GENG 8010–Part 1: Elements of Differential and Difference Equations

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Representation of DEs using differential operator I

A system of differential equations may be given or come out of a modeling exercise. This discussion shows how the equation can be represented in a compact form. Define the operator D as

$$D[u(t)] = \frac{d}{dt}[u(t)]$$

If c_1 and c_2 are constants, the operator exhibits the following properties

$$D^m(D^n u) = D^{m+n}u = \frac{d^{m+n}u}{dt^{m+n}}$$

$$D^m(c_1u_1 + c_2u_2) = c_1D^m u_1 + c_2D^m u_2$$

Representation of DEs using differential operator II

$$(D + c_1)(D + c_2)u = [D^2 + (c_1 + c_2)D + c_1c_2]u$$

where n, and m are non-negative integers.

While mostly the operator D can be treated as an algebraic quantity, **be cautious that in general it does not commute wrt functions, i.e.,**

$$D(tu) \neq t(Du)$$

$$D(u_1u_2) \neq u_1(Du_2)$$

Representation of DEs using differential operator III

Given the above, consider the differential equation

$$\begin{aligned} a_n\frac{d^ny}{dt^n} + a_{n-1}\frac{d^{n-1}}{dt^{n-1}} + \ldots + a_1\frac{dy}{dt} + a_0y &= b_m\frac{d^mu}{dt^m} + \ldots + b_1\frac{du}{dt} + b_0u \\ &= F(t) \end{aligned} \tag{1.1}$$

Recall that if the above is to be linear a_i and b_j's cannot be a function of u or y but can be a function of t. Also, for a **fixed** linear system, these coefficients have to be constant.

Representation of DEs using differential operator IV

Now, in compact form (3) can be written as

$$(a_n D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0) y(t) = (b_m D^m + \ldots + b_1 D + b_0) u(t) = F(t)$$

If this is a fixed system, then

$$A(D) y(t) = B(D) u(t) = F(t)$$

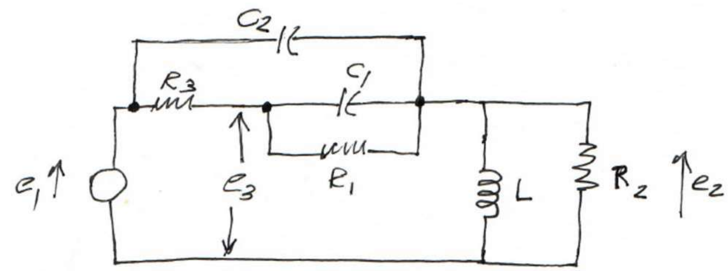
for a **varying linear system**, we have

$$A(D,t) y(t) = B(D,t) u(t) = F(t)$$

Representation of DEs using differential operator V

Example–

Consider the following circuit. Get a differential equation relating e_2 to e_1 assuming all elements have a unity value.



Representation of DEs using differential operator VI

$$\left[C_1 \frac{de_3}{dt} + \left(\frac{1}{R_1} + \frac{1}{R_3}\right) e_3\right] - \left[C_1 \frac{de_2}{dt} + \frac{1}{R_1} e_2\right] = \frac{1}{R_3} e_1$$

$$-\left[C_1 \frac{de_3}{dt} + \frac{1}{R_1} e_3\right] + \left[(C_1 + C_2) \frac{de_2}{dt} + \left(\frac{1}{R_1} + \frac{1}{R_2}\right) e_2 + \frac{1}{L} \int e_2 dt\right] = C_2 \frac{de_1}{dt}$$

differentiate the second equation above to get rid of the integral and use the short form notation

Representation of DEs using differential operator VII

$$(D+2)e_3 - (D+1)e_2 = e_1$$

$$-(D^2 + D)e_3 + (2D^2 + 2D + 1)e_2 = (D^2)e_1$$

multiplying the first eq. by $(D^2 + D)$ and the second one by $(D+2)$ and adding, eliminates e_3 and we get

$$(D^3 + 4D^2 + 4D + 2)e_2 = (D^3 + 3D^2 + D)e_1$$

The above is a single third order differential equation that relates the output voltage to the input voltage. Recall from circuit theory that we have three energy storage elements in this circuit, and hence, third order differential equation that described the linear fixed system.

What is a solution? I

Definition–A function ϕ, defined over $a < x < b$, is called a solution of a differential equation described by

$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}\right)$$

if the n derivatives of y exist on $a < x < b$, and

$$\frac{d^n \phi(x)}{dx^n} = f\left(x, \phi(x), \frac{d\phi(x)}{dx}, \frac{d^2\phi(x)}{dx^2}, \ldots, \frac{d^{n-1}\phi(x)}{dx^{n-1}}\right)$$

$\forall x \in a < x < b$.

What is a solution? II

Example–Consider the differential equation

$$\frac{dy}{dt} = t\sqrt{y}$$

Is $y = \frac{1}{16}t^4$ a solution?

For y to be a solution, it has to satisfy the differential equation. So plug the solution into the differential Eq.

$$\frac{dy}{dt} = \frac{4}{16}t^3 = t \times \sqrt{\frac{1}{16}t^4} \Longrightarrow \frac{1}{4}t^3 = \frac{1}{4}t^3$$

So it checks and the given y is a solution.

What is a solution? III

Remark

Normally we associate an interval for the solution of the differential equation as was done in the definition above. Such an interval is called **domain of the solution** or **interval of the definition**.

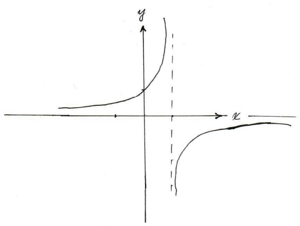
A solution that is identically zero over the interval of definition, is called a **Trivial Solution**.

What is a solution? IV

Example–

Consider the graph of

$$y = \frac{1}{1-x}$$



shown here. It can be verified that y given, is a solution to nonlinear differential equation

$$\frac{dy}{dx} = y^2$$

As you can see the differential equation has two continuous on two different intervals (domain of solution) $(-\infty, 1)$ and on $(1, \infty)$.

E & U–first order equation I

Theorem 2.1.1

Consider

$$\frac{dy}{dx} = f(x,y) \tag{A}$$

Define T to be a rectangular region defined by $|x - x_0| \leq a$ and $|y - y_0| \leq b$, with the point (x_0, y_0) at the center. Suppose that f and $\frac{\partial f}{\partial y}$ are continuous functions of x and y in T. Under these conditions $\exists$ an interval about $x_0 \ni |x - x_0| \leq d$, and a function $y(x)$ which satisfies:

1. *$y = y(x)$ is a solution of (A) on the interval $|x - x_0| \leq d$*
2. *On the interval $|x - x_0| \leq d$, $y(x)$ satisfies $|y(t) - y_0| \leq b$*
3. *At $x = x_0, y = y(x_0) = y_0$*
4. *$y(x)$ is unique on the interval $|x - x_0| \leq d$, that is, only $y(x)$ satisfies the above three conditions.*

E & U–first order equation II

Remark

1. The interval $|x - x_0| \leq d$ may or may not be smaller than $|x - x_0| \leq a$ over which we imposed conditions on $f(x,y)$.
2. Basicaly, the theorem says that if the function $f(x,y)$ is sufficiently well behaved near the point (x_0, y_0) then (A) has a solution that passes through (x_0, y_0) and that solution is **unique** near (x_0, y_0)

E & U–first order equation III

Theorem 2.1.2

Note that for a linear case, we can prove a stronger result than in the nonlinear case. Consider the linear diff. eq.

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Assume that P and Q are continuous functions on the interval $a < x < b$, and that $x = x_0$ is any number in that interval. If y_0 is an arbitrary real number , $\exists$ a **unique** *solution $y(x)$ that satisfies the initial condition $y(x_0) = y_0$. Moreover this solution satisfies the diff. eq. throughout the interval $a < x < b$.*

Solution by direct integration I

In this discussion we will study first order equations of the form

$$M(x,y)dx + N(x,y)dy = 0$$

where M and N can be functions of both x and y. We start with some simple equations that can be put in the form

$$A(x)dx + B(y)dy = 0 \tag{B}$$

where the variables can be separated.

A solution can then be readily obtained. The trick is to find a function F whose total differential is $A(x)dx + B(y)dy$. Then $F = c$ where c is an arbitrary constant is the desired result.

Solution by direct integration II

Example 1– Obtain the solution of the following

$$\frac{dy}{dx} = x^2$$

integrating

$$y = \int x^2 dx \Rightarrow \boxed{y = \frac{1}{3}x^3 + c}$$

Note as well that in terms of the discussion on previous page $F \triangleq y - \frac{1}{3}x^3 = c$ and verify that the total differential

$$dF = dy - x^2 dx = 0$$

which is in the form (B).

Solution by direct integration III

Example 2–Consider

$$\frac{dy}{dx} = e^{2x}$$

Integrate to get

$$y = \frac{1}{2}e^{2x} + c$$

as a solution. If we were told that $x = 0, y = 1$, then

$$y = \frac{1}{2}(e^{2x} + 1)$$

is the solution.

Solution via integration–extensions to more general forms I

Let us consider a more general form than (B) given by

$$M(x, y)dx + N(x, y)dy = 0 \qquad \text{(C)}$$

where the separation of variables as in the previous cases may not be possible. Suppose that a function $F(x, y)$ can be found $\ni$

$$dF = Mdx + Ndy \qquad (3.1)$$

then $F(x, y) = c$ **implicity** defines a set of solutions to (C), and clearly, $dF = 0 = Mdx + Ndy$.

Solution via integration–extensions to more general forms II

Remark

The above discussion leads to the following

1. Under what conditions M, N, and F exists?
2. If the conditions are satisfied, determine F

If $\exists F \ni$ (3.1) is the total differential of F, then (C) is an **exact equation**.

Solution via integration–extensions to more general forms III

Theorem 3.1.1

If $M, N, \frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous functions of x, and y, then a necessary and sufficient condition that

$$M(x,y)dx + N(x,y)dy = 0$$

be an exact equation is that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution via integration–extensions to more general forms IV

Exact differential equation

Equation

$$M(x,y)dx + N(x,y)dy = 0$$

is called an **exact differential equation** if its differential

$$dF = F_x dx + F_y dy$$

is **exactly** $Mdx + Ndy$.

Solution via integration–extensions to more general forms I

Example–Consider

$$3x(xy-2)dx + (x^3+2y)dy = 0$$

Clearly this is in a more complicated form than (B). Note

$$\frac{\partial M}{\partial y} = 3x^2, \quad \frac{\partial N}{\partial x} = 3x^2$$

Therefore the equation is exact, and solution is given by $F = c$, where

$$\frac{\partial F}{\partial x} = M = 3x^2y - 6x$$

$$\frac{\partial F}{\partial y} = N = x^3 + 2y$$

Solution via integration–extensions to more general forms II

Integrating the first of the above two with respect to x while holding y constant gives

$$F = x^3y - 3x^2 + R(y) \qquad R(y) \triangleq \text{integration constant}$$

Now from the second of the two above

$$\frac{\partial F}{\partial y} = x^3 + R'(y) = x^3 + 2y \Longrightarrow R'(y) = 2y \rightarrow R(y) = y^2$$

So

$$F = x^3y - 3x^2 + y^2$$

Finally the set of solutions that satisfy the above differential equation is given by $F = c$

$$\boxed{x^3y - 3x^2 + y^2 = c}$$

Solution via integration–extensions to more general forms III

Example

Solve

$$\left(2x^3 - xy^2 - 2y + 3\right) dx - \left(x^2y + 2x\right) dy = 0$$

Note

$$\frac{\partial M}{\partial y} = -2xy - 2 = \frac{\partial N}{\partial x}$$

so the equation is exact, and the set of solution to it is given by $F = c$, where

$$\frac{\partial F}{\partial x} = 2x^3 - xy^2 - 2y + 3$$

$$\frac{\partial F}{\partial y} = -x^2y - 2x$$

Solution via integration–extensions to more general forms

start with the simpler second equation to get

$$F = -\frac{1}{2}x^2y^2 - 2xy + R(x)$$

using this and the first equation gives

$$-xy^2 - 2y + R'(x) = 2x^3 - xy^2 - 2y + 3 \longrightarrow R'(x) = 2x^3 + 3$$

$$R(x) = \frac{1}{2}x^4 + 3x$$

therefore,

$$-\frac{1}{2}x^2y^2 - 2xy + \frac{1}{2}x^4 + 3x = \frac{1}{2}c \longrightarrow \boxed{x^4 - x^2y^2 - 4xy + 6x = c}$$

The integrating factor method I

- You saw that a solution to 1^{st} order exact equation can be obtained by integration.
- If an equation is not exact, we can try to make it exact, through a factor called integrating factor.
  - Very little can be done in general.
  - There are certain special cases where a means for finding an integrating factor is available.

The integrating factor method: linear case I

Consider the linear differential equation

$$A(x)\frac{dy}{dx} + B(x)y = C(x)$$

Divide by $A(x)$ to get

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{3.2}$$

which is in the standard form.

The integrating factor method: linear case II

Assume that $\exists$ an integrating factor $v(x) > 0$,

$$v(x)\left[\frac{dy}{dx} + P(x)y\right] = v(x)Q(x)$$

$$v(x)dy + [v(x)P(x)y - v(x)Q(x)]\,dx = 0$$

$$Mdx + Ndy = 0$$

The integrating factor method: linear case III

Again, we wish to have an exact equation of the form

$$Mdx + Ndy = 0$$

as in Theorem 3.1.1, where now

$$M = vPy - vQ \qquad \text{and} \qquad N = v$$

and v, P and Q are functions of x.

Recall that for the equation to be exact Theorem 3.1.1 requires that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The integrating factor method: linear case IV

then, we must have

$$vP = \frac{dv}{dx} \Rightarrow Pdx = \frac{dv}{v}$$

$$\ln v = \int Pdx \Rightarrow \boxed{v = e^{\int Pdx}}$$

So if (3.2) has a positive integrating factor independent of y, then the above is it. Now multiply (3.2) by this integrating factor

$$e^{\int Pdx}\frac{dy}{dx} + Pe^{\int Pdx}y = Qe^{\int Pdx}$$

$$\frac{d}{dx}\left(ye^{\int Pdx}\right) = Qe^{\int Pdx}$$

The integrating factor method: linear case V

The right side is a function of x only, hence, the equation is exact.

$$ye^{\int P(x)dx} = \int Q(x)e^{\int P(x)dx}dx + c$$

Letting $g(x) = e^{\int P(x)dx}$, we get

$$y(x) = \frac{1}{g(x)}\left(\int Q(x)g(x)dx + c\right)$$

The integrating factor method: linear case VI

Remark (1)

Let us summarize

1. Put the equation in standard form $\frac{dy}{dx} + Py = Q$
2. Integrating factor is $e^{\int Pdx}$.
3. Apply the integrating factor.
4. Solve to get y.

The integrating factor method: linear case VII

Remark (2)

1. In evaluating $e^{\int P(x)dx}$ it is not necessary to include a constant of integration since $e^{\int P(x)dx + c} = e^c e^{\int P(x)dx} = ke^{\int P(x)dx}$
2. The integral on the right can be **difficult** to solve.
3. The solution, if found, **will give complimentary as well as particular solution** of diff. eq.
4. The method of solution will **work for both time varying as well as time invariant** systems.

The integrating factor method: linear case VIII

Example–Solve the differential equation

$$2(y - 4t^2)dt + tdy = 0$$

Re-write as

$$\frac{dy}{dt} + \frac{2}{t}y = 8t$$

Integrating factor is

$$e^{\int \frac{2}{t}dt} = e^{2\ln|t|} = e^{\ln t^2} = t^2$$

multiplying both side by the IF

$$t^2\frac{dy}{dt} + 2ty = 8t^3$$

or

$$\frac{d}{dt}(t^2 y) = 8t^3$$

The integrating factor method: linear case IX

$$\frac{d}{dt}(t^2 y) = 8t^3$$

intergrading both side gives

$$t^2 y = 2t^4 + c \longrightarrow \boxed{y(t) = 2t^2 + \frac{1}{t^2}c}$$

By substituting this solution into the differential equation, you can see that it satisfied the equation.

The integrating factor method: linear case X

Example Let us find integrating factor to find the solution of

$$\frac{dy}{dx} - y = e^x \sin x$$

IF $e^{\int(-1)dx} = e^{-x}$, that would give

$$y = \frac{1}{e^{-x}}\left(\int e^{-x}\left(e^x \sin x\right)dx + c\right) = e^x\left(\int \sin x dx + c\right)$$

which results in the solution

$$\boxed{y = e^x(-\cos x + c)}$$