TMA4285 Timeseries, Exercise 3

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Abstract

This article presents the basics of time series analysis, and applies it to model monthly average atmospheric CO2 levels observed at Mauna Loa Observatory, Hawaii, U.S. As carbon dioxide is a green house

Introduction

Carbon dioxide levels have been monitored since 1958 at the Mauna Loa Observatory, located in the Pacific Ocean on top of Hawaii's biggest volcano. Co2 is a green-house gas, and as humans burn fossile fuels, more is added to the atmosphere. In this article we will use theory on time series to model Co2 levels measured at the observatory, with the goal of forecasting future levels. We will focus on monthly averages as we are interested in predicting the long-term trend.

Theory

We note that large parts of the theory presented here, is based on Brockwell and Davis.

A timeseries can be considered as a stochastic process, a set of random variables indexed in accordance to their occurrence relative to the other measurements. We will organize the finite timeseries we will deal with into vectors: $\mathbf{X} = (X_1, X_2, \ldots, X_n)^T$. A realization of a timeseries is an observation of the the real values of the time series, and will be denoted as $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$

The covariance function of two elements of $\mathbf{X}$ is defined as:

$$\gamma_X(r, s) = Cov(X_r, X_s) = E((X_r - E(X_r))(X_s - E(X_s))$$

In general the covariance function tells us how observations correlate to each other.

A time series is weakly stationary if:

- $E(X_t)$ is independent of t
- $\gamma_X(t + h, t)$ is independent of t at each h value.

In a weakly stationary timesries the correlation between two observations would only be depended the relative time between the observation of the repsective observations.

The auto correlation function at a selected lag h is:

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)}$$

Where lagging a time-series is shifting it forward by a selected amount of indexes.

The estimator of the autocovariance of a realized time-series, referred to as the sample co-variance is:

$$\hat{\gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-|h|} (x_{t+|h|} - \bar{x})(x_t - \bar{x})$$

For white noise time-series sample autocorrolation $\gamma(h) \sim N(0, \frac{1}{n})$ distribution. Under general assumptions this estimator is for large samples close to unbiased.

The time series we will observe in the data-analysis later, is classically decomposed in the following manner:

$$X_t = m_t + s_t + Y_t, \quad t = 1, \dots, n$$

Where $EY_t = 0$, $s_{t+d} = s_t$ and $\sum_{j=1}^{d} s_j = 0.m$ is referred to as the trend of the time-series, and s_t is the seasonal trend. The noise Y_t is often assumed to be identically independently distributed (iid) white noise:

$$Y_t \sim WN(0, \sigma^2)$$

Subtracting the trend estimation, will yield as timeseries without a trend. A trend can also be eliminated from a series by differencing:

$$\nabla X_t = X_t - X_{t-1} = (1 - B)X_t$$

This is referred to as lag-1 difference, where B is the backward shift operator.

$$BX_t = X_{t-1}$$

Can difference n-times by:

$$\nabla^n X_t = (1 - B)^n X_t$$

Some data, as stated earlier, show a seasonal trend, this can e.g. be that months of different years correlate, applying a lag-d difference operator is a way of reducing such a case to a stationary process, a lag-d operator can be defined as:

$$\nabla_d X_t = X_t - X_{t-d} = (1 - B^d)X_t$$

A time series $\mathbf{X}$ is a linear process if:

$$X_t = \sum_{j=-\infty}^{\infty} \phi_j Y_j$$

with $\sum_{j=-\infty}^{\infty} |\phi_j| < \infty$. Where $\mathbf{Y}$ is white noise.

ARMA and ARIMA

This brings us to the ARMA and SARIMA models, the latter which will be a central part in our analysis. An $ARMA(p, q)$ model is defined as the following:

$$\phi(B)X_t = \theta(B)Z_t$$

where Z_t is white noise, iid with variance σ^2

and the SARIMA;

$$\phi(B)\Phi(B)X_t = \theta(B)\Theta(B)Z_t$$

More specifically we have:

$$ARMA(p, q): \quad Y_t = (1 - B)^d X_t$$

$$SARIMA(p, d, q) \times (P, D, Q)_s: \quad Y_t = (1 - B)^d (1 - B^s)^D X_t$$

Where:

$$\theta(z) = 1 + \theta_1 z + \dots + \theta_q z^q$$

$$\Theta(z) = 1 + \Theta_1 z + \dots + \Theta_Q z^Q$$

$$\phi(z) = 1 - \phi_1 z + \dots + \phi_p z^p$$

$$\Phi(z) = 1 - \Phi_1 z + \dots + \Phi_P z^P$$

Multiplying the polynomials together, this is equivalent to a $ARMA(p + sP, q + sQ)$ model, and fitting it would thus in essence be equivalent to fitting an ARMA model.

The innovations algorithm

To fit the model one can use the innovations algorithim. Suppose $\{X_t\}$ is zero mean, with $E(X_t)^2 < \infty$ for all t and suppose

$$E(X_iX_j) = \kappa(i, j)$$

Denote the best linear predictors, and their mean squared erros as:

$$\hat{X}_n = \begin{cases} 0, n = 1 \\ P_{n-1}X_n, n \geq 2 \end{cases}$$

$$\upsilon_n = E(X_{n+1} - P_nX_{n+1})^2$$

Where P_i is the projection onto the space spanned by the i first elements of the timeseries in the $L^2(\Omega)$ hilbert-space with inner-product:

$$< X_i, X_j >= E(X_iX_j)$$

Define the innovations as:

$$U_n = X_n - \hat{X}_n$$

and let $\mathbf{U}$ be a vector with the innovations ranging from 1 to n as its elements. As all X_n are assumed to be the result of some linear combination of the preceding values there exists a matrix A such that

$$\mathbf{U}_n = A_n\mathbf{X}_n$$

$$A = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ a_{11} & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1,-n-1} & a_{n-1,-n-2} & a_{n-1,-n-3} & \dots & 1 \end{bmatrix}$$

This matrix is invertible, as it is is unit lower triangular (it has a non-zero determinant of 1). The inverse will be on the following form:

$$C_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \theta_{11} & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \theta_{n-1,-n-1} & \theta_{n-1,-n-2} & \theta_{n-1,-n-3} & \dots & 1 \end{bmatrix}$$

$$A_nC_n = C_nA_n = I_n$$

This yields:

$$\hat{\mathbf{X}}_n = \mathbf{X}_n - \mathbf{U}_n = C_n\mathbf{U}_n - \mathbf{U}_n = C_n\mathbf{X}_n - C_n\hat{\mathbf{X}}_n - \mathbf{X}_n + \hat{\mathbf{X}}_n = (C_n - I_n)(\mathbf{X}_n - \hat{\mathbf{X}}_n)$$

Define

$$\Theta_n = C_n - I_n$$

$$\Rightarrow \hat{\mathbf{X}}_n = \Theta_n(\mathbf{X}_n - \hat{\mathbf{X}}_n)$$

Combining this with the fact that:

$$\mathbf{X}_n = C_n(\mathbf{X}_n - \hat{\mathbf{X}}_n)$$

solving for the coefficients above can easily done numerically, Brockwell and Davis suggest calculating recursively yielding the innovation algorithm:

$$\upsilon_0 = \kappa(1, 1)$$

$$\theta_{n,n-k} = \upsilon_k^{-1}(\kappa(n+1,k+1) - \sum_{j=0}^{k-1} \theta_{k,k-j}\theta$$

$$\upsilon_n = \kappa(n+1,n+1) - \sum_{j=0}^{n-1} \theta_{n,n-j}^2 \upsilon_j$$

## ARIMA and the innovation algorithm Forecasting ARMA process using the innovation algorithim. Assume we have an ARMA process with known (p,q) values. We transform the process into:

$$\begin{cases} W_t = \sigma^{-1} X_t, \quad t = 1, \dots, m \\ W_t = \sigma^{-1}\phi(B) X_t, \quad t > m \end{cases}$$

$m = max(p,q)$ The autocovariance can then be found using:

$$\kappa(i,j) = \begin{cases} \sigma^{-2}\gamma_X(i-j), \quad 1 \le i,j \le m \\ \sigma^{-2}(\gamma_x(i-j) - \sum_{i=1}^{p} \phi_r \gamma_X(r - |i-j|)), \quad min(i,j) \le m < max(i,j) \le 2m \\ \sum_{r=0}^{q} \theta_r \theta_{r+|i-j|}, \quad min(i,j) > m, \\ 0 \end{cases}$$

Where applying the innovation algorithim to W yields:

$$\begin{cases} \hat{W}_{n+1} = \sum_{j=1}^{n} \theta_{nj}(W_{n+1-j} - \hat{W}_{n+1-j}), \quad 1 \le n < m \\ \hat{W}_{n+1} = \sum_{j=1}^{q} \theta_{nj}(W_{n+1-j} - \hat{W}_{n+1-j}), \quad n \ge m \end{cases}$$

This enables us to write X_t as a linear combination of the W_t-s vice versa.

We note:

$$\hat{W}_{n+1} = P_n W_{n+1}$$

$$\hat{X}_{n+1} = P_n X_{n+1}$$

Where P is a projection in the hilbert space $L^2(\Omega)$.

Using this we get

$$\begin{cases} \hat{W}_t = \sigma^{-1}\hat{X}_t, \quad 1 \le t \le m \\ \hat{W}_t = \sigma^{-1}(\hat{X}_t - \phi_1 X_{t-1} - \dots - \phi_p X_{t-p}), \quad t > m \end{cases}$$

Combining this we get:

$$X_t - \hat{X}_t = \sigma(W_t - \hat{W}_t)$$

which in turns yield:

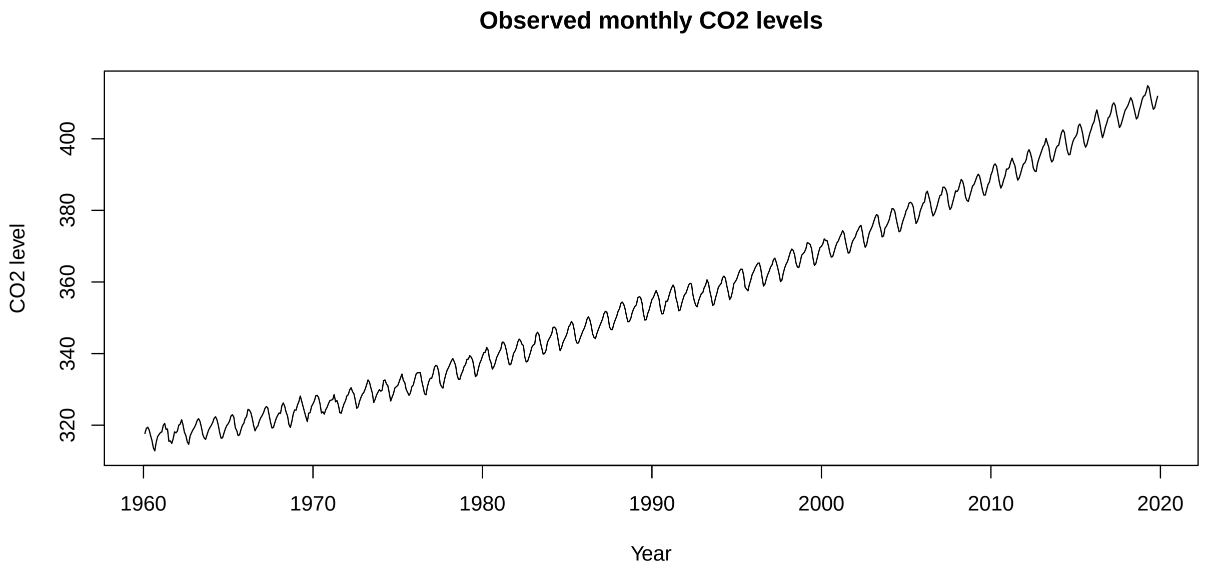
$$\hat{X}_{n+1} = \begin{cases} \sum_{j=1}^{n} \theta_{nj}(X_{n+1-j} - \hat{X}_{n+1-j}), \quad 1 \le t < m \\ \phi_1 X_n \dots + \phi_p X_{n+1+p} + \sum_{j=1}^{q} \theta_{nj} X \quad t \ge m \end{cases}$$

Model comparison, finding values for p and q

Selecting different values for p and q would yield different models. Using AICC and the AIC criterion is one way of selecting among these. The methods both base themselves on the idea of scoring on the likelihood of seeing the realized values given the model and then to punish the complex models more. We prefer simple models as they are less prone to over-fitting the data. I.e. will mean square error not be able to pick up upon this, so more complex models will always be favored, which is why we should use other criteria as AIC or AICC.

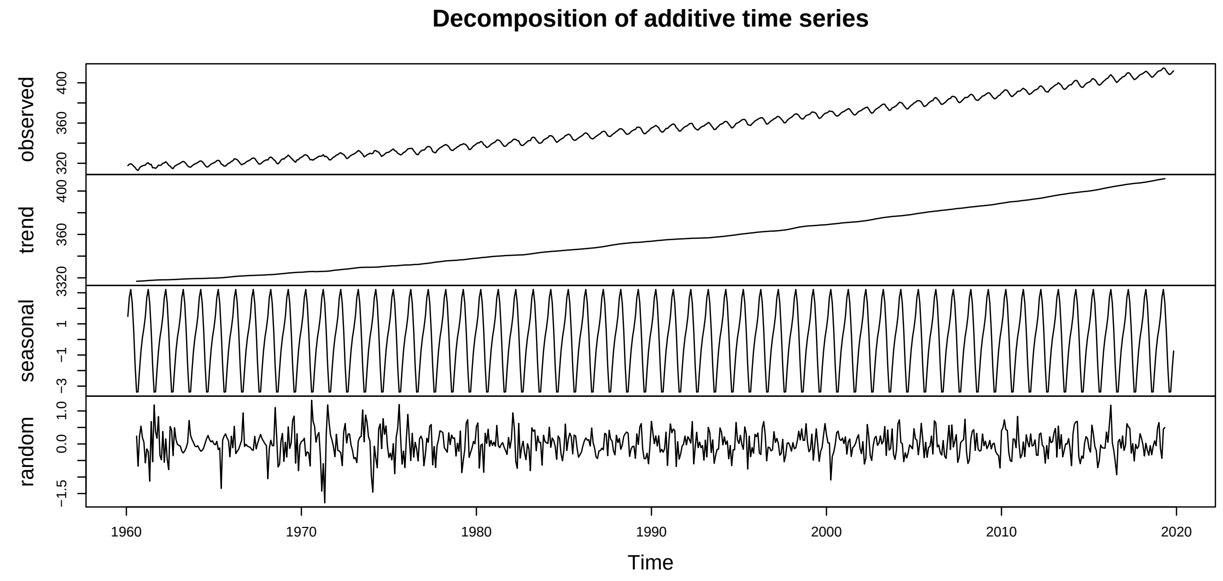
Data Analysis

We start by creating simple time series plot to get an overview.



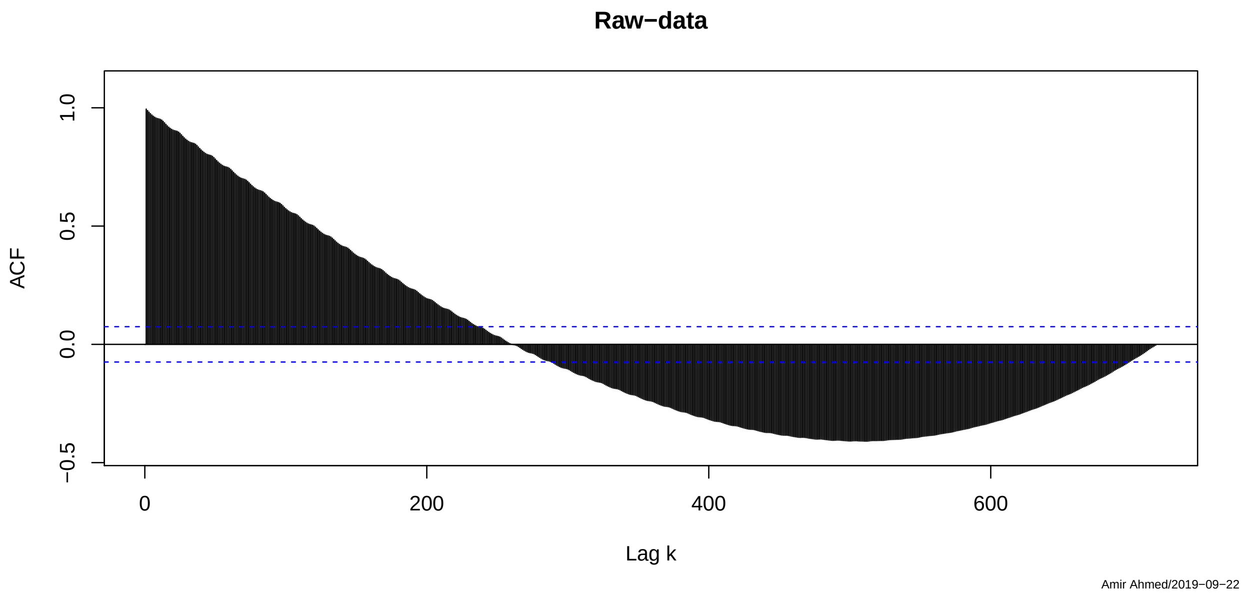
As expected we see an increase. There also seems to be a seasonal trend.

We can decompse the timeseries to get a closer look.

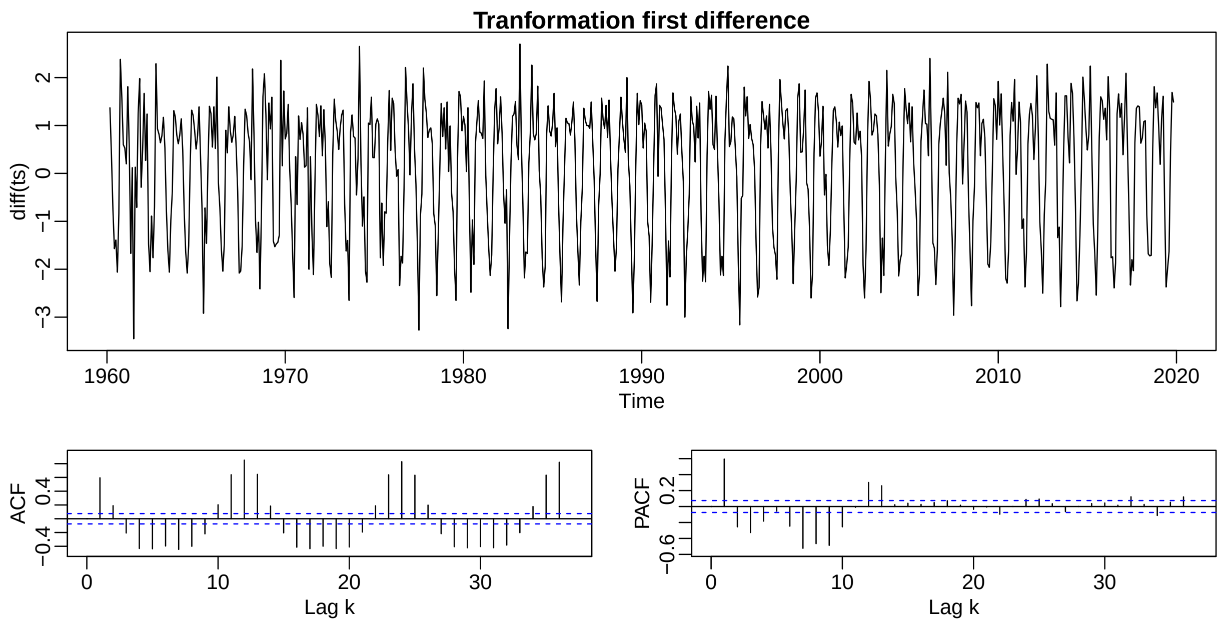


With this we see a much clearer sesonal trend.

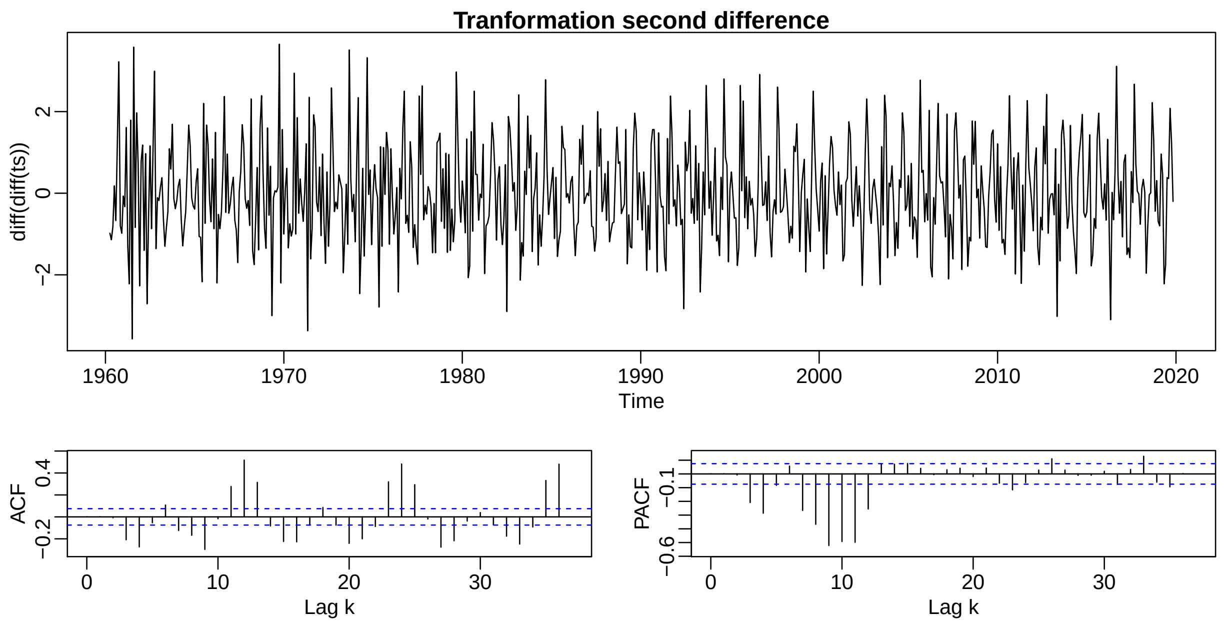
We plot the autoccorlation plot for the time series



Definitely not stationary, apply differencing (∇) to see if it gets any better.

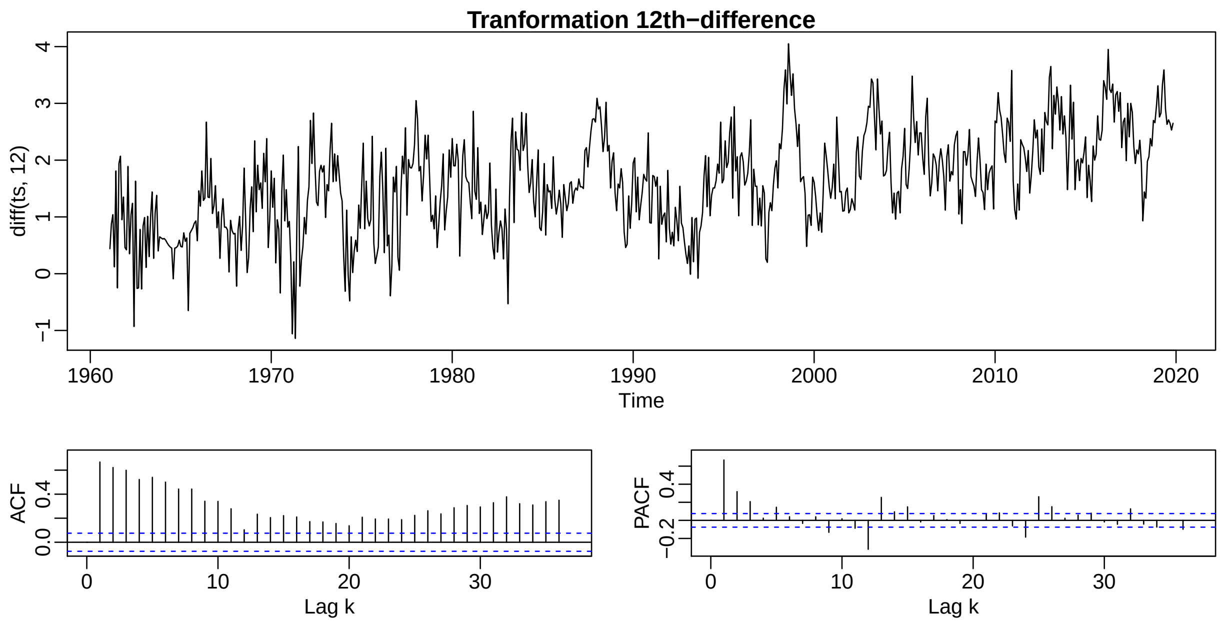


Applying ∇^2:

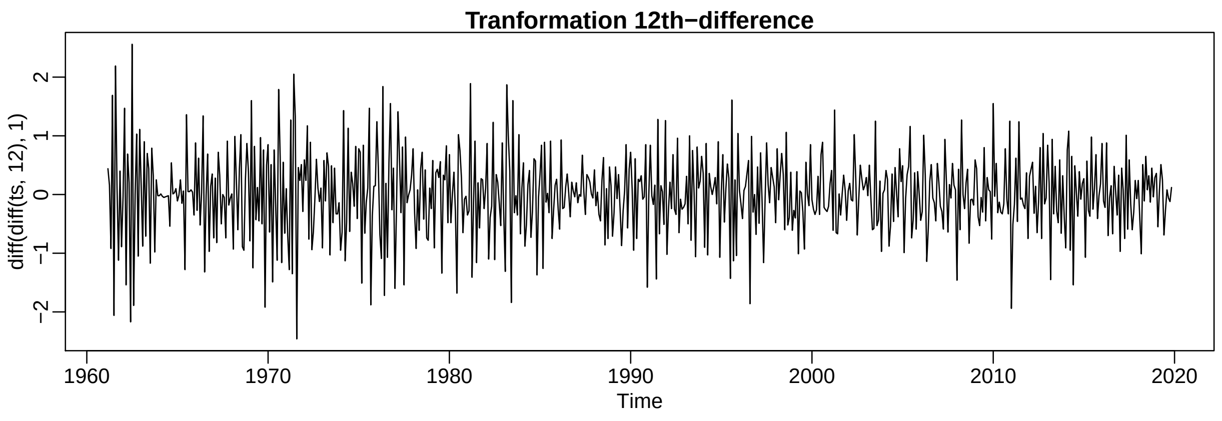


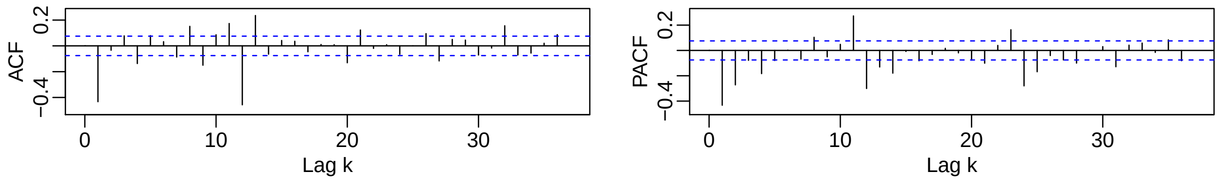
Does not seem to get any better.

We try capturing the seasonal trend. We expect that a season last for a year, so 12 months seems like a good starting point for differencing. Applying ∇^{12} we get:



With this it seems we have removed the seasonal component. But it is clearly not completly stationary. We try applying $\nabla\nabla^{12}$ to capture the lag:





Which seems somewhat stationary, we could increase the lag to get a better fit, but as we prefer simple models, and we want to avoid overfitting, we decide to use $\nabla\nabla^{12}$ to tranform.

We will later see that the acf plot improve form.

There parameters results in a SARIMA model.

(Coefficient can i.e. be fit using innovations algoritihm. But that might not be the method impletemented in the package).

To find the best choice of parameters we compare AIC and AICC of different models.

and $p = 4, d = 1, P = 5, Q = 1$ seems to be best choice of the remaining parameters, where we get AIC and AICcc values of 1.27

With this chose of parameters our model would be:

$$\phi(B)\Phi(B)X_t = \theta(B)\Theta(B)Z_t$$

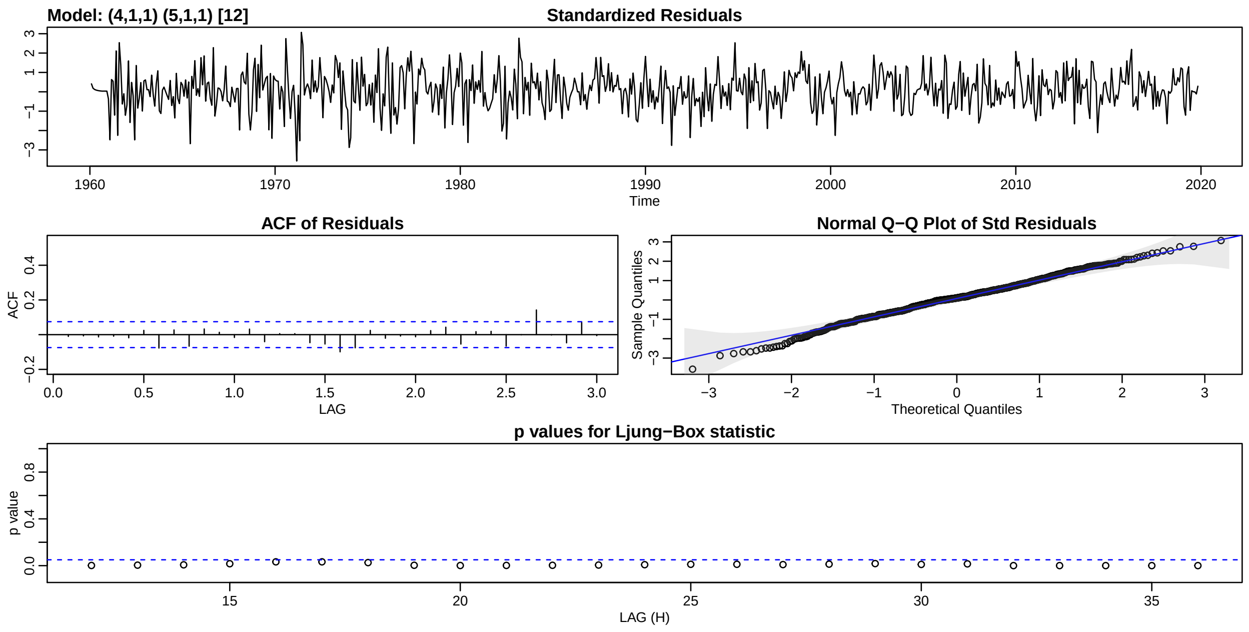
$$\theta(z) = 1 + \theta_1 z$$

$$\Theta(z) = 1 + \Theta_1 z$$

$$\phi(z) = 1 - \phi_1 z + \cdots + \phi_4 z^4$$

$$\Phi(z) = 1 - \Phi_1 z + \cdots + \Phi_5 z^5$$

Fitting the model with the selected parameters, we get the following diagnostic:



```
## initial  value -0.393684
## iter   2 value -0.661896
## iter   3 value -0.765862
## iter   4 value -0.788071
## iter   5 value -0.791576
## iter   6 value -0.792059
## iter   7 value -0.793117
## iter   8 value -0.793789
## iter   9 value -0.794061
## iter  10 value -0.794077
## iter  11 value -0.794083
## iter  12 value -0.794092
## iter  13 value -0.794133
## iter  14 value -0.794148
## iter  15 value -0.794151
## iter  16 value -0.794151
## iter  17 value -0.794151
## iter  17 value -0.794151
## iter  17 value -0.794151
## final  value -0.794151
## converged
## initial  value -0.787101
## iter   2 value -0.788459
## iter   3 value -0.790011
## iter   4 value -0.790812
## iter   5 value -0.791655
## iter   6 value -0.791907
## iter   7 value -0.791946
## iter   8 value -0.792014
## iter   9 value -0.792215
## iter  10 value -0.792420
## iter  11 value -0.792517
## iter  12 value -0.792534
## iter  13 value -0.792536
## iter  14 value -0.792536
## iter  15 value -0.792536
## iter  15 value -0.792536
## iter  15 value -0.792536
## final  value -0.792536
## converged
```

And the following coefficients:

```
##        phi1        phi2        phi3        phi4      theta1        Phi1
## -0.05851883 -0.06976959 -0.02473830 -0.11904962 -0.51345326 -0.01650475
##        Phi2        Phi3        Phi4        Phi5      Theta1
## -0.09926963  0.01414698 -0.11440507 -0.07581437 -0.82967466
```

The standardized residuals seems to be random, and there are not clear trend. The residuals seems homoscedastic and are centered at zero, which is a good indcation that our models assumptions are held by the data.

Looking at the autocorrolation plot, there does not seem to be any lag corrolating that much. Still, there are a few outliers, but that is often on 12+ months and can be considered to be well within the range of some family wise error rate.

Points in the qq-plot also seems to align with the normal line quite well, which indicate a good fit for our model.

All in all, the model seem to fit the data quite well.

Confidence interval for coefficients

We use bootstrapping to create confidence intervals for the coeffiecient. In using bootstraping we create a simulated data-set by drawing at random from our original data set, and fit a model using this. We repeat this, and in the end get a set of possible coefficient values, which we can use to confidence intervals.

For the coefficients we get the following bootstrapped confidence intervals (95%) using 200 samples:

```
##                   phi1        phi2        phi3       phi4     theta1
## lower 2.5% -0.1822252 -0.28560066 -0.27060160 -0.2218933 -0.7799848
## upper 2.5%  0.4505167 -0.05617251 -0.03621915 -0.0521043 -0.2397842
##                   Phi1        Phi2        Phi3        Phi4        Phi5
## lower 2.5% -0.1527267 -0.17364720 -0.12560727 -0.11921476 -0.10586522
## upper 2.5%  0.0596933  0.05461404  0.07821557  0.05451245  0.04177681
##                 Theta1
## lower 2.5% -0.9480253
## upper 2.5% -0.7909041
```

Forecasting and bootstrapping

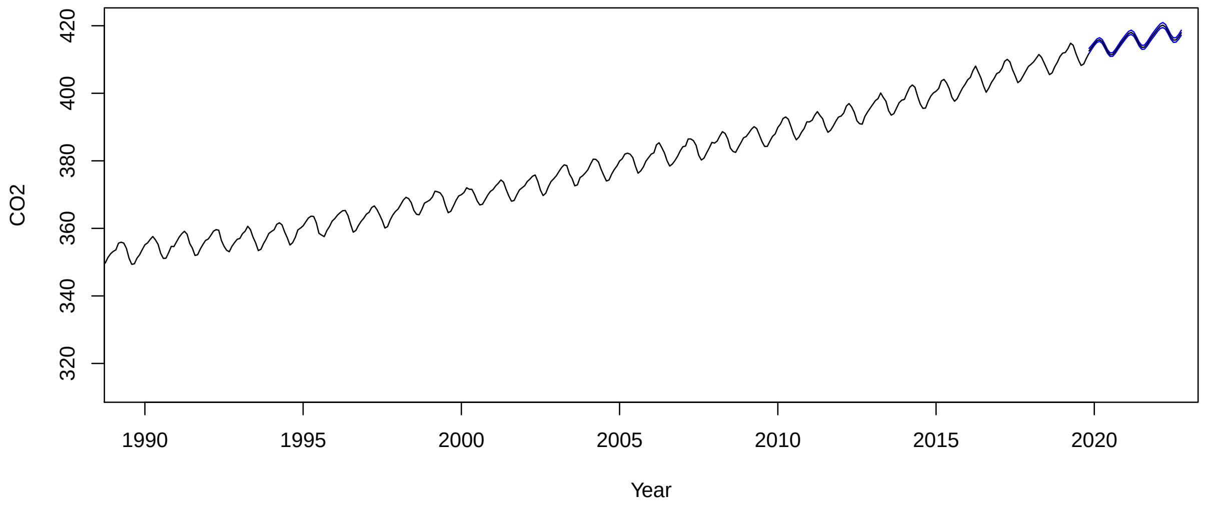
Bootstrapping can also be used to obtain prediction interval. The ones gained with traditional methods is often too narrow. (REFERENCE) We will use a a blocked bootstrap, we select adjunct sections the time series at random and join the selections together.

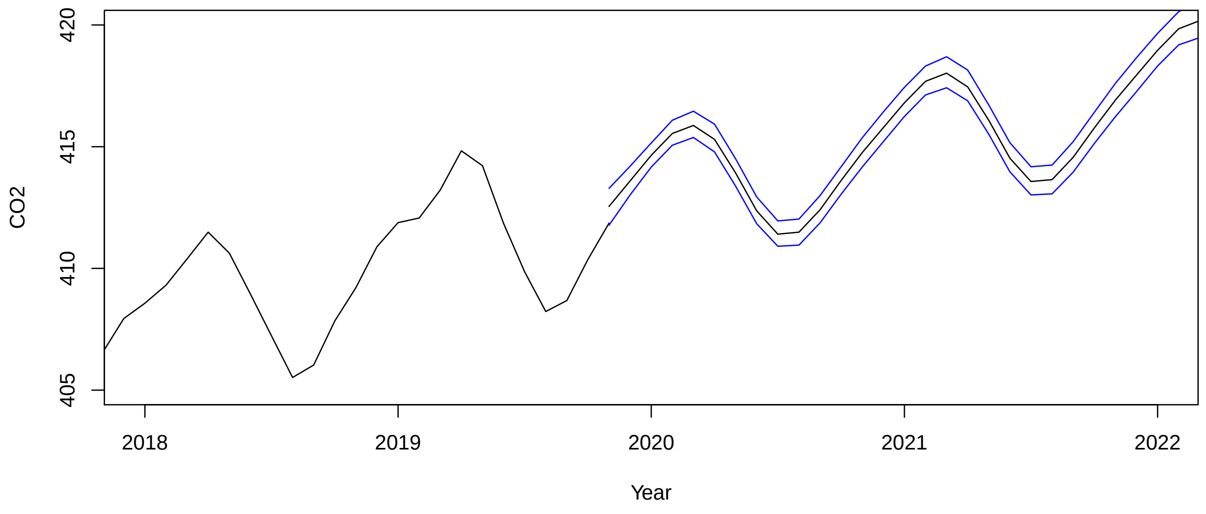
The method used in this section is adapted from (REFERENCE) chapter on bootstrapping and bagging.

```
## Error in autoplot(ts.bootstrap, colour = TRUE): object 'ts.bootstrap' not found
```

```
## $y
## [1] "Bootstrapped series"
##
## attr(,"class")
## [1] "labels"
```

We the fit different models, on 500 bootstrap samples:





Discussion

The model seem to predict an increase of the co2 levels in the next few years, and the data seems to fit the model assumptions. And seem to be quite confident on its predictions.

Conclusion

References