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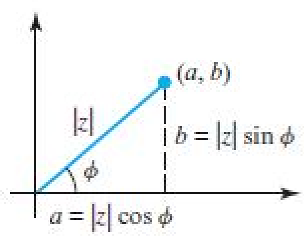
Different form of Complex Number

Polar Form of a Complex Number: If $z = a + ib$ is a nonzero complex number, and if φ is an angle from the real axis to the vector *z*, then, as suggested in given Figure , the real and imaginary parts of *z* can be expressed as

$$a = |z|\cos\varphi \quad \text{and} \quad b = |z|\sin\varphi$$



Thus the complex number $z = a + ib$ can be expressed as

$$z = a + ib = |z|\cos\varphi + i|z|\sin\varphi = |z|(\cos\varphi + i\sin\varphi)$$

$$\text{i.e. } z = |z|(\cos\varphi + i\sin\varphi) \quad \ldots\ldots\ldots (1)$$

which is called a ***polar form*** of *z*.

Argument: The angle φ in the equation (1) is called an ***argument*** of *z*. It can be find that, by using the formula,

$$\tan\varphi = \frac{b}{a} \Rightarrow \varphi = \tan^{-1}\left(\frac{b}{a}\right) \quad \ldots\ldots\ldots (2)$$

which is called a ***polar form*** of *z*. The angle φ in this formula is called an ***argument*** of *z*. The argument of *z* is not unique because we can add or subtract any multiple of 2π to it to obtain a different argument of *z*. However, there is only one argument whose radian measure satisfies

$$-\pi < \varphi < \pi$$

This is called the ***principal argument*** of *z*.

General Argument is defined by, $Arg(z) = 2n\pi + \varphi$.

Example: Express $z = 1 - \sqrt{3}i$ in polar form using the principal argument.

Solution: The modulus of z is

$$|z| = \sqrt{1^2 + \left(-\sqrt{3}\right)^2} = \sqrt{4} = 2$$

Since, $a = 1$ and $b = -\surd 3$, so that

$$1 = 2\cos\varphi \text{ and } -\surd 3 = 2\sin\varphi$$

and this implies that

$$\cos\varphi = \frac{1}{2} \text{ and } \sin\varphi = -\frac{\sqrt{3}}{2}$$

Thus the principal argument is

$$\varphi = \tan^{-1}\left(-\frac{\sqrt{3}}{2}\Big/\frac{1}{2}\right) = \tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$$

Thus, a polar form of z is

$$z = |z|\left(\cos\varphi + i\sin\varphi\right) = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) \quad \text{(ans)}$$

DeMoivre's Formula: If n is a positive integer, and if z is a nonzero complex number with polar form

$$z = |z|\left(\cos\varphi + i\sin\varphi\right)$$

then raising z to the nth power yields

$$z^n = \underbrace{z \cdot z \cdot \cdots \cdot z}_{n \text{ factors}} = |z|^n[\cos(\underbrace{\phi + \phi + \cdots + \phi}_{n \text{ terms}})] + i[\sin(\underbrace{\phi + \phi + \cdots + \phi}_{n \text{ terms}})]$$

which we can write more succinctly as

$$z^n = |z|^n\left(\cos n\varphi + i\sin n\varphi\right)$$

In the special case where $|z| = 1$ this formula simplifies to

$$z^n = \left(\cos n\varphi + i\sin n\varphi\right)$$

which, using the polar form for z, becomes

$$\left(\cos\varphi + i\sin\varphi\right)^n = \cos n\varphi + i\sin n\varphi$$

This result is called ***DeMoivre's formula***, named for the French mathematician Abraham de Moivre (1667–1754).

Example: If $z = 1 - \sqrt{3}i$, then evaluate z^3.

Solution: Given that,

$$z = 1 - \sqrt{3}i$$

The modulus of z is

$$|z| = \sqrt{1^2 + \left(-\sqrt{3}\right)^2} = \sqrt{4} = 2$$

Since, $a = 1$ and $b = -\sqrt{3}$, so that

$$1 = 2\cos\varphi \text{ and } -\sqrt{3} = 2\sin\varphi$$

and this implies that

$$\cos\varphi = \frac{1}{2} \text{ and } \sin\varphi = -\frac{\sqrt{3}}{2}$$

Thus the principal argument is

$$\varphi = \tan^{-1}\left(-\frac{\sqrt{3}}{2} \Big/ \frac{1}{2}\right) = \tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$$

Thus, a polar form of z is

$$z = |z|\left(\cos\varphi + i\sin\varphi\right) = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$$

Since, $z = |z|\left(\cos\varphi + i\sin\varphi\right) = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

Hence, $z^3 = |z|^3\left(\cos\varphi + i\sin\varphi\right)^3 = 2^3\left(\cos 3\varphi + i\sin 3\varphi\right)$

$$= 8\left(\cos\frac{3\pi}{3} - i\sin\frac{3\pi}{3}\right) = 8\left(\cos\pi - i\sin\pi\right) = 8\left((-1) - i.0\right) = -8 \text{ (ans)}$$

Euler's Formula: If θ is a real number, say the radian measure of some angle, then the ***complex exponential*** function $e^{i\theta}$ is defined to be

$$e^{i\theta} = \cos\theta + i\sin\theta$$

which is sometimes called ***Euler's formula***, named for the Swiss mathematician Leonhard Euler (1707–1783).

Multiplying both side by $|z|$, we get

$$|z|e^{i\theta} = |z|\left(\cos\theta + i\sin\theta\right)$$

Here, $|z|e^{i\theta}$ is known as the Euler form of a complex number.

Example: If $z = 1 - \sqrt{3}i$, then evaluate Euler form of z.

Solution: Given that,

$$z = 1 - \sqrt{3}i$$

The modulus of z is

$$|z| = \sqrt{1^2 + \left(-\sqrt{3}\right)^2} = \sqrt{4} = 2$$

Since, $a = 1$ and $b = -\sqrt{3}$, so that

$$1 = 2\cos\varphi \text{ and } -\sqrt{3} = 2\sin\varphi$$

and this implies that

$$\cos\varphi = \frac{1}{2} \text{ and } \sin\varphi = -\frac{\sqrt{3}}{2}$$

Thus the principal argument is

$$\varphi = \tan^{-1}\left(-\frac{\sqrt{3}}{2}\Big/\frac{1}{2}\right) = \tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$$

Thus, a polar form of z is

$$z = |z|\left(\cos\varphi + i\sin\varphi\right) = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$$

Thus, Euler form of z is

$$z = |z|\left(\cos\varphi + i\sin\varphi\right) = |z|e^{i\varphi} = 2e^{-\frac{i\pi}{3}} \text{ (Ans.)}$$

Assignment

1. Find the principle and general argument of the following complex number and also evaluate the Euler form.

   i. $z = -1 - \sqrt{3}i$ ii. $z = -\sqrt{3} - i$ iii. $z = \sqrt{2} + \sqrt{6}i$ iv. $z = -\sqrt{3}i$

2. Evaluate z^3, z^5, z^9 of the following complex number by using DeMoivre's formula,

   i. $z = i$ ii. $z = \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{\sqrt{2}}i$