Part I-Outline II

- Complex conjugate roots
- Solution of the non-homogeneous equation
 - Method of undetermined coefficients
 - Variation of parameters
 - Green Functions
- 8 Laplace transforms
 - Definition and transforms
 - Existence and Properties of $\mathcal{L}\{f(t)\}$
 - System engineering review
 - Response of system
 - Resonance
- Oifference equations
 - Difference and anti-difference operators
 - Solution of difference equation
 - System engineering concepts



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Part I-Outline III

- Z transform
 - Definitions, transforms, properties
 - ullet Applications of ${\mathscr Z}$



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The integrating factor method: linear case XIII

integrating (the right side by parts)[†]

$$xy^{3}e^{-y} = -2 \int y^{2}e^{-y}dy$$
$$= 2y^{2}e^{-y} + 4ye^{-y} + 4e^{-y} + c$$

and the implicit solution is given by

$$xy^3 = 2y^2 + 4y + 4 + ce^y$$

$$\frac{1}{\int x^2 e^{ax} dx = e^{ax} \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right)}$$



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The integrating factor method: nonlinear case I

Let us start again with

$$M(x,y)dx + N(x,y)dy = 0 (3.3)$$

Suppose u(x) is an integrating factor for (3.3), and $\frac{\partial u}{\partial y}=0$, and $\frac{\partial u}{\partial x}=\frac{du}{dx}$. Multiplying the (3.3) with u(x) gives

$$uMdx + uNdy = 0$$

If the above is to be exact then $\frac{\partial}{\partial y}(uM) = \frac{\partial}{\partial x}(uN)$, and

$$u\frac{\partial M}{\partial y} + M\frac{\partial u}{\partial y} = u\frac{\partial N}{\partial x} + N\frac{\partial u}{\partial x}$$



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The integrating factor method: nonlinear case IV

Example-Solve for the solution of

$$y(x + y + 1)dx + x(x + 3y + 2)dy = 0$$

Calculate

$$\frac{\partial M}{\partial y} = x + 2y + 1$$
 $\frac{\partial N}{\partial x} = 2x + 3y + 2$

and

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -x - y - 1$$

and

$$\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = -\frac{x + y + 1}{x(x + 3y + 2)}$$

Note that the above right side is not a function of x only as required in Rule 1.



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The integrating factor method: nonlinear case V

So now calculate

$$\frac{1}{M} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = -\frac{x + y + 1}{y(x + y + 1)} = -\frac{1}{y}$$

So now we see that the above right side is a function of y only and therefore Rule 2 applies and $e^{\ln|y|} = |y|$ is the IF. In other words

$$IF = \begin{cases} y & if \quad y > 0 \\ -y & if \quad y < 0 \end{cases}$$

In either case

$$(xy^2 + y^3 + y^2)dx + (x^2y + 3xy^2 + 2xy)dy = 0$$



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More on determination of an integrating factor

consider four commonly encountered exact differentials below

$$d(xy) = xdy + ydx \qquad d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$$
$$d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2} \qquad d\left(\tan^{-1}\frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$$

based on the above, group the equation as

$$ydx + (x + x^3y^2)dy = (ydx + xdy) + x^3y^2dy = d(xy) + x^3y^2dy = 0$$

or

$$\frac{1}{(xy)^3}d(xy)+\frac{1}{y}dy=0$$

which is integrable, so

$$-\frac{1}{2x^2y^2} + \ln|y| + \ln|c| = 0 \Rightarrow \boxed{2x^2y^2 \ln|cy| = 1}$$



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Another useful trick I

Sometimes a change of variables can result in a transformation of the differential equation in a solvable form. The differential equation itself can be explored for this purpose. Let's illustrate this through an example.

Example- Solve

$$(x + 2y - 1)dx + 3(x + 2y)dy = 0$$

Notice the term x + 2y has occurred twice. So let's define

$$v = x + 2y \Rightarrow dx = dv - 2dy$$

and subsititue to get

$$(v-1)(dv-2dy)+3vdy=0$$

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Bernoulli's Equation I

Bernoulli's Equation is a well know equation that is of the class of equations we have considered, and is given by

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{3.4}$$

- We have dealt already with the case of n=1, in (B) for which the variables are separable.
- So consider the cases when $n \neq 1$. (3.4) can be put into that form.

$$y^{-n}dy + Py^{-n+1}dx = Qdx$$



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Bernoulli's Equation II

Let

$$r = v^{-n+1}$$

$$dr = (1-n)y^{-n}dy \Longrightarrow dy = \frac{1}{(1-n)y^{-n}}dr$$

Therefore

$$dr + (1-n)Prdx = (1-n)Qdx$$

Hence, the Bernoulli Equation can be reduced with a change of variable to a standard form that is easily solvable using the techniques already discussed.



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Nature of solution(s)- I

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We already established that any *n*th order linear differential equation can be written as

$$(a_nD^n + a_{n-1}D^{n-1} + \ldots + a_1D + a_0)y(t) = F(t)$$

The above is called a **inhomogeneous differential equation**. Setting the right side equal to 0, we get the homogeneous differential equation

$$(a_n D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0) y(t) = 0$$
 (4.1)

It turns out that the above differential equation will have at most, nlinearly independent solutions.



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Nature of solution(s)- II

Theorem 4.1.1

A necessary and sufficient condition for the n solutions of (4.1) to be independent is that their Wronskian does not vanish. That is if y_1, y_2, \dots, y_n represent the n solutions, the Wronskian W(t) is nonzero, where

$$W(t) = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ Dy_1 & Dy_2 & \dots & Dy_n \\ \dots & \dots & \dots & \dots \\ D^{n-1}y_1 & D^{n-1}y_2 & \dots & D^{n-1}y_n \end{vmatrix}$$

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Linear differential equations Solutions and independent solutions

Nature of solution(s)-V

Corollary 4.1.3

to Theorem 4.1.2 is

- **1** A constant multiple $y = c_1y_1(x)$ of a solution $y_1(x)$ of a homogeneous linear differential equation is also a solution.
- ② A homogeneous linear differential equation always possesses the trivial solution y = 0.



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