University of St.Gallen

Matlab project

# Portfolio Choice

*Alexander Steeb - [17-614-611]*
*Jonas Gartenmeier - [13-612-700]*

supervised by
Peter H. Gruber

December 15, 2017

# 1 Overview

Portfolio selection is one of the fundamental topics of financial theory. Every investor searches for "the optimal portfolio" to reduce risk measured through volatility while simultaneously increasing returns. The first known instance of portfolio selection can be traced back as far as 400 B.C. At this time, the Babylonian Rabbi Isaac bar Aha stated that "one should always divide his wealth into three parts: a third in land, a third in merchandise, and a third ready to hand." (Source: Babylonian Talmud: Tractate Baba Mezi'a, folio 42a). Although this method of portfolio selection, also known as the 1/N rule, is still applicable today, many more sophisticated techniques have evolved since then.

Especially nowadays in the era of robo-advisors which promise high returns in combination with low risk and ETFs – that reduce transaction costs drastically - portfolio optimization seems more than ever like the holy grail of asset management. Furthermore, buzzwords like Markowitz portfolio, mean-variance optimization and efficient frontier are nowadays common not only in financial theory but also in marketing material targeted directly at end users.

In this paper, we use Matlab to calculate the efficient frontier for different investment universes to find optimal portfolios. We calculate the efficient frontier both with the classical method and with a Monte Carlo simulation and contrast both approaches. For our investment universe, we first use data for the Dow Jones Industrial Average and then extend our analysis to the DAX as well. We calculate not only efficient portfolios but also efficient frontiers and tangency portfolios for different assets, time horizons and investors.

The rest of the paper is structured as follows. First, we will briefly introduce the problem and discuss the underlying theory of portfolio optimization. Next, we will describe the data we used and how we acquired it. The third part describes the methodology we used and is primarily concerned with the Matlab implementation of the respective portfolio optimization, efficient frontier and tangency portfolio. After the methodology, we will present our results and conclude with the comparison of both methods.

# 2 Theory

In this section, we want to give a brief overview of the theoretical concepts on which our analysis is based. First, we will look at the theory behind the classical method of portfolio optimization and then we will describe how the Monte Carlo approach makes use of both the classical way and Monte Carlo simulations to calculate more robust portfolios.

Fundamentally, portfolio optimization can be achieved using two different methods. On the one hand, one can apply an analytical approach to calculate the one and only optimal portfolio for a given set of constraints and a given investment universe. This method is based on the theoretical work on portfolio optimization by Markowitz and others and is not only theoretically rigorous but also relatively easy to apply. On the other hand, one can use a Monte Carlo approach in combination with the classical method to find the optimal portfolio with the help of randomization and many iterations. This method should theoretically lead to more robust portfolio weights and can be especially helpful if the existing data is scarce.

## 2.1 The Classical Way

The classical way of portfolio optimization was introduced by Harry Markowitz in 1952. In 1990, he received the Nobel prize for his work on portfolio selection. The concept is simple and best described by Markowitz himself: "Given a target return, the investor should choose the portfolio that minimizes her risk and satisfies the target return." (Source: Portfolio Selection: Diversification of Investments, 1959). The classical Markowitz approach is therefore a simple minimization problem. One minimizes the variance for a given target return to find the optimal portfolio for this target return. Of course, one can also apply this concept to find the portfolio with the highest return given a specified variance. Two interesting portfolios that can also be found through this are the minimum-variance-portfolio and the mean-variance-portfolio, also known as the tangency portfolio. However, in this paper we concentrate on minimizing the expected variance for a given target return. The minimization problem is therefore as follows.

Minimize variance given a target portfolio return $\mu^*$ :

$$\min \tfrac{1}{2}\, w^T\, \Sigma\, w \quad \text{s.t.} \quad w^T u = 1 \quad \text{and} \quad w^T \mu = \mu^*$$

with:

$\Sigma =$ variance-covariance matrix,
$w =$ vector of weights,
$\mu =$ vector of mean returns,
$u =$ unit vector of the same length as $w$.

This is a natural way of defining the problem if the investor wants to grow his wealth to some desired size $W^* = \mu^* w^*$ (e.g. for a major purchase). The portfolio variance is:

$$\Sigma_{Portfolio} = w^T\, \Sigma\, w$$

The first constraint ensures that the sum of all weights is equal to 1 and therefore that all funds are invested. Note that this constraint allows short selling. The second constraint ensures that the expected return of the found portfolio equals the target return.

Through constrained optimization with Lagrange multipliers we can transform the given minimization problem and find the following solution:

$$w^* = \tfrac{1}{D}\,(B\Sigma^{-1}u - A\Sigma^{-1}\mu + \mu^*(C\Sigma^{-1}\mu - A\Sigma^{-1}u))$$

$$\text{with} \quad A = u^T\Sigma^{-1}\mu, \quad B = \mu^T\Sigma^{-1}\mu, \quad C = u^T\Sigma^{-1}u, \;\; D = BC - A^2$$

This is precisely the mathematical formula that we also used in our Matlab implementation.

## 2.2 The Monte Carlo Approach

The Monte Carlo approach makes use of the formulas above but adds a Monte Carlo simulation to achieve more robust portfolio weights. Simply put, the idea is to first generate $N$ random return series with the same characteristics (mean and covariance) as the historical return series. We used the Cholesky decomposition to generate those simulated return series out of normally distributed random numbers. Next, we calculate the optimal portfolio for each of those $N$ random return series. Finally, we can take the average of those $N$ optimal portfolios to arrive at one final optimal portfolio.

## 2.3 The Efficient Frontier

Since all portfolios can be defined by the two measures expected return and standard deviation we can plot all efficient portfolios. Such a plot is also called the efficient frontier. It displays the optimal portfolio for all given expected returns or standard deviations. Portfolios above this frontier are not achievable with the given assets and portfolios below this frontier are inferior and irrational compared to the efficient frontier.

## 2.4 The Tangency Portfolio

In the third part of this analysis we calculate tangency portfolios. Tangency portfolios are those portfolios on the efficient frontier with the highest slope and therefore the highest sharpe ratio. In the case of a portfolio optimization with a risk free asset the tangency portfolio is the point on the efficient frontier that is tangential to a line starting at the *y-axis* wherever the risk free rate is.

The formula to derive at this tangency portfolio is as follows:

$$\omega_T = \frac{X}{1_N{}'X} = \frac{V^{-1}(E - R_f 1_N)}{1_N{}'\,V^{-1}(E - R_f 1_N)}$$

# 3 Data

The main objective of this paper is to apply and contrast two different methods of portfolio optimization. We are therefore not primarily interested in finding the one optimal portfolio for a specific asset universe. Consequently, we use data from the constituents of the DJIA and the DAX since it is easily available and the companies are well known and sufficiently large.

To acquire the data, we use the user written function *get_yahoo_stockdata3* by Captain Awesome (version 3.0). [Source: `https://tinyurl.com/yczmoru2`]

The input parameters for the function are the following: *ticker, d1, d2, freq*

Additionally, we also needed the daily FX-rate for USD/EUR and EUR/USD. This exchange rate is used when we calculate the daily prices of the DJIA stocks for a German-based investor and vice versa. We used the user written function *hist_stock_data* (version 1.6) from Josiah Renfree to get this data. We used a different function to obtain the FX-rates since it was not possible to obtain both the FX-rate and the stock data with only one of both functions. [Source: `https://tinyurl.com/yan6whey`]

The input parameters for the function are the following: *start_date, end_date, varargin*

Once we have gathered all the date we need we can transform it into a user-friendly matrix format. To achieve this, we use two different functions. The first function *transform_stock_data* transforms the stock data into a matrix, the second function *transform_FX_rate* the FX-rate. Both functions work in a similar way and simply transform the data structure that both data gathering functions output into a matrix format.

# 4 Methodology

Our analysis can be divided into three parts. In the first part, we calculate the efficient frontier with both methods for the constituents of the DJIA and a subset of five randomly selected DJIA stocks. In the second part, we calculate the efficient frontier for a German-based and a US-based investor. They both share the same investment universe consisting of both the constituents of the DJIA and the DAX. However, due to the different currencies of the two investors we first have to calculate the EUR and USD denominated prices for all assets. Therefore, exchange rate fluctuations and their effects are incorporated in the price data. In the third part, we calculate the efficient frontier including the tangency portfolio. We again use the DJIA data but calculate the tangency portfolio for the last trading day of the years 2016, 2015 and 2014.

The functions that we used are the following:

**portfolio_choice**
This is the main body of our program. All other functions are embedded in this script.

**[mu, sigma] = calc_mu_sigma(stock_return)**
This function is used to calculate the mean return vector and the variance-covariance matrix for a given return matrix *stock_return*. The two outputs are therefore the mean return vector *mu* and the variance-covariance matrix *sigma*.

**[weight_optimal] = calc_opt_weight(mu, sigma, target)**
This function calculates the optimal portfolio weights for a given mean return vector, the variance-covariance matrix and target return. It uses exactly the formulas stated in part X.Y. The output is the vector *weight_optimal* consisting of $n$ weights for $n$ stocks.

**[return_ef_classic, std_ef_classic] = calc_ef_classic(mu, sigma)**
This function calculates the optimal weights for a given *mu* and *sigma* for a range of target values. The range is automatically chosen and consists of 100 steps. The output consists of two vectors of the same

length that contain the expected return and standard deviation of the different optimal portfolios on the efficient frontier. Additionally, a graph of the efficient frontier is displayed.
The method behind this formula works as follows. First, a suitable range of target values is automatically defined based on the average of the expected return vector. Next, the function *calc_opt_weight* is used to calculate a matrix consisting of the optimal weights for all values of the target return vector. Finally, for each of these optimal portfolios the expected return and standard deviation is calculated. The resulting vectors *return_ef_classic* and *std_ef_classic* are outputted.

**[return_ef_MC, std_ef_MC] = calc_ef_MC(mu, sigma, steps, iterations)**
This function calculates the efficient frontier based on the Monte Carlo method described above. Like the function for the classical efficient frontier, a range of target returns is first defined and then the optimal portfolio weights for each of those points on the efficient frontier are calculated. The range of target returns is again automatically defined based on the mean expected return over all assets and consist of 100 evenly spaced values.
Unlike the classical efficient frontier this function then uses the Monte Carlo method to calculate the optimal portfolio for each of the target returns. Two additional inputs are therefore needed. The input steps refers to the number of random returns that are generated for each asset. The input iterations refers to how many portfolios are calculated based on those randomly generated returns. Thus, the predefined version with 252 steps and 1000 iterations indicates that 1000 optimal portfolios based on 252 randomly generated returns for each asset are calculated for each point on the efficient frontier. The output consists of two vectors containing the expected return and standard deviation for each of the optimal portfolios that constitute the efficient frontier. Additionally, a graph displaying the calculated efficient frontier is shown.

**[target_return_tp, tangency_line_tp] = calc_tp(mu, sigma)**
This function calculates the tangency portfolio for a given *mu* and *sigma* with the formula stated above. A risk-free rate of 1 % per year is assumed and can be changed inside the function if necessary. The tangency portfolio and the corresponding line starting at the risk free portfolio is then displayed in a graph. The two output vectors *target_return_tp* and *tangency_line_tp* refer to each point on the line. This function is designed to be overlaid on a chart with an efficient frontier.

# 5 Results

## 5.1 Part 1 - 30 and 5 DJIA stocks

In this section, we present the results of our analysis. All returns are annualized. First, we looked at a portfolio consisting of 30 DJIA Stocks between 01.11.2012 and 01.11.2017. The respective returns of these 30 stocks can be seen in figure 1. The graph shows that the returns are well behaved and do not show a time trend. However, we can observe that the variance is not uniform during this time period.

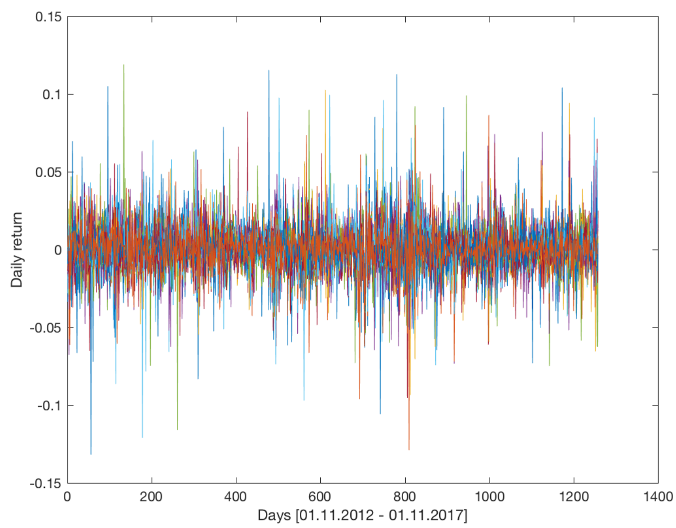



Figure 1: Daily returns for the DJIA-listed stocks

The efficient frontier for these returns calculated with the classical method can be seen in figure 2. Additionally, figure 2 also includes a scatter plot of the mean return and standard deviation of the 30 DJIA stocks. This graph clearly shows the effect of diversification. An optimal portfolio on the efficient frontier allows us to realize higher returns with a lower standard deviation than any single stock does. The minimum variance portfolio seems to generate an annualized return of 10% with a standard deviation of approximately 0.1. Figure 3 displays the efficient frontier for the same universe of assets and the same time horizon but calculated with the Monte Carlo method. It can be seen that the key points of the efficient frontier are about the same for both methods. However, due to the randomization, the later method produces an efficient frontier that is not smooth.

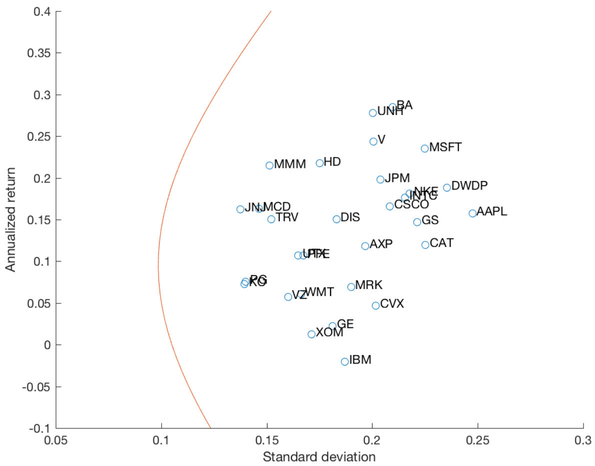



Figure 2: Scatter plot and efficient frontier for 30 DJIA stocks

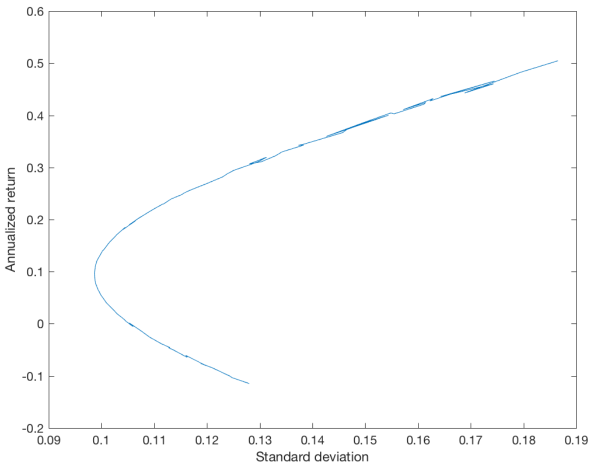
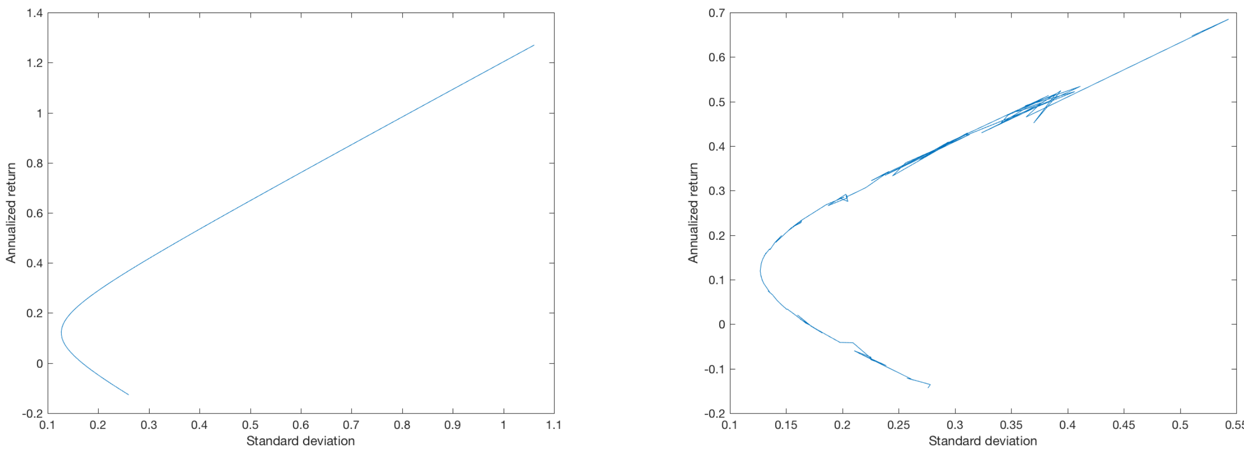



Figure 3: Monte Carlo efficient frontier for 30 DJIA stocks

Next, we randomly chose five stocks out of the 30 DJIA constituents and performed both methods again. Figure 4 displays the results. The two efficient frontiers are clearly different and more precisely inferior to the previous ones consisting of 30 stocks. Additionally, the efficient frontier of the Monte Carlo method is now even less smooth compared to figure 3.



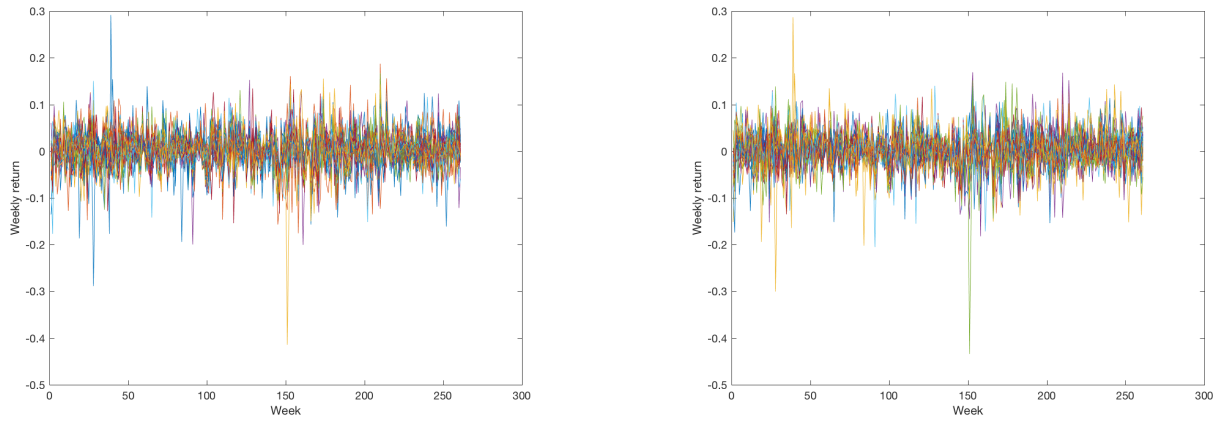

(a) Classical method

(b) Monte Carlo method

Figure 4: Efficient frontier for the five randomly selected stocks

## 5.2 Part 2 - Comparison between US- and German-based investor

In this part we looked at a portfolio consisting of 60 stocks from the German DAX and the DJIA. We calculated both the EUR denominated and USD denominated weekly prices of this portfolio to analyse if the efficient frontier would be different with regards to the location of the investor. Figure 5 displays the respective returns of such a portfolio for the two investors.




(a) German based investor (b) US based investor

Figure 5: Weekly returns of the portfolio consisting of 60 stocks for the two investors

Figure 6 displays the efficient frontier for both the US based and the German investor. Both frontiers have been calculated with the classical method. The efficient frontier of the US investor is clearly superior to the one of the German investor. Hence, the US-based investor can achieve the same target return as the German investor with a lower associated standard deviation. The exchange rate therefore has a significant effect on the optimal portfolio and with it the efficient frontier.

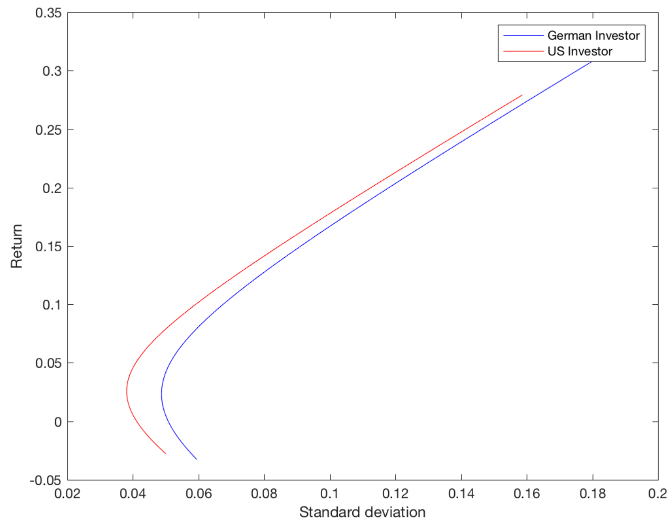



Figure 6: Efficient frontier for both investors

### 5.3 Part 3 - Tangency portfolios over time

Finally, in this section we calculate the tangency portfolio for the last trading day of the three consecutive years 2014, 2015 and 2016. The asset universe consists of the 30 DJIA constituents. Figure 7 displays the results. While the three efficient frontiers are similar we can observe some variation. For instance, the tangency portfolio for the year 2014 is significantly better with regards to return and risk than the two other portfolios. This could indicate that either the overall performance of the 30 DJIA stocks was better in 2014 compared to the following two years or that the correlation between the 30 stocks was lower in 2014, which would allow for a better diversification.

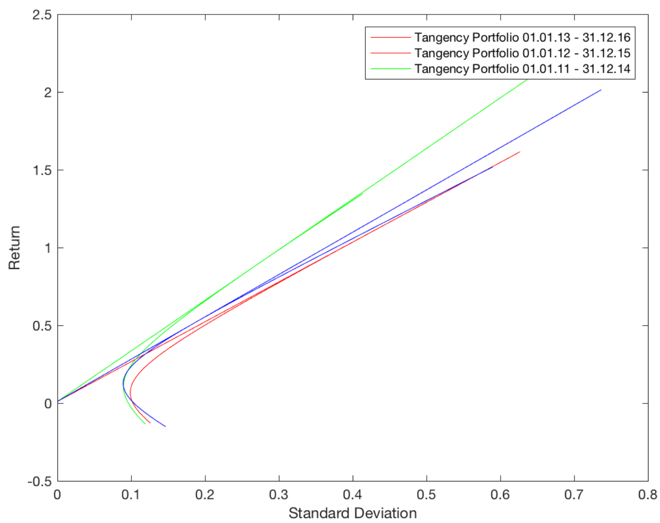



Figure 7: Efficient frontier and tangency line for the last trading day of three consecutive years

# 6 Conclusion

Both methods of portfolio optimization have their merits and drawbacks. The classical method is elegant and simple to apply. It is also highly efficient since only a few calculation steps are necessary. One drawback of the classical method is that it can produce extreme results under certain circumstances. For instance, the weights are often extreme and since short-selling is allowed in our general example there are often extreme negative weights. A normal investor would probably never enter such extreme short positions due to their high risk. Additionally, already slight changes in the expected return vector or variance-covariance matrix can have an extreme impact on the composition of the optimal portfolio. This would mean that an investor who precisely follows this classical method would have to reallocate large portions of his portfolio every time a slight change occurs. Due to transaction costs, this is of course not feasible.

The Monte Carlo method can be applied to mitigate some of those drawbacks. For example, the optimal portfolios are often more robust and short positions are not as extreme when calculated with the Monte Carlo method. For an investor this means that he could apply this method of portfolio optimization without the necessity for extreme short positions and large asset reallocations. However, compared to the classical method, the Monte Carlo method requires the specification of hyper parameters. Those hyper parameters are the number of iterations and the number of forecasted steps for each asset. Since there is no clear solution to optimize those parameters there is some room to debate. Furthermore, the Monte Carlo method requires a lot of computational power due to the many iterations that are necessary. This can of course be a limiting factor.

One limitation that applies to both methods is that past performance is never a guarantee for future results. Every investor that uses portfolio theory to optimize his returns should be aware of this simple fact. Additionally, there exist many more measures of risk that are important to look at besides standard deviation. For example, additional risk measures like semivariance, Value-at-Risk or Expected Shortfall can help to better evaluate the risk profile of a portfolio.