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Euler Theorem, De Moivre's theorem and Cauchy-Riemann Equation

Statement of Euler Theorem: Euler's law states that 'For any real number θ, $e^{i\theta} = \cos\theta + i\sin\theta$ where, e= base of natural logarithm

i= imaginary unit

θ = angle in radians

This complex exponential function is sometimes denoted cis θ ("cosine plus i sine"). The formula is still valid if θ is a complex number.

Let z be a non-zero complex number; we can write z in the polar form as, $z = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta}$, where $|z|$ is the modulus and θ is argument of z.

Mathematical Statement De Moivre's theorem: For any real number x, we have

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

OR

$$\left(e^{i\theta}\right)^n = e^{in\theta}$$

Where n is a positive integer and " i " is the imaginary part, and $i = \sqrt{(-1)}$. Also assume $i^2 = -1$.

Cauchy-Riemann Equation: If a function $f(z) = u + iv$ is complex differentiable, then its real and imaginary parts satisfy the following equations,

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}; \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}. \end{aligned} \quad \ldots\ldots\ldots \text{(i)}$$

This equation is known as the Cauchy-Riemann equation.

And the complex derivative $f'(z)$ is then given by

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Analytic function: If a function satisfied the Cauchy-Riemann equation, then the function is called analytic function.

Examples: Prove the function $f(z) = z^2$ satisfies the Cauchy-Riemann equations, and also find its complex derivatives.

Solution: Given that,

$$f(z) = z^2 \dots\dots\dots \text{(i)}$$

Here $z = x + iy$, so $z^2 = (x+iy)^2 = x^2 + 2xiy + (iy)^2 = x^2 + 2xyi - y^2 = x^2 - y^2 + 2xyi \dots\dots\dots$ (ii)

By comparing equation (ii) with $f(z) = u + iv$, we get

$$\begin{aligned} u &= x^2 - y^2; \\ v &= 2xy; \end{aligned} \dots\dots\dots \text{(iii)}$$

Now by partially differentiating with respect to *x* and *y*, we get

$$\begin{aligned} &\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2 - y^2) = 2x; \quad \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2 - y^2) = -2y; \\ &\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(2xy) = 2y; \qquad \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(2xy) = 2x; \end{aligned} \dots\dots\dots \text{(iv)}$$

From equation (iv), we have

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}; \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}. \end{aligned}$$

Which satisfied the C-R equation. (Proved)

And the complex derivative $f'(z)$ is

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + 2yi = 2(x+iy) = 2z$$

Or

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = 2x - (-2y)i = 2(x+iy) = 2z \text{. (Ans)}$$

Assignment

1. State the Euler Theorem and De Moivre's theorem.
2. Prove the following function satisfies the Cauchy-Riemann equations, and also find its complex derivatives.
   i. $f(z) = e^{z}$
   ii. $f(z) = 2z^{3} - 4z + 1; \quad z = x + iy$
   iii. $f(z) = \dfrac{\overline{z}}{|z|}; \quad z = x + iy$
3. Check each function for analyticity by using the Cauchy-Riemann equation,
   i. $f(z) = e^{z^{2}}$
   ii. $f(z) = |z|^{2}$
   iii. $f(z) = \overline{z}$
   iv. $f(z) = \dfrac{z}{|z|}; \quad z = x + iy$