Quantum information scrambling in three-qubit random mixed states

A Project submitted to the School of Physics, University of Hyderabad

in partial fulfillment for the award of degree of

Master of Science in Physics

By

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Declaration

I hereby declare that, this project entitled **'Quantum information scrambling in three-qubit random mixed states'** is carried out by me at the School of Physics, University of Hyderabad under the supervision of Prof. V. Subrahmanyam.

No part of this project has been previously submitted for a degree or diploma or any other qualification at this University or any other.

Rodge Amey Suryakant
21PHMP26

Certificate

This is to certify that, this project entitled **'Quantum information scrambling in three-qubit random mixed states'** is carried out by Rodge Amey Suryakant (21PHMP26) under my supervision at the School of Physics, University of Hyderabad, in partial fulfilment of the requirements for the award of the degree of **Master of Science in Physics**. No part of this project has been previously submitted for a degree or diploma or any other qualification at this university or any other.

Prof. V. Subrahmanyam
Project Supervisor

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Abstract

Quantum information scrambling refers to the rapid and complex spreading of quantum information throughout a quantum system. It is a phenomenon that occurs in certain quantum systems, such as black holes or strongly interacting many-body systems, where information becomes highly entangled and mixed. Research on quantum information scrambling aims to understand the fundamental principles behind this phenomenon and its implications for quantum information processing, quantum gravity, and the study of complex quantum systems. This paper delves into the investigation of the behaviour of tripartite mutual information (TMI) as a measure of information scrambling, specifically focusing on three-qubit random mixed states. To create these random mixed states, random numbers are generated using the method of inverse transform sampling from various distributions. The average TMI for random mixed states is then examined in relation to a parameter of the chosen distribution, revealing different behaviours. The variation of average tripartite mutual information for real as well as complex parameters of random mixed state has been studied by varying corresponding parameter of various distribution functions.

Keywords: Tripartite mutual information, Information scrambling, random mixed states.

1 Introduction

Classically, data scrambling is the process of deleting or obfuscating the data to preserve its confidentiality. This is an irreversible process, so the original data cannot be retained from scrambled data. Scrambling of quantum information is the spreading or dispersal of quantum information, it describes the propagation and effective loss of initial local information in quantum many-body dynamics. Scrambling is usually considered as a property of a state and a state is said to be scrambled if the information of the state cannot be learned from local measurements.

Tripartite mutual Information (TMI) is used to study the information scrambling for three-qubit random mixed states. To generate the random mixed states, here random numbers are generated from standard distribution functions using the method of inverse transform sampling and 8x8 random hermitian matrices are constructed using these random numbers. The behaviour of TMI for these random mixed states is studied by varying the parameter of the distribution functions.

2 Out of time order correlations and Tripartite Mutual Information

The simplest measures of scrambling are expectation values of the product of operators at different times, the lowest order of it is called out-of-time order correlations (OTOC). It is defined by the two commuting operators W_t and V as,

$$F(t) = < W_t^{\dagger} V^{\dagger} W_t V >, \tag{1}$$

where $W_t = U(-t)WU(t)$ is the Heisenberg operator obtained by time evolution. In a many-body system with non-trivial interacting Hamiltonian H, $F(t)$ diagnoses the spread of quantum information by measuring how quickly two commuting operators W and V fail to commute [5].

Another measure to quantify the scrambling is the study of mutual information between systems. The quantum mutual information for two systems A and B in a state ρ_{AB} is defined as,

$$I(A : B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \tag{2}$$

where $S(\rho_A)$, $S(\rho_B)$ and $S(\rho_{AB})$ are the von Neumann entanglement entropies (where $S(\rho) = -Tr\rho \log_2 \rho$) [3]. The mutual information is a non-negative quantity and follows $0 \leq I(A : B) \leq 2$ by the concavity property of von Neumann entropy. $I(A : B) = 0$, for pure bipartite separable states and $I(A : B) = 2$, for pure bipartite entangled states. In classical information theory, tripartite mutual information (TMI) between random variables A, B and C is defined as,

$$I(A : B : C) = I(A : B) + I(A : C) - I(A : BC), \tag{3}$$

I(A:B) represents the mutual information shared between A and B, I(A:C) represents the mutual information shared between A and C, while I(A:BC) represents the mutual information shared between A and BC as a system together. In terms of Von-Neumann entropy

TMI is given as,

$$I(A:B:C) = S(\rho_A) + S(\rho_B) + S(\rho_C) - S(\rho_{AB}) - S(\rho_{BC}) - S(\rho_{CA}) + S(\rho_{ABC}), \quad (4)$$

Tripartite mutual information is identically zero if ρ_{ABC} represents a pure state, this is because any partition between the subparts A and BC will lead to the same von Neumann entropy, i.e. $S(\rho_A) = S(\rho_{BC})$ and similar for other bi-partitions. In the case of a tripartite mixed state or a many-qubit pure state with at least four qubits, TMI becomes a non-trivial significant measure. TMI being positive indicates that A with B and A with C share some vague information as compared to A with BC as a system. However, if TMI is negative, the sum of the information shared between A and B, A and C is smaller than the information shared between A and BC as a system, which is the signature of loss of information and is termed as information scrambling. Thus the scrambling of the quantum information can also be quantified by this observable independent quantity called tripartite mutual information

3 Illustration of scrambling and TMI

To illustrate the positive and negative nature of TMI, consider the following three-qubit mixed state,

$$\rho = \frac{p}{3} \left|100 + 010 + 001\right\rangle \left\langle 100 + 010 + 001\right| + q \left|111\right\rangle \left\langle 111\right| + (1 - p - q) \left|000\right\rangle \left\langle 000\right|, \quad (5)$$

where $p, q \geq 0$ and $p + q \leq 1$. Tripartite mutual information is calculated for this state with different values of p and q, the behaviour is shown in the density plot in Fig. 1 as a function of p and q. This state becomes a pure state when $p = 0, q = 1$ and $p = 1, q = 0$, thus $I(A : B : C)$ becomes zero. It is observed that the sign of TMI for this state depends on the values of p and q, particularly TMI is negative for $p \gtrsim 0.2$.

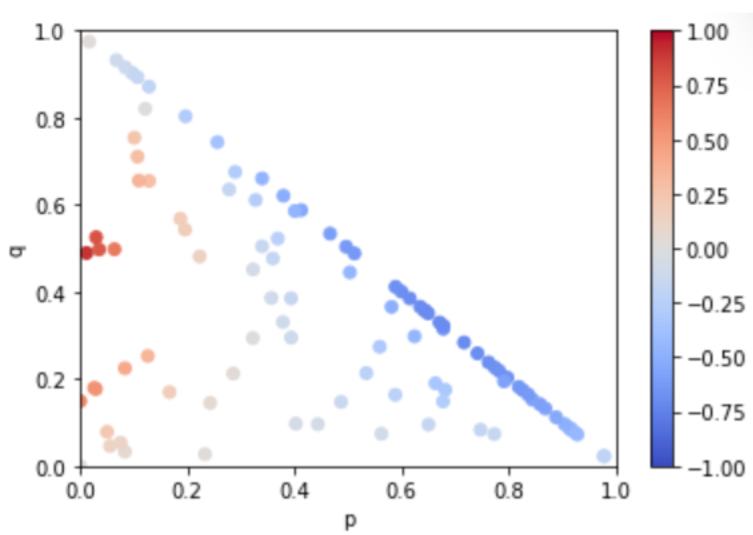


Figure 1: Density plot of TMI $I(A : B : C)$ between three spins for the state given in Eq.(5) as a function of p and q [4].

4 Behaviour of TMI for three-qubit random mixed states

To generate random three-qubit mixed states, random numbers are generated using a method called inverse transform sampling [1]. It is a method for generating random numbers from any probability distribution by finding its cumulative distribution function (CDF) $F(x)$ and then taking the inverse of it i.e. $F^{-1}(X)$[2]. The CDF for a random variable X is $F_x = P(X \leq x)$, where X is a real-valued variable and P is the probability that X will have a value less than or equal to x. For example, to generate random numbers from Gaussian distribution $X \sim G(\mu, \sigma)$

$$F(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp -\frac{(x-\mu)^2}{2\sigma^2} \tag{6}$$

CDF is given as,

$$F_X(x) = \frac{1}{2}(1 + erf(\frac{x-\mu}{\sqrt{2}\sigma}) = u(say) \tag{7}$$

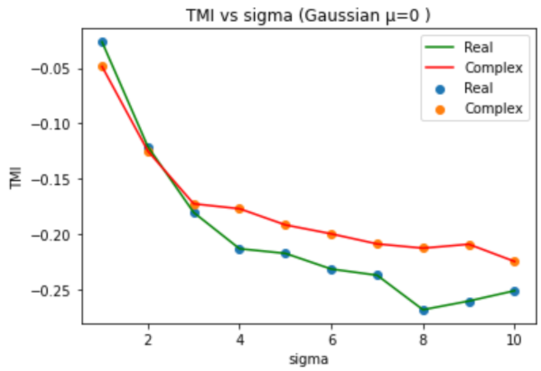
where erf(x) is an error function of x which is defined as $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp -t^2 dt$. Further inverse CDF is given as,

$$F_X^{-1}(x) = \mu + \sqrt{2}\sigma erf^{-1}(2u-1) \tag{8}$$

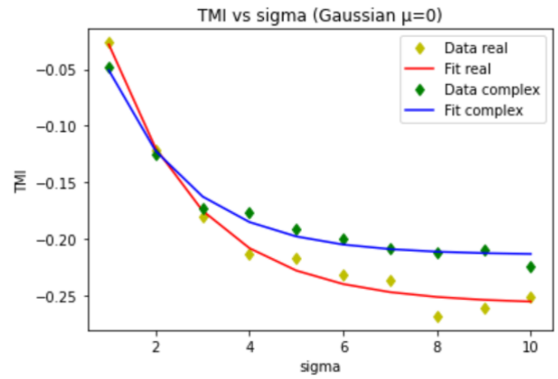
which gives desired sampling formula and thus random numbers x are generated which follows the Gaussian distribution with mean μ and standard deviation σ.

4.1 Random mixed states generated from Gaussian distribution

Random numbers are generated from a gaussian distribution by the method of inverse transform sampling as discussed above. Further, to create random mixed states, 8×8 random hermitian matrix is constructed by considering real and complex parameters. In the case of 8×8 hermitian matrix with real parameters, 35 independent parameters are required, while for an 8×8 hermitian matrix with complex parameters, 63 independent parameters in total are required to define a three-qubit random mixed state. By specifying particular values of μ and σ for a gaussian distribution, random mixed states are generated and the average tripartite mutual information $I(A:B:C)$ is calculated by considering both real and complex parameters. Fig. 2 shows the behaviour of TMI for the different values of σ keeping μ constant. The green curve represents the variation of average TMI with σ for a three-qubit random mixed state with real parameters and the red curve represents the same with complex parameters. From Fig. 2(b) it is observed that this variation of TMI can be fitted to an exponentially decaying curve. As observed from these graphs, the variation of TMI with σ follows the same pattern for both real and complex parameters and moreover, average TMI is negative in both the cases representing the loss of information.



(a) TMI vs σ



(b) TMI vs σ fitted to exponential decay function

Figure 2: Variation of TMI with σ (keeping μ constant).

4.2 Random mixed states generated from exponential distribution

Next, random numbers are generated from an exponential distribution $F(x) \propto \exp(-\alpha|x|)$ and thus three-qubit random mixed states are constructed. The behaviour of TMI with parameter α of an exponential distribution is shown in Fig. 3. A blue curve shows the behaviour of TMI with α for real parameters of random mixed states, while the red curve represents the same for complex parameters. It is seen that for both cases average TMI is negative and it becomes more and more negative as parameter α increases up to 6, and further TMI becomes less negative as α increases up to 10.

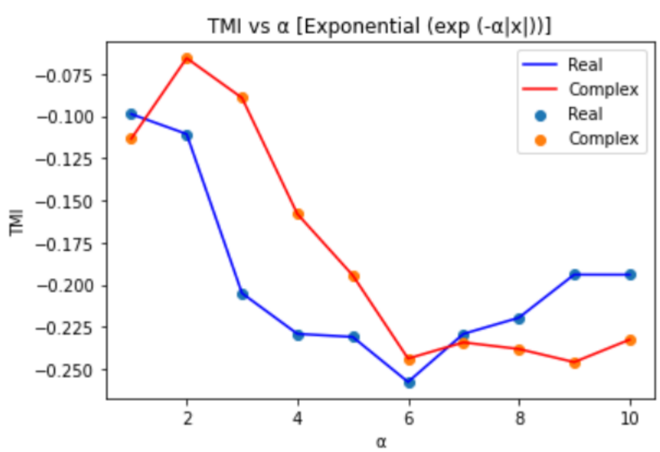


Figure 3: Variation of TMI with α

4.3 Random mixed states generated from uniform distribution

Further, a simple uniform (constant) distribution $F(x) = a$ is used to generate random numbers. The behaviour of TMI is studied by taking a as a parameter for this distribution. Fig. 4 shows the variation in average TMI with parameter a. It is seen that this variation is more or less similar for real and complex parameters of random mixed states, at $a = 1$ the value of TMI is approximately the same for both real and complex parameters and further as a increases the TMI goes on becoming less negative.

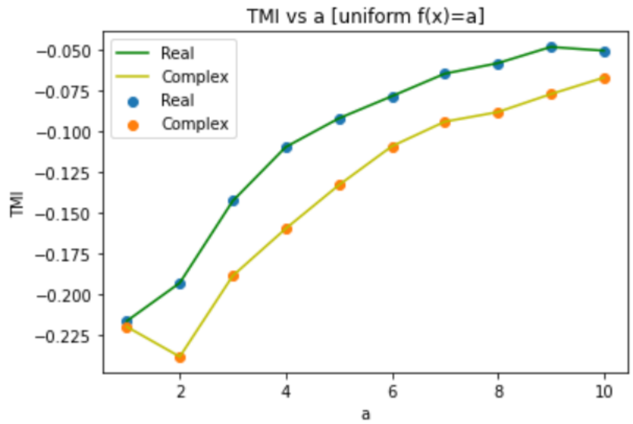


Figure 4: Variation of TMI vs a

4.4 Random mixed states generated from $F(x) = Ax^{|k|}$

Afterwards, random numbers are generated from $F(x) = Ax^{|k|}$, where A is the normalization constant and k is the parameter used. In this case, the TMI shows an interesting behaviour (Fig. 5), at the value of k between 0 and 1, the average TMI has a negative value for both real and complex parameters of three-qubit random mixed states. At $k = 2$, TMI starts becoming positive, and in the case of real parameters, TMI becomes more and more positive as k increases beyond 2. On the other hand, for complex parameters also TMI starts becoming positive at $k \approx 3$, and the value of TMI increases till ≈ 0.1 as k is increased further. The important point to note here is that, in contrast to other distributions, the TMI exhibits positive values ($k \gtrsim 3$) for both the scenarios of real and complex parameters in three-qubit random mixed states. This observation indicates that there is no loss of information; instead, there is a sharing of redundant information.

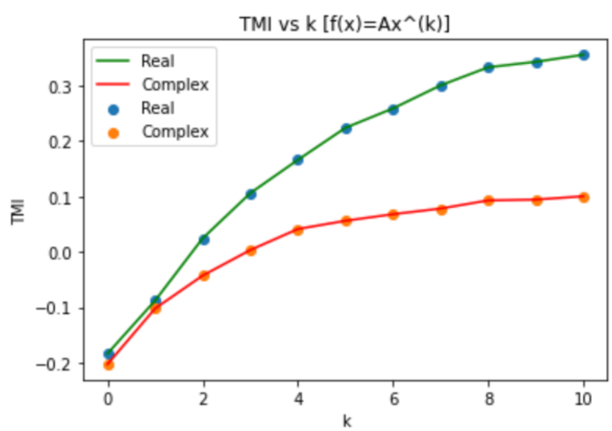


Figure 5: Variation of TMI vs k

5 Conclusion

The nature of tripartite mutual information (TMI) has been studied for three-qubit random mixed states. To construct random mixed states, random numbers were generated from various distributions using the method called inverse transform sampling. For both real and complex parameters of three-qubit random mixed states, the variation of average TMI with a parameter of a given distribution shows different kinds of behaviours. These behaviours show that the values of TMI do depend on the value of the parameter used to generate random numbers. For most of the cases, the value of TMI is negative, showing the loss of information. Moreover, for complex parameter three-qubit random mixed states, values of TMI are more negative as compared to the real parameter three-qubit random mixed states. Further, delving into additional theoretical approaches will enable a comprehensive understanding of these behaviours, shedding light on their underlying physical significance. All the computational work has been done using Python programming language with basic libraries such as NumPy, SciPy.

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