Pingala Series

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***Abstract*—This manual provides a simple introduction to Transforms**

1 JEE 2019

Let α and β ($\alpha > \beta$) be the roots of the equation $z^2 - z - 1 = 0$. Define,

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \tag{1.1}$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \tag{1.2}$$

Verify the following using a python code.

1.1
$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{1.3}$$

1.2
$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{1.4}$$

1.3
$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \tag{1.5}$$

1.4
$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \tag{1.6}$$

The following Python code completes these checks

```
import numpy as np
import matplotlib.pyplot as plt

#1.1
n = 100
alpha = (1 + np.sqrt(5))/2
beta = (1 - np.sqrt(5))/2
k = np.linspace(1, n, n)
a = (alpha**k - beta**k)/(alpha - beta)
ca = np.cumsum(a)
if (np.allclose(ca[:98], a[2:] - 1)): print("1.1 correct")
else: print("1.1 incorrect")

#Checking by plotting
# plt.plot(ca)
# plt.plot(a)
# plt.show()

#1.2
t = 10**k
ta = a*(1/t)
eps = 1e-6
ans = 10/89
sa = np.cumsum(ta)
if (abs(sa[-1] - ans) < eps): print("1.2 correct")
else: print("1.2 incorrect")

#1.3
b = a[2:] + a[:98]
b = np.pad(b, (1, 0), 'constant', constant_values=(1, 0))
b_new = alpha**k + beta**k
if (np.allclose(b, b_new[:99])): print("1.3 correct")
else: print("1.3 incorrect")

#1.4
tb = b*(1/t[:99])
eps = 1e-6
ans = 8/89
sb = np.cumsum(tb)
if (abs(sb[-1] - ans) < eps): print("1.4 correct")
else: print("1.4 incorrect")
```

2 Pingala Series

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^{+}(z) = \sum_{n=0}^{\infty} x(n) z^{-n}, \quad z \in \mathbb{C} \tag{2.1}$$

2.2 The *Pingala* series is generated using the difference equation

$$x(n+2) = x(n+1) + x(n), \quad x(0) = x(1) = 1, n \geq 0 \tag{2.2}$$

Generate a stem plot for $x(n)$.
Solution: The following code generates the plot

```
import numpy as np
import matplotlib.pyplot as plt

N=15
x=np.zeros(N)

x[0]=1
x[1]=1

for i in range(2,N):
    x[i]=x[i−1]+x[i−2]

plt.stem(range(N),x)
plt.ylabel("$x(n)$")
plt.grid()
plt.show()
```

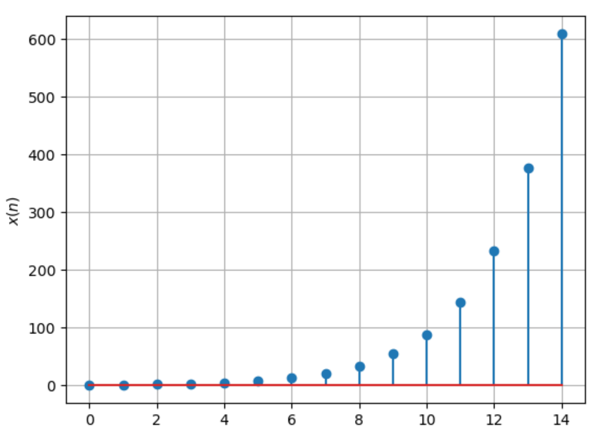

Fig. 2.1: Plot of $x(n)$

2.3 Find $X^{+}(z)$.
Solution: Taking the one-sided Z-transform on both sides of (2.2),

$$\mathcal{Z}^{+}\left[x(n+2)\right] = \mathcal{Z}^{+}\left[x(n+1)\right] + \mathcal{Z}^{+}\left[x(n)\right] \tag{2.3}$$

$$z^2 X^{+}(z) - z^2 x(0) - z x(1) = z X^{+}(z) - z x(0) + z X^{+}(z) \tag{2.4}$$

$$\left(z^2 - z - 1\right) X^{+}(z) = z^2 \tag{2.5}$$

$$X^{+}(z) = \frac{1}{1 - z^{-1} - z^{-2}} \tag{2.6}$$

$$= \frac{1}{\left(1 - \alpha z^{-1}\right)\left(1 - \beta z^{-1}\right)}, \quad |z| > \alpha \tag{2.7}$$

2.4 Find $x(n)$.
Solution: Expanding $X^{+}(z)$ in (2.7) using partial fractions, we get

$$X^{+}(z) = \frac{1}{(\alpha - \beta) z^{-1}} \left[\frac{1}{1 - \alpha z^{-1}} - \frac{1}{1 - \beta z^{-1}}\right] \tag{2.8}$$

$$= \frac{1}{(\alpha - \beta)} \sum_{n=0}^{\infty} \left(\alpha^n - \beta^n\right) z^{-n+1} \tag{2.9}$$

$$= \sum_{n=1}^{\infty} \frac{\alpha^n - \beta^n}{\alpha - \beta} z^{-n+1} \tag{2.10}$$

$$= \sum_{k=0}^{\infty} \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} z^{-k} \tag{2.11}$$

where $k := n + 1$. Thus,

$$x(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} u(n) = a_{n+1} u(n) \tag{2.12}$$

2.5 Sketch

$$y(n) = x(n-1) + x(n+1), \quad n \geq 0 \tag{2.13}$$

Solution: The following Python code plots the $y(n)$

```
import numpy as np
import matplotlib.pyplot as plt

N=14

x=[1,1]

for i in range(N):
    x.append(x[−1]+x[−2])
```

```
y=np.add(x[:N], x[2:])
plt.stem(range(N),y)
plt.grid()
plt.ylabel("$y(n)$")
plt.show()
```

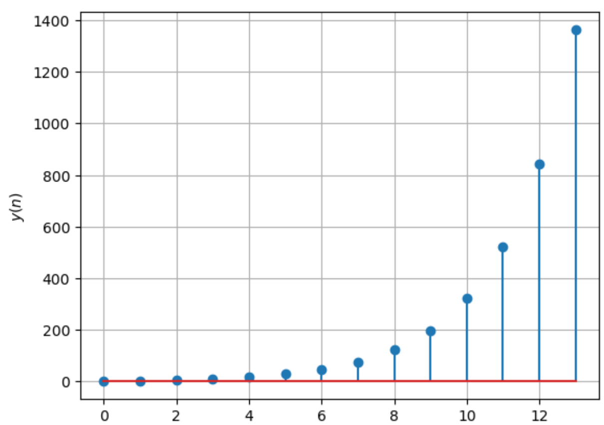

Fig. 2.2: Plot of $y(n)$

2.6 Find $Y^+(z)$.
Solution: Taking the one-sided Z-transform on both sides of (2.13),

$$\mathcal{Z}^+[y(n)] = \mathcal{Z}^+[x(n+1)] + \mathcal{Z}^+[x(n-1)] \tag{2.14}$$

$$Y^+(z) = zX^+(z) - zx(0) + z^{-1}X^+(z) + zx(-1) \tag{2.15}$$

$$= \frac{z+z^{-1}}{1-z^{-1}-z^{-2}} - z \tag{2.16}$$

$$= \frac{1+2z^{-1}}{1-z^{-1}-z^{-2}}, \quad |z| > \alpha \tag{2.17}$$

since $x(n) = 0 \ \forall \ n < 0$.

2.7 Find $y(n)$. **Solution:** Using (2.7),

$$Y^+(z) = (1+2z^{-1}) \sum_{n=0}^{\infty} x(n)z^{-n} \tag{2.18}$$

$$= \sum_{n=0}^{\infty} x(n)z^{-n} + \sum_{n=1}^{\infty} 2x(n-1)z^{-n} \tag{2.19}$$

$$= x(0) + \sum_{n=1}^{\infty} (x(n) + 2x(n-1))\, z^{-n} \tag{2.20}$$

Thus, $y(0) = x(0) = 1$ and for $n \geq 1$, using the fact that α and β are the roots of the equation $z^2 - z - 1 = 0$,

$$y(n) = \frac{\left(\alpha^{n+1} - \beta^{n+1}\right) + (2\alpha^n + 2\beta^n)}{\alpha - \beta} \tag{2.21}$$

$$= \frac{\left(\alpha^{n+2} - \beta^{n+2}\right) + (\alpha^n + \beta^n)}{\alpha - \beta} \tag{2.22}$$

$$= \frac{\left(\alpha^{n+2} - \beta^{n+2}\right) - \alpha\beta(\alpha^n + \beta^n)}{\alpha - \beta} \tag{2.23}$$

$$= \frac{(\alpha - \beta)\left(\alpha^{n+1} + \beta^{n+1}\right)}{\alpha - \beta} \tag{2.24}$$

$$= \alpha^{n+1} + \beta^{n+1} \tag{2.25}$$

Thus, $y(n) = \alpha^{n+1} + \beta^{n+1}$ for $n \geq 0$ as $\alpha + \beta = 1$. Comparing (2.22) with the definition of b_n, we see that $y(n) = b_{n+1}$. Hence, $b_n = \alpha^n + \beta^n$.

3 Power of the Z transform

3.1 Show that

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) = x(n) * u(n-1) \tag{3.1}$$

Solution: From (2.12), and noting that $x(n) = 0 \ \forall \ n < 0$,

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) \tag{3.2}$$

$$= \sum_{k=-\infty}^{n-1} x(k) \tag{3.3}$$

$$= \sum_{k=-\infty}^{\infty} x(k)u(n-1-k) \tag{3.4}$$

$$= x(n) * u(n-1) \tag{3.5}$$

3.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \tag{3.6}$$

can be expressed as

$$[x(n+1) - 1]\, u(n) \tag{3.7}$$

Solution: From (2.12),

$$a_{n+2} - 1 = [x(n+1) - 1], \quad n \geq 0 \tag{3.8}$$

and so, using the definition of $u(n)$,

$$a_{n+2} - 1 = [x(n+1) - 1]\, u(n) \tag{3.9}$$

3.3 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+(10) \quad (3.10)$$

Solution:

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{a_{k+1}}{10^k} \quad (3.11)$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} \quad (3.12)$$

$$= \frac{1}{10} X^+(z) \quad (3.13)$$

$$= \frac{1}{10} \times \frac{100}{89} = \frac{10}{89} \quad (3.14)$$

3.4 Show that

$$\alpha^n + \beta^n, \quad n \geq 1 \quad (3.15)$$

can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \quad (3.16)$$

and find $W(z)$.

Solution: Putting $n = k + 1$ in (3.15) and using the definition of $u(n)$,

$$\alpha^n + \beta^n = \left(\alpha^{k+1} + \beta^{k+1}\right) u(k) \quad (3.17)$$

Hence, (3.15) can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) = y(n) \quad (3.18)$$

Therefore,

$$W(z) = Y(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \quad (3.19)$$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \quad (3.20)$$

Solution:

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{b_{k+1}}{10^k} \quad (3.21)$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} \quad (3.22)$$

$$= \frac{1}{10} Y^+(z) \quad (3.23)$$

$$= \frac{1}{10} \times \frac{120}{89} = \frac{12}{89} \quad (3.24)$$

3.6 Solve the JEE 2019 problem.

Solution: We know that

$$\sum_{k=1}^{n} a_k = x(n) * u(n-1) \quad (3.25)$$

But

$$x(n) * u(n-1) \overset{\mathcal{Z}}{\rightleftharpoons} X(z) z^{-1} U(z) \quad (3.26)$$

$$= \frac{z^{-1}}{\left(1 - z^{-1} - z^{-2}\right)\left(1 - z^{-1}\right)} \quad (3.27)$$

$$= z \left[\frac{1}{1 - z^{-1} - z^{-2}} - \frac{1}{1 - z^{-1}}\right] \quad (3.28)$$

$$\overset{\mathcal{Z}}{\rightleftharpoons} z \sum_{n=0}^{\infty} (x(n) - 1) z^{-n} \quad (3.29)$$

$$= \sum_{n=0}^{\infty} (x(n) - 1) z^{-n+1} \quad (3.30)$$

$$= \sum_{n=0}^{\infty} (x(n+1) - 1) z^{-n} \quad (3.31)$$

$$(3.32)$$

From (2.12), we get

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1 \quad (3.33)$$

We have already established the remaining options in order in the problems (3.3), (2.7), (3.5). Therefore, options 1, 2, and 3 are correct and option 4 is incorrect.