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Laplace Equation and Harmonic Function

Laplacian: The second order partial differential operator is called the Laplacian. Mathematically it is defined by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Laplace's Equation in two dimensions: The second order partial differential equation is called the Laplace equation in two dimensions.

Mathematically it is defined by

$$\nabla^2 u = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

where, $u(x, y)$ is the real valued function.

Harmonic Function: Any function of two variables *x* and *y* is said to be harmonic on a domain D, if throughout D it has second order continuous partial derivatives and satisfies the Laplace's equation.

Example: Examine the following function *u* is harmonic or not, if it is harmonic then **Construct** the conjugate harmonic function *v*.

$$u(x, y) = 2x + 3xy^2 - x^3$$

Solution: Given that,

$$u(x, y) = 2x + 3xy^2 - x^3 \ldots \ldots \ldots \text{(i)}$$

Now partially differentiating (i) with respect to *x* and *y*, we get

$$\frac{\partial u}{\partial x} = u_x = \frac{\partial}{\partial x}\{u(x,y)\} = \frac{\partial}{\partial x}\left(2x+3xy^2-x^3\right) = 2+3y^2-3x^2 \ldots\ \ldots\ \ldots\ \text{(ii)}$$

$$\frac{\partial u}{\partial y} = u_y = \frac{\partial}{\partial y}\{u(x,y)\} = \frac{\partial}{\partial y}\left(2x+3xy^2-x^3\right) = 6xy \ldots\ \ldots\ \ldots\ \text{(iii)}$$

Again partially differentiating equation (ii) with respect to *x* and equation (iii) with respect to *y*, we get,

$$\frac{\partial^2 u}{\partial x^2} = u_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(2+3y^2-3x^2\right) = -6x \ldots\ \ldots\ \ldots\ \text{(iv)}$$

$$\frac{\partial^2 u}{\partial y^2} = u_{yy} = \frac{\partial}{\partial y}\left\{\frac{\partial u}{\partial y}\right\} = \frac{\partial}{\partial y}(6xy) = 6x \ldots\ \ldots\ \ldots\ \text{(v)}$$

Now by adding equation (iv) and (v), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6x+6x = 0$$

Which satisfies the Laplace's equation so the function is said to be a harmonic function.

For conjugate harmonic:

We have the C-R equation,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ldots\ \ldots\ \ldots\ \text{(vi)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \ldots\ \ldots\ \ldots\ \text{(vii)}$$

From (vi), we can write

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \frac{\partial v}{\partial y} = 2+3y^2-3x^2 \ \text{[Using (ii)]}$$

$$\Rightarrow \frac{\partial v}{\partial y} = 2+3y^2-3x^2$$

$$\Rightarrow \partial v = \int\left(2+3y^2-3x^2\right)\partial y$$

$$\Rightarrow v = 2y+3\frac{y^3}{3}-3x^2y+C(x)$$

$$\therefore v = 2y + y^3 - 3x^2y + C(x) \ldots \ldots \ldots \text{(viii)}$$

Partially differentiating equation (viii) w.r.t *x*, we get

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left\{2y + y^3 - 3x^2y + C(x)\right\} = -6xy + C'(x) \ldots \ldots \ldots \text{(ix)}$$

From equation (vii)

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow 6xy = -\left\{-6xy + C'(x)\right\} \text{ [Using (iii) and (ix)]}$$

$$\Rightarrow 6xy = 6xy - C'(x)$$

$$\Rightarrow C'(x) = 0$$

$$\Rightarrow \frac{dC(x)}{dx} = 0$$

$$\Rightarrow dC(x) = 0$$

$$\therefore C(x) = c \text{ [By integrating]}$$

Thus the conjugate harmonic function of u is

$$\therefore v = 2y + y^3 - 3x^2y + c \text{ (Ans.)}$$

Assignment

1. **Examine** the following function *u* is harmonic or not, if it is harmonic then **Construct** the conjugate harmonic function *v*. Also **Setup** the complex variable $f(z) = u + iv$.

   i. $u(x,y) = 2xy + 3xy^2 - 2y^3$

   ii. $u(x,y) = e^{-x}(x\cos y - y\sin y)$

   iii. $u(x,y) = \frac{1}{2}\ln(x^2 + y^2)$

   iv. $u(x,y) = xe^x\cos y - ye^x\sin y$

   v. $u(x,y) = e^{x^2-y^2}\cos 2xy$