# Digital Signal Processing

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Contents

***Abstract*—This manual provides a simple introduction to digital signal processing.**

## 1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

## 2 Digital Filter

2.1 Download the sound file from

```
wget https://github.com/yashrajput22/EE3900
    −22/blob/master/codes/Section−2/
    Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem **??** in the spectrogram and play. Observe the spectrogram. What do you find?
**Solution:** There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
**Solution:**

```
import soundfile as sf
from scipy import signal

#read .wav file
input_signal,fs = sf.read("Sound_Noise.wav
    ")
#sampling frequency of Input signal
sampl_freq=fs
# print("Sample Frequency ",sampl_freq)
#order of the filter
order=4
#cutoff frquency 4kHz
cutoff_freq=4000.0
#digital frequency
Wn=2*cutoff_freq/sampl_freq
# b and a are numerator and
    denominatorpolynomials respectively
b, a = signal.butter(order,Wn, 'low')
#filter the input signal with butterworth filter
# output_signal = signal.filtfilt(b, a,
    input_signal)
output_signal = signal.lfilter(b, a,
    input_signal)
#write the output signal into .wav file
sf.write('Sound_With_ReducedNoise.wav',
    output_signal, fs)
```

2.4 The output of the python script in Problem **??** is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem **??**. What do you observe?
**Solution:** The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

## 3 Difference Equation

3.1 Let

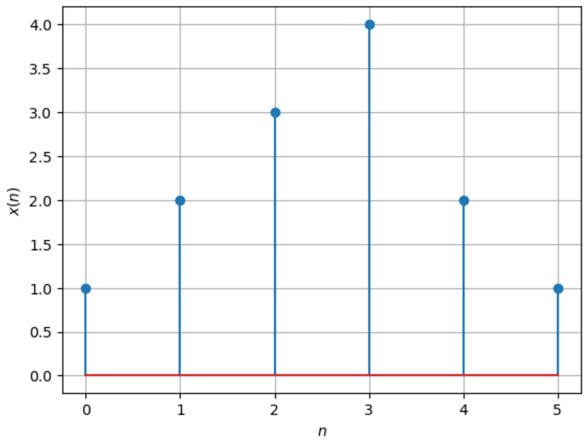
$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.
**Solution:**

```
import numpy as np
import matplotlib.pyplot as plt

x=np.array([1.0,2.0,3.0,4.0,2.0,1.0])

plt.stem(range(0,len(x)),x)
plt.ylabel("$x(n)$")
```

```
plt.xlabel("$n$")
plt.grid()
plt.show()
```

The above code yields

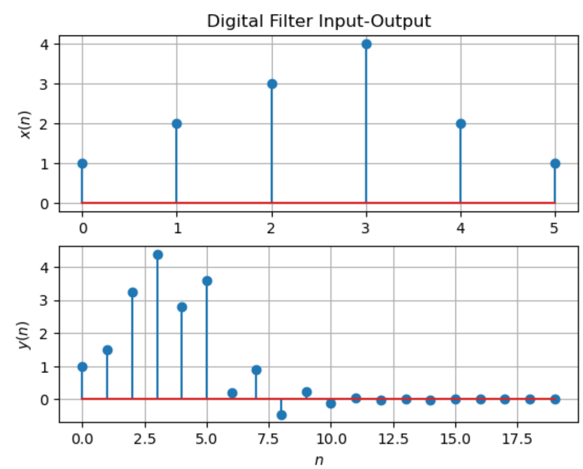



3.2 Let

$$y(n)+\frac{1}{2}y(n-1)=x(n)+x(n-2),$$
$$y(n)=0,n<0 \quad (3.2)$$

Sketch $y(n)$.
**Solution:**

```
import numpy as np
import matplotlib.pyplot as plt
#If using termux
import subprocess
import shlex
#end if

x=np.array([1.0,2.0,3.0,4.0,2.0,1.0])
k = 20
y = np.zeros(20)


y[0] = x[0]
y[1] = -0.5*y[0]+x[1]

for n in range(2,k-1):
        if n < 6:
                y[n] = -0.5*y[n-1]+x[n]+x
                    [n-2]
        elif n > 5 and n < 8:
                y[n] = -0.5*y[n-1]+x[n-2]
        else:
                y[n] = -0.5*y[n-1]
print(y)

#subplots
plt.subplot(2, 1, 1)
plt.stem(range(0,6),x)
plt.title('Digital Filter Input-Output')
plt.ylabel('$x(n)$')
plt.grid()# minor


plt.subplot(2, 1, 2)
plt.stem(range(0,k),y)
plt.xlabel('$n$')
plt.ylabel('$y(n)$')
plt.grid()# minor

#If using termux
# plt.savefig('../figs/xnyn.pdf')
# plt.savefig('../figs/xnyn.eps')
# subprocess.run(shlex.split("termux-open ../
    figs/xnyn.pdf"))
#else
plt.show()
```



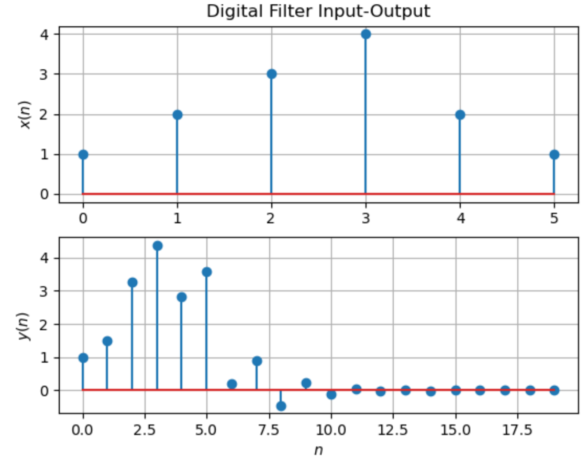

3.3 Repeat the above exercise using a C code.
**Solution:** The following C code generates data and saves it to a .dat file

```
#include <stdlib.h>
#include <stdio.h>
#define k 20

int main(){
    double x[6]={1,2,3,4,2,1};
    double y[k]={0};
    y[0]=x[0];
    y[1]=x[1]- 0.5*y[0];
    for(int i=2;i<k;i++){
```

```
        if (i<6)
            y[i]=−0.5∗y[i−1] + x[i]+x[i−2];
        else if(i<8)
            y[i]=−0.5∗y[i−1] +x[i−2];
        else
            y[i]=−0.5∗y[i−1];
    }
    int axes_x[sizeof(x)/sizeof(int)]={0};
    int axes_y[k]={0};

    FILE∗ fpy;
    fpy=fopen("3_3_data_y.dat","w");
    for(int i=0;i<k;i++){
        fprintf(fpy,"%f\n",y[i]);
    }
    fclose(fpy);
    FILE∗ fpx;
    fpx=fopen("3_3_data_x.dat","w");
    for(int i=0;i<6;i++){
        fprintf(fpx,"%f\n",x[i]);
    }
    fclose(fpx);
    return 0;
}
```

The following Python code sketches $x(n)$ and $y(n)$

```
import numpy as np
import matplotlib.pyplot as plt

y=np.loadtxt('3_3_data_y.dat',dtype="
    double")
x=np.loadtxt('3_3_data_x.dat',dtype="
    double")

plt.subplot(2,1,1)
plt.stem(range(0,len(x)),x)
plt.title("Digital Filter Input−Output")
plt.ylabel("$x(n)$")
plt.grid()

plt.subplot(2,1,2)
plt.stem(range(0,len(y)),y)
plt.ylabel("$y(n)$")
plt.xlabel("$n$")
plt.grid()
plt.show()
```




## 4 Z-transform

4.1 The $Z$-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

**Solution:** From (**??**),

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.5}$$

resulting in (**??**). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.6}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem **??**.
**Solution:**

$$Z(x(n)) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.7}$$

$$= x(0)z^{0} + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3}+ \tag{4.8}$$

$$x(4)z^{-4} + x(5)z^{-5}$$
$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \tag{4.9}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.10}$$

from (**??**) assuming that the $Z$-transform is a linear operation.

**Solution:**

Applying (**??**) in (**??**),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.11}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.12}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.13}$$

and show that the $Z$-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.14}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.15}$$

**Solution:** It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.16}$$

and from (**??**),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.17}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.18}$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.19}$$

**Solution:**

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} \tag{4.20}$$

$$= \sum_{n=-\infty}^{\infty} u(n)(az^{-1})^n \tag{4.21}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n, \quad \left|az^{-1}\right| < 1 \tag{4.22}$$

$$\tag{4.23}$$

$$= \frac{1}{1 - az^{-1}}, \quad |a| < |z| \tag{4.24}$$

using the fomula for the sum of an infinite geometric progression.

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.25}$$

Plot $\left|H\left(e^{\jmath\omega}\right)\right|$. Comment. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $x(n)$.

**Solution:** The graph is symmetric and periodic. It is achieves a high of value 4 and a minimum value between 0 - 0.5. It is bounded between (0, 4) with period of $2\pi$

$$H\left(e^{j\omega}\right) = \frac{1 + e^{-2j\omega}}{1 + \frac{e^{-j\omega}}{2}} \tag{4.26}$$

$$\implies \left|H\left(e^{j\omega}\right)\right| = \frac{\left|1 + e^{-2j\omega}\right|}{\left|1 + \frac{e^{-j\omega}}{2}\right|} \tag{4.27}$$

$$= \frac{\left|1 + e^{2j\omega}\right|}{\left|e^{2j\omega} + \frac{e^{j\omega}}{2}\right|} \tag{4.28}$$

$$= \frac{\left|1 + \cos 2\omega + j \sin 2\omega\right|}{\left|e^{j\omega} + \frac{1}{2}\right|} \tag{4.29}$$

$$= \frac{\left|4\cos^2(\omega) + 4j\sin(\omega)\cos(\omega)\right|}{\left|2e^{j\omega} + 1\right|} \tag{4.30}$$

$$= \frac{\left|4\cos(\omega)\right|\left|\cos(\omega) + j\sin(\omega)\right|}{\left|2\cos(\omega) + 1 + 2j\sin(\omega)\right|} \tag{4.31}$$

$$\therefore \left|H\left(e^{j\omega}\right)\right| = \frac{\left|4\cos(\omega)\right|}{\sqrt{5 + 4\cos(\omega)}} \tag{4.32}$$

The following code plots $\left|H\left(e^{j\omega}\right)\right|$

```
import numpy as np
```

```
import matplotlib.pyplot as plt

#DTFT
def H(z):
        num = np.polyval([1,0,1],z**(-1))
        den = np.polyval([0.5,1],z**(-1))
        H = num/den
        return H

#Input and Output
omega = np.linspace(-3*np.pi,3*np.pi,100)

#subplots
plt.plot(omega, abs(H(np.exp(1j*omega))))
plt.title('Filter Frequency Response')
plt.xlabel('$\omega$')
plt.ylabel('$|H(e^{\jmath\omega})| $')
plt.grid()# minor

plt.show()
```

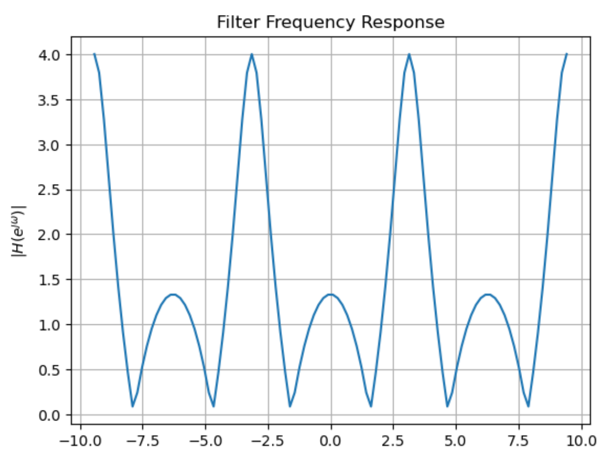



Fig. 4.6: $h(n)$ as the inverse of $H(z)$

4.7 Express $h(n)$ in terms of $H(e^{\jmath\omega})$.

**Solution:**

$$H\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n} \tag{4.33}$$

$$\int_{-\pi}^{\pi} H\left(e^{j\omega}\right) e^{j\omega k} d\omega = \sum_{n=-\infty}^{\infty} h(n) \int_{-\pi}^{\pi} e^{-j\omega n} e^{j\omega k} d\omega \tag{4.34}$$

(4.35)

$$\int_{-\pi}^{\pi} e^{\jmath\omega(n-k)} d\omega = \begin{cases} 2\pi & n = k \\ 0 & \text{otherwise} \end{cases} \tag{4.36}$$

$$\int_{-\pi}^{\pi} H\left(e^{j\omega}\right) e^{j\omega k} d\omega = h(n)\, 2\pi \tag{4.37}$$

$$\int_{-\pi}^{\pi} H\left(e^{j\omega}\right) e^{j\omega k} d\omega = 2\pi h(n) \tag{4.38}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} H\left(e^{j\omega}\right) e^{j\omega k} d\omega = h(n) \tag{4.39}$$

## 5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (**??**).
**Solution:**

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2} z^{-1}} \tag{5.2}$$

Let $z^{-1} = x$,then, by polynomial long division we get

$$\begin{array}{r} 2x - 4 \\ \frac{1}{2}x + 1 \overline{) \; x^2 \qquad + 1} \\ -x^2 - 2x \\ \hline -2x + 1 \\ 2x + 4 \\ \hline 5 \end{array}$$

$$\implies (1 + z^{-2}) = (\frac{1}{2}z^{-1} + 1)(2z^{-1} - 4) + 5 \tag{5.3}$$

$$\implies \frac{(1 + z^{-2})}{\frac{1}{2}z^{-1} + 1} = (2z^{-1} - 4) + \frac{5}{\frac{1}{2}z^{-1} + 1} \tag{5.4}$$

$$\implies H(z) = (2z^{-1} - 4) + \frac{5}{\frac{1}{2}z^{-1} + 1} \tag{5.5}$$

Now, consider $\frac{5}{\frac{1}{2}z^{-1}+1}$
The denominator $\frac{1}{2}z^{-1} + 1$ can be expressed as sum of an infinite geometric progression, which as its first term equal to 1 and common ratio $\frac{-1}{2}z^{-1}$
Therefore, we can write $\frac{5}{\frac{1}{2}z^{-1}+1}$ as $5\left(1 + \left(\frac{-1}{2}z^{-1}\right) + \left(\frac{-1}{2}z^{-1}\right)^2 + \left(\frac{-1}{2}z^{-1}\right)^3 + \left(\frac{-1}{2}z^{-1}\right)^4 + \dots\right)$
Therefore, H(z) can be given by,

$$H(z) = (2z^{-1} - 4) + \frac{5}{\frac{1}{2}z^{-1} + 1} \tag{5.6}$$

$$\tag{5.7}$$

$$= 2z^{-1} - 4 + 5 + \frac{-5}{2}z^{-1} + \frac{5}{4}z^{-2} + \frac{-5}{8}z^{-3} + \frac{5}{16}z^{-4} + \dots \tag{5.8}$$

$$\implies H(z) = 1z^0 + \frac{-1}{2}z^{-1} + \frac{5}{4}z^{-2} + \frac{-5}{8}z^{-3} + \frac{5}{16}z^{-4} + \dots \tag{5.9}$$

Comparing the above expression to (**??**) we get h(n) for n<5 as,

$$h(0) = 1 \tag{5.10}$$

$$h(1) = \frac{-1}{2} \tag{5.11}$$

$$h(2) = \frac{5}{4} \tag{5.12}$$

$$h(3) = \frac{-5}{8} \tag{5.13}$$

$$h(4) = \frac{5}{16} \tag{5.14}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.15}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (**??**).
**Solution:** From (**??**),

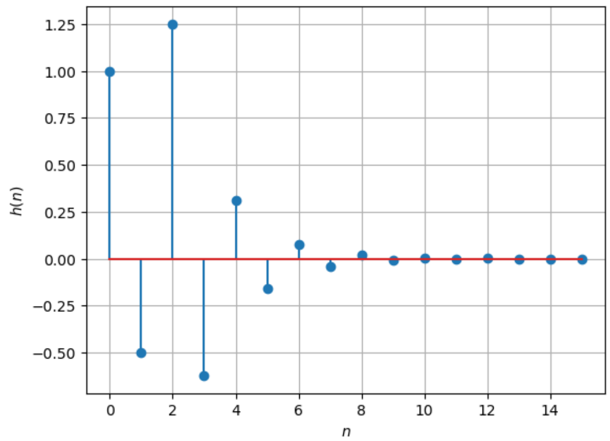
$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}}, \quad \left|\frac{1}{2}\right| < |z| \tag{5.16}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.17}$$

using (**??**) and (**??**).

5.3 Sketch $h(n)$. Is it bounded? Justify theoritically.
**Solution:** The following code plots Fig. **??**.

```
import numpy as np
import matplotlib.pyplot as plt

n=np.arange(14)
un=(−1/2)**n

hn1=np.pad(un,(0,2),'constant',
    constant_values=(0))
hn2=np.pad(un,(2,0),'constant',
    constant_values=(0))

hn=hn1+hn2

plt.stem(range(0,len(hn)),hn)
plt.grid()
plt.ylabel("$h(n)$")
plt.xlabel("$n$")
plt.show()
```




From (??) we know that

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.18)$$

Implies we can write that

$$h(n) = \begin{cases} 0 & ,n < 0 \\ \left(\frac{-1}{2}\right)^n & ,0 \le n < 2 \\ 5\left(\frac{-1}{2}\right)^n & ,n \ge 2 \end{cases} \quad (5.19)$$

A sequence is said to be bounded when

$$|x_n| \le M, \forall n \in \mathcal{N} \quad (5.20)$$

Now consider (??),
For n<0,

$$|h(n)| \le 0 \quad (5.21)$$

For $0 \le n < 2$,

$$|h(n)| = (\frac{1}{2})^n \quad (5.22)$$
$$\implies |h(n)| \le 1 \quad (5.23)$$

For $n \ge 2$,

$$|h(n)| = 5(\frac{1}{2})^n \quad (5.24)$$
$$\implies |h(n)| \le 5 \quad (5.25)$$

From above we can say that,

$$M = max\{0, 1, 5\} \quad (5.26)$$
$$= 5 \quad (5.27)$$

Therefore since $M$ exists and is a real value, we can say that h($n$) is bounded.

5.4 Convergent? Justify using the ratio test.
**Solution:**
Yes, it is convergent. We can clearly see in the plot it is not tending to infinite and remain finite.
For large $n$, we see that

$$h(n) = \left(-\frac{1}{2}\right)^n + \left(-\frac{1}{2}\right)^{n-2} \quad (5.28)$$
$$= \left(-\frac{1}{2}\right)^n (4+1) = 5\left(-\frac{1}{2}\right)^n \quad (5.29)$$
$$\implies \left|\frac{h(n+1)}{h(n)}\right| = \frac{1}{2} \quad (5.30)$$

and therefore, $\lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| = \frac{1}{2} < 1$. Hence, we see that $h(n)$ converges.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \quad (5.31)$$

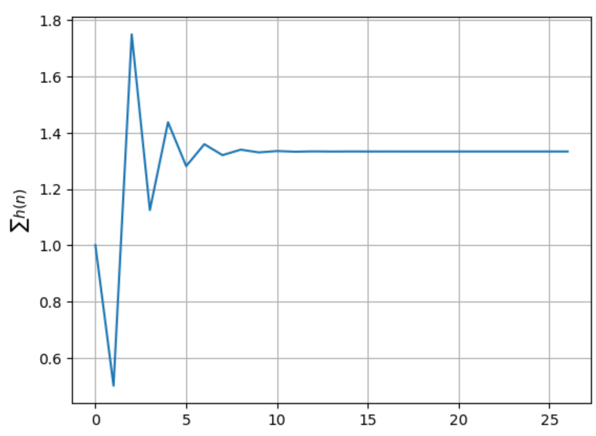
Is the system defined by (??) stable for the impulse response in (??)?
**Solution:** By using h(n) from 5.3

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.32)$$
$$= \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.33)$$
$$= \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^n u(n) + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.34)$$
$$= \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^n + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^{n-2} \quad (5.35)$$
$$(5.36)$$
$$= \frac{2}{3} + \frac{2}{3} < \infty \quad (5.37)$$
$$(5.38)$$
$$= \frac{4}{3} < \infty \quad (5.39)$$

5.6 Verify the above result using a python code.
**Solution:** The following code computes and plots at each n. We can see that the sum converges to a constant value as n tends to infinity.

```
import numpy as np
import matplotlib.pyplot as plt
```

```
from sympy import N

n=np.arange(25)
un=(−1/2)∗∗n

hn1=np.pad(un,(0,2),'constant',
    constant_values=(0))
hn2=np.pad(un,(2,0),'constant',
    constant_values=(0))

hn=hn1+hn2

nh=len(hn)
sum_hn=np.zeros(nh)
sum_hn[0]=hn[0]
for i in range(1,nh):
    sum_hn[i]=sum_hn[i−1]+hn[i]

plt.plot(range(len(hn)),sum_hn)
plt.ylabel("$\sum{h(n)}$")
plt.xlabel("$n$")
plt.grid()
plt.show()
```




5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \quad (5.40)$$

This is the definition of $h(n)$.
**Solution:** The following code plots Fig. **??**. Note that this is the same as Fig. **??**.

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2) \quad (5.41)$$

$$H(z) + \frac{1}{2}z^{-1}H(z) = 1 + z^{-2} \quad (5.42)$$

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}}, \quad \left|\frac{1}{2}\right| < |z| \quad (5.43)$$

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.44)$$

```
import numpy as np
import matplotlib.pyplot as plt

n=np.arange(14)
un=(−1/2)∗∗n

hn1=np.pad(un,(0,2),'constant',
    constant_values=(0))
hn2=np.pad(un,(2,0),'constant',
    constant_values=(0))

hn=hn1+hn2

plt.stem(range(0,len(hn)),hn)
plt.grid()
plt.ylabel("$h(n)$")
plt.xlabel("$n$")
plt.show()
```

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \quad (5.45)$$

Comment. The operation in (**??**) is known as *convolution*.
**Solution:** The following code plots $y(n)$.

```
import numpy as np
import matplotlib.pyplot as plt
from sympy import N

n=np.arange(25)
un=(−1/2)∗∗n

hn1=np.pad(un,(0,2),'constant',
```

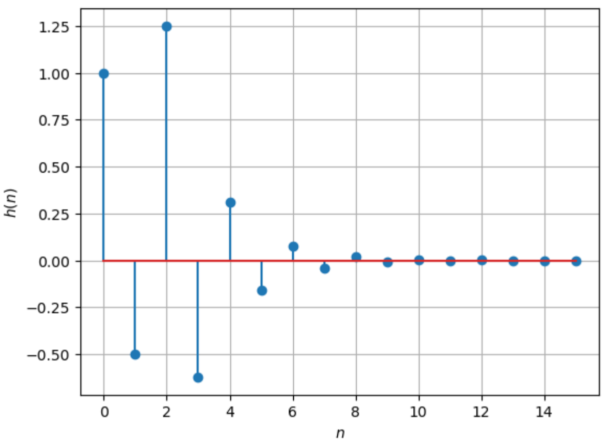



Fig. 5.7: $h(n)$ from the definition

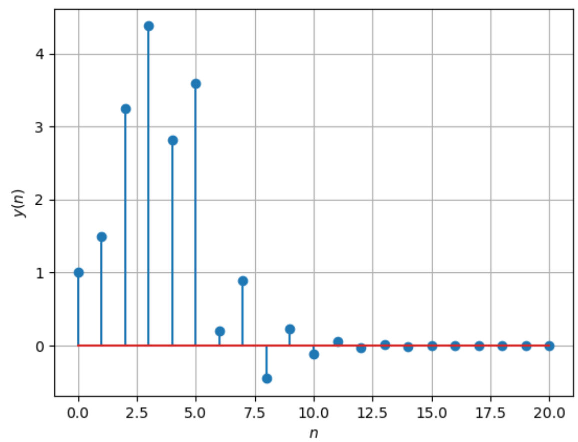



Fig. 5.8: $h(n)$ from the definition

```
    constant_values=(0))
hn2=np.pad(un,(2,0),'constant',
    constant_values=(0))

hn=hn1+hn2

nh=len(hn)
sum_hn=np.zeros(nh)
sum_hn[0]=hn[0]
for i in range(1,nh):
    sum_hn[i]=sum_hn[i-1]+hn[i]

plt.plot(range(len(hn)),sum_hn)
plt.ylabel("$\sum{h(n)}$")
plt.xlabel("$n$")
plt.grid()
plt.show()
```

5.9 Express the above convolution using a Teoplitz matrix.
**Solution:** We know that from, (**??**),

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (5.46)$$

This can also be wrtten as a matrix-vector multiplication given by the expression,

$$y = T(h) * x \quad (5.47)$$

In the equation (**??**), $T(h)$ is a Teoplitz matrix.

The equation (**??**) can be expanded as,

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} \quad (5.48)$$

$$\mathbf{y} = \begin{pmatrix} h_1 & 0 & . & . & . & 0 \\ h_2 & h_1 & . & . & . & 0 \\ h_3 & h_2 & h_1 & . & . & 0 \\ . & . & . & . & . & . \\ h_{n-1} & h_{n-2} & h_{n-3} & . & . & 0 \\ h_n & h_{n-1} & h_{n-2} & . & . & h_1 \\ 0 & h_n & h_{n-1} & h_{n-2} & . & h_2 \\ . & . & . & . & . & . \\ 0 & . & . & . & 0 & h_{n-1} \\ 0 & . & . & . & 0 & h_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} \quad (5.49)$$

5.10 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \quad (5.50)$$

**Solution:** From (**??**), we substitute $k := n - k$ to get

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \quad (5.51)$$

$$= \sum_{n-k=-\infty}^{\infty} x(n-k)\,h(k) \quad (5.52)$$

$$= \sum_{k=-\infty}^{\infty} x(n-k)\,h(k) \quad (5.53)$$

## 6 DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.
**Solution:**
From **??**, we know that ,

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{6.2}$$

Here, let, $\omega = e^{-j2\pi k}$. Then,

$$X(k) = 1 + 2\omega^{\frac{1}{5}} + 3\omega^{\frac{2}{5}} + 4\omega^{\frac{3}{5}} + 2\omega^{\frac{4}{5}} + \omega \tag{6.3}$$

Similarly, we know from (**??**),

$$h(n) = \begin{cases} 0 & , n < 0 \\ \left(\frac{-1}{2}\right)^n & , 0 \leq n < 2 \\ 5\left(\frac{-1}{2}\right)^n & , n \geq 2 \end{cases} \tag{6.4}$$

Now,again let, $\omega = e^{-j2\pi k}$. Then,

$$H(k) = 1 + \frac{-1}{2}\omega^{\frac{1}{5}} + \frac{5}{4}\omega^{\frac{2}{5}} + \frac{-5}{8}\omega^{\frac{3}{5}} + \frac{5}{16}\omega^{\frac{4}{5}} + \frac{-5}{32}\omega \tag{6.5}$$

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.6}$$

**Solution:**
Now, from (**??**) and (**??**), we know $X(k) and H(k)$. Now, given that,

$$Y(k) = X(k) * H(k) \tag{6.7}$$

$$Y(k) = (1 + 2\omega^{\frac{1}{5}} + 3\omega^{\frac{2}{5}} + 4\omega^{\frac{3}{5}} + 2\omega^{\frac{4}{5}} + \omega) * \\ (1 + \frac{-1}{2}\omega^{\frac{1}{5}} + \frac{5}{4}\omega^{\frac{2}{5}} + \frac{-5}{8}\omega^{\frac{3}{5}} + \frac{5}{16}\omega^{\frac{4}{5}} + \frac{-5}{32}\omega) \tag{6.8}$$

$$Y(k) = 1 + \frac{3}{2}\omega^{\frac{1}{5}} + \frac{13}{4}\omega^{\frac{2}{5}} + \frac{35}{8}\omega^{\frac{3}{5}} + \frac{45}{16}\omega^{\frac{4}{5}} \\ \frac{115}{32}\omega^{\frac{5}{5}} + \frac{1}{8}\omega^{\frac{6}{5}} + \frac{25}{32}\omega^{\frac{7}{5}} - \frac{5}{8}\omega^{\frac{8}{5}} \\ - \frac{5}{32}\omega^{5} \tag{6.9}$$

where, $\omega = e^{-j2k\pi}$

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k) \cdot e^{j2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \tag{6.10}$$

**Solution:** The following code plots Fig. **??** and computes $X(k)$ and $Y(k)$. Note that this is the same as $y(n)$ in Fig. **??**.

```
import numpy as np
import matplotlib.pyplot as plt

N=14
xtemp=np.array([1.0,2.0,3.0,4.0,2.0,1.0])
x=np.pad(xtemp, (0,8), 'constant',
    constant_values=(0))
n=np.arange(N)

X=np.zeros(N) + 1j*np.zeros(N)
for k in range(N):
    for i in range(N):
        X[k]+=x[i]*np.exp(-1j*2*np.pi*k*i/
            N)

un=(-1/2)**n
hn1=np.pad(un,(0,2),"constant",
    constant_values=(0))
hn2=np.pad(un,(2,0),"constant",
    constant_values=(0))
hn=hn1+hn2
H=np.zeros(N)+ 1j*np.zeros(N)
for k in range(N):
    for i in range(N):
        H[k]+=hn[i]*np.exp(-1j*2*np.pi*k*i
            /N)

Y=np.zeros(N)+1j*np.zeros(N)
for k in range(N):
    Y[k]=X[k]*H[k]
y=np.real(Y)
plt.stem(n,y)
plt.ylabel("$Y(k)$")
plt.xlabel("$k$")
plt.grid()
plt.show()
```

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
**Solution:** The following python codes compute $X(k), H(k)$ and $y(n)$ through FFT and IFFT.

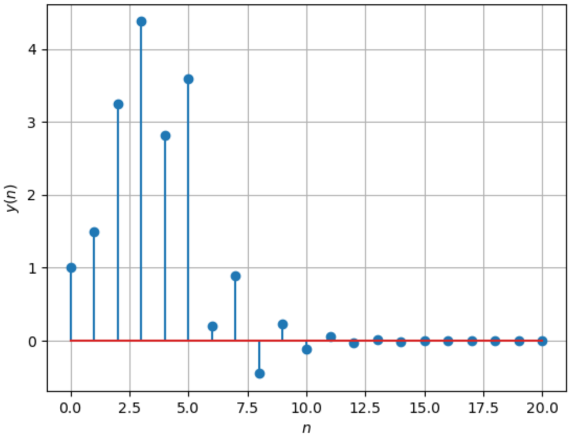



Fig. 6.3: $y(n)$ from the DFT

```
from scipy.fft import fft, ifft
import numpy as np
import matplotlib.pyplot as plt

x=np.array([1.0,2.0,3.0,4.0,2.0,1.0])
x=np.pad(x,(0,8),'constant',constant_values
      =(0))
N=14
n=np.arange(N)
un=(-1/2)**n
hn1=np.pad(un,(0,2),'constant',
      constant_values=(0))
hn2=np.pad(un,(2,0),'constant',
      constant_values=(0))
hn=hn1+hn2

X=fft(x)
H=fft(hn[:N])
Y=np.zeros(N)+1j*np.zeros(N)
for i in range(N):
      Y[i]=X[i]*H[i]
y=ifft(Y)

plt.stem(range(0,N),np.real(X))
plt.title("Using FFT")
plt.ylabel("$X(k)$")
plt.grid()
plt.show()

plt.stem(range(0,N),np.real(H))
plt.title("Using FFT")
plt.ylabel("$H(k)$")
plt.grid()
plt.show()

plt.stem(range(0,N),np.real(y))
plt.title("Using FFT and IFFT")
plt.ylabel("$y(n)$")
plt.grid()
plt.show()
```

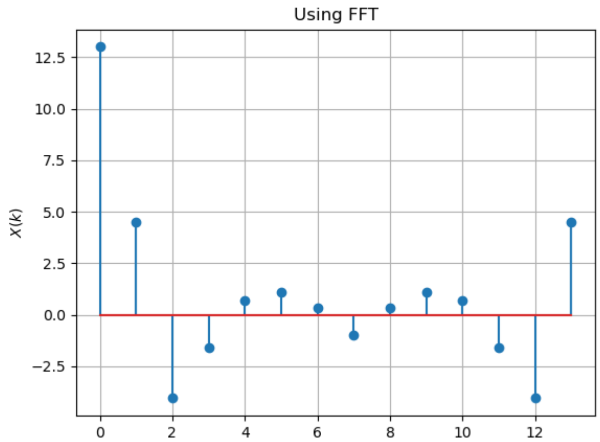



Fig. 6.4: $X(k)$ from the FFT

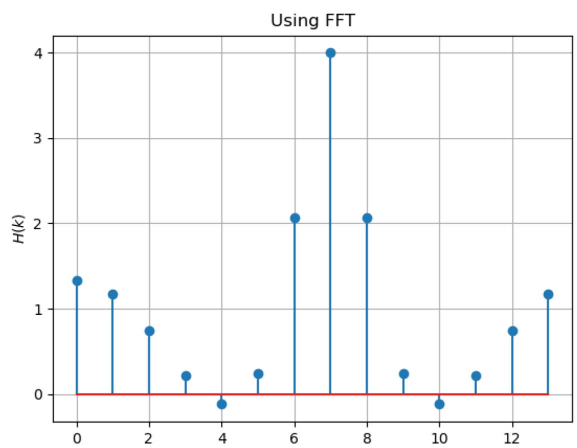



Fig. 6.4: $H(k)$ from the FFT

6.5 Wherever possible, express all the above equations as matrix equations.
**Solution:** We use the DFT Matrix, where

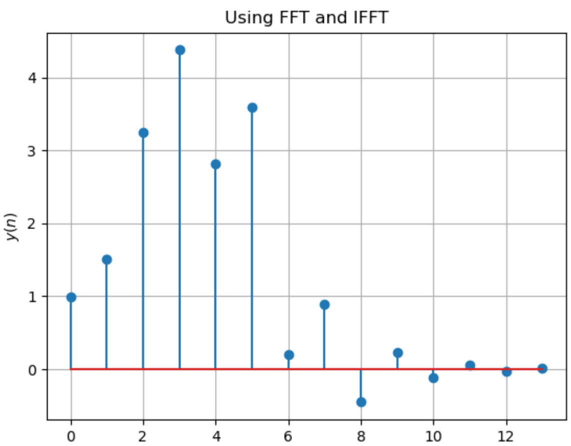



Fig. 6.4: $y(n)$ from the IFFT

$\omega = e^{-\frac{j2k\pi}{N}}$, which is given by

$$\mathbf{W} = \begin{pmatrix} \omega^0 & \omega^0 & \dots & \omega^0 \\ \omega^0 & \omega^1 & \dots & \omega^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \omega^0 & \omega^{N-1} & \dots & \omega^{(N-1)(N-1)} \end{pmatrix} \tag{6.11}$$

i.e. $W_{jk} = \omega^{jk}$, $0 \leq j, k < N$. Hence, we can write any DFT equation as

$$\mathbf{X} = \mathbf{W}\mathbf{x} = \mathbf{x}\mathbf{W} \tag{6.12}$$

where

$$\mathbf{x} = \begin{pmatrix} x(0) \\ x(1) \\ \vdots \\ x(n-1) \end{pmatrix} \tag{6.13}$$

Using (**??**), the inverse Fourier Transform is given by

$$\mathbf{x} = \mathcal{F}^{-1}(\mathbf{X}) = \mathbf{W}^{-1}\mathbf{X} = \frac{1}{N}\mathbf{W}^{\mathbf{H}}\mathbf{X} = \frac{1}{N}\mathbf{X}\mathbf{W}^{\mathbf{H}} \tag{6.14}$$

$$\implies \mathbf{W}^{-1} = \frac{1}{N}\mathbf{W}^{\mathbf{H}} \tag{6.15}$$

where $H$ denotes hermitian operator. We can rewrite (**??**) using the element-wise multiplication operator as

$$\mathbf{Y} = \mathbf{H} \cdot \mathbf{X} = (\mathbf{W}\mathbf{h}) \cdot (\mathbf{W}\mathbf{x}) \tag{6.16}$$

The plot of $y(n)$ using the DFT matrix in Fig. (**??**) is the same as $y(n)$ in Fig. (**??**).

```
import numpy as np
from numpy.fft import fft, ifft
import matplotlib.pyplot as plt
#If using termux
#import subprocess
#import shlex
#end if

N = 14
n = np.arange(N)
fn=(-1/2)**n
hn1=np.pad(fn, (0,2), 'constant',
    constant_values=(0,0))
hn2=np.pad(fn, (2,0), 'constant',
    constant_values=(0,0))
h = hn1+hn2
xtemp=np.array([1.0,2.0,3.0,4.0,2.0,1.0])
x=np.pad(xtemp, (0,10), 'constant',
    constant_values=(0))
dftmtx = fft(np.eye(len(x)))
X = x@dftmtx
H = h@dftmtx
Y = H*X
invmtx = np.linalg.inv(dftmtx)
y = (Y@invmtx).real
#plots
plt.stem(range(0,16),y)
plt.xlabel('$n$')
plt.ylabel('$y(n)$')
plt.grid()# minor

#If using termux
#plt.savefig('../figs/6_5.png')
#subprocess.run(shlex.split("termux-open ../
    figs/yndft.pdf"))
#else
plt.show()
```

## 7 FFT

1. The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{7.1}$$

2. Let

$$W_N = e^{-j2\pi/N} \tag{7.2}$$

Then the $N$-point *DFT matrix* is defined as

$$\mathbf{F}_N = [W_N^{mn}], \quad 0 \leq m, n \leq N-1 \tag{7.3}$$

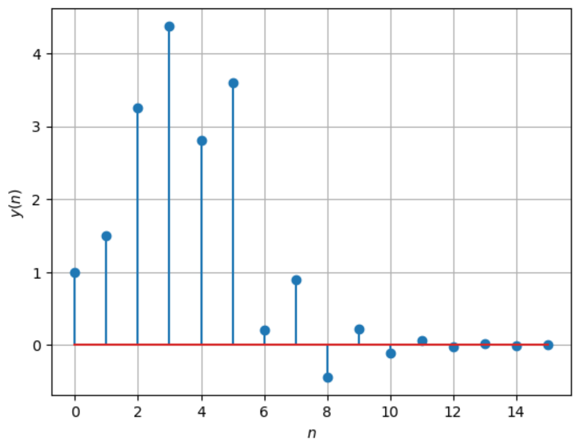



Fig. 6.5: $y(n)$ using the DFT matrix

where $W_N^{mn}$ are the elements of $\mathbf{F}_N$.

3. Let
$$\mathbf{I}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^2 & \mathbf{e}_4^3 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.4}$$
be the $4 \times 4$ identity matrix. Then the 4 point *DFT permutation matrix* is defined as
$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.5}$$

4. The 4 point *DFT diagonal matrix* is defined as
$$\mathbf{D}_4 = diag \begin{pmatrix} W_4^0 & W_N^1 & W_N^2 & W_N^3 \end{pmatrix} \tag{7.6}$$

5. Show that
$$W_N^2 = W_{N/2} \tag{7.7}$$
**Solution:** We write
$$W_N^2 = \left(e^{-\frac{j2\pi}{N}}\right)^2 = e^{-\frac{j2\pi}{N/2}} = W_{N/2} \tag{7.8}$$

6. Show that
$$\mathbf{F}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.9}$$
**Solution:** Observe that for $n$ in N,$W_4^{4n} = 1$ and $W_4^{4n+2} = -1$. Using (**??**),
$$\mathbf{D}_2\mathbf{F}_2 = \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} \tag{7.10}$$
$$= \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^0 & W_4^2 \end{bmatrix} \tag{7.11}$$
$$= \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^1 & W_4^3 \end{bmatrix} \tag{7.12}$$
$$\implies -\mathbf{D}_2\mathbf{F}_2 = \begin{bmatrix} W_4^2 & W_4^6 \\ W_4^3 & W_4^9 \end{bmatrix} \tag{7.13}$$
and
$$\mathbf{F}_2 = \begin{pmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{pmatrix} \tag{7.14}$$
$$= \begin{pmatrix} W_4^0 & W_4^0 \\ W_4^0 & W_4^2 \end{pmatrix} \tag{7.15}$$
Hence,
$$\mathbf{W}_4 = \begin{pmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^2 & W_4^1 & W_4^3 \\ W_4^0 & W_4^4 & W_4^2 & W_4^6 \\ W_4^0 & W_4^6 & W_4^3 & W_4^9 \end{pmatrix} \tag{7.16}$$
$$= \begin{bmatrix} \mathbf{I}_2\mathbf{F}_2 & \mathbf{D}_2 F_2 \\ \mathbf{I}_2\mathbf{F}_2 & -\mathbf{D}_2 F_2 \end{bmatrix} \tag{7.17}$$
$$= \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & \mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \tag{7.18}$$
Multiplying (**??**) by $\mathbf{P}_4$ on both sides, and noting that $\mathbf{W}_4\mathbf{P}_4 = \mathbf{F}_4$ gives us.

7. Show that
$$\mathbf{F}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N \tag{7.19}$$
**Solution:** Observe that for even $N$ and letting $\mathbf{f}_N^i$ denote the $i^{\text{th}}$ column of $\mathbf{F}_N$, from (**??**) and (**??**),
$$\begin{pmatrix} \mathbf{D}_{N/2}\mathbf{F}_{N/2} \\ -\mathbf{D}_{N/2}\mathbf{F}_{N/2} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_N^2 & \mathbf{f}_N^4 & \dots & \mathbf{f}_N^N \end{pmatrix} \tag{7.20}$$
and
$$\begin{pmatrix} \mathbf{I}_{N/2}\mathbf{F}_{N/2} \\ \mathbf{I}_{N/2}\mathbf{F}_{N/2} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_N^1 & \mathbf{f}_N^3 & \dots & \mathbf{f}_N^{N-1} \end{pmatrix} \tag{7.21}$$
Thus,
$$\begin{bmatrix} \mathbf{I}_2\mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2 \\ \mathbf{I}_2\mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix}$$
$$= \begin{pmatrix} \mathbf{f}_N^1 & \dots & \mathbf{f}_N^{N-1} & \mathbf{f}_N^2 & \dots & \mathbf{f}_N^N \end{pmatrix} \tag{7.22}$$

and so,

$$\begin{bmatrix}\mathbf{I}_{N/2} & \mathbf{D}_{N/2}\\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2}\end{bmatrix}\begin{bmatrix}\mathbf{F}_{N/2} & 0\\ 0 & \mathbf{F}_{N/2}\end{bmatrix}\mathbf{P}_N$$
$$= \begin{pmatrix}\mathbf{f}_N^1 & \mathbf{f}_N^2 & \dots & \mathbf{f}_N^N\end{pmatrix} = \mathbf{F}_N \tag{7.23}$$

8. Find

$$\mathbf{P}_4\mathbf{x} \tag{7.24}$$

**Solution:** We have,

$$\mathbf{P}_4\mathbf{x} = \begin{pmatrix}\mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4\end{pmatrix}\begin{pmatrix}x(0)\\x(1)\\x(2)\\x(3)\end{pmatrix} = \begin{pmatrix}x(0)\\x(2)\\x(1)\\x(3)\end{pmatrix} \tag{7.25}$$

9. Show that

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \tag{7.26}$$

where $\mathbf{x}, \mathbf{X}$ are the vector representations of $x(n), X(k)$ respectively.
**Solution:** Writing the terms of $X$,

$$X(0) = x(0) + x(1) + \ldots + x(N-1) \tag{7.27}$$

$$X(1) = x(0) + x(1)e^{-\frac{\jmath 2\pi}{N}} + \ldots + \\ + x(N-1)e^{-\frac{\jmath 2(N-1)\pi}{N}} \tag{7.28}$$

$$\vdots$$

$$X(N-1) = x(0) + x(1)e^{-\frac{\jmath 2(N-1)\pi}{N}} + \ldots + \\ + x(N-1)e^{-\frac{\jmath 2(N-1)(N-1)\pi}{N}} \tag{7.29}$$

Clearly, the term in the $m^{\text{th}}$ row and $n^{\text{th}}$ column is given by ($0 \le m \le N-1$ and $0 \le n \le N-1$)

$$T_{mn} = x(n)e^{-\frac{\jmath 2mn\pi}{N}} \tag{7.30}$$

and so, we can represent each of these terms as a matrix product

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \tag{7.31}$$

where $\mathbf{F}_N = \left[e^{-\frac{-\jmath 2mn\pi}{N}}\right]_{mn}$ for $0 \le m \le N-1$ and $0 \le n \le N-1$.

10. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix}X(0)\\X(1)\\X(2)\\X(3)\end{bmatrix} = \begin{bmatrix}X_1(0)\\X_1(1)\\X_1(2)\\X_1(3)\end{bmatrix} + \begin{bmatrix}W_8^0 & 0 & 0 & 0\\ 0 & W_8^1 & 0 & 0\\ 0 & 0 & W_8^2 & 0\\ 0 & 0 & 0 & W_8^3\end{bmatrix}\begin{bmatrix}X_2(0)\\X_2(1)\\X_2(2)\\X_2(3)\end{bmatrix} \tag{7.32}$$

$$\begin{bmatrix}X(4)\\X(5)\\X(6)\\X(7)\end{bmatrix} = \begin{bmatrix}X_1(0)\\X_1(1)\\X_1(2)\\X_1(3)\end{bmatrix} - \begin{bmatrix}W_8^0 & 0 & 0 & 0\\ 0 & W_8^1 & 0 & 0\\ 0 & 0 & W_8^2 & 0\\ 0 & 0 & 0 & W_8^3\end{bmatrix}\begin{bmatrix}X_2(0)\\X_2(1)\\X_2(2)\\X_2(3)\end{bmatrix} \tag{7.33}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix}X_1(0)\\X_1(1)\end{bmatrix} = \begin{bmatrix}X_3(0)\\X_3(1)\end{bmatrix} + \begin{bmatrix}W_4^0 & 0\\ 0 & W_4^1\end{bmatrix}\begin{bmatrix}X_4(0)\\X_4(1)\end{bmatrix} \tag{7.34}$$

$$\begin{bmatrix}X_1(2)\\X_1(3)\end{bmatrix} = \begin{bmatrix}X_3(0)\\X_3(1)\end{bmatrix} - \begin{bmatrix}W_4^0 & 0\\ 0 & W_4^1\end{bmatrix}\begin{bmatrix}X_4(0)\\X_4(1)\end{bmatrix} \tag{7.35}$$

$$\begin{bmatrix}X_2(0)\\X_2(1)\end{bmatrix} = \begin{bmatrix}X_5(0)\\X_5(1)\end{bmatrix} + \begin{bmatrix}W_4^0 & 0\\ 0 & W_4^1\end{bmatrix}\begin{bmatrix}X_6(0)\\X_6(1)\end{bmatrix} \tag{7.36}$$

$$\begin{bmatrix}X_2(2)\\X_2(3)\end{bmatrix} = \begin{bmatrix}X_5(0)\\X_5(1)\end{bmatrix} - \begin{bmatrix}W_4^0 & 0\\ 0 & W_4^1\end{bmatrix}\begin{bmatrix}X_6(0)\\X_6(1)\end{bmatrix} \tag{7.37}$$

$$P_8\begin{bmatrix}x(0)\\x(1)\\x(2)\\x(3)\\x(4)\\x(5)\\x(6)\\x(7)\end{bmatrix} = \begin{bmatrix}x(0)\\x(2)\\x(4)\\x(6)\\x(1)\\x(3)\\x(5)\\x(7)\end{bmatrix} \tag{7.38}$$

$$P_4\begin{bmatrix}x(0)\\x(2)\\x(4)\\x(6)\end{bmatrix} = \begin{bmatrix}x(0)\\x(4)\\x(2)\\x(6)\end{bmatrix} \tag{7.39}$$

$$P_4\begin{bmatrix}x(1)\\x(3)\\x(5)\\x(7)\end{bmatrix} = \begin{bmatrix}x(1)\\x(5)\\x(3)\\x(7)\end{bmatrix} \tag{7.40}$$

Therefore,

$$\begin{bmatrix}X_3(0)\\X_3(1)\end{bmatrix} = F_2\begin{bmatrix}x(0)\\x(4)\end{bmatrix} \tag{7.41}$$

$$\begin{bmatrix}X_4(0)\\X_4(1)\end{bmatrix} = F_2\begin{bmatrix}x(2)\\x(6)\end{bmatrix} \tag{7.42}$$

$$\begin{bmatrix}X_5(0)\\X_5(1)\end{bmatrix} = F_2\begin{bmatrix}x(1)\\x(5)\end{bmatrix} \tag{7.43}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.44}$$

**Solution:** We write out the values of performing an 8-point FFT on $\mathbf{x}$ as follows.

$$X(k) = \sum_{n=0}^{7} x(n) e^{-\frac{\jmath 2kn\pi}{8}} \tag{7.45}$$

$$= \sum_{n=0}^{3} \left( x(2n) e^{-\frac{\jmath 2kn\pi}{4}} + e^{-\frac{\jmath 2k\pi}{8}} x(2n+1) e^{-\frac{\jmath 2kn\pi}{4}} \right) \tag{7.46}$$

$$= X_1(k) + e^{-\frac{\jmath 2k\pi}{4}} X_2(k) \tag{7.47}$$

where $\mathbf{X}_1$ is the 4-point FFT of the even-numbered terms and $\mathbf{X}_2$ is the 4-point FFT of the odd numbered terms. Noticing that for $k \ge 4$,

$$X_1(k) = X_1(k-4) \tag{7.48}$$

$$e^{-\frac{\jmath 2k\pi}{8}} = -e^{-\frac{\jmath 2(k-4)\pi}{8}} \tag{7.49}$$

we can now write out $X(k)$ in matrix form as in (**??**) and (**??**). We also need to solve the two 4-point FFT terms so formed.

$$X_1(k) = \sum_{n=0}^{3} x_1(n) e^{-\frac{\jmath 2kn\pi}{8}} \tag{7.50}$$

$$= \sum_{n=0}^{1} \left( x_1(2n) e^{-\frac{\jmath 2kn\pi}{4}} + e^{-\frac{\jmath 2k\pi}{8}} x_2(2n+1) e^{-\frac{\jmath 2kn\pi}{4}} \right) \tag{7.51}$$

$$= X_3(k) + e^{-\frac{\jmath 2k\pi}{4}} X_4(k) \tag{7.52}$$

using $x_1(n) = x(2n)$ and $x_2(n) = x(2n+1)$. Thus we can write the 2-point FFTs

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.53}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.54}$$

Using a similar idea for the terms $X_2$,

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.55}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.56}$$

But observe that from (**??**),

$$\mathbf{P}_8 \mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \tag{7.57}$$

$$\mathbf{P}_4 \mathbf{x}_1 = \begin{pmatrix} \mathbf{x}_3 \\ \mathbf{x}_4 \end{pmatrix} \tag{7.58}$$

$$\mathbf{P}_4 \mathbf{x}_2 = \begin{pmatrix} \mathbf{x}_5 \\ \mathbf{x}_6 \end{pmatrix} \tag{7.59}$$

where we define $x_3(k) = x(4k)$, $x_4(k) = x(4k+2)$, $x_5(k) = x(4k+1)$, and $x_6(k) = x(4k+3)$ for $k = 0, 1$.

11. For

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.60}$$

compute the DFT using (**??**)
**Solution:** Download the Python code from

```
$ wget https://raw.githubusercontent.com/
    samar2605/EE3900/master/filter/codes/
    A1_7_11.py
```

12. Write a C program to compute the 8-point FFT.
**Solution:** The C code for the above two problems can be downloaded from

```
$ wget https://raw.githubusercontent.com/
    samar2605/EE3900/master/filter/codes/
    A1_7_13.c
```

## 8 Exercises

Answer the following questions by looking at the python code in Problem **??**.

8.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem **??** is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \tag{8.1}$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.

```python
import soundfile as sf
from scipy import signal, fft
import numpy as np
from numpy.polynomial import Polynomial
    as P
from matplotlib import pyplot as plt

def myfiltfilt(b, a, input_signal):
    X = fft.fft(input_signal)
    w = np.linspace(0, 1, len(X) + 1)
    W = np.exp(2j*np.pi*w[:-1])
    B = (np.absolute(np.polyval(b,W)))**2
    A = (np.absolute(np.polyval(a,W)))**2
    Y = B*(1/A)*X
    return fft.ifft(Y).real

#read .wav file
input_signal,fs = sf.read('/media/beyonder/
    DATA/Sem-5/EE3900/Assignment_1/
    codes/Sound_Noise.wav')

#sampling frequency of Input signal
sampl_freq=fs

#order of the filter
order=4

#cutoff frquency 4kHz
cutoff_freq=4000.0

#digital frequency
Wn=2*cutoff_freq/sampl_freq

# b and a are numerator and denominator
    polynomials respectively
b, a = signal.butter(order, Wn, 'low')

#filter the input signal with butterworth filter
output_signal = signal.filtfilt(b, a,
    input_signal)
#output_signal1 = signal.lfilter(b, a,
    input_signal)
os1 = myfiltfilt(b, a, input_signal)
x_plt = np.arange(len(input_signal))
#Verify outputs by plotting
plt.plot(x_plt[:100], output_signal[:100], 'b.'
    )
plt.plot(x_plt[:100], os1[:100], 'r.')
plt.grid()
plt.show()
```

8.2 Repeat all the exercises in the previous sections for the above $a$ and $b$.
**Solution:** For the given values, the difference equation is

$$\begin{aligned} &y(n) - (4.44)\, y(n-1) + (8.78)\, y(n-2) \\ &- (9.93)\, y(n-3) + (6.90)\, y(n-4) \\ &- (2.93)\, y(n-5) + (0.70)\, y(n-6) \\ &- (0.07)\, y(n-7) = \left(5.02 \times 10^{-5}\right) x(n) \\ &+ \left(3.52 \times 10^{-4}\right) x(n-1) + \left(1.05 \times 10^{-3}\right) x(n-2) \\ &+ \left(1.76 \times 10^{-3}\right) x(n-3) + \left(1.76 \times 10^{-3}\right) x(n-4) \\ &+ \left(1.05 \times 10^{-3}\right) x(n-5) + \left(3.52 \times 10^{-4}\right) x(n-6) \\ &+ \left(5.02 \times 10^{-5}\right) x(n-7) \end{aligned} \tag{8.2}$$

From (**??**), we see that the transfer function can be written as follows

$$H(z) = \frac{\sum_{k=0}^{N} b(k) z^{-k}}{\sum_{k=0}^{M} a(k) z^{-k}} \tag{8.3}$$

$$= \sum_i \frac{r(i)}{1 - p(i) z^{-1}} + \sum_j k(j) z^{-j} \tag{8.4}$$

where $r(i)$, $p(i)$, are called residues and poles respectively of the partial fraction expansion of $H(z)$. $k(i)$ are the coefficients of the direct polynomial terms that might be left over. We can now take the inverse $z$-transform of (**??**) and get using (**??**),

$$h(n) = \sum_i r(i) [p(i)]^n u(n) + \sum_j k(j) \delta(n-j) \tag{8.5}$$

Substituting the values,

$$\begin{aligned} h(n) &= [(2.76)\,(0.55)^n \\ &+ (-1.05 - 1.84\jmath)\,(0.57 + 0.16\jmath)^n \\ &+ (-1.05 + 1.84\jmath)\,(0.57 - 0.16\jmath)^n \\ &+ (-0.53 + 0.08\jmath)\,(0.63 + 0.32\jmath)^n \\ &+ (-0.53 - 0.08\jmath)\,(0.63 - 0.32\jmath)^n \\ &+ (0.20 + 0.004\jmath)\,(0.75 + 0.47\jmath)^n \\ &+ (0.20 - 0.004\jmath)\,(0.75 - 0.47\jmath)^n] u(n) \\ &+ \left(-6.81 \times 10^{-4}\right) \delta(n) \end{aligned} \tag{8.6}$$

The values $r(i)$, $p(i)$, $k(i)$ and thus the impulse response function are computed and plotted at

```
import soundfile as sf
import matplotlib.pyplot as plt
from scipy import signal
from scipy import vectorize as vec
import numpy as np

#read .wav file
input_signal,fs = sf.read('/media/beyonder/
    DATA/Sem-5/EE3900/Assignment_1/
    codes/Sound_Noise.wav')

#sampling frequency of Input signal
sampl_freq=fs

#order of the filter
order=7

#cutoff frquency 4kHz
cutoff_freq=4000.0

#digital frequency
Wn=2*cutoff_freq/sampl_freq

# b and a are numerator and denominator
    polynomials respectively
b, a = signal.butter(order, Wn, 'low')

# get partial fraction expansion
r, p, k = signal.residuez(b, a)

#number of terms of the impulse response
sz = 50
sz_lin = np.arange(sz)

def rp(x):
    return r@(p**x).T

rp_vec = vec(rp, otypes=['double'])

h1 = rp_vec(sz_lin)
k_add = np.pad(k, (0, sz - len(k)), 'constant
    ', constant_values=(0,0))
h = h1 + k_add
plt.stem(sz_lin, h)
plt.xlabel('n')
plt.ylabel('h(n)')
plt.grid()
plt.plot()
plt.savefig('/media/beyonder/DATA/Sem-5/
    EE3900/Assignment_1/figures/
    Figure_8_1_new.png')
```

The filter frequency response is plotted at

```
import numpy as np
import matplotlib.pyplot as plt
from scipy import signal
import soundfile as sf
#if using termux
#import subprocess
#import shlex
#end if

#read .wav file
input_signal,fs = sf.read('/media/beyonder/
    DATA/Sem-5/EE3900/Assignment_1/
    codes/Sound_Noise.wav')

#sampling frequency of Input signal
sampl_freq=fs

#order of the filter
order=4

#cutoff frquency 4kHz
cutoff_freq=4000.0

#digital frequency
Wn=2*cutoff_freq/sampl_freq

# b and a are numerator and denominator
    polynomials respectively
b, a = signal.butter(order, Wn, 'low')

#DTFT
def H(z):
        num = np.polyval(b,z**(-1))
        den = np.polyval(a,z**(-1))
        H = num/den
        return H

#Input and Output
omega = np.linspace(0,np.pi,100)

#subplots
plt.plot(omega, abs(H(np.exp(1j*omega))))
plt.xlabel('$\omega$')
plt.ylabel('$|H(e^{\jmath\omega})| $')
plt.grid()# minor
```

```
#if using termux
plt.savefig('/media/beyonder/DATA/Sem−5/
    EE3900/Assignment_1/figures/
    Figure_8_2.png')
#subprocess.run(shlex.split("termux−open ../
    figs/dtft.pdf"))
#else
plt.show()
```

Observe that for a series $t_n = r^n$, $\frac{t_{n+1}}{t_n} = r$. By the ratio test, $t_n$ converges if $|r| < 1$. We note that observe that $|p(i)| < 1$ and so, as $h(n)$ is the sum of convergent series, we see that $h(n)$ converges. From Fig. (**??**), it is clear that $h(n)$ is bounded. From (**??**),

$$\sum_{n=0}^{\infty} h(n) = H(1) = 1 < \infty \tag{8.7}$$

Therefore, the system is stable. From $h(n)$ is negligible after $n \geq 64$, and we can apply a 64-bit FFT to get y(n). The following code uses the DFT matrix to generate $y(n)$.

```
import soundfile as sf
import matplotlib.pyplot as plt
from scipy import signal
from scipy import vectorize as vec
import numpy as np

#read .wav file
input_signal,fs = sf.read('/media/beyonder/
    DATA/Sem−5/EE3900/Assignment_1/
    codes/Sound_Noise.wav')

#sampling frequency of Input signal
sampl_freq=fs

#order of the filter
order=7

#cutoff frquency 4kHz
cutoff_freq=4000.0

#digital frequency
Wn=2*cutoff_freq/sampl_freq

# b and a are numerator and denominator
    polynomials respectively
b, a = signal.butter(order, Wn, 'low')
output_signal = signal.filtfilt(b, a,
    input_signal)

# get partial fraction expansion
r, p, k = signal.residuez(b, a)
#number of terms of the impulse response
sz = 64
sz_lin = np.arange(sz)

dftmtx = np.fft.fft(np.eye(sz))
invmtx = np.linalg.inv(dftmtx)
def rp(x):
    return r@(p**x).T

rp_vec = vec(rp, otypes=['double'])

h1 = rp_vec(sz_lin)
k_add = np.pad(k, (0, sz − len(k)), 'constant
    ', constant_values=(0,0))
h = h1 + k_add
H = h@dftmtx
X = input_signal[:sz]@dftmtx
Y = H*X
y = (Y@invmtx).real
plt.stem(np.arange(sz), y[:sz])
plt.xlabel('n')
plt.ylabel('y(n)')
plt.grid()
plt.plot()
plt.savefig('/media/beyonder/DATA/Sem−5/
    EE3900/Assignment_1/figures/
    Figure_8_3.png')
```

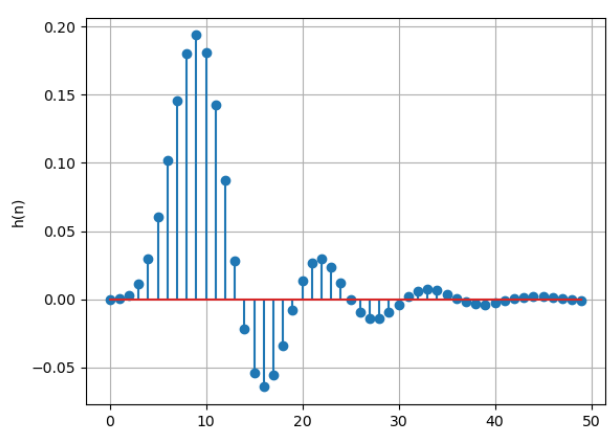



Fig. 8.2: Plot of $h(n)$

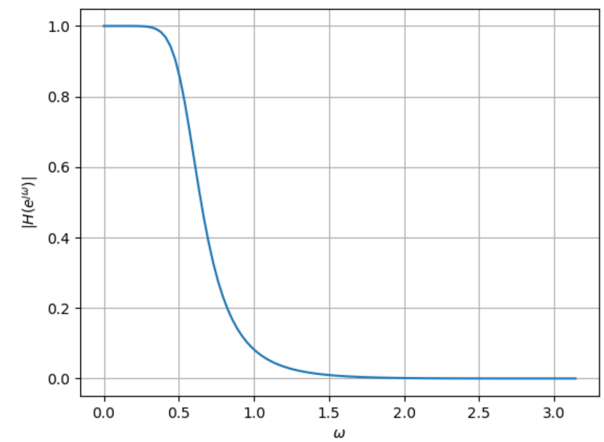



Fig. 8.2: Filter frequency response

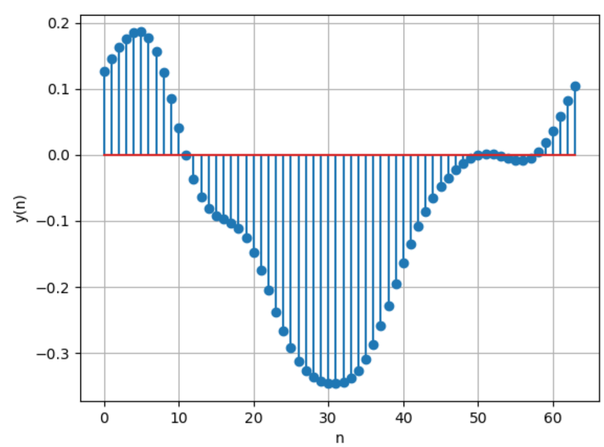



Fig. 8.2: Plot of $y(n)$

8.3 What is the sampling frequency of the input signal?
**Solution:** Sampling frequency(fs)=44.1kHZ.

8.4 What is type, order and cutoff-frequency of the above butterworth filter
**Solution:** The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.

8.5 Modifying the code with different input parameters and to get the best possible output. **Solution:** A better filtering was found on setting the order of the filter to be 7.