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Fourier Transforms

Finite Fourier Sine transform: The finite Fourier sine transform of $F(x)$ in the interval $0 < x < l$, is defined as

$$f_s(n) = \int_0^l F(x)\sin\frac{n\pi x}{l}\,dx,$$ where n is an integer.

The function $F(x)$ is called the inverse finite Fourier sine transform of $f_s(n)$, and is given by

$$F(x) = \frac{2}{l}\sum_{n=1}^{\infty} f_s(n)\sin\frac{n\pi x}{l}.$$

Finite Fourier cosine transform: The finite Fourier cosine transform of $F(x)$ in the interval $0 < x < l$, is defined as

$$f_c(n) = \int_0^l F(x)\cos\frac{n\pi x}{l}\,dx,$$ where n is an integer.

The function $F(x)$ is called the inverse finite Fourier cosine transform of $f_c(n)$, and is given by

$$F(x) = \frac{1}{l}f_c(0) + \frac{2}{l}\sum_{n=1}^{\infty} f_c(n)\cos\frac{n\pi x}{l}.$$

Infinite Fourier Sine transform: The infinite Fourier sine transform of $F(x)$ in the interval $0 < x < \infty$, is defined as

$$f_s(n) = \int_0^\infty F(x)\sin nx\,dx,$$ where n is an integer.

The function $F(x)$ is called the inverse infinite Fourier sine transform of $f_s(n)$, and is given by

$$F(x) = \frac{2}{\pi}\int_0^\infty f_s(n)\sin nx\,dn.$$

Infinite Fourier cosine transform: The infinite Fourier cosine transform of $F(x)$ in the interval $0 < x < \infty$, is defined as

$$f_c(n) = \int_0^\infty F(x)\cos nx\,dx,$$ where n is an integer.

The function $F(x)$ is called the inverse infinite Fourier cosine transform of $f_c(n)$, and is given by

$$F(x) = \frac{2}{\pi}\int_0^\infty f_c(n)\cos nx dn .$$

Example-01: Evaluate the Fourier cosine transform of $\sin kx$, $0 < x < \pi$.

Solution: Given that,

$$F(x) = \sin kx \dots \dots \dots \text{(i)}$$

By the definition of Fourier cosine transform of $F(x)$ in the interval $0 < x < \pi$, we have

$$f_c(n) = \int_0^l F(x)\cos\frac{n\pi x}{l}dx$$

$$\Rightarrow f_c(n) = \int_0^\pi \sin kx \cos\frac{n\pi x}{\pi}dx$$

$$\Rightarrow f_c(n) = \int_0^\pi \sin kx \cos nx dx$$

$$\Rightarrow f_c(n) = \frac{1}{2}\int_0^\pi \left[\sin(k+n)x + \sin(k-n)x\right]dx$$

$$\Rightarrow f_c(n) = \frac{1}{2}\left[\frac{-\cos(k+n)x}{k+n} + \frac{-\cos(k-n)x}{k-n}\right]_0^\pi$$

$$\Rightarrow f_c(n) = -\frac{1}{2}\left[\left\{\frac{\cos(k+n)\pi}{k+n} + \frac{\cos(k-n)\pi}{k-n}\right\} - \left\{\frac{\cos 0}{k+n} + \frac{\cos 0}{k-n}\right\}\right]$$

$$\Rightarrow f_c(n) = -\frac{1}{2}\left[\left\{\frac{\cos k\pi\cos n\pi - \sin k\pi \sin n\pi}{k+n} + \frac{\cos k\pi \cos n\pi + \sin k\pi \sin n\pi}{k-n}\right\} - \left\{\frac{1}{k+n} + \frac{1}{k-n}\right\}\right]$$

$$\Rightarrow f_c(n) = -\frac{1}{2}\left[\left\{\frac{\cos k\pi\cos n\pi}{k+n} + \frac{\cos k\pi\cos n\pi}{k-n}\right\} - \left\{\frac{k-n+k+n}{(k+n)(k-n)}\right\}\right]$$

$$\Rightarrow f_c(n) = -\frac{1}{2}\left[\cos k\pi\cos n\pi\left\{\frac{1}{k+n} + \frac{1}{k-n}\right\} - \left\{\frac{2k}{k^2-n^2}\right\}\right]$$

$$\Rightarrow f_c(n) = -\frac{1}{2}\left[\cos k\pi\cos n\pi\left\{\frac{2k}{k^2-n^2}\right\} - \left\{\frac{2k}{k^2-n^2}\right\}\right]$$

$$\Rightarrow f_c(n) = -\frac{1}{2}\left\{\frac{2k}{k^2-n^2}\right\}[\cos k\pi \cos n\pi - 1]$$

$$\Rightarrow f_c(n) = \frac{k}{n^2-k^2}\left[(-1)^n \cos k\pi - 1\right]$$

$$\therefore f_c(n) = \frac{k}{n^2-k^2}\left[(-1)^n \cos k\pi - 1\right]. \text{ (Ans.)}$$

Example-02: Evaluate the Fourier sine transform of $e^{-x},\ 0 \le x$.

Solution: Given that,

$$F(x) = e^{-x} \ldots \ldots \ldots \text{(i)}$$

By the definition of Fourier sine transform of $F(x)$ for $0 \le x$, we have

$$f_s(n) = \int_0^\infty F(x)\sin nx dx$$

$$\Rightarrow f_s(n) = \int_0^\infty e^{-x}\sin nx dx$$

$$\Rightarrow f_s(n) = \left[\frac{e^{-x}}{(-1)^2+n^2}(-\sin nx - n\cos nx)\right]_0^\infty$$

$$\Rightarrow f_s(n) = -\left[\frac{e^{-x}}{1+n^2}(\sin nx + n\cos nx)\right]_0^\infty$$

$$\Rightarrow f_s(n) = -\left[0 - \frac{1}{1+n^2}(\sin 0 + n\cos 0)\right]$$

$$\Rightarrow f_s(n) = -\left[-\frac{n}{1+n^2}\right]$$

$$\therefore f_s(n) = \frac{n}{1+n^2}. \text{ (Ans.)}$$

Complex Fourier transform: The complex Fourier transform of $F(x)$ is defined as

$$f(n) = \int_{-\infty}^{\infty} F(x)e^{-inx}dx.$$

Example-01: Evaluate the complex Fourier transform of $F(x) = e^{-a|x|};\ a > 0.$.

Solution: Given that,

$$F\left(x\right)=e^{-a|x|}\ \ldots\ \ldots\ \ldots\ \text{(i)}$$

We have, $|x|=\begin{cases}-x & \text{if } x<0\\ x & \text{if } x>0\end{cases}$

By the definition of Fourier transform, we have

$$f\left(n\right)=\int_{-\infty}^{\infty}F\left(x\right)e^{-inx}dx$$

$$\Rightarrow f\left(n\right)=\int_{-\infty}^{\infty}e^{-a|x|}e^{-inx}dx$$

$$\Rightarrow f\left(n\right)=\int_{-\infty}^{0}e^{-a(-x)}e^{-inx}dx+\int_{0}^{\infty}e^{-ax}e^{-inx}dx$$

$$\Rightarrow f\left(n\right)=\int_{-\infty}^{0}e^{ax-inx}dx+\int_{0}^{\infty}e^{-ax-inx}dx$$

$$\Rightarrow f\left(n\right)=\int_{-\infty}^{0}e^{(a-in)x}dx+\int_{0}^{\infty}e^{-(a+in)x}dx$$

$$\Rightarrow f\left(n\right)=\left[\frac{e^{(a-in)x}}{(a-in)}\right]_{-\infty}^{0}-\left[\frac{e^{-(a+in)x}}{(a+in)}\right]_{0}^{\infty}$$

$$\Rightarrow f\left(n\right)=\frac{1}{(a-in)}\left[e^{0}-e^{-\infty}\right]-\frac{1}{(a+in)}\left[e^{-\infty}-e^{0}\right]$$

$$\Rightarrow f\left(n\right)=\frac{1}{(a-in)}-\frac{1}{(a+in)}(-1)$$

$$\Rightarrow f\left(n\right)=\frac{1}{(a-in)}+\frac{1}{(a+in)}$$

$$\Rightarrow f\left(n\right)=\frac{a+in+a-in}{(a-in)(a+in)}$$

$$\Rightarrow f\left(n\right)=\frac{2a}{a^{2}+n^{2}}$$

$$\therefore\ f(n) = \frac{2a}{a^2 + n^2}\text{ . (Ans.)}$$

Assignment

1. **Evaluate** the Fourier (sine/cosine) transform of the followings;
   - **i.** $F(x) = 2x; \quad 0 < x < 4$
   - **ii.** $F(x) = e^{mx}; \quad 0 < x < \pi$
   - **iii.** $F(x) = \sin mx; \quad 0 < x < \pi$
   - **iv.** $F(x) = \cos mx; \quad 0 < x < \pi$
2. **Evaluate** the Fourier (complex) transform of the followings;
   - **i.** $F(x) = e^{-|x|}; \quad -\infty < x < \infty$