GENG 8010–Part 2 - Elements of Applied Linear Algebra

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   - Change of bases
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Outline of part 2– II

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Left and right inverses– I

Left Inverse

A matrix $\boldsymbol{A} \in \Re^{mxn}$ that satisfies $\boldsymbol{A}_L\boldsymbol{A} = \boldsymbol{I}$ is called the **left inverse** of $\boldsymbol{A}$, and $\boldsymbol{A}$ is said to be **left invertible**. Note that the left inverse in this case will be of dimension nxm.

Remark

The left inverse of a matrix does not have to be unique, and a matrix that has more than one left inverse, will have infinitely many.

Left and right inverses– II

Example–consider

$$\boldsymbol{A} = \begin{bmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{bmatrix}$$

The following two matrices are both left inverses of $\boldsymbol{A}$

$$\boldsymbol{A}_L^1 = \frac{1}{2}\begin{bmatrix} 0 & -1 & 6 \\ 0 & 1 & -4 \end{bmatrix} \qquad \boldsymbol{A}_L^2 = \frac{1}{3}\begin{bmatrix} -11 & -7 & -2 \\ 7 & 5 & 1 \end{bmatrix}$$

In both of the above cases

$$\boldsymbol{A}_L^{1,2}\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Left and right inverses– III

Recall that $\boldsymbol{A}\boldsymbol{x} = 0$ has a trivial solution, i.e. $\boldsymbol{x} = 0$ if its nullity, $\gamma(\boldsymbol{A}) = 0$ in which case that implies that $\boldsymbol{A}$ is full rank. Also knowing that $\boldsymbol{A}_L\boldsymbol{A} = \boldsymbol{I}$ means

$$\boldsymbol{A}_L\boldsymbol{A}\boldsymbol{x} = \boldsymbol{I}\boldsymbol{x} = \boldsymbol{x} = \boldsymbol{0}$$

which means that given $\boldsymbol{A}_L$ exists, $\boldsymbol{A}$ would be full rank and that columns of $\boldsymbol{A}$ are linearly independent.

Remark

1. If matrix $\boldsymbol{A}$ has a left inverse, then the columns of $\boldsymbol{A}$ are linearly independent.
2. The above implies that *a matrix with a left inverse is either square or tall.*

Left and right inverses– IV

Now going back to the problem of finding the solution of

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$$

If $\boldsymbol{A}$ is square, then

$$\boldsymbol{y} = \boldsymbol{A}^{-1}\boldsymbol{x}$$

and we have a unique solution for $\boldsymbol{x}$. However,

If $\boldsymbol{A}$ is tall, then

$$\boldsymbol{A}_L\boldsymbol{y} = \boldsymbol{A}_L(\boldsymbol{A}\boldsymbol{x}) = \boldsymbol{I}\boldsymbol{x} = \boldsymbol{x} \qquad \therefore \boxed{\boldsymbol{x} = \boldsymbol{A}_L\boldsymbol{y}}$$

Left and right inverses– V

Example–Find the solution $\boldsymbol{x}$ for

$$\begin{bmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$$

from the previous example, recall that both of the following are the left inverses of $\boldsymbol{A}$

$$\boldsymbol{A}_L^1 = \frac{1}{2}\begin{bmatrix} 0 & -1 & 6 \\ 0 & 1 & -4 \end{bmatrix} \qquad \boldsymbol{A}_L^2 = \frac{1}{3}\begin{bmatrix} -11 & -7 & -2 \\ 7 & 5 & 1 \end{bmatrix}$$

you can verify that

$$\boxed{\boldsymbol{x} = \boldsymbol{A}_L^1 \boldsymbol{y} = \boldsymbol{A}_L^2 \boldsymbol{y} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

Left and right inverses– VI

Right Inverse

A matrix $\boldsymbol{A} \in \Re^{mxn}$ that satisfies $\boldsymbol{A}\boldsymbol{A}_R = \boldsymbol{I}$ is called the **right inverse** of $\boldsymbol{A}$, and $\boldsymbol{A}$ is said to be **right invertible**. Note that the right inverse in this case will be of dimension nxm.

Left and right inverses– VII

Remark

1. Consider

$$\boldsymbol{A}\boldsymbol{A}_R = \boldsymbol{I} \Longrightarrow (\boldsymbol{A}\boldsymbol{A}_R)^T = \boldsymbol{I} \Longrightarrow \underbrace{\boldsymbol{A}_R^T}_{\boldsymbol{B}_L} \underbrace{\boldsymbol{A}^T}_{\boldsymbol{B}} = \boldsymbol{I}$$

   From the above it can be concluded that $\boldsymbol{A}$ is right invertible, then $\boldsymbol{A}^T$ must be left invertible.

2. From the above and previous discussion on left invertability of a matrix, we conclude that $\boldsymbol{A}$ is invertible if its rows are linearly independent.
3. From (2), we conclude that right invertible matrices are square or wide.

Left and right inverses– VIII

Example–Find the solution $\boldsymbol{x}$ for

$$\begin{bmatrix} -3 & 4 & 1 \\ -4 & 6 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} -6 \\ -9 \end{bmatrix}$$

From the last example, we can readily write down

$$\boldsymbol{A}_R^1 = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ -1 & 1 \\ 6 & -4 \end{bmatrix} \qquad \boldsymbol{A}_R^2 = \frac{1}{3}\begin{bmatrix} -11 & 7 \\ -7 & 5 \\ -2 & 1 \end{bmatrix}$$

$$\boxed{\boldsymbol{x} = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ -1 & 1 \\ 6 & -4 \end{bmatrix}\begin{bmatrix} -6 \\ -9 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}} \;;\; \boxed{\boldsymbol{x} = \frac{1}{3}\begin{bmatrix} -11 & 7 \\ -7 & 5 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -6 \\ -9 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}$$

2 equations and 3 unknowns $\Longrightarrow$ infinitely many other solutions.

Left and right inverses– IX

Remark

1. If $\boldsymbol{A}$ has both left and right inverse (say $\boldsymbol{B}$ and $\boldsymbol{C}$, respectively), then

$$\boldsymbol{BA} = \boldsymbol{I} \qquad \text{and} \quad \boldsymbol{AC} = \boldsymbol{I}$$

$$\boldsymbol{C} = \boldsymbol{IC} = (\boldsymbol{BA})\boldsymbol{C} = \boldsymbol{BI} = \boldsymbol{B}$$

so, in this case the left and right inverse are the same and the matrix must be square.

2. For a square matrix the following statements are equivalent (i.e. if any hold, others hold)
   - $\boldsymbol{A}$ is invertible.
   - Columns of $\boldsymbol{A}$ are linearly independent.
   - Rows of $\boldsymbol{A}$ are linearly independent.
   - $\boldsymbol{A}$ has a left and right inverse and they are equal.

Least square solution of $Ax = y$ – I

Consider again the equation

$$\boldsymbol{Ax} = \boldsymbol{y} \qquad \boldsymbol{A} \in \Re^{mxn}$$

Let us now try to determine the *"best"* solution to an inconsistent set as described above. To do this, let us define the *"best"* solution in the *least square* sense. That is, let us try to solve the following optimization problem

$$\begin{aligned} min_{\boldsymbol{x}} J = \|\boldsymbol{Ax} - \boldsymbol{y}\|^2 &= (\boldsymbol{Ax} - \boldsymbol{y})^T(\boldsymbol{Ax} - \boldsymbol{y}) \\ &= (\boldsymbol{x}^T\boldsymbol{A}^T - \boldsymbol{y}^T)(\boldsymbol{Ax} - \boldsymbol{y}) \\ &= \boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{Ax} - \boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{y} - \boldsymbol{y}^T\boldsymbol{Ax} + \boldsymbol{y}^T\boldsymbol{y} \end{aligned}$$

to find the minimum of J take $\frac{\partial}{\partial \boldsymbol{x}}$ and set equal to zero

Least square solution of $Ax = y$ – II

$$\boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{A} + \boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{A} - \boldsymbol{y}^T\boldsymbol{A} - \boldsymbol{y}^T\boldsymbol{A} = 0 \Longrightarrow \boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{y}$$

$$\boxed{\boldsymbol{x}^o = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{y}}$$

Where it is assumed that $\boldsymbol{A}$ is full rank, and hence, $\boldsymbol{A}^T\boldsymbol{A}$ is invertible.

Remark

1. Note that the *nxm* matrix $\boldsymbol{A}_L = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T$ is the left inverse of $\boldsymbol{A}$, since $\boldsymbol{A}_L\boldsymbol{A} = \boldsymbol{I}$.
2. The least square solution to $\boldsymbol{Ax} = \boldsymbol{y}$ when $\boldsymbol{A}$ is tall is $\boldsymbol{x}^o = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{A}_L\boldsymbol{y}$.

State estimation– I

Consider the measurement equation

$$\boldsymbol{z} = \boldsymbol{Hx} + \boldsymbol{v} \qquad \boldsymbol{x} \in \Re^n, \boldsymbol{z} \in \Re^m, m \gg n, \boldsymbol{v} \quad \text{is noise or error}$$

The objective is to obtain an optimum estimate of $\boldsymbol{x}$, i.e., $\hat{\boldsymbol{x}}$.

Note that the actual measurement should be $\boldsymbol{y} = \boldsymbol{Hx}$, however due to measurement errors or noise in the sensor collecting the data, at each sampling interval we obtain $\boldsymbol{z}$ which is corrupted by the error/noise. Hence we want

$$\hat{\boldsymbol{y}} = \boldsymbol{H}\hat{\boldsymbol{x}}$$

State estimation– II

Define the estimation error as

$$\boldsymbol{e} = \boldsymbol{z} - \hat{\boldsymbol{y}}$$

Obviously, the objective should be to obtain the best estimate of $\hat{\boldsymbol{x}}$ so that the estimation error is minimized in some sense. Therefore define a quadratic cost function as

$$\begin{aligned} J = \frac{1}{2}\boldsymbol{e}^T\boldsymbol{e} &= \frac{1}{2}(\boldsymbol{z} - \boldsymbol{H}\hat{\boldsymbol{x}})^T(\boldsymbol{z} - \boldsymbol{H}\hat{\boldsymbol{x}}) \\ &= \boldsymbol{z}^T\boldsymbol{z} - \boldsymbol{z}^T\boldsymbol{H}\hat{\boldsymbol{x}} - \hat{\boldsymbol{x}}^T\boldsymbol{H}^T\boldsymbol{z} + \hat{\boldsymbol{x}}^T\boldsymbol{H}^T\boldsymbol{H}\hat{\boldsymbol{x}} \end{aligned}$$

The objective is now to obtain the optimal estimate $\hat{\boldsymbol{x}}$ which would minimize the cost function J, and hence, minimizes the estimation error $\boldsymbol{e}$.

State estimation– III

By taking $\frac{\partial J}{\partial \hat{\boldsymbol{x}}}$ and setting it to zero, we obtain

$$\frac{\partial J}{\partial \hat{\boldsymbol{x}}} = \frac{1}{2}\left(-2\boldsymbol{z}^T\boldsymbol{H} + 2\hat{\boldsymbol{x}}^T\boldsymbol{H}^T\boldsymbol{H}\right) = 0 \Longrightarrow \boldsymbol{H}\boldsymbol{z} = \boldsymbol{H}^T\boldsymbol{H}\hat{\boldsymbol{x}}$$

$$\boxed{\hat{\boldsymbol{x}} = (\boldsymbol{H}^T\boldsymbol{H})^{-1}\boldsymbol{H}^T\boldsymbol{z}}$$

and this gives

$$\boxed{\hat{\boldsymbol{y}} = \boldsymbol{H}(\boldsymbol{H}^T\boldsymbol{H})^{-1}\boldsymbol{H}^T\boldsymbol{z}}$$

Remark

An estimate can be obtained with $m = n$ measurements. However, for a better estimate and for averaging to occur, we would ideally need a larger sample of measurement and $m \gg n$. Therefore, we wish for $\boldsymbol{H}$ to be tall rather than square.

State estimation– IV

Example–

The above has many useful applications. To illustrate one, suppose we have collected a bunch of data from an experiment and wish to fit a cubic polynomial to the data, i.e.,

$$y(t) = \alpha_1 + \alpha_2 t + \alpha_3 t^2 + \alpha_4 t^3$$

This problem can be cast into the formulation. Assuming that our data is contained in a vector $\boldsymbol{z}$, we reformulate the problem as a least square problem just discussed.

State estimation– V

$$\boldsymbol{z} = \begin{bmatrix} 1 & t_1 & t_1^2 & t_1^3 \\ 1 & t_2 & t_2^2 & t_2^3 \\ 1 & t_3 & t_3^2 & t_3^3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & t_m & t_m^2 & t_m^3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{bmatrix}$$

It is now possible to use least square estimation to find the optimum α, i.e. $\hat{\alpha}$.

In our case, consider the set of data given in the table

State estimation– VI

Time	y(t)	Time	y(t)
0	-226.1	11	366.6
1	27.17	12	328.2
2	227.6	13	313.8
3	330.7	14	323.2
4	430.8	15	326.9
5	494.1	16	358.1
6	519.2	17	460.2
7	518.9	18	587.7
8	453.4	19	797
9	424.5	20	1007
10	423.9		

Table 1: A set of noisy data

State estimation– VII

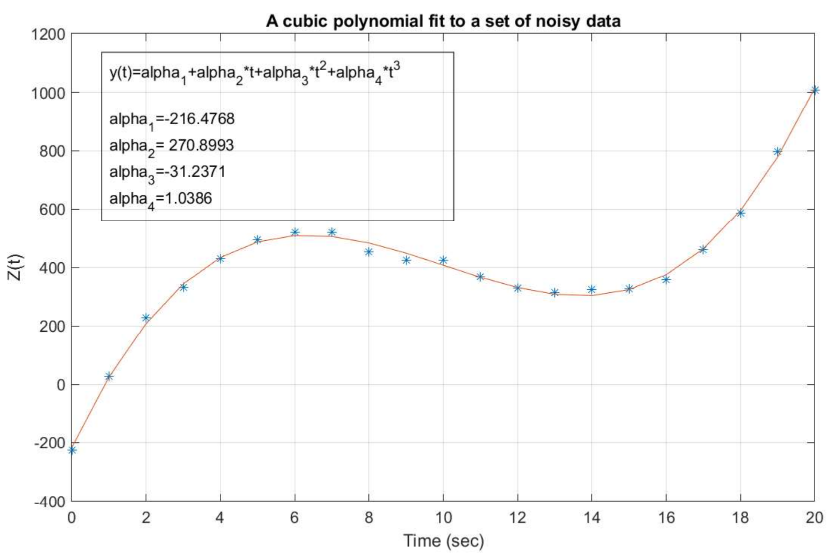
The data ranges from $t = 0$ to $t = 20$ seconds and has been sampled each second for a total of 21 data points. Therefore

$$\boldsymbol{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 20 & 400 & 8000 \end{bmatrix}_{21 \times 4}$$

Now

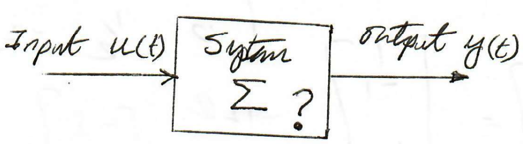
$$\hat{\alpha} = (\boldsymbol{H}^T \boldsymbol{H})^{-1} \boldsymbol{H}^T \boldsymbol{z} = \begin{bmatrix} -216.4768 \\ 270.8993 \\ -31.2371 \\ 1.0386 \end{bmatrix}$$

State estimation– VIII



Dynamical system identification– I

Example–As a second example, consider the problem of identification of an unknown dynamical system



Let us assume that the system is described by the following second order difference equation with unknown parameter which we wish to identify using a set of input/output data.

Dynamical system identification– II

$$y(k+2) + a_1 y(k+1) + a_0 y(k) = b_2 u(k+2) + b_1 u(k+1) + b_0 u(k)$$

where $u(k)$ is the system's input at time instant k, and $y(k)$ is the output. Assume further that the system is *relaxed and causal* and that $y(-i) = u(-i) = 0 \forall i$. Then

$$\begin{aligned}
y(0) &= b_2 u(0) \\
y(1) &= -a_1 y(0) - b_2 u(1) + b_1 u(0) \\
y(2) &= -a_1 y(1) - a_0 y(0) + b_2 u(2) + b_1 u(1) + b_0 u(0) \\
y(3) &= -a_1 y(2) - a_0 y(1) + b_2 u(3) + b_1 u(2) + b_0 u(1) \\
\vdots &= \qquad \vdots
\end{aligned}$$

Dynamical system identification– III

The above can be put into the following matrix formulation

$$\begin{bmatrix} 0 & 0 & u(0) & 0 & 0 \\ -y(0) & 0 & u(1) & u(0) & 0 \\ -y(1) & -y(0) & u(2) & u(1) & u(0) \\ -y(2) & -y(1) & u(3) & u(2) & u(1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} a_1 \\ a_0 \\ b_2 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ \vdots \end{bmatrix}$$

Note that the above is in the form of

$$\boldsymbol{Ax} = \boldsymbol{y}$$

Dynamical system identification– IV

If $\boldsymbol{A}$ is tall, it has a solution of the form

$$\boldsymbol{x} = \boldsymbol{A}_L \boldsymbol{y} = (\boldsymbol{A}^T \boldsymbol{A})^{-1} \boldsymbol{A}^T \boldsymbol{y}$$

Remark

I the set of measurement are consistent, then five measurements will do the job. However, if the data contains error, then more measurements are needed in order to get a good estimate of the system's parameters.

Dynamical system identification– V

Now consider the following consistent set of measured data

k	$u(k)$	$y(k)$
0	1	-1
1	2	4
2	3	-2
3	4	15
4	5	-13
5	6	50
6	7	-68
7	8	177
8	9	-303
9	10	668

Dynamical system identification– VI

Given the consistent set of measurements

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 \\ -4 & 1 & 3 & 2 & 1 \\ 2 & -4 & 4 & 3 & 2 \\ -15 & 2 & 5 & 4 & 3 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} -1 \\ 4 \\ -2 \\ 15 \\ -13 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} a_1 \\ a_0 \\ b_2 \\ b_1 \\ b_0 \end{bmatrix} = \boldsymbol{A}^{-1} \begin{bmatrix} -1 \\ 4 \\ -2 \\ 15 \\ -13 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 5 \\ -3 \end{bmatrix}$$

Now consider the case of noisy measurement, and consider the following set of data obtained from a second order system

Dynamical system identification– VII

k	$u(k)$	Noisy output $y(k)$
0	1	-0.7810
1	2	3.8281
2	3	-0.7113
3	4	14.0468
4	5	-8.5347
5	6	44.0117
6	7	-52.5617
7	8	150.4161
8	9	-245.5048
9	10	557.3905

Dynamical system identification– VIII

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1.0000 & 0 & 0 \\ 0.7810 & 0 & 2.0000 & 1.0000 & 0 \\ -3.8281 & 0.7810 & 3.0000 & 2.0000 & 1.0000 \\ 0.7113 & -3.8281 & 4.0000 & 3.0000 & 2.0000 \\ -14.0468 & 0.7113 & 5.0000 & 4.0000 & 3.0000 \\ 8.5347 & -14.0468 & 6.0000 & 5.0000 & 4.0000 \\ -44.0117 & 8.5347 & 7.0000 & 6.0000 & 5.0000 \\ 52.5617 & -44.0117 & 8.0000 & 7.0000 & 6.0000 \\ -150.4161 & 52.5617 & 9.0000 & 8.0000 & 7.0000 \\ 245.5048 & -150.4161 & 10.0000 & 9.0000 & 8.0000 \end{bmatrix}$$

$$\boldsymbol{x} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{y} = \begin{bmatrix} 1.0975 \\ -1.8071 \\ -0.7810 \\ 4.6403 \\ -2.2203 \end{bmatrix}$$

Dynamical system identification– IX

Thus our system is identified to be described by

$$\begin{aligned} & y(k+2) + 1.0975y(k+1) - 1.8070y(k) \\ & \quad = -0.7810u(k+2) + 4.6403u(k+1) - 2.2202u(k) \end{aligned}$$

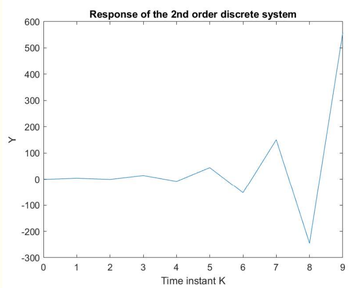
Dynamical system identification– X

Remark

Note that the above difference equation has a characteristic equation

$$\beta^2 + 1.0975\beta - 1.8070 = 0$$

with roots at -2 and 0.9, and unstable, as can be seen



Dynamical system identification– XI

Example 4.2.1

Consider now the dynamical system described by the following third order difference equation

$$y(k+3) + 1.1y(k+2) + 0.09y(k+1) - 0.445y(k) = \sin 4k \qquad \text{(A)}$$

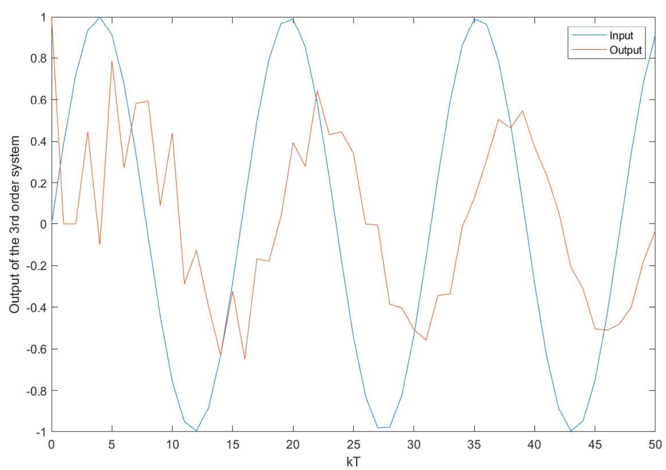
Also $y(0) = 1$.

The characteristic polynomial for the above difference equation are given by the roots of

$$\beta^3 + 1.1\beta^2 + 0.09\beta - 0.445 = 0$$

resulting in three roots at $\beta_1 = 0.5$, and $\beta_{2,3} = -0.8 \pm j0.5$

Dynamical system identification– XII



Dynamical system identification– XIII

A set of noisy measurement from the above system is given, and suppose that we have decided to identify a second order (reduced) order model to represent the data. Therefore, we plan to model the system as

$$y_m(k+2) + a_1 y_m(k+1) + a_0 y_m(k) = b_0 u(k)$$

As such, we can see that

$$\begin{aligned}
y(0) &= 1 \\
y(1) &= -a_1 y(0) \\
y(2) &= -a_1 y(1) - a_0 y(0) + b_0 u(0) \\
y(3) &= -a_1 y(2) - a_0 y(1) + b_0 u(1) \\
y(4) &= -a_1 y(3) - a_0 y(2) + b_0 u(2) \\
\vdots &= \qquad \vdots
\end{aligned}$$

Dynamical system identification– XIV

or

$$\begin{bmatrix} 0 & 0 & 0 \\ -y(0) & 0 & 0 \\ -y(1) & -y(0) & u(0) \\ -y(2) & -y(1) & u(1) \\ -y(3) & -y(2) & u(2) \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} a_1 \\ a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ \vdots \end{bmatrix}$$

Finally,

$$\begin{bmatrix} a_1 \\ a_0 \\ b_0 \end{bmatrix} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{y} = \begin{bmatrix} 0.0851 \\ -0.3798 \\ 0.4432 \end{bmatrix}$$

Hence, the reduced order identified model is

$$y_m(k+2) + 0.0851y_m(k+1) - 0.3798y_m(k) = 0.4432u(k) \qquad \text{(B)}$$

Dynamical system identification– XV

The above reduced order model has a characteristic polynomial given by

$$\beta_m^2 + 0.0851\beta_m - 0.3798 = 0$$

with roots of the reduced order model at $\beta_{m1} = -0.6603$ and $\beta_{m2} = 0.5752$.

Dynamical system identification– XVI

Remark

It is desired for the reduced order model (B) to more or less capture the dynamics of the third order system (A). The plot confirms this.