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Basic Concept of Fourier analysis

Fourier analysis: In mathematics, Fourier analysis is the study of the way general functions may be represented or approximated by sums of simpler trigonometric functions. Fourier analysis grew from the study of Fourier series, and is named after Joseph Fourier, who showed that representing a function as a sum of trigonometric functions greatly simplifies the study of heat transfer.

Fourier Series

General form of Fourier series: Let $F(x)$ satisfy the following conditions;

- $F(x)$ is defined in the interval $-l < x < l$
- $F(x)$ and $F'(x)$ are sectionally continuous in $-l < x < l$
- $F(x+2l) = F(x)$, i.e. $F(x)$ is periodic with period $2l$.

Then at every point of continuity, we have

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \text{ ……… (i)}$$

Where,

$$a_0 = \frac{1}{l}\int_{-l}^{l} F(x)\,dx; \qquad a_n = \frac{1}{l}\int_{-l}^{l} F(x)\cos\frac{n\pi x}{l}\,dx; \qquad b_n = \frac{1}{l}\int_{-l}^{l} F(x)\sin\frac{n\pi x}{l}\,dx \text{ ……… (ii)}$$

Example: Construct the Fourier series of $F(x) = e^x; \quad -\pi < x < \pi$.

Solution: Given that

$$F(x) = e^x; \quad -\pi < x < \pi \text{ ……… (i)}$$

We have the Fourier series

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \text{ ……… (ii)}$$

Here $a_0 = \frac{1}{l}\int_{-l}^{l} F(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{\pi}\left[e^x\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left(e^{\pi} - e^{-\pi}\right)$ ……… (iii)

And $\quad a_n = \frac{1}{l}\int_{-l}^{l} F(x)\cos\frac{n\pi x}{l}dx$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos\frac{n\pi x}{\pi}dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nxdx$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{e^x}{1^2+n^2}(n\sin nx+\cos nx)\right]_{-\pi}^{\pi} \quad \left[\therefore \int e^{ax}\cos bxdx = \frac{e^{ax}}{a^2+b^2}(b\sin bx + a\cos bx)\right]$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{e^{\pi}}{1+n^2}(n\sin \pi n+\cos \pi n) - \frac{e^{-\pi}}{1+n^2}(n\sin(-\pi n)+\cos(-\pi n))\right]$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{e^{\pi}}{1+n^2}\left(0+(-1)^n\right) - \frac{e^{-\pi}}{1+n^2}\left(0+(-1)^n\right)\right]$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{(-1)^n e^{\pi}}{1+n^2} - \frac{(-1)^n e^{-\pi}}{1+n^2}\right]$$

$$\Rightarrow a_n = \frac{(-1)^n}{1+n^2}\frac{1}{\pi}\left(e^{\pi}-e^{-\pi}\right) \ \ldots\ldots\ldots \text{(iv)}$$

And $\quad b_n = \frac{1}{l}\int_{-l}^{l} F(x)\sin\frac{n\pi x}{l}dx$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin\frac{n\pi x}{\pi}dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nxdx$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[\frac{e^x}{1^2+n^2}(\sin nx-n\cos nx)\right]_{-\pi}^{\pi} \quad \left[\therefore \int e^{ax}\sin bxdx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[\frac{e^{\pi}}{1+n^2}(\sin \pi n-n\cos \pi n) - \frac{e^{-\pi}}{1+n^2}(\sin(-\pi n)-n\cos(-\pi n))\right]$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[\frac{e^{\pi}}{1+n^2}\left(0-n(-1)^n\right) - \frac{e^{-\pi}}{1+n^2}\left(0-n(-1)^n\right)\right]$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[\frac{-(-1)^n ne^{\pi}}{1+n^2} + \frac{(-1)^n ne^{-\pi}}{1+n^2}\right]$$

$$\Rightarrow b_n = -\frac{(-1)^n n}{1+n^2}\frac{1}{\pi}\left(e^{\pi} - e^{-\pi}\right) \text{ ... ... (v)}$$

Now using equations (iii), (iv) and (v) in equation (ii), we get

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$\Rightarrow F(x) = \frac{1}{2\pi}\left(e^{\pi} - e^{-\pi}\right) + \sum_{n=1}^{\infty}\left(\frac{(-1)^n}{1+n^2}\frac{1}{\pi}\left(e^{\pi} - e^{-\pi}\right)\cos\frac{n\pi x}{\pi} + \frac{-(-1)^n n}{1+n^2}\frac{1}{\pi}\left(e^{\pi} - e^{-\pi}\right)\sin\frac{n\pi x}{\pi}\right)$$

$$\Rightarrow F(x) = \frac{1}{2\pi}\left(e^{\pi} - e^{-\pi}\right) + \frac{\left(e^{\pi} - e^{-\pi}\right)}{\pi}\sum_{n=1}^{\infty}\left\{\frac{(-1)^n}{1+n^2}\right\}\left(\cos nx - n\sin nx\right)$$

$$\therefore F(x) = \frac{e^{\pi} - e^{-\pi}}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left\{\frac{(-1)^n}{1+n^2}\right\}\left(\cos nx - n\sin nx\right)\right] \text{ (Ans.)}$$

Complex form of Fourier series: In complex notation, the Fourier series and coefficients can be written as

$$F(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}} \text{ ... ... (i)}$$

Where $C_n = \frac{1}{2l}\int_{-l}^{l} F(x) e^{-\frac{in\pi x}{l}} dx$.

ODD and EVEN Functions:

- A function $F(x)$ is called odd if and only if $F(-x) = -F(x)$.

  Example: Let $F(x) = x^3$, then $F(-x) = (-x)^3 = -x^3 = -F(x)$; thus the function $F(x)$ is an odd function.

- A function $F(x)$ is called even if and only if $F(-x) = F(x)$.

  Example: Let $F(x) = x^2$, then $F(-x) = (-x)^2 = x^2 = F(x)$; thus the function $F(x)$ is an even function.

Fourier series of Odd and Even Function:

- If $F(x)$ is odd function of period $2l$, then the Fourier series is defined by

$$F(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \;\ldots\ldots\ldots\; (1)$$

Where $a_0=0$, $a_n=0$ and

$$b_n=\frac{2}{l}\int_0^l F(x)\sin\frac{n\pi x}{l}dx$$

This series (1) is called **half-range Fourier Cosine series**.

❖ If $F(x)$ is even function of period $2l$, then the Fourier series is defined by

$$F(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \;\ldots\ldots\ldots\; (2)$$

Where, $b_n=0$ and

$$a_0=\frac{2}{l}\int_0^l F(x)dx\,; \qquad a_n=\frac{2}{l}\int_0^l F(x)\cos\frac{n\pi x}{l}dx$$

This series (2) is called **half-range Fourier Cosine series.**

Example: Demonstrate the half-range Fourier cosine series of $F(x)=x^2; \quad -\pi<x<\pi$.

Solution: Given that,

$$F(x)=x^2; \quad -\pi<x<\pi \;\ldots\ldots\ldots\; \text{(i)}$$

Here, $F(-x)=(-x)^2=x^2=F(x)$ therefore $F(x)$ is an even function.

So, we have the half-range Fourier cosine series for even function as

$$F(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \;\ldots\ldots\ldots\; \text{(ii)}$$

Where,

$$a_0=\frac{2}{l}\int_0^l F(x)dx=\frac{2}{\pi}\int_0^{\pi} x^2dx=\frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}=\frac{2}{3\pi}\left(\pi^3-0\right)=\frac{2\pi^2}{3} \;\ldots\ldots\ldots\; \text{(iii)}$$

And $\quad a_n=\frac{2}{l}\int_0^l F(x)\cos\frac{n\pi x}{l}dx$

$$\Rightarrow a_n=\frac{2}{\pi}\int_0^{\pi} x^2\cos\frac{n\pi x}{\pi}dx$$

$$\Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx dx$$

$$\Rightarrow a_n = \frac{2}{\pi}\left[x^2\int \cos nx dx - \int\left\{\frac{d}{dx}\left(x^2\right)\int \cos nx dx\right\}dx\right]_0^{\pi}$$

$$\Rightarrow a_n = \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \int 2x\left(\frac{\sin xn}{n}\right)dx\right]_0^{\pi}$$

$$\Rightarrow a_n = \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} - \frac{2}{n}\int x \sin xn dx\right]_0^{\pi}$$

$$\Rightarrow a_n = \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} - \frac{2}{n}\left\{x\left(-\frac{\cos nx}{n}\right) + \int \frac{\cos nx}{n}dx\right\}\right]_0^{\pi}$$

$$\Rightarrow a_n = \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} - \frac{2}{n}\left\{x\left(-\frac{\cos nx}{n}\right) + \frac{1}{n}\int \cos nx dx\right\}\right]_0^{\pi}$$

$$\Rightarrow a_n = \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} - \frac{2}{n}\left\{x\left(-\frac{\cos nx}{n}\right) + \frac{1}{n}\frac{\sin nx}{n}\right\}\right]_0^{\pi}$$

$$\Rightarrow a_n = \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} + 2x\frac{\cos nx}{n^2} - 2\frac{\sin nx}{n^3}\right]_0^{\pi}$$

$$\Rightarrow a_n = \frac{2}{\pi}\left[\left\{\pi^2\frac{\sin n\pi}{n} + 2\pi\frac{\cos n\pi}{n^2} - 2\frac{\sin n\pi}{n^3}\right\} - \left\{0.\frac{\sin 0}{n} + 2.0.\frac{\cos 0}{n^2} - 2\frac{\sin 0}{n^3}\right\}\right]$$

$$\Rightarrow a_n = \frac{2}{\pi}\left[\left\{0 + \frac{2\pi}{n^2}(-1)^n - 0\right\} - \{0 + 0 - 0\}\right]$$

$$\Rightarrow a_n = (-1)^n\frac{4}{n^2} \;\dots\;\dots\;\dots\;\text{(iv)}$$

Now from equation (ii), we get

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} = \frac{1}{2}\frac{2\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2}\cos\frac{n\pi x}{\pi} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2}\cos nx$$

$$\therefore F(x) = \frac{\pi^2}{3} + 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \frac{\cos 4x}{4^2} - \dots\dots\dots\right]$$

Assignment

1. **Construct** the Fourier series of the following function:

   i. $F(x)=\begin{cases}0, & -\pi \le x \le 0 \\ x, & 0 \le x \le \pi\end{cases}$

   ii. $F(x)=\begin{cases}-1, & -\pi \le x \le 0 \\ +1, & 0 \le x \le \pi\end{cases}$

   iii. $F(x)=e^{mx}, \quad -\pi \le x \le \pi$

2. **Demonstrate** the half-range Fourier cosine/sine series of the following function:

   i. $F(x)=x; \quad -\pi \le x \le \pi$.

   ii. $F(x)=\sin mx; \quad -\pi \le x \le \pi$

   iii. $F(x)=\cos mx; \quad -\pi \le x \le \pi$

Application

Fourier analysis is a mathematical technique that is widely used in various fields, including software engineering. Fourier analysis is used to decompose a complex signal into its constituent frequencies. This technique is useful in software engineering because many software systems and applications involve the processing and analysis of signals, such as sound, images, and data.

Here are some specific applications of Fourier analysis in software engineering:

- Signal processing: Fourier analysis is widely used in signal processing, which involves the manipulation of signals to extract useful information. In software engineering, signal processing is often used in applications such as speech recognition, image processing, and data analysis. Fourier analysis is used to decompose the signal into its constituent frequencies, which can then be analyzed and manipulated.
- Compression: Fourier analysis is also used in data compression, which is the process of reducing the size of data without losing important information. Fourier analysis can be used to identify redundant or unnecessary information in a signal, which can then be removed to reduce the size of the data.
- Filtering: In software engineering, filtering is often used to remove noise or unwanted information from a signal. Fourier analysis can be used to identify the frequency components of the signal that contain the noise or unwanted information. A filter can then be applied to remove these components and improve the quality of the signal.
- Pattern recognition: Fourier analysis can be used in pattern recognition applications to analyze signals and identify patterns. For example, Fourier analysis can be used to analyze the frequency components of a sound signal and identify the specific sound or voice.

Overall, Fourier analysis is a powerful tool that is widely used in software engineering to analyze and manipulate signals. Its applications include signal processing, compression, filtering, and pattern recognition, among others.