GENG 8010–Part 1: Elements of Differential and Difference Equations

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Difference equations I

When dealing with discrete-time systems, difference equations will arise.

In this context, the shift operator is defined as

$$Ey(k) = y(k+1), \quad E^2y(k) = y(k+2), \quad E^ny(k) = y(k+n)$$

The difference operator is defined Δ is defined as

$$\Delta y(k) = y(k+1) - y(k) = (E-1)y(k) \text{ so } \longrightarrow \Delta = E - 1$$

$$\Delta^2 y(k) = y(k+2) - 2y(k+1) + y(k) \text{ and so on}$$

Difference equations II

The operators E, and Δ obey the algebraic laws, i.e.

$$E[cy(k)] = cEy(k)$$
$$E^m[y(k) + z(k)] = E^m y(k) + E^m z(k)$$
$$E^m E^n y(k) = E^n E^m y(k) = E^{n+m} y(k)$$

The operators generally commute with each other but not with functions, i.e.

$$E[y(k)z(k)] \neq y(k)Ez(k)$$

Difference equations III

Properties of linear difference equations

An n-th order linear difference equation can be written as

$$\left(a_n E^n + a_{n-1} E_{n-1} + \ldots + a_1 E + a_0\right) y(k) = F(k) \tag{9.1}$$

where a_i are defined for all k. a_i's can be a function of k in which case the equation is (time) varying or they can be constants in which case the system is fixed or time invariant. As opposed to differential equations, the order of the difference equation is the difference between the highest and lowest power of E. For instance $(E^2 + 2E)y(k)$ is a first order difference equation.

If $F(k)$ in 9.1 is zero, then we have a **homogeneous difference equation**.

Difference equations IV

Difference equations are also called **Recurrence formulas** since

$$y(k+n) = -\frac{1}{a_n}\left[a_{n-1}y(k+n-1) + \ldots + a_1 y(k+1) + a_0 y(k) - F(k)\right]$$

That is knowing $y(0), \ldots, y(n-1)$, then $y(k), \forall k \geq n$ could be found through recursion from the above. So as opposed to differential equation, solution of a difference equation can be found from the equation itself given$y(0), \ldots, y(n-1)$.

Difference equations V

Theorem 9.1.1

A homogeneous difference equation of n^{th} order with $a_1 \neq 0$, and $a_n \neq 0$ has n-linearly independent solutions $y_1(k), \ldots, y_n(k)$ iff

$$C(k) = \begin{vmatrix} y_1 & y_2 & \ldots & y_n \\ Ey_1 & Ey_2 & \ldots & Ey_n \\ \ldots & \ldots & \ldots & \ldots \\ E^{n-1}y_1 & E^{n-1}y_2 & \ldots & E^{n-1}y_n \end{vmatrix} \neq 0$$

where $C(k)$ is analogous to Wronskian $W(t)$ given in Theorem 4.1.1, and is called the ***Casorati's determinant****.*

Difference equations VI

General solution of difference equation

The most general solution of equation (9.1) is

$$y = y_c + y_p$$

where

$$y_c = c_1 y_1(k) + c_2 y_2(k) + \ldots + c_n y_n(k)$$

is the **homogeneous or complementary solution** and $y_p(k)$ is the **particular solution** and is any solution that satisfies 9.1, and the n constants in y_c have to be determined using initial or boundary conditions.

Homogeneous solution of difference equations I

Consider the n-th order equation

$$(a_n E^n + a_{n-1}E_{n-1} + \ldots + a_1 E + a_0)\, y(k) = 0$$

By analogy to differential equations, assume a solution $y_c(k) = \beta^k$ which must satisfy the equation, and where β is a complex constant to be found. Noting that $E\beta^k = \beta^{k+1}$, and $E^2\beta^k = \beta^{k+2}$, and so on, we have

$$a_n\beta^{k+n} + a_{n-1}\beta^{k+n-1} + \ldots + a_1\beta^{k+1} + a_0\beta^k = 0$$

Homogeneous solution of difference equations II

taking out and canceling β^k gives

Homogeneous solution of difference equation

$$a_n\beta^n + a_{n-1}\beta^{n-1} + \ldots + a_1\beta + a_0 = 0 \tag{9.2}$$

Equation (9.2) is referred to as **Characteristic or Auxilary equation** for the difference equation.

The CE is a polynomial of n roots and depending on the nature of the roots, $y_c(k)$ is determined.

Homogeneous solution of difference equations III

Case I– Roots $\beta_1, \beta_2, \ldots, \beta_n$ are real and distinct, in which case

$$\boxed{y_c(k) = \alpha_1\beta_1^k + \alpha_2\beta_2^k + \ldots + \alpha_n\beta_n^k}$$

Example–Solve

$$(2E^3 + 7E^2 + 7E + 2)y(k) = 0$$

with $y(k) = \beta^k$ we get the CE to be
$2\beta^3 + 7\beta^2 + 7\beta + 2 = (2\beta+1)(\beta+2)(\beta+1) = 0$ and therefore
$\beta_1 = -\frac{1}{2}, \beta_2 = -2, \beta_3 = -1$, Hence,

$$\boxed{y_c(k) = \alpha_1\left(\frac{-1}{2}\right)^k + \alpha_2(-2)^k + \alpha_3(-1)^k}$$

Homogeneous solution of difference equations IV

Case II– β_1 is repeated m times and the remaining roots are real and distinct. in this case

$$\boxed{y_c(k) = \alpha_1\beta_1^k + \alpha_2 k\beta_1^k + \ldots + \alpha_m k^{m-1}\beta_1^k + \alpha_{m+1}\beta_{m+1}^k + \ldots + \alpha_n\beta^k}$$

Example–Solve

$$(2E^3 + 5E^2 + 4E + 1)y(k) = 0$$

CE is $2\beta^3 + 5\beta^2 + 4\beta + 1 = 0$ which has roots at $\frac{-1}{2}, -1, -1$. Therefore,

$$\boxed{y_c(k) = \alpha_1\left(\frac{-1}{2}\right)^k + \alpha_2(-1)^k + \alpha_3 k(-1)^k}$$

Homogeneous solution of difference equations V

Case III–Roots appear in complex conjugate pairs.If there are roots at, say, $\beta_1 = ae^{j\theta}$ and $\beta_2 = ae^{-j\theta}$, then

$$\beta_1^k = a^k e^{jk\theta} = a^k(\cos k\theta + j\sin k\theta)$$
$$\beta_2^k = a^k e^{-jk\theta} = a^k(\cos k\theta - j\sin k\theta)$$

$$\boxed{y_c(k) = c_1 a^k\cos k\theta + c_2 a^k\sin k\theta + \alpha_3\beta_3^k + \ldots + \alpha_n\beta_n^k}$$

Nonhomogeneous solution of difference equations I

As in the case of differential equations, There are two methods for finding $y_p(k)$:

1. Method of undetermined coefficients
   - Note that this method works when repeated application of E to forcing function $F(k)$ produces a finite number of linearly independent terms. Examples of possible $F(k)$ are $k^m + \ldots + b_1 k + b_0$, $\sin\theta k$, $\cosh\theta k$, etc.
2. Variation of parameters (parallel process to finding a solution for differential equations but will not be pursued).

Nonhomogeneous solution of difference equations II

Example–Solve the following by the method of undetermined coefficients:

$$(E^2 + 2E + 1)y(k) = k(-1)^k$$

CE is

$$\beta^2 + 2\beta + 1 = 0$$

with $\beta_{1,2} = -1$, $y_c(k) = \alpha_1(-1)^k + \alpha_2 k(-1)^k$. Since $F(k)$ contains $(-1)^k$ which is also in the homogeneous solution, rather than choosing a form as such $(Ak + B)(-1)^k$, we choose

Nonhomogeneous solution of difference equations III

$$y_p(k) = (Ak^3 + Bk^2)(-1)^k$$

and

$$\begin{aligned} Ey_p &= -\left[A(k+1)^3 + B(k+1)^2\right](-1)^k \\ E^2 y_p &= \left[A(k+2)^3 + B(k+2)^2\right](-1)^k \end{aligned}$$

substituting into the equation and equating results in $A = \frac{1}{6}$, and $B = -\frac{1}{2}$, and complete solution is then

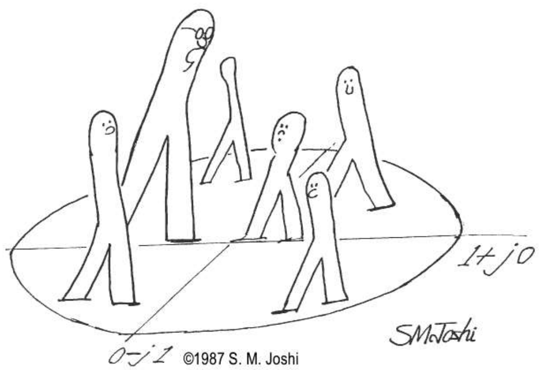
$$\boxed{y(k) = \left(\alpha_1 + \alpha_2 k - \frac{1}{2}k^2 + \frac{1}{6}k^3\right)(-1)^k}$$

System concepts I

Stability of linear discrete-time systems

1. Roots of CE are the poles of the system. Poles that lie within a unit circle produce terms in the homogeneous equation that converge to zero as $k \longrightarrow \infty$ (stable).
2. Roots outside the unit circle produce terms of growing magnitude without bound as k increases (unstable).
3. Simple roots on the unit circle produce terms of constant magnitude (stable or marginally stable), but repeated roots on the unit circle will produce terms that increase without limit as k increases (unstable).

System concepts II



"And then in the seventies came the digital revolution, and they confined us to this silly circle!"

System concepts III

Much in the same way as in continuous time systems, the input-output relation between in a discrete-time system can be represented as

$$y(k) = \mathscr{H}_D u(k)$$

consider now the discrete-Delta function defined as

$$\delta(k) = \begin{cases} 1, & \text{if } k = 0. \\ 0, & \text{if } k \neq 0. \end{cases}$$

System concepts IV

Now represent the input to the system as

$$u(k) = \sum_{m=-\infty}^{\infty} u(m)\delta(k-m)$$

$$\begin{aligned} y(k) = \mathscr{H}_D u(k) &= \mathscr{H}_D \left(\sum_{m=-\infty}^{\infty} u(m)\delta(k-m) \right) \\ &= \sum_{m=-\infty}^{\infty} u(m)g(k,m) \end{aligned}$$

System concepts V

Weighting sequence and response

$g(k,m)$ is the **impulse response** or **weighting sequence**.

$$\boxed{y(k) = \sum_{m=-\infty}^{\infty} u(m)g(k,m)}$$ Response for noncausal time varying systems

$$\boxed{y(k) = \sum_{m=k_0}^{k} u(m)g(k,m)}$$ $k > m$, Response for causal time varying systems with input at k_0

$$\boxed{y(k) = \sum_{m=k_0}^{k} u(m)g(k-m)}$$ $k > m$, for time invariant systems

Introduction & practical sampling I

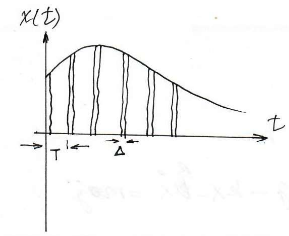
Note that if there was such a device as an ideal sampler, a sampled form of a continuous signal $x(t)$ would be

$$x^*(t) = \sum_{k=0}^{\infty} x(kT)\delta(t - kT)$$

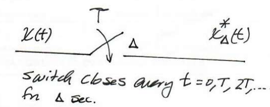
But an ideal sampler device is just that, IDEAL!

Introduction & practical sampling II

Consider a sampled data system where the sampling duration Δ is much smaller than the sampling period and the largest time constant of the input signal $x(t)$, and as such can be approximated by a series of flat topped pulses as



$$x^*_{\Delta}(t) = \begin{cases} x(kT), & \text{if } kT \le t \le kT + \Delta \\ 0, & \text{if } kT + \Delta \le t \le (k+1)T \end{cases}$$



$$x^*_{\Delta}(t) = \sum_{k=0}^{\infty} x(kT)\left(u(t - kT) - u(t - kT - \Delta)\right)$$

Introduction & practical sampling III

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$$\begin{aligned} X^*_{\Delta}(s) &= \mathscr{L}\left\{\sum_{k=0}^{\infty} x(kT)\left(u(t - kT) - u(t - kT - \Delta)\right)\right\} \\ &= \sum_{k=0}^{\infty} x(kT)\left(\frac{e^{-kTs}}{s} - \frac{e^{-(kT+\Delta)s}}{s}\right) \\ &= \sum_{k=0}^{\infty} x(kT)\left(\frac{(1 - e^{-\Delta s})e^{-kTs}}{s}\right) \end{aligned}$$

If $\Delta \longrightarrow 0$, then

Introduction & practical sampling IV

$$1 - e^{-\Delta s} \approx 1 - \left(1 - \Delta s + \frac{(\Delta s)^2}{2} - \dots\right) = \Delta s \qquad e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!}$$

$$X^*_{\Delta}(s) = \sum_{k=0}^{\infty} x(kT)\Delta e^{-kTs}$$

but $\mathscr{L}^{-1}\left(e^{-kTs}\right) = \delta(t - kT)$, therefore

$$\boxed{x^*_{\Delta}(t) \approx \Delta \sum_{k=0}^{\infty} x(kT)\delta(t - kT) \quad \text{train of impulses with strength of } \Delta x(kT)}$$

Introduction & practical sampling V

Remark

The above means that the finite pulse width sampler can be replaced with an ideal sampler ($\Delta = 0$) cascaded by a gain term with an attenuation of Δ. The output of an ideal sampler is given by

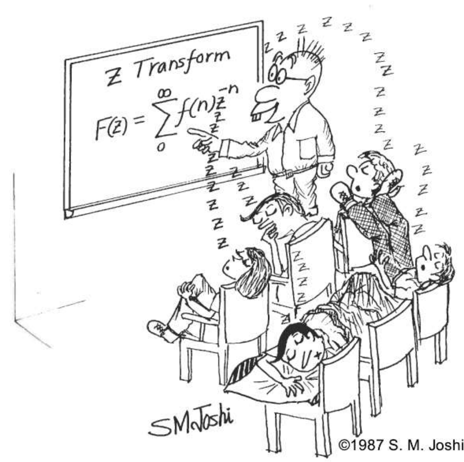
$$x^*(t) \triangleq x(t)\delta_T(t) \text{ where } \delta_T(t) \triangleq \sum_0^{\infty} \delta(t - kT)$$

$$x^*(t) = \sum_{k=0}^{\infty} x(kT)\delta(t - kT)$$

$$\mathscr{L}[x^*(t)] = X^*(s) = \sum_{k=0}^{\infty} x(kT)e^{-(kTs)}$$

Infinite series involves factors of e^{sT} and its powers make $\mathscr{L}^{-1}$ difficult.

Introduction & practical sampling VI



Introduction & practical sampling VII

$\mathscr{Z}$ transform definition

Define

$$z \triangleq e^{Ts} \longrightarrow s = \frac{1}{T}\ln z \quad \text{and then}$$

$$X^*\left(s = \frac{1}{T}\ln z\right) \triangleq X(z) = \sum_{k=0}^{\infty} x(kT)z^{-k} \qquad \text{One-sided } \mathscr{Z} \text{ transform}$$

The above converges absolutely for $|z| > c$, where c is the radius of convergence.

Summary steps

1. Sample $x(t)$ to get $x^*(t)$
2. Obtain $\mathscr{L}(x^*(t)) = \sum_{k=0}^{\infty} x(kT)e^{-(kTs)}$
3. Find $X(z) = \sum_{k=0}^{\infty} x(kT)z^{-k}$

Introduction & practical sampling VIII

Theorem 10.1.1

Ratio Test–*For the series $f_1(z) + f_2(z) + \ldots + f_n(z) + \ldots$, let*

$$\lim_{n\to\infty}\left|\frac{f_{n+1}(z)}{f_n(z)}\right| = |r(z)|$$

Then the given series converges absolutely for those values of z for which $0 \le |r(z)| < 1$ and diverges for those values of z for which $|r(z)| > 1$. The values of z for which $|r(z)| = 1$ form the boundary of the region of convergence of the series, and at those points the ratio test provides no information about the convergence or divergence of the series.

Introduction & practical sampling IX

Theorem 10.1.2

Two functions of time have the same $\mathscr{Z}$ transformation if and only if they are identical $\forall t = nT(n = 0, 1, 2, \ldots)$.

Remark

This theorem tells us that $\mathscr{Z}$ transform is not unique, and that $F(z)$ contains no information about $f(t)$ except at sampling instants.

Introduction & practical sampling X

Example–Find $\mathscr{Z}(u(t))$

$$u^*(t) = \sum_{k=0}^{\infty} u(kT)\delta(t - kT) = \sum_{k=0}^{\infty} \delta(t - kT)$$

$$U^*(s) = \sum_{k=0}^{\infty} e^{-(kTs)} \implies U(z) = \sum_{k=0}^{\infty} z^{-k}$$

$$U(z) = 1 + z^{-1} + z^{-2} + \ldots \qquad \text{multiply by } z - 1$$

$$(z-1)U(z) = z + 1 + z^{-1} + z^{-2} + \ldots - 1 - z^{-1} - z^{-2} - \ldots$$

$$(z-1)U(z) = z$$

$$\boxed{U(z) = \frac{z}{z-1}}$$

Series converges for $|z| > 1$

Introduction & practical sampling XI

Consider the following series:

$$S_k = \sum_{k=0}^{\infty} cr^k = c + cr + cr^2 + cr^3 + \cdots + cr^{k-1}$$

$$rS_k = cr + cr^2 + cr^3 + cr^4 + \cdots + cr^k$$

$$S_k - rS_k = c - cr^k \implies S_k = \frac{c(1 - r^k)}{1 - r} \qquad r \neq 1$$

if $k \to \infty$ and $|r| < 1$, then

$$S_k = \frac{c}{1 - r}$$

Introduction & practical sampling XII

Example–Find $\mathscr{Z}(e^{-at}u(t))$

$$x^*(t) = \sum_{k=0}^{\infty} e^{-akT}\delta(t - kT) \implies X^*(s) = \sum_{k=0}^{\infty} e^{-akT}e^{-kTs}$$

$$X^*(z) = \sum_{k=0}^{\infty} e^{-akT}z^{-k} = \sum_{k=0}^{\infty} \left(e^{-aT}z^{-1}\right)^k$$

In the above $c = 1$ and $r = e^{-aT}z^{-1}$, then

$$\boxed{X(z) = \frac{1}{1 - e^{-aT}z^{-1}} = \frac{z}{z - e^{-aT}}}$$

Introduction & practical sampling XIII

Example–Find $\mathscr{Z}(\beta^t)$

Let $\beta = e^{-a}$ in the last example to get

$$\beta^t \longleftrightarrow \frac{z}{z-\beta^T} \quad \text{for } T=1 \quad \boxed{\beta^t \longleftrightarrow \frac{z}{z-\beta}}$$

Example–Note that

$$\mathscr{Z}\left(e^{j\omega t}\right) = \mathscr{Z}\left(\cos\omega t + j\sin\omega t\right) = \mathscr{Z}(\cos\omega t) + j\mathscr{Z}(\sin\omega t)$$

$$e^{j\omega t} \longleftrightarrow \frac{z}{z-e^{j\omega T}} = \frac{z}{z-(\cos\omega T + j\sin\omega T)}$$
$$= \frac{z^2 - z\cos\omega T + jz\sin\omega T}{z^2 + \sin^2\omega T + \cos^2\omega T - 2z\cos\omega T}$$

Introduction & practical sampling XIV

$$\mathscr{Z}\left(e^{j\omega t}\right) = \mathscr{Z}\left(\cos\omega t + j\sin\omega t\right) = \mathscr{Z}(\cos\omega t) + j\mathscr{Z}(\sin\omega t)$$

$$e^{j\omega t} \longleftrightarrow \frac{z}{z-e^{j\omega T}} = \frac{z^2 - z\cos\omega T + jz\sin\omega T}{z^2 - 2z\cos\omega T + 1}$$

$$\therefore \quad \boxed{\cos\omega t \longleftrightarrow \frac{z(z-\cos\omega T)}{z^2 - 2z\cos\omega T + 1}} \quad \boxed{\sin\omega t \longleftrightarrow \frac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1}}$$

Some properties of $\mathscr{Z}$: I

1. Linearity property–

$$\begin{aligned}
\mathscr{Z}[a_1f_1(t) + a_2f_2(t)] &= a_1\mathscr{Z}[f_1(t)] + a_2\mathscr{Z}[f_2(t)] \\
&= \sum_{k=0}^{\infty}[a_1f_1(kT) + a_2f_2(kT)]z^{-k} \\
&= a_1\sum_{k=0}^{\infty}f_1(kT)z^{-k} + a_2\sum_{k=0}^{\infty}f_2(kT)z^{-k} \\
&= a_1F_1(z) + a_2F_2(z)
\end{aligned}$$

Some properties of $\mathscr{Z}$: II

2. Shifting property– As an example let's find $\mathscr{Z}[f(t+mT)]$

$$\begin{aligned}
\mathscr{Z}[f(t+mT)] &= \sum_{k=0}^{\infty} f(kT+mT)z^{-k} \qquad \text{let } i=k+m \\
&= \sum_{i=m}^{\infty} f(iT)z^{-i+m} = z^m\sum_{i=m}^{\infty} f(iT)z^{-i} \\
&= z^m\sum_{i=0}^{\infty} f(iT)z^{-i} - z^m\sum_{i=0}^{m-1} f(iT)z^{-i} \\
&= z^mF(z) - \sum_{i=0}^{m-1} f(iT)z^{m-i}
\end{aligned}$$

$$\therefore \boxed{f(t+mT) \longleftrightarrow z^mF(z) - z^mf(0) - z^{m-1}f(T) - \ldots - zf(mT-T)}$$

Some properties of $\mathscr{Z}$: III

3. $\sum_{k=0}^{\infty} f_1(t-kT) f_2(kT) \longleftrightarrow F_1(z)F_2(z)$
4. $t^m f(t) \longleftrightarrow (-Tz)^m \frac{d^m}{dz} F(z)$
5. $\frac{f(t)}{t} \longleftrightarrow -\int \frac{1}{T}\frac{F(z)}{z} dz$
6. $e^{-at} f(t) \longleftrightarrow F\left(ze^{aT}\right)$ where $|e^{aT} z| < R^-$
7. $\frac{\partial}{\partial \alpha} f(t,\alpha) \longleftrightarrow \frac{\partial}{\partial \alpha} F(z,\alpha)$
8. $f(0) = \lim_{z\to\infty} F(z)$
   Initial Value Theorem: provided that the limit exist.
9. $f(\infty) = \lim_{z\to 1}(1-z^{-1})F(z)$
   Final Value Theorem: if $(1-z^{-1})F(z)$ is analytic for $|z| > 1$.

Solution of difference equations I

$\mathscr{Z}$ transform

$\mathscr{Z}$ transform is a useful tool for solution of constant coefficient difference (discrete) equations that arise in many fields, including signal processing, digital control, manufacturing systems, operations research, etc., where a phenomenon physically and naturally is described by a difference equation or is transformed into a discrete-time process.

Solution of difference equations II

Example– Solve $(E+1)y(k) = (-1)^k$ with $y(0) = 1$.
Taking $\mathscr{Z}$ transform

$$zY(z) - zY(0) + Y(z) = \frac{z}{z+1}$$

$$(z+1)Y(z) = z + \frac{z}{z+1} = \frac{z^2+2z}{z+1}$$

$$\frac{Y(z)}{z} = \frac{K_1}{z+1} + \frac{K_2}{(z+1)^2} \qquad K_1 = 1, K_2 = 1$$

$$Y(z) = \frac{z}{z+1} + \frac{z}{(z+1)^2}$$

$$y(k) = (-1)^k - k(-1)^k$$

where $k(-1)^k \leftrightarrow \frac{-z}{(z+1)^2}$

Solution of difference equations III

Example–Consider a second order discrete time system described by

$$(E^2 + aE - b)y(k) = u(k)$$

1. What is the transfer function of this system if $a = 1, b = 2$?
2. What is the weighting sequence of this system?
3. What is the response of the system to unit step input if $y(0) = 0, y(1) = 1$?

Taking the $\mathscr{Z}$ we get

$$z^2 Y(z) - z^2 y(0) - zy(1) + zY(z) - 2Y(z) = U(z)$$

$$(z^2 + z - 2)Y(z) = z + U(z)$$

Solution of difference equations IV

Note $G(z) = \frac{Y(z)}{U(z)}$, so

$$\boxed{G(z) = \frac{1}{z^2 + z - 2} = \frac{1}{(z+2)(z-1)}}$$

now $g(k) = {}^{-1}G(z)$, and

$$\frac{G(z)}{z} = \frac{-\frac{1}{2}}{z} + \frac{\frac{1}{6}}{z+2} + \frac{\frac{1}{3}}{z-1}$$

$$\boxed{g(k) = -\frac{1}{2}\delta(k) + \frac{1}{6}(-2)^k + \frac{1}{3}}$$

If the input is the unit step, we get

$$(z^2 + z - 2)Y(z) = z + \frac{z}{z-1}$$

Solution of difference equations V

$$Y(z) = \frac{z}{(z+2)(z-1)} + \frac{z}{(z+2)(z-1)^2} = \frac{z^2}{(z+2)(z-1)^2}$$

$$\frac{Y(z)}{z} = \frac{z}{(z+2)(z-1)^2} = \frac{\frac{1}{3}}{(z-1)^2} + \frac{\frac{2}{9}}{z-1} - \frac{\frac{2}{9}}{z+2}$$

$$\boxed{y(k) = \frac{2}{9} + \frac{1}{3}k - \frac{2}{9}(-2)^k}$$