A greedy sampling approach for the projection basis construction in parametric model order reduction for structural dynamics models

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Abstract
Over the past decades models of structural systems have moved from single-query applications in uniform domain towards a design process where multiple queries of a multi-domain model are required for different design parameters. In order to limit the computational burden associated with these multi-query applications, parametric model order reduction (pMOR) becomes an indispensable tool. These techniques allow a low(er) cost evaluation of the original model (e.g. a finite element model) for a range of design parameters.

This paper investigates the possibility of applying a greedy algorithm to generate a reduced order basis (ROB) from sampled eigenvectors for structural dynamics applications. The resulting ROB can be used in projection-based pMOR techniques and in the numerical example it is used to reduce an affine representation of a finite element (FE).

1 Introduction

The large use of simulation tools is growing through the years in parallel to the advances in computational power available to engineers. These simulations often involve numerical models used to approximate physical phenomena governed by partial differential equations. As a result of the discretization of these PDEs, the generated models can be very large. In the worst scenario, these large models may require a computational effort that goes beyond the available hardware possibilities making practical simulations impossible to simulate practically. In order to offset this rapid rise in computational load, model order reduction techniques can be exploited to lower this load. Model order reduction (MOR) techniques are used in many stages of problem modelling with *a priori* or *a posteriori* reduction techniques [2, 3].

Reduction is often obtained through projection onto a subspace of reduced size with respect to the original domain. The shape and dimension of the subspace are parameter dependent and each subspace is part of the same manifold of the solutions associated with the model. Dealing efficiently with this manifold is an open challenge and different techniques can be employed depending on the characteristics of the manifold.

Given a parametric manifold $\mathcal{M}(\mu)$, where μ represents the parameter configuration, the most revealing characteristic is the Kolmogorov n-width [4]. This n-width tells how smoothly the subspaces contained in the manifold vary with respect to the input parameters. If the Kolmogorov n-width is small, there is a low variability of the subspaces in $\mathcal{M}(\mu)$ and a common subspace obtained as the union of local subspaces would be of small size. On the other hand, when the Kolmogorov n-width is large, the subspaces vary

considerably from one parameter to another and a global reduced order basis (ROB) would be large. This would compromise the tradeoff between accuracy and performance and techniques based on partitioning of the parameter domain and manifold interpolation techniques are more suitable than a single global space.

Regardless of the characteristic of the manifold, if an a posteriori technique based on projection is employed for the reduction, it is important to have, in all portions of the parameter space of interest, the smallest global ROB keeping the desired level of accuracy. A very popular technique to generate a ROB is the proper orthogonal decomposition (POD) [3]. This technique assumes knowledge of the result for a representative number of configurations of the system and therefore requires the original system to be evaluated a sufficient amount of times. The state vectors gathered during the reference simulations can be reduced through singular value decomposition (SVD) of this space where only the most dominant components are kept.

As an alternative to POD, greedy algorithms can be used to select an appropriate reduction basis in a more flexible fashion [4]; this method does not rely on the existence of a sufficient number of solutions of the system but, by means of an efficient error estimator, choses the parameter configurations at which calculating new vectors having the accuracy as objective function for the optimization. By means of the optimization approach, the greedy algorithm offers some important information on the quality of the reduction as well as the optimal projection subspace for given parameter range. The greedy algorithm is a constitutive part of the Reduced Basis methods [5] from which the conceptual scheme of implementation is borrowed for the parametric model order reduction employed in this paper.

The main goal of this work is adapting the greedy algorithms for cases where the basis vectors are eigenmodes. It is important to recognise that the frequency response of a system can be accurately described in a bounded range by a modal model if this contains a sufficient number of modes also above the frequency range of interest. This fact influences the construction of the greedy algorithm hereafter proposed. Using a single eigenvector at a time as in standard “greedy” to expand the reduced order basis may bring the algorithm to divergence. Therefore at each iteration of the greedy algorithm, not one but a set of modes will be added. This requires some other modifications to the original techniques in order to maintain the benefits of the greedy without compromising accuracy.

The other challenge of the proposed methodology is finding a suitable error estimator. In this paper it is shown that the residual of the eigenvalue problem is indeed appropriate for such purpose. It must however be emphasised that it is not possible to associate the residual with the approximation error of the eigenvalues or eigenvectors separately which makes the interpretation of residual as an error more complex.

The article is organized as follows: chapter 2 presents the pMOR scheme adopted; in chapter 3, a background on greedy algorithms and error estimators is given and the developed method is presented; chapter 4 shows a numerical example obtained on the gantry bridge model of [1].

2 The reduction scheme

Model order reduction procedures based on projection consist of distinct operations to be executed:

(i) generation of a Full Order Model (FOM);
(ii) generation of the global ROB;
(iii) projection of the FOM on the subspace spanned by the ROB to generate a ROM;
(iv) simulation of the ROM.

These operations can be executed offline (executed only once in the preprocessing) or online (executed at each calculation). The ideal configuration for optimal reduction is when operations (i-iii) are all made offline and only (iv), the simulation, is performed online. This can for example be achieved by employing a reduced basis (RB) method [5]. These methods require that the FOM can be expressed as an affine function of the parameters.

In [1], an affine representation of the system is proposed and used for a RB method for a model made of 3D beam elements. The affine representation is obtained from analytical considerations about the mechanism of deformation of beams to obtain a better-than-polynomial description. Hereafter, the same reduction scheme is adopted albeit improvements to the method are presented in section 3 through the construction of an optimal ROB via a greedy algorithm.

The affine representation of a general parametric matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is expressed as

$$\boldsymbol{A}(\mu) = \sum_{i=1}^{q} A_i \cdot f_i(\mu) \tag{1}$$

where the A_i represent the matrices of coefficient that are constant and parameter independent, and the corresponding f_i are scalar basic functions of the parameters. Given a general parametric n degrees-of-freedom (DOF) system in matrix form without damping

$$\sum_{n=0}^{p} K(\mu) \cdot x + \sum_{n=0}^{p} M(\mu) \cdot \ddot{x} = B \cdot u \tag{2}$$

where $K(\mu), M(\mu) \in R^{n\times n}$, are respectively the parametric stiffness and mass matrices, $x \in \mathbb{R}^n$ is the vector of nodal displacements and $B \in \mathbb{R}^{n\times m}$ is the force application matrix for the m input forces u. Using the affine representation of equation **Error! Bookmark not defined.**(1), the system matrices can then be represented as

$$\sum_{n=0}^{p} K_n \cdot f_n^K(\mu) \cdot x + \sum_{n=0}^{p} M_n \cdot f_n^M(\mu) \cdot \ddot{x} = B \cdot u. \tag{3}$$

Equation (3), can be reduced through a Galerkin projection on the subspace spanned by the ROB indicated as $V \in \mathbb{R}^{n\times q}$ with q generalized coordinates.

$$\mathrm{V}^T \cdot \sum_{n=0}^{p} K_n \cdot f_n^K(\mu) \cdot \mathrm{V} \cdot x_r + \mathrm{V}^T \cdot \sum_{n=0}^{p} M_n \cdot f_n^M(\mu) \cdot \mathrm{V} \cdot \ddot{x}_r = \mathrm{V}^T \cdot Bu. \tag{4}$$

Thanks to the affine representation, the projection of the constant matrices can be done offline with enhancements of the computational performance. The resulting ROM is

$$\sum_{n=0}^{p} K_{n,r} \cdot f_n^K(\mu) \cdot x_r + \sum_{n=0}^{p} M_{n,r} \cdot f_n^M(\mu) \cdot \ddot{x}_r = \mathrm{V}^T \cdot Bu. \tag{5}$$

Equation (4) represents already a ROM but this implies that the projection of the full system matrices is performed in the online phase prior to simulation. With equation (5), the affine representation of the system is exploited and the constant matrices K_n and M_n of equation (4) can be reduced offline to generate the reduced constant matrices $K_{n,r}$ and $M_{n,r}$.

The projection space V will be generated through a greedy algorithm developed ad-hoc to work with eigenvectors as described in the following section.

3 The greedy algorithm

3.1 General background

Greedy algorithms were introduced in the seventies as optimization techniques [6]. They increased their popularity in many field because, despite the fact they do not necessarily find a global optimal solution they succeed in finding local optima in a relatively short time [4].

In the field of MOR, greedy algorithms are used to construct subspaces adding iteratively new vectors to the basis. The objective function is the accuracy of the solution coming from the ROM therefore the method will seek where, in the parameter range of interest, the solution at iteration $q-1$ is worst for the existent ROB V_{q-1} and add new information using the FOM. The general workflow of a greedy algorithm is the following:

(i) selection of the parameter domain and sampling;
(ii) initialization of the procedure with a nominal basis vector;
(iii) employing an a posteriori error estimator to localize the combination that yields the worst result
(iv) updating of the available set of basis vectors.

The method will iterate between (iii) and (iv) until the tolerance threshold set for the error estimator is matched. The procedure has several crucial characteristics: if N is the number of samples chosen for the ROB generation, the error estimator will be called N times per iteration. Therefore, the error estimator has to be cheap to compute and not be dependent on the complexity of the FOM.

An error estimator is an indicator of the error committed with the ROM with respect to the FOM. Error estimators are used in many MOR schemes to generate adaptively the ROB. As discussed above, the error estimator should be fast to compute as it is called many times in the ROM creation algorithm and should be asymptotically correct with respect to the actual error [4].

Given a continuous and coercive parametric variational problem, a relation between error and residual can be obtained in matrix form as

$$r(v,\mu) = \mathbb{A}e(v,\mu) \tag{6}$$

where $\mathbb{A}$ is the system matrix. If $\mathbb{A}$ is non-singular both sides can be pre-multiplied by $\mathbb{A}^{-1}$. Calculating the norm on both sides and exploiting the triangle inequality

$$\|e\| \leq \|A^{-1}\| \cdot \|r\|. \tag{7}$$

This important result allows to solve the local minimization of the error using an error estimator based on the residual.

In the next session a specific error estimator to be used in the construction of ROB is presented.

3.2 The proposed greedy algorithm for eigenvector basis

The concept behind the methodology presented here follows that of RB methods. However, the error estimator has to take information from the residual of an eigenvalue problem. To ensure the generality of the method some analytical considerations are made.

Given a parameter domain of interest that is limited as $\mu_{low} \leq \mu \leq \mu_{high}$, the procedure starts choosing as initial ROB a truncated set of eigenvectors of a nominal configuration of the parameters indicated as V_0. At each iteration i, the system is updated by a set of new vectors chosen according to the error estimator. The new ROB V_i is then obtained as the combination of the old ROB V_{i-1} and this new set of vectors.

The residual of an eigenvalue problem similarly to equation (6) can be expressed as

$$r = \left(K_i - M_i\hat{\lambda}_{ij}\right)\hat{X}_{ij} \tag{8}$$

where λ_{ij} and X_{ij} come from the ROM, while K_i and M_i are the FOM matrices. If $\hat{\lambda}$ and $\hat{X}$ are an eigenvalue and corresponding eigenvector for the problem of equation (8), the residual term will be zero. In the general case, $\hat{\lambda}$ and $\hat{X}$ are not expected to be a solution of equation (8) therefore a residual $r \neq 0$ is obtained. From equation (8) it is not straight forward to invert and obtain an error bound as it is in equation (7). In fact equation (8) has two error source and requires finer considerations: for very accurate approximation of the eigenvectors the residual may still be very high if the paired approximated eigenvalue is far from the actual eigenvalue. Vice versa, having a precise eigenvalue, therefore a small residual, does not necessarily guarantee a good approximation for the eigenvector.

Proving that the residual of equation (8) can be used as error estimator for a greedy algorithm is possible introducing the theory on matrix perturbations. Investigations on eigenvalue and eigenvector sensitivity due to matrix perturbation are particularly relevant for the scope of this work [7].

The accuracy of the approximated eigenvalues depends on the conditioning of the diagonalization matrix. If this results ill-conditioned, the residual norm will not represent a thoroughly trustable error estimator. This is formalized in a theorem proved in [7] that is here recalled.

Theorem 1. Given a semisimple matrix $\mathbb{A} \in \mathbb{C}^{n\times n}$, suppose $V^{-1}\mathbb{A}\mathrm{V} = D$, where D is a diagonal matrix and V is non-singular. Given a perturbation of the matrix $\delta\mathbb{A}$, $\hat{\lambda}$ is an eigenvalue of the matrix $\mathbb{A} + \delta\mathbb{A}$. It follows that $\mathbb{A}$ has an eigenvalue λ such that

$$\left|\hat{\lambda} - \lambda\right| \leq cond(V)\|\delta\mathbb{A}\|_p \tag{9}$$

where $cond(V)$ represents the condition number of the matrix V and $\|\cdot\|_P$ represents the p-norm and $1 \leq p \leq \infty$. This tells that, the approximation of the eigenvalues is good if the condition number of the matrix of eigenvectors V is good and the perturbation of the system $\mathbb{A}$ is small. The other requirement is on the eigenvectors whose sensitivity assessment results to be more complex. A condition number for eigenvectors can be obtained using a Schur-like decomposition of the system (see chapter 6 of [7]). A condition number estimator for the eigenvectors is just given by the quantity $\|\mathbb{A}^{-1}\|$.

Table 1 shows the greedy algorithm proposed. To favor readability the error estimation is here based on only one eigenvalue pair but the essence of the algorithm does not change if the estimate is done using more pairs.

The matrices K_i and M_i may be large depending on the complexity of the model. Therefore it is convenient precomputing them as this will bring important improvements of computational costs also in the preprocessing time.

Unlike other greedy approaches present in literature where a single vector contribution per iteration is usually added, this procedure requires to add a set of vectors at each iteration. This is done to improve the convergence of the method. The added set corresponds to the parameter configuration whose largest amount of information is missing.

The drawback of adding a set of vector is that some of the information spanned by the added vectors is probably already available in the temporary global ROB and has to be filtered out. Therefore, the “union operation” of line 12 is done through SVD.

The algorithm of table 1 is implemented and discussed in the numerical example of section 4.

1:	Function $\boldsymbol{V_g}$ = GreedyAlgorithm($\boldsymbol{\mu}, \boldsymbol{V_{\mu_0}}, \boldsymbol{toll}$)
2:	$\boldsymbol{V_g} = V_{\mu_0}$
3:	**while** $MaxR > toll$
4:	**for** $i = 1{:}N$
5:	$[K_i, M_i]$ = getFOMmatrices($\boldsymbol{\mu_i}$)
6:	$K_r = V_g^T K_i V_g; \quad M_r = V_g^T M_i V_g$
7:	$[X_r, \lambda_r] = eig(K_r, M_r, 1^{st}\ mode)$
8:	$r_i = K_i V_g X_r - M_i V_g X_r \lambda_r$
9:	**end for**
10:	$[MaxR, ID] = \max(r)$
11:	$\left[V_{\mu_{ID}}, \sim\right] = eig(K, M)_{\mu_{ID}}$
12:	$\boldsymbol{V_g} = V_g \cup V_{\mu_{ID}}$
13:	**end while**
14:	**end function**

Table 1: Greedy approach based on the residual of one eigenvector

4 Numerical application to a FE model of a gantry bridge

A gantry bridge is a tool used to maneuver objects in a 3D space. For this application, the gantry consists of a framework of beams. The system is initially modeled using an FE method followed by the generation of an affine model with 1835 DOF. The details of the derivation of the affine functions used in this work can be found in [1]. For this example the gantry is considered made of a uniform linear-elastic material and constrained in the 4 nodes at the bottom corners of the structure preventing any rigid motion. The gantry is loaded along the central vertical axis of symmetry with a force pointing downwards. The parameters of the model are the dimension of the cross-sections of the constitutive beams being h, w and t (figure 1).

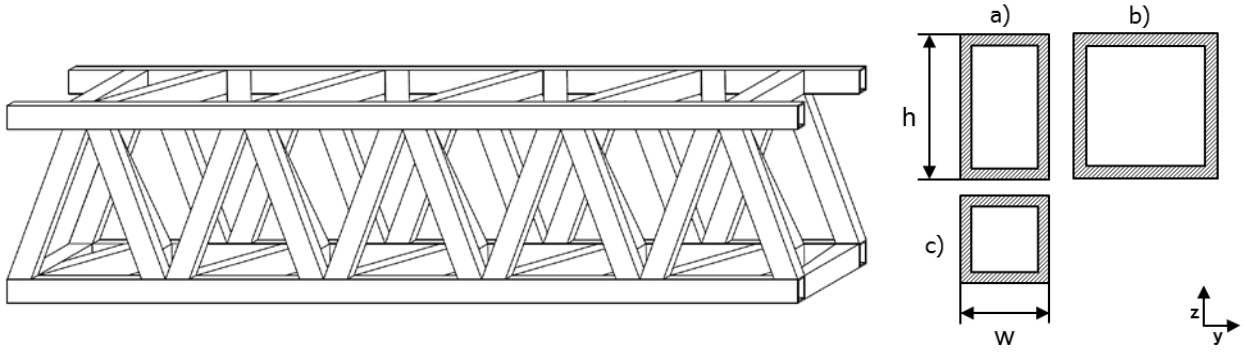

Figure 1: Gantry bridge of the laser-cutting machine. On the right hand side, the cross-section of the constitutive beams are represented [1].

The greedy algorithm presented in section 3 yields the required accuracy in 8 iterations. The resulting ROB counts 195 generalized coordinates. In figure 2, the representation of the sample points selected through the greedy algorithm is given.

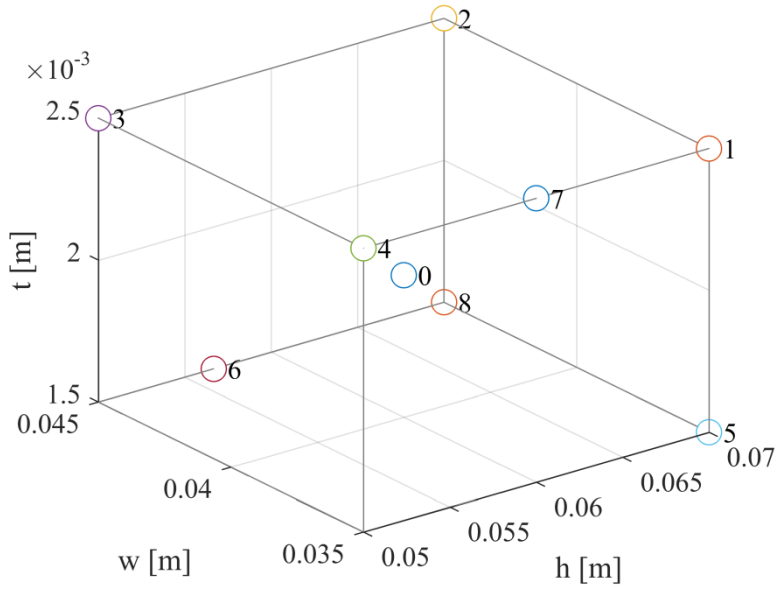

Figure 2: The prism represents the parameter domain that is gridded and used to construct the global ROB through the greedy algorithm presented in section 3. The enumerated points represent the parameter configuration selected by the greedy algorithm at each iteration.

To assess the reduction accuracy, a number of parameter configurations is taken and compared with the result of the full order model. In figure 3 the decay of the residual and of the actual error are plotted in logarithmic scale. Although the results seem very promising, it should be underlined that, the error in figure 3 only accounts for the error committed on the eigenvalue and not the error on the eigenvectors. Figure 4 shows the collocated frequency response functions for the load case described and for all the parameters of the model; the smooth variation of the response functions with the parameters is a good indication of a small Kolmogorov n-width for the manifold associated with the problem. Finally in Figure 5, the error between the FRFs computed with the ROM and the FOM are plotted.

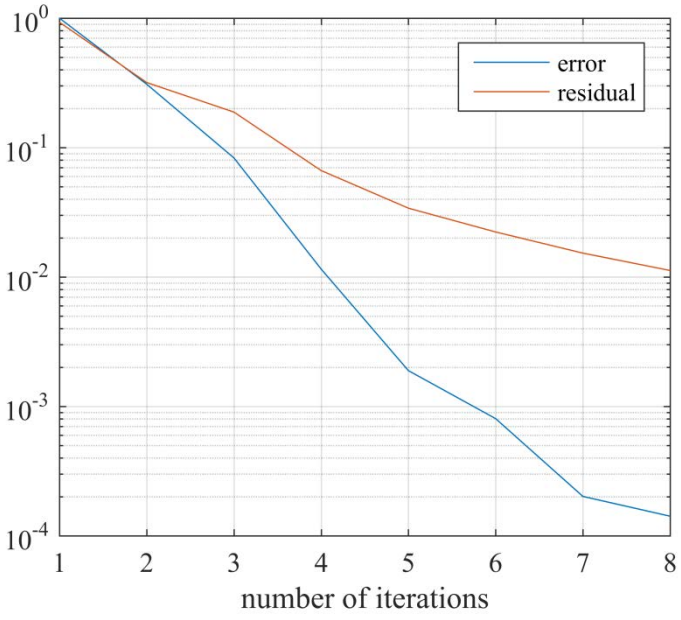

Figure 3: normalized value of the actual error and of the residual at each iteration.

The good results for eigenvalues of Figure 3 could be expected as they are aligned with the theoretical discussion of section 3.2. Nevertheless, the error representation of Figure 5 shows that also the FRF can be represented very accurately in the chosen parameter range.

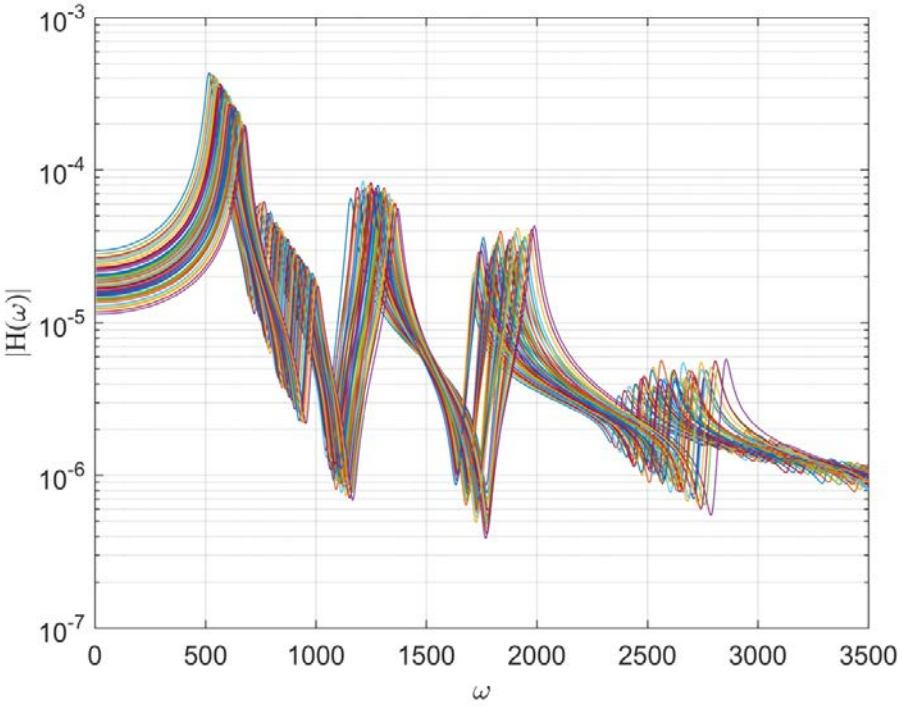

Figure 4: Collocated frequency response functions of the gantry bridge at the center point where the load is imposed. Each response function line is generate using the ROM and corresponds to a different parameter configuration in the range of interest.

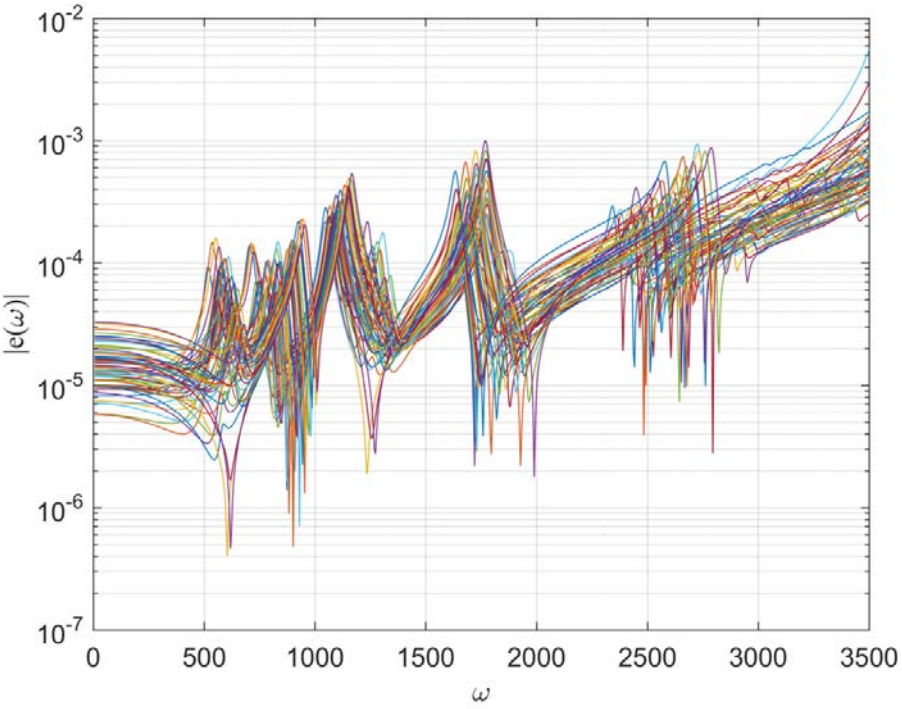

Figure 5: error distribution for the FRFs plotted in figure 3. $e(\omega) = \left|\frac{y_{ROM}(\omega) - y_{FOM}(\omega)}{y_{FOM}(\omega)}\right|$.

5 Conclusions

This work presents a technique for the reduced order basis construction for parametric structural dynamics problems using a greedy algorithm based on normal modes.

The error estimator developed in section 3 allows an interesting implementation of the greedy algorithm when eigenvector basis are used. In fact, the greedy approach to construct the basis offers some accuracy estimation before simulation otherwise unavailable for purely modal basis. Therefore, the proposed methodology sums up the advantages of incremental improvements offered by the greedy algorithm to the popular modal reduction techniques.

Some of the theoretical investigations made through the paper have been further analyzed by means of the numerical example reflecting the potential of the methodology.

Nevertheless, a number of further improvements will be presented in future research. The temporary global ROB is updated with a full local ROB at each iteration of the greedy regardless of the fact that not all the new vectors being added may be needed to improve the solution. Including only the strictly necessary vectors would allow a finer optimization of the number of generalized coordinates with improvements in memory management and performances of the simulation.

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