Numerical Representations

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How Are Numbers Represented

It’s all 0s & 1s. How do you represent 123?

Binary Notation

$$(b_4 b_3 b_2 b_1 b_0)_2 = b_4 2^4 + b_3 2^3 + b_2 2^2 + b_1 2^1 + b_0 2^0 =$$
$$16 b_4 + 8 b_3 + 4 b_2 + 2 b_1 + b_0$$

Common Number Sizes

- Byte: 8 bits, 0 to 255 ($2^8 - 1$).
- Short: 16 bits, 0 to 65535 ($2^{16} - 1$).
- 32-bit int: 32 bits, 0 to 4294967295 ($2^{32} - 1$).
- 64-bit int: 64 bits, 0 to 18446744073709551615 ($2^{64} - 1$).

Bit-wise operations

1. NOT(A): true if A is not true (~A)
2. AND(A,B): true if A is true and B is true (A & B)
3. OR(A,B): true if either A or B are true (A | B)
4. XOR(A,B): true if one is true and the other is false A ^ B

```python
def fact(N):
    if N == 0: return 1
    return N * fact(N-1)

print  fact(100)
```

Prints out
93326215443944152681699238856266700490715968264381621468592963895217599993229915608941463976156518286253697920827223758251185210916864000000000000000000000000L

What about negative numbers?

- Sign bit
- Biasing
- Ones' complement
- Twos' complement

Sign Bit

$$(sb_4b_3b_2b_1b_0)_2 = (-1)^s \left(b_4 2^4 + b_3 2^3 + b_2 2^2 + b_1 2^1 + b_0 2^0\right)$$

Biasing

Have a bias B, so that the number n is representated as unsigned($n + B$).

One's Complement

If $(b_k b_{k-1} \cdots b_1 b_0)_2$ is some number n, then we represent $-n$ by $(\tilde{b}_k \tilde{b}_{k-1} \cdots \tilde{b}_1 \tilde{b}_0)_2$ is some number n, then we represent $-n$ by

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$(00000011)_2$ is 3
$(11111100)_2$ is -3

$(00001111)_2$ is 31
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$(00000011)_2$ is 3
$(11111100)_2$ is -3

$(00001111)_2$ is 31
$(11110000)_2$ is -31

$(00000000)_2$ is 0
$(11111111)_2$ is -0

One's Complement

If $(b_k b_{k-1} \cdots b_1 b_0)_2$ is some number n, then we represent $-n$ by $(\tilde{b}_k\tilde{b}_{k-1} \cdots \tilde{b}_1\tilde{b}_0)_2$ is some number n, then we represent $-n$ by

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$(11111111)_2$ is -0

Ones' complement is not actually used in any modern machine.

Twos' Complement

Image from Wikipedia
Metaphor from Steve Heller

Twos' Complement

$(11111111)_2$ is -1
$(11111110)_2$ is -2
$(11111101)_2$ is -3

Ranges

- 8 bits: -128 to 127.
- 16 bits: -32768 to 32767.
- 32 bits: -2147483648 to 2147483647.
- 64 bits: -9223372036854775808 to 9223372036854775807.

Fractional Numbers

What about fractional numbers?

- Fixed point
- Floating point

Fixed Point

Given a fixed base B, then an integer n really represents the number $n * 2^B$.

Floating Point

60221417930303030303030303

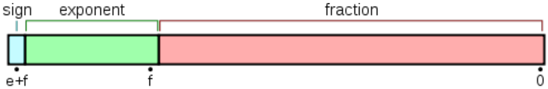
Floating Point

602214179303030303030303

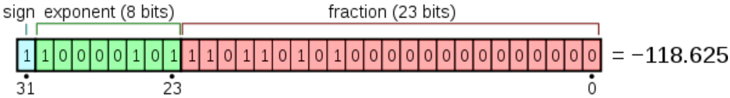
$6.022 * 10^{23}$

Floating Point Representation

$$(-1)^s * c * 2^q$$



IEEE-754



sign
exponent (8 bits)
fraction (23 bits)
1 1 0 0 0 0 1 0 1 1 1 0 1 1 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0
= −118.625
31
23
0

IEEE-754 Formats

- 32-bit floats: 1 sign bit, 23 bit fraction, 8 bit exponent.
- 64-bit floats: 1 sign bit, 52 bit fraction, 11 bit exponent.
- Non-standard 80-bit floats: 1 sign bit, 64 bit fraction, 15 bit exponent.

Ranges

- 32-bit float: $\pm 1.1810^{-38}$ to $\pm 3.410^{38}$.
- 64-bit float: $\pm 210^{-308}$ to $\pm 1.8^{308}$.

Limited Precision

```
print 0.3 * 3
print (0.3 * 3) == .9
```

prints

```
.9
False
```

```
print 1.1 * 0 == 0.0
print 1.1 * 1 == 1.1
print 1.1 * 2 == 2.2
print 1.1 * 3 == 3.3
print 1.1 * 4 == 4.4
print 1.1 * 5 == 5.5
print 1.1 * 6 == 6.6
print 1.1 * 7 == 7.7
print 1.1 * 8 == 8.8
print 1.1 * 9 == 9.9
print 1.1 *10 == 11
```

```
print 1.1 * 0 == 0.0      #  True
print 1.1 * 1 == 1.1      #  True
print 1.1 * 2 == 2.2      #  True
print 1.1 * 3 == 3.3      #  False
print 1.1 * 4 == 4.4      #  True
print 1.1 * 5 == 5.5      #  True
print 1.1 * 6 == 6.6      #  False
print 1.1 * 7 == 7.7      #  False
print 1.1 * 8 == 8.8      #  True
print 1.1 * 9 == 9.9      #  True
print 1.1 *10 == 11       #  True
```

You never compare two floating-point numbers for equality!

```python
x = 0.0
while x < big_number:
    ... # x is unchanged in here!
    x += 1.
```

Can this go into an infinite loop?

```
x = 0.0
while x < big_number:
    ... # x is unchanged in here!
    x += 1.
```

Can this go into an infinite loop?
Yes, it can!

Overflow & Underflow

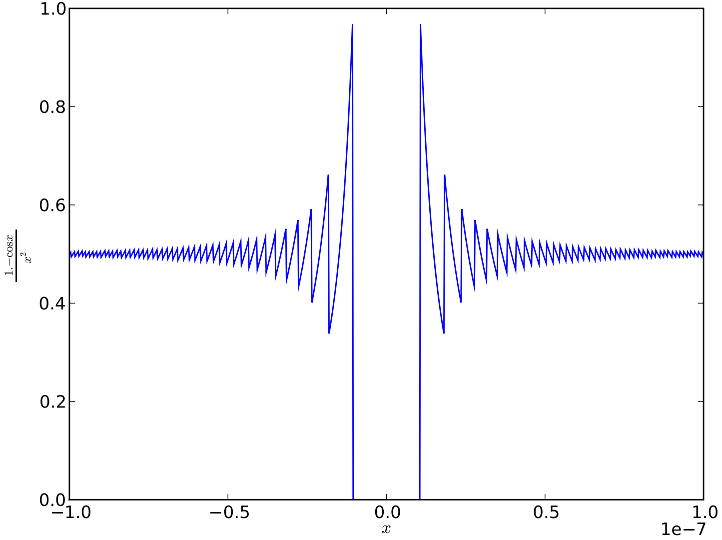
When numbers are too big, we say they overflow.
When they are too small, we say they underflow.

Catastrophic Cancellation

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2}$$

(Example from “Introduction to Programming in Java”)

Catastrophic Cancellation



Be Careful

- Use existing implementations of algorithms instead of rolling your own.
- Don't trust your instincts.

Some Special Numbers

- -0: minus zero.
- $\pm\infty$
- NaN: Not a Number

NaN

```
A = float('NaN')
```

```
print A == A
```

prints False!!