GENG 8010–Part 1: Elements of Differential and Difference Equations

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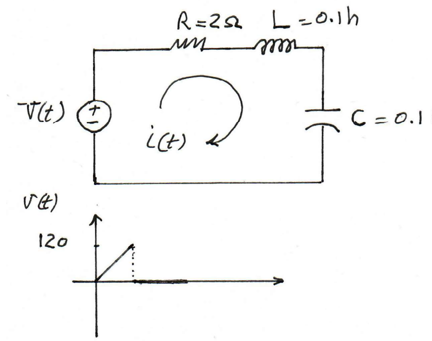
Properties and further definitions XIII

Integro–differential equations– Let us illustrate this class of problems with the following simple RLC circuit

The circuit is described by

$$v(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt$$

The above is an integro-differential equation.



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$$v(t) = 120t - 120tu(t-1)$$

Before taking Laplace Transform, we shall write $v(t)$ as:

$$\begin{aligned} v(t) &= 120t - 120(t-1+1)u(t-1) \\ &= 120t - 120(t-1)u(t-1) - 120u(t-1) \end{aligned}$$

giving

$$V(s) = 120\left(\frac{1}{s^2} - e^{-s}\left[\frac{1}{s^2} + \frac{1}{s}\right]\right)$$

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Taking the Laplace transform of the integro-differential equation gives (with $i(0) = 0$)

$$2I(s) + 0.1sI(s) + 10\frac{I(s)}{s} = 120\left[\frac{1}{s^2} - \left(\frac{e^{-s}}{s^2} + \frac{e^{-s}}{s}\right)\right]$$

$$\begin{aligned} I(s) = 1200\Bigg[&\frac{0.01}{s} - \frac{0.01}{s+10} - \frac{0.1}{(s+10)^2} - \frac{0.01}{s}e^{-s} + \frac{0.01}{s+10}e^{-s} \\ &+ \frac{0.1}{(s+10)^2}e^{-s} - \frac{1}{(s+10)^2}e^{-s}\Bigg] \end{aligned}$$

Properties and further definitions XVI

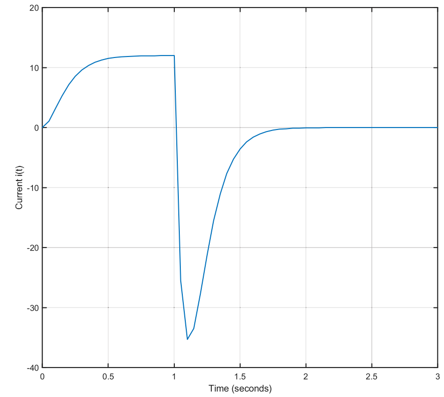
leading to

$$\begin{aligned} i(t) &= 12\left[1-u(t-1)\right] - 12\left[e^{-10t} - e^{-10(t-1)}u(t-1)\right] \\ &\quad - 120te^{-10t} - 1080(t-1)e^{-10(t-1)}u(t-1) \end{aligned}$$

$$i(t) = \begin{cases} 12 - 12e^{-10t} - 120te^{-10t}, & 0 \le t < 1 \\ -12e^{-10t} + 12e^{-10(t-1)} - 120te^{-10t} - 1080(t-1)e^{-10(t-1)}, & t \ge 1 \end{cases}$$

Properties and further definitions XVII

The MATLAB response of the current in response to the applied voltage (solution of the differential equation for the given forcing function) is given below.



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Theorem 8.2.7

Transform of an integral– *Let $f(t)$ be of exponential order and $\mathscr{L}\{f(t)\} = F(s)$, then*

$$\mathscr{L}\{\int_0^t f(\tau)d\tau\} = \frac{F(s)}{s} \qquad s > k$$

Theorem 8.2.8

Integral of a transform– *Let $\frac{f(t)}{t}$ be of exponential order and $\mathscr{L}\{\frac{f(t)}{t}\} = G(s)$, for $s > k$ and $\mathscr{L}\{f(t)\} = F(s)$, then*

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(x)dx$$

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Theorem 8.2.9

Property of Dirac Delta Function– *Assuming $f(t)$ is defined and integrable and is continuous in the vicinity of a, then*

$$\int_0^\infty f(t)\delta(t-a)dt = f(a)$$

$$\mathscr{L}\{\delta(t-a)\} = e^{-as}$$

$$\mathscr{L}\{\delta(t)\} = 1$$

Summary: advantages of Laplace transform

1) Solution is routine an progresses systematically; 2) it gives the total solution; and 3) initial conditions are taken care of in the process.

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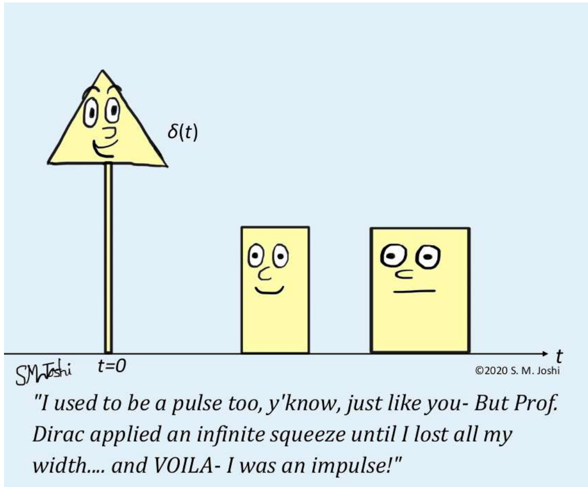


Some system engineering concepts I

Some defentions

[System] *A system is a collection of things which are related in such a way which make sense to think of them as whole.*

[Single vs. multivariable] *A system is single variable iff* $\exists$ *one input and one output. Otherwise, it is multivariable.*

[Relaxedness] *A system is relaxed at time* t_0 *iff the output* $y_{[t_0,\infty]}$ *is solely and uniquely defined by input* $u_{[t_0,\infty]}$.

[Memoryless] *A system is zero memory if* $y(t_1)$ *depends only on* $u(t_1)$.

[Memory] *A system has "memory" if* $y(t_1)$ *depends on* $u(t_1)$ *as well as* $u(t)$, $t < t_1$.

Some system engineering concepts II

Remark

If a system has memory, it is impossible to determine $y(t)$ without assuming the system has been relaxed or at rest. In that case,

$$y(t) = \mathscr{H}u(t)$$

where $\mathscr{H}$ is some operator mapping the input space into output space. $\mathscr{H} : \{u(t)\} \longrightarrow \{y(t)\}$, or $\mathscr{H}$ is the **mathematical model of the system**.

Some system engineering concepts III

[Linearity] *A system is linear iff*

$$\mathscr{H}\left[(a_1u_1(t) + a_2u_2(t)\right] = a_1\mathscr{H}u_1(t) + a_2\mathscr{H}u_2(t)$$

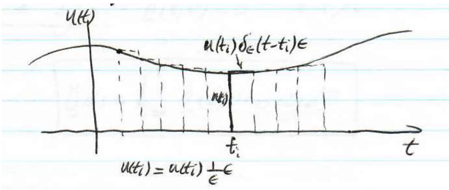
[Causality] *A system is causal or non-anticipatory if the output of the system at time* t *does not depend on inputs applied after* t.

[Time invariance] *If the system's characteristic remains the same with time, then its is <u>time invariant or stationary</u>*

$$\mathscr{H}\mathscr{S}_\tau u(t) = \mathscr{S}_\tau\mathscr{H}u(t)$$

where $\mathscr{S}_\tau$ *is the shifting operator, i.e.,* $\mathscr{S}_\tau u(t) = u(t+\tau)$. *Otherwise, the system is <u>time varying</u>.*

Some system engineering concepts IV



Consider the input $u(t)$ as in the figure on the left. Every piecewise continuous function can be approximated by a series of pulse functions, and

$$u(t) \cong \sum_i u(t_i)\delta_\epsilon(t-t_i)\epsilon$$

$$y(t) = \mathscr{H}u(t) \cong \sum_i \mathscr{H}u(t_i)\delta_\epsilon(t-t_i)\epsilon$$

Some system engineering concepts V

As $\epsilon \longrightarrow 0$, the approximation will tend to exact equality and the summation will become an integration

$$y(t) = \int_{-\infty}^{\infty} \mathscr{H}\delta(t-\tau)u(\tau)d\tau$$

$$h(t,\tau) = \mathscr{H}\delta(t-\tau)$$

Response for time-varying non-causal single variable system

$$y(t) = \int_{-\infty}^{\infty} h(t,\tau)u(\tau)d\tau$$

for a causal system $h(t,\tau) = 0 \;\forall\; \tau > t$, so

Some system engineering concepts VI

Response for time-varying causal single variable system

$$y(t) = \int_{-\infty}^{t} h(t,\tau)u(\tau)d\tau$$

If the system is relaxed at t_0 at which time the input is applied

Response for time-varying relaxed causal single variable system

$$y(t) = \int_{t_0}^{t} h(t,\tau)u(\tau)d\tau$$

Some system engineering concepts VII

Theorem 8.3.1

Relaxed system– *A system is relaxed at t_0 iff $u_{[t_0,\infty)} = 0$ implies* $y_{[t_0,\infty)} = 0$.

For linear time invariant (LTI) systems $h(t,\tau) = h(t-\tau)$, so

Response for LTI relaxed causal single variable system

$$y(t) = \int_{t_0}^{t} h(t-\tau)u(\tau)d\tau$$

Some system engineering concepts VIII

Assume $t_0 = 0$, also let $t - \tau = \lambda \longrightarrow -d\tau = d\lambda$ and at $\tau = 0 \to t = \lambda$ and $\tau = t \to \lambda = 0$ which gives.

$$y(t) = \int_{\lambda}^{0} h(\lambda)u(t-\lambda)(-d\lambda) = \int_{0}^{\lambda} h(\lambda)u(t-\lambda)d\lambda$$

Response of LTI single variable system

$$y(t) = \int_{0}^{t} h(\tau)u(t-\tau)d\tau$$

Compare to previous discussion involving the Green Function

Some system engineering concepts IX

Taking the Laplace transform of the above

I/O relation through transfer function

$$Y(s) = H(s)U(s)$$

and

$$H(s) = \frac{Y(s)}{U(s)} = \mathscr{L}\{h(t)\}$$

First order response I

Consider the following first order system

$$\frac{dx(t)}{dt} + a_0 x(t) = b_0 r(t)$$

Taking Laplace transform

$$sX(s) - x(0) + a_0 X(s) = b_0 R(s) \Longrightarrow (s + a_0)X(s) = b_0 R(s) + x(0)$$

Response of 1st order system

$$X(s) = \underbrace{\frac{b_0}{s+a_0}R(s)}_{\text{zero state response}} + \underbrace{\frac{x(0)}{s+a_0}}_{\text{zero input response}}$$

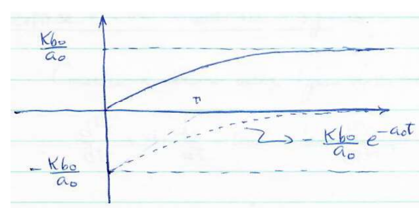
First order response II

$$H(s) = \mathscr{L}\left\{\frac{output}{input}\right\} = \frac{X(s)}{R(s)} = \frac{b_0}{s+a_0}, \quad with \;\; x(0) = 0$$

Considering $r(t) = Ku(t) \Longrightarrow R(s) = \frac{K}{s}$

If $x(0) = 0$

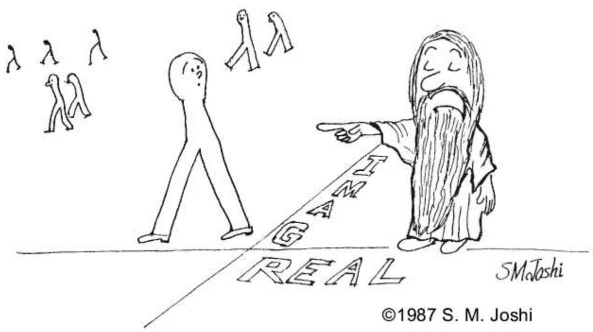
$$X(s) = \frac{Kb_0}{s(s+a_0)} = \frac{Kb_0/a_0}{s} + \frac{-Kb_0/a_0}{s+a_0}$$



$$\boxed{x(t) = \frac{Kb_0}{a_0}\left(1 - e^{-a_0 t}\right)}$$

If $a_0 > 0$ we say that the system is **stable** and $-a_0$ is the **pole** of the system.

First order response III

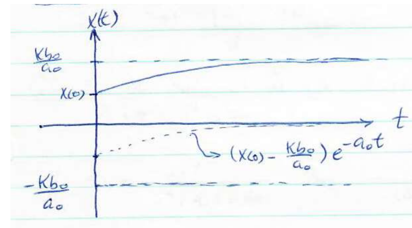


"Go West, young man!"

First order response IV

<u>If $x(0) \neq 0$</u>

$$X(s) = \frac{Kb_0}{s(s+a_0)} + \frac{x(0)}{s+a_0} = \frac{Kb_0}{a_0}\left(\frac{1}{s}\right) + \left(x(0) - \frac{Kb_0}{a_0}\right)\left(\frac{1}{s+a_0}\right)$$



$$\boxed{x(t) = \frac{Kb_0}{a_0} + \left(x(0) - \frac{Kb_0}{a_0}\right) e^{-a_0 t}}$$

<u>Remark</u>– In both cases $x_{ss} = \frac{Kb_0}{a_0}$.

<u>Time Constant</u>–The value of time that makes the exponent of e equal to -1 is called the time constant τ, therefore $\tau = \frac{1}{a_0}$ is the time interval over which the exponential decays by a factor of $\frac{1}{e} = 0.368$.

Second order response I

Consider a second order system described by

$$\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0 x = b_1\frac{dr}{dt} + b_0 r$$

$$X(s) = \underbrace{\frac{b_1 s + b_0}{s^2 + a_1 s + a_0}}_{=\text{ H(s)}} R(s) + \frac{\text{1st order polynomial involving IC}}{s^2 + a_1 s + a_0}$$

The characteristic polynomial for this system is

$$\Delta(s) = s^2 + a_1 s + a_0 = 0 \quad \text{with} \quad s_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}$$

<u>Case 1</u>–$a_1^2 > 4a_0 \Longrightarrow$ distinct and real roots s_1, s_2, in which case,

Second order response II

Overdamped system response

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

<u>Example</u>–

$$H(s) = \frac{12}{s^2 + 4s + 3} \quad \text{with} \quad R(s) = \frac{1}{s} \quad \text{and} \quad I.C. = 0$$

$$x(t) = \mathscr{L}^{-1}\left\{\frac{4}{s} + \frac{2}{s+3} - \frac{6}{s+1}\right\}$$

$$\boxed{x(t) = 4 + 2e^{-3t} - 6e^{-t}}$$

Second order response III

Case 2–$a_1^2 = 4a_0 \implies$ repeated roots $s_{1,2} = s_1$, in which case,

$$X(s) = \frac{s+b}{(s-s_1)^2} = \frac{c_1}{s-s_1} + \frac{c_2}{(s-s_1)^2}$$

Critically damped system response

$$x(t) = c_1 e^{s_1 t} + c_2 t e^{s_1 t}$$

Example–

$$H(s) = \frac{9}{s^2+6s+9} \quad \text{with} \quad R(s) = \frac{1}{s} \quad \text{and} \quad I.C. = 0$$

$$X(s) = \frac{1}{s} - \frac{1}{s+3} - \frac{3}{(s+3)^2}$$

Second order response IV

$$\boxed{x(t) = 1 - (1+3t)e^{-3t}}$$

Case 3–$a_1^2 < 4a_0 \implies$ complex conjugate pairs $s_{1,2} = \sigma \pm j\omega$, in which case,

$$X(s) = \frac{s+b}{(s+\sigma)^2 + \omega^2}$$

note that if

$$X(s) = \frac{N}{(s+\sigma-j\omega)(s+\sigma+j\omega)} = \frac{K_1}{s+\sigma-j\omega} + \frac{K_1^*}{s+\sigma+j\omega}$$

and if $K_1 = Ae^{j\theta}$ then $x(t) = 2Ae^{-\sigma t}\cos(\omega t + \theta)$

Underdamped system response

$$x(t) = 2Ae^{-\sigma t}\cos(\omega t + \theta)$$

Second order response V

Example–

$$H(s) = \frac{-3s+17}{s^2+2s+17} \quad \text{with} \quad R(s) = \frac{1}{s} \quad \text{and} \quad I.C. = 0$$

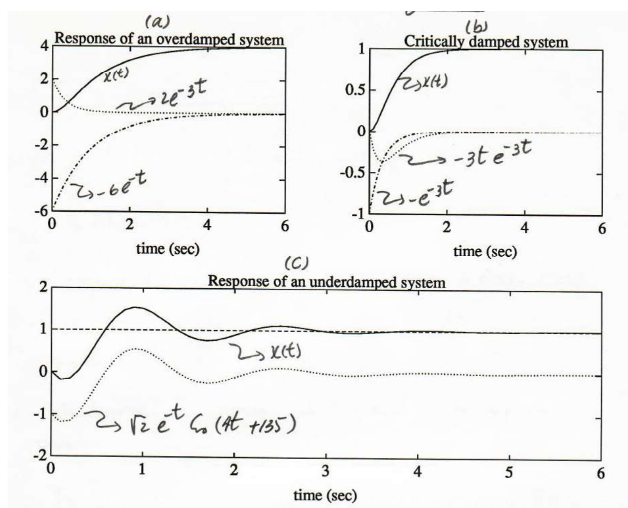
$$X(s) = \frac{-3s+17}{s(s^2+2s+17)} = \frac{1}{s} - \frac{s+5}{(s+1)^2+(4)^2}$$
$$= \frac{1}{s} - \left(\frac{K_1}{s+1-j4} + \frac{K_1^*}{s+1+j4}\right)$$

This gives $K_1 = \sqrt{\frac{1}{2}}\underline{/45^\circ}$

$$\boxed{x(t) = 1 + \sqrt{2}e^{-t}\cos(4t+135)}$$

The response of the three example cases are plotted below

Second order response VI



Second order response VII