \ni \boldsymbol{Ax} = \boldsymbol{y}\}$$

Solutions of $Ax = y$– III

Theorem 3.1.1

$\mathscr{R}(\boldsymbol{A})$ is a subspace of $(\mathscr{F}^m, \mathscr{F})$

Proof.

If $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathscr{R}(\boldsymbol{A}) \implies \exists \boldsymbol{x}_1, \boldsymbol{x}_2 \in (\mathscr{F}^n, \mathscr{F}) \ni, \boldsymbol{y}_1 = \boldsymbol{Ax}_1, \boldsymbol{y}_2 = \boldsymbol{Ax}_2$.

We wish to show that $\forall \alpha_1, \alpha_2 \in \mathscr{F}, \boldsymbol{y} = \alpha_1 \boldsymbol{y}_1 + \alpha_2 \boldsymbol{y}_2 \in \mathscr{R}(\boldsymbol{A})$.

$\boldsymbol{y} = \alpha_1 \boldsymbol{y}_1 + \alpha_2 \boldsymbol{y}_2 = \alpha_1 \boldsymbol{Ax}_1 + \alpha_2 \boldsymbol{Ax}_2 = \boldsymbol{A}(\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2)$ and since $(\mathscr{F}^n, \mathscr{F})$ is a vector space, then $\exists$ $\boldsymbol{x} = \alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2 \in (\mathscr{F}^n, \mathscr{F})$ $\forall \alpha_1, \alpha_2 \in \mathscr{F}$, and thus, $\boldsymbol{y} \in \mathscr{R}(\boldsymbol{A})$ ∎

Solutions of $Ax = y$– IV

Let $\boldsymbol{A} = [\boldsymbol{a}_1 \;\; \boldsymbol{a}_2 \cdots \boldsymbol{a}_n]$, then $\boldsymbol{y}$ can be written as

$$\boldsymbol{y} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Solutions of $Ax = y$– V

based on the above formulation, let us now make some observations and note that one leads from the other:

1. $\mathscr{R}(\boldsymbol{A})$ is the set of $\boldsymbol{y}$'s $\ni \boldsymbol{Ax} = \boldsymbol{y}$.
2. $\mathscr{R}(\boldsymbol{A})$ is the set of $\boldsymbol{y}$'s obtained by allowing $x_1, x_2, \cdots, x_n$ "ranging" through all their possible values in $\mathscr{F}^n$
3. $\mathscr{R}(\boldsymbol{A})$ is the set of all linear combinations of the columns of $\boldsymbol{A}$.
4. Dimension of $\mathscr{R}(\boldsymbol{A})$ is equal to the maximum number of linearly independent columns in $\boldsymbol{A}$.

Solutions of $Ax = y$– VI

Consider now the following definition.

Rank

Rank of a matrix $\boldsymbol{A}$, represented by $\rho(\boldsymbol{A})$, is the maximum number of linearly independent columns in $\boldsymbol{A}$ or equal to the dimension of the range space of $\boldsymbol{A}$, i.e. $\rho(\boldsymbol{A}) = \dim \mathscr{R}(\boldsymbol{A})$. Hence, for an nxm matrix

$$\rho(\boldsymbol{A}) \leq \min(n, m)$$

Theorem 3.1.2

A square matrix is **nonsingular** *or has a non-zero determinant, iff it is of full rank.*

Solutions of $Ax = y$– VII

Example–consider $\boldsymbol{A}$ where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 1 & -1 & 2 \end{bmatrix}$$

if you calculate $\det \boldsymbol{A}$ you will find that it is zero. Hence this matrix is **singular**.

$$\boldsymbol{A} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore we conclude $\rho(\boldsymbol{A}) = 2$ which also implies that the matrix has a zero determinant and is singular.

Solutions of $Ax = y$– VIII

Theorem 3.1.3

Consider $\mathbf{Ax} = \mathbf{y}$ where $\mathbf{A} : (\mathscr{F}^n, \mathscr{F}) \longrightarrow (\mathscr{F}^m, \mathscr{F})$.

1. *Given $\mathbf{A}$, and $\underline{\underline{a}}$ vector $\mathbf{y} \in (\mathscr{F}^m, \mathscr{F}), \exists\, \mathbf{x} \in (\mathscr{F}^n, \mathscr{F}) \ni \mathbf{Ax} = \mathbf{y}$ iff $\mathbf{y} \in \mathscr{R}(\mathbf{A})$ or $\rho(\mathbf{A}) = \rho([\mathbf{A}\ \ \mathbf{y}])$.*
2. *Given $\mathbf{A}$, $\forall \mathbf{y} \in (\mathscr{F}^m, \mathscr{F}), \exists\, \mathbf{x} \in (\mathscr{F}^n, \mathscr{F}) \ni \mathbf{Ax} = \mathbf{y}$ iff the mapping is <u>onto</u>, i.e., $\mathscr{R}(\mathbf{A}) = (\mathscr{F}^m, \mathscr{F})$ or $\rho(\mathbf{A}) = m$.*

Solutions of $Ax = y$– IX

Proof.

1. Follows by the definition of $\mathscr{R}(\mathbf{A})$.
2. $\rho(\mathbf{A}) = \dim \mathbf{A}$. Since $\mathscr{R}(\mathbf{A}) \subset (\mathscr{F}^m, \mathscr{F})$ if $\rho(\mathbf{A}) = m$, then $\mathscr{R}(\mathbf{A}) = (\mathscr{F}^m, \mathscr{F})$, and so then $\forall \mathbf{y} \in (\mathscr{F}^m, \mathscr{F})\ \exists$, an $\mathbf{x} \in (\mathscr{F}^n, \mathscr{F})\ \ni \mathbf{Ax} = \mathbf{y}$.

   If $\rho(\mathbf{A}) < m$, then $\exists$ at least one nonzero vector $\mathbf{y} \in (\mathscr{F}^m, \mathscr{F}), \mathbf{y} \notin \mathscr{R}(\mathbf{A})$ for which $\exists$ no $\mathbf{x} \ni \mathbf{Ax} = \mathbf{y}$. Therefore we must have $\mathscr{R}(\mathbf{A}) = (\mathscr{F}^m, \mathscr{F})$ or $\rho(\mathbf{A}) = m$.

Solutions of $Ax = y$– X

Null space and nullity

The **null (or kernel) space** of a linear operator $\mathbf{A}$ is the set $\mathscr{N}(\mathbf{A})$ defined as

$$\mathscr{N}(\mathbf{A}) = \{\text{all } \mathbf{x} \in (\mathscr{F}^n, \mathscr{F}), \ni \mathbf{Ax} = \mathbf{0}\}$$

The dimension of $\mathscr{N}(\mathbf{A})$ is called the **nullity** of $\mathbf{A}$ and is denoted by $\gamma(\mathbf{A}) = \dim \mathscr{N}(\mathbf{A})$.

Remark

If $\gamma(\mathbf{A}) = 0$, then the only solution to $\mathbf{Ax} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. If on the other hand $\gamma(\mathbf{A}) = k$, then there are k linearly independent solution to $\mathbf{Ax} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.

Solutions of $Ax = y$– XI

Lemma 3.1.4 (Fundamental Lemma)

Let $\mathbf{A}$ be an nxn matrix. Then the homogeneous equation

$$\mathbf{Ax} = \mathbf{0}$$

has a nonzero solution iff $\mathbf{A}$ is singular.

<u>**Example**</u>–Find $\rho, \mathscr{N}, \gamma$ for $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad \text{verify } \ \rho(\mathbf{A}) = 2$$

$$\mathbf{Ax} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Longrightarrow x_1 = 0\ ;\ x_2 = 0$$

and $\gamma(\mathbf{A}) = 0$, so the null space only contains the zero vector.

Solutions of $Ax = y$– XII

Example–Consider $\boldsymbol{A} : (\Re^5, \Re) \longrightarrow (\Re^3, \Re)$ where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 2 \\ -1 & 3 & 2 & -4 & 6 \end{bmatrix}$$

Verify that $\rho(\boldsymbol{A}) = 2$ and that the first two columns of $\boldsymbol{A}$ are linearly independent. Note

$$Row3 = -Row1 + 3Row2$$

As a result of the above

$$\begin{aligned} x_1 + x_3 + x_4 &= 0 \\ x_2 + x_3 - x_4 + 2x_5 &= 0 \end{aligned}$$

Solutions of $Ax = y$– XIII

a set of solution is

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

The above vectors are linearly independent and every $\boldsymbol{x}$ for which $\boldsymbol{Ax} = \boldsymbol{0}$ must be a linear combination of the above vectors. Therefore, these vectors form a basis for $\mathscr{N}(\boldsymbol{A})$, and $\gamma(\boldsymbol{A}) = 3$.

Solutions of $Ax = y$– XIV

Theorem 3.1.5

Let $\boldsymbol{A}$ be an mxn matrix, then $\rho(\boldsymbol{A}) + \gamma(\boldsymbol{A}) = n$.

Corollary 3.1.6

The number of linearly independent solutions of $\boldsymbol{Ax} = \boldsymbol{0}$ is equal to $n - \rho(\boldsymbol{A})$, where n is the number of columns in $\boldsymbol{A}$.

Solutions of $Ax = y$– XV

Theorem 3.1.7 (Solutions of $\boldsymbol{Ax} = \boldsymbol{y}$)

Consider the set of algebraic equation described by

$$\boldsymbol{Ax} = \boldsymbol{y} \qquad \boldsymbol{A} \in \Re^{mxn}; \boldsymbol{x} \in \Re^n, \boldsymbol{y} \in \Re^m$$

Let $\boldsymbol{x}_s$ be a solution and suppose $\gamma(\boldsymbol{A}) = n - \rho(\boldsymbol{A})$ is the nullity of $\boldsymbol{A}$. If $\gamma = 0$, then $\boldsymbol{x}_s$ is unique. If however $\gamma > 0$, then $\forall \alpha_i \in \mathscr{F}$

$$\boldsymbol{x} = \boldsymbol{x}_s + \alpha_1 \boldsymbol{n}_1 + \alpha_2 \boldsymbol{n}_2 + \cdots + \alpha_\gamma \boldsymbol{n}_\gamma$$

represent solutions where $\boldsymbol{n}_i \in \mathscr{N}(\boldsymbol{A})$, $i = 1, 2, \cdots, \gamma$

Proof.

$$\boldsymbol{Ax} = \boldsymbol{Ax}_s + \cancelto{0}{\sum_{i=1}^{\gamma} \alpha_i \boldsymbol{An}_i} = \boldsymbol{y}$$

Solutions of $Ax = y$– XVI

Example 3.1.8

Consider
$$\begin{pmatrix} 1 & 2 & -2 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 1 & -1 & -2 \end{pmatrix} \boldsymbol{x} = \begin{pmatrix} 5 \\ -2 \\ 2 \\ 1 \end{pmatrix}$$

verify the followings:

$$\rho(\boldsymbol{A}) = 3 \Longrightarrow \gamma(\boldsymbol{A}) = 4 - \rho(\boldsymbol{A}) = 1 \text{ and } \boldsymbol{n} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \in \mathcal{N}(\boldsymbol{A})$$

Solutions of $Ax = y$– XVII

Furthermore,
$$\boldsymbol{x}_s = \begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \end{pmatrix}$$

is a solution. Hence, the above has infinitely many solutions and they are given by

$$\boxed{\boldsymbol{x} = \begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}} \quad \text{where } \alpha \in \Re$$