Lecture Notes 13th June 2023

- Power Series

- 2 Questions for Midterm – 2
- 1 Question for Final Exam

Theory:

A numerical series is of the form:

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$$

Example:

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$$

- This is a geometric series
- Sum of an infinite geometric series:

$$\frac{1}{1-r}$$

- For the example above:

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \approx \frac{1}{1-\frac{1}{2}} \approx 2$$

- If you get an answer – The series is said to be convergent.
- If you don't get an answer – The series is said to be divergent.
- The example above is convergent.

$$\sum_{k=1}^{\infty} k = 1 + 2 + 3 + \cdots$$

- This example is divergent.

Power Series:

A power series (in one variable) is an infinite series of the form:

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots$$

- A power series is an infinite polynomial.
- The most important example of Power Series is called **Taylor Series**.

Idea:

- The easiest function in mathematics/engineering is a polynomial.
- With Taylor Series we can approximation a function to a polynomial.

Remark:

- Any nice function f(x) has a Taylor Series approximation.
- Nice: All derivatives at x_0 exist.
- Such a function is called a Smooth Function.

Taylor Series of f(x) at center x_0:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x - x_0)^n$$

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots$$

- Taylor Series is a special case of Power Series.

Find Taylor Series of:

Example 1:

$$f(x) = e^{x};\ x_0 = 0$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(x_0)}{n!}(x - x_0)^{n}$$

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots$$

$$f(x) = e^{x}; f(0) = 1$$

$$f'(x) = e^{x}; f'(0) = 1$$

$$f''(x) = e^{x}; f''(0) = 1$$

$$e^{x} = f(0) + \frac{f'(0)}{1!}(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 + \cdots$$

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^{n}}{n!}$$

Application of Power Series:

- Calculator/Computer.

Example 2: Find $\int e^{x^2} dx$

By Taylor Series,

$$e^x = f(0) + \frac{f'(0)}{1!}(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$x ==> x^2$$

$$e^{x^2} = 1 + x^2 + \frac{{x^2}^2}{2!} + \frac{{x^3}^2}{3!} + \cdots$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

$$\int e^{x^2} dx = \int 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

$$\int e^{x^2} dx = x + \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42} + \cdots$$

This is how a Calculator/Computer computes $\int e^{x^2} dx$

Application of Power Series:

- Solving ODEs.

Theorem:

$$y'' + p(x)y' + q(x)y = 0$$

If $p(x)$ and $q(x)$ have power series approximations then the solution of ODE, y, has a power series approximation.

Example 3: Solve using power series method

$$y'' + y = 0;\ x_0 = 0$$

Here, $p(x)$ and $q(x)$ are polynomials. Therefore, they have power series approximations

Therefore, solution 'y' of the ODE also has a power series approximation

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

$$y = \sum_{n=0}^{\infty} a_n x^n \dots (x_0 = 0)$$

$$y' = \sum_{n=1}^{\infty} a_n . n. x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n . n(n-1). x^{n-2}$$

Substituting these in the given ODE we get,

$$\sum_{n=2}^{\infty} a_n . n(n-1). x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2}\,.\,(n+2)(n+1).\,x^n + \sum_{n=0}^{\infty} a_n\, x^n = 0 \;\; \dots [\sum_{n=2}^{\infty} a_n = \sum_{n=0}^{\infty} a_{n+2}]$$

$$\sum_{n=0}^{\infty} [a_{n+2}\,.\,(n+2)(n+1) + a_n]x^n = 0 \;\; \dots [\sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} a_n + b_n]$$

$$a_{n+2}.\,(n+2)(n+1) + a_n = 0$$

$$n = 0;\; 2a_2 + a_0 = 0;\; a_2 = -\frac{1}{2}a_0$$

$$n = 1;\; 6a_3 + a_1 = 0;\; a_3 = -\frac{1}{6}a_1$$

$$n = 2;\; 12a_4 + a_2 = 0;\; a_4 = -\frac{1}{12}a_2 = \frac{1}{24}a_0$$

$$n = 3;\; 20a_5 + a_3 = 0;\; a_5 = -\frac{1}{20}a_3 = \frac{1}{120}a_1$$

$$\vdots$$

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$

$$y = a_0 + a_1x + (-\frac{1}{2}a_0)x^2 + (-\frac{1}{6}a_1)x^3 + (\frac{1}{24}a_0)x^4 + (\frac{1}{120}a_1)x^5 + \cdots$$

$$y = a_0(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots) + a_1(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots)$$

$$y = a_0(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots) + a_1(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots)$$

$$y = a_0 \cos x + a_1 \sin x$$

Useful Techniques:

$$\sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} a_n + b_n$$

$$\sum_{n=2}^{\infty} a_n = \sum_{n=0}^{\infty} a_{n+2}$$

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_{n-1} = a_{-1} + a_0 + a_1 + a_2 + \cdots ; \text{If } a_{-1} = 0$$