Operations Research - Lecture 1

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- **Operations** (or **Operational**) **Research** (**OR**) is an interdisciplinary branch of (applied) mathematics originated in UK during World War II;
- It started with the development of a radar defense system for the Royal Air Force;
- The term Operations Research is attributed to a RAF official after the initiation of teams to do *operational researches/analysis* on the communication system and the control of a radar station;
- Some believe that Charles Babbage (1791-1871) is the founding father of **OR** because he did researches on sorting and transportation of mail in England (which established the modern postal system - Penny Post in England).

- The new approach of picking an "operational" system and conducting "research" on how to make it run more efficiently soon started to expand into other arenas of the war.
- **OR** grew rapidly as many scientists realized that the principles that they had applied to solve problems for the military were equally applicable to many problems in the civilian sector.
- After the war, the ideas advanced in military operations were adapted to improve efficiency and productivity in the civilian sector.

- **OR** *is a discipline that deals with the application of advanced analytic methods to help make better decisions. Employing techniques from other mathematical sciences, such as mathematical modeling, statistical analysis, and mathematical optimization,* **OR** *arrives at optimal or near-optimal solutions to complex decision-making problems.* (INFORMS site)
- **OR**, *application of scientific methods to the management and administration of organized military, governmental, commercial, and industrial processes.* (Encyclopædia Britannica)

To take advantage of the **OR** usefulness one must

- learn the *standard mathematical models* and *OR techniques*
- determine the solutions by using *algorithms*,
- understand innovative methods and practical issues related to the *use and development of computer implementations*.

Students need to know how to

- identify problems that the methods of **OR** can solve;
- structure the problems into standard mathematical models;
- apply or/and develop computational tools to solve the problems.

The aim of our **OR** course is to give the master student:

- a good foundation in the mathematics of **OR** and
- an appreciation of its potential applications.

The principal phases for implementing OR in practice include:

1. **Definition of the problem.**
2. **Construction of the model.**
3. **Model solution.**
4. **Validation of the model.**
5. **Implementation of the solution.**

1. **Definition of the problem.** Requires answering questions like:
   - What are the decision **alternatives**?
   - Under what **restrictions** is the decision made?
   - What is an appropriate **objective function** for evaluating the alternatives?

2. **Construction of the model.** The resulted model can fit one of the standard mathematical models or, if the model is too complex, it can be simplified using an heuristic approach, simulation, or a combination of these.

3. Model solution.

- This phase entails the use of well-defined optimization algorithms. A solution of the model is **feasible** if it satisfies all the constraints and is called **optimal** if, in addition of being feasible, it yields the best (maximum or minimum) value of the objective function.
- Another aspect of this phase is **sensitivity analysis**: obtaining additional information about the behavior of the optimum solution when the model undergoes some parameter changes.

4. Validation of the model. Model validity checks whether (or not) the proposed model does what it purports to do. As a common method: the model is valid if, under similar input conditions, it reasonably duplicates past performance. (We may use simulation as an independent tool for verifying the output of the mathematical model, if no historical data are available.)

5. Implementation of the solution. The translation of the results into understandable operating instructions to be issued to the people who will administer the recommended system.

The most prominent of the OR techniques is linear programming or LP. LP was designed for models with linear objective function and linear constraints; integer programming is a technique for LP problems in which some variables assume integer values.

- Linear Programming algorithms: Simplex/Dual Simplex.
- Integer Programming algorithms: Branch and Bound (B&B) algorithm, Cutting Plane algorithm.
- Interior Point algorithm is a general method for solving LP problems.

The simplex method was invented and developed by George Dantzig in 1947, based on his work for the U.S. Air Force. Even earlier, in 1939, L. V. Kantorovich (who was charged with the reorganization of the timber industry in the U.S.S.R.), formulated a restricted class of linear programs and a method for finding their solution.

Following *sciencing.com*:

- Food and Agriculture:
  - Farmers apply linear programming techniques to their work. By determining what crops they should grow, the quantity of it and how to use it efficiently, farmers can increase their revenue.
  - In nutrition, linear programming provides a powerful tool to aid in planning for dietary needs. In order to provide healthy, low-cost food baskets for needy families, nutritionists can use linear programming.
- Applications in Engineering:
  - Engineers also use linear programming to help solve design and manufacturing problems.
  - For example, in airfoil meshes, engineers seek aerodynamic shape optimization. This allows for the reduction of the drag coefficient of the airfoil.

- Transportation Optimization:
  - Transportation systems rely upon linear programming for cost and time efficiency.
  - Bus and train routes must factor in scheduling, travel time and passengers. Airlines use linear programming to optimize their profits according to different seat prices and customer demand. Airlines also use linear programming for pilot scheduling and routes.
- Efficient Manufacturing:
  - Manufacturing requires transforming raw materials into products that maximize company revenue. Each step of the manufacturing process must work efficiently to reach that goal.
  - For example, raw materials must past through various machines for set amounts of time in an assembly line.

- Energy Industry:
  - Linear programming provides a method to optimize the electric power system design.
  - It allows for matching the electric load in the shortest total distance between generation of the electricity and its demand over time.
- *Delta Airlines* uses linear and integer programming in its Coldstart project to solve its fleet assignment problem. The problem is to match aircraft to flight legs and fill seats with paying passengers.
- *LibbeyOwens Ford* utilizes a large-scale linear programming model to achieve integrated production, distribution and inventory planning for its glass products. Schedulers and planners in the flat glass products group must coordinate production schedules for more than 200 different glass products.

- Some of the *general theoretical applications* of Linear/Integer Programming: the transportation model and its variants, the assignment model, the transshipment model, network models (The shortest-route problem, the maximal flow model, the critical path method (CPM)), the set-covering problem, the fixed-charge problem, the traveling salesperson (TSP) problem, capital budgeting.
- *Problems of interest to computer scientists* where linear/integer programming can be fruitful applied: maximum flow, rank aggregation, combinatorial (reverse) auctions, Markov decision processes, multi-agent systems, secret sharing schemes, linear time secure cryptography.

A manager of an oil refinery has 8 million barrels of crude oil A and 5 millions barrels of crude oil B allocated for production during the coming month. These resources can be used to make either gasoline, which sells for \$38 per barrel, or home heating oil, which sells for \$35 per barrel. There are three production processes with the following characteristics

	Process 1	Process 2	Process 3
input crude A	3	1	5
input crude B	5	1	3
output gasoline	4	1	3
output heating oil	3	1	4
Cost (\$)	51	11	40

All quantities are in barrels. Formulate a linear programming problem that would help the manager to maximise net revenue over the next month.

The corresponding LP (OR) model has three basic components:

1. the decision variables that we see to determine:

- $x_1 =$ barrels of crude oil A used in the first process;
- $x_2 =$ barrels of crude oil A used in the second process;
- $x_3 =$ barrels of crude oil A used in the third process;

From these values the quantities of crude oil B used are easily computed:

- $\frac{5}{3}x_1 =$ barrels of crude oil B will be used in the first process;
- $\frac{1}{1}x_2 =$ barrels of crude oil B will be used in the second process;
- $\frac{3}{5}x_3 =$ barrels of crude oil B will be used in the third process;

2. **the objective (goal) that we need to optimize:**
   - the revenue: $38 \cdot \left(\frac{4}{3}x_1 + \frac{1}{1}x_2 + \frac{3}{5}x_3\right) + 35 \cdot \left(\frac{3}{3}x_1 + \frac{1}{1}x_2 + \frac{4}{5}x_3\right) = \frac{257}{3}x_1 + 73x_2 + \frac{254}{5}x_3$.
   - the costs: $\frac{51}{3}x_1 + 11x_2 + \frac{40}{5}x_3 = 17x_1 + 11x_2 + 8x_3$
   - letting z represent the total profit (in \$), the objective will be
   - to maximize $z = \frac{206}{3}x_1 + 62x_2 + \frac{214}{5}x_3$.
3. **the constraints that the solution must satisfy:**
   - restrictions associated to current month availabilities of crude oil: $x_1 + x_2 + x_3 \leqslant 8,000,000$ (crude oil A); $\frac{5}{3}x_1 + x_2 + \frac{3}{5}x_3 \leqslant 5,000,000$ (crude oil B).
   - The non-negativity restrictions: $x_1 \geqslant 0$, $x_2 \geqslant 0$, $x_3 \geqslant 0$.

The complete model becomes:

$$
\begin{aligned}
\text{maximize} \quad & z = \frac{206}{3}x_1 + 62x_2 + \frac{214}{5}x_3 \\
\text{subject to} \quad & \\
& x_1 + x_2 + x_3 \leqslant 8,000,000 \\
& \frac{5}{3}x_1 + x_2 + \frac{3}{5}x_3 \leqslant 5,000,000 \\
& x_1, x_2 \geqslant 0
\end{aligned}
$$

A company produces two kind of products. A product of the first type requires 1/4 hours of assembly labor, 1/8 hours of testing, and \$1.2 worth of raw materials. A product of the second type 1/3 hours of assembly, 1/3 hours of testing, and \$0.9 worth of raw materials. Given the existing work force, there can be at most 90 hours of assembly labor and 80 hours of testing, each day. Products of the first and second type have a market value of \$9 and \$8, respectively.

(i) Formulate a linear programming problem that can be used to maximize the daily profit of the company.

(ii) Consider the following two modifications to the original problem

  (1) Suppose that up to 50 hours of overtime assembly labor can be scheduled, at a cost of \$7 per hour.

  (2) Suppose that the raw material supplier provides a 10% discount if the daily bill is above \$300.

Which of the above two elements can be easily incorporated into the linear programming formulation and how? If one or both are not easily to incorporate, indicate how you might nevertheless solve the problem.

1. the decision variables of the model are:

- x_1 = quantity of the first product;
- x_2 = quantity of the second product.

2. the objective is to maximize the profit which is the difference between the revenues and costs:

- revenues: $9x_1 + 8x_2$;
- costs: $1.2x_1 + 0.9x_2$.
- the objective is to maximize $z = (9x_1 + 8x_2) - (1.2x_1 + 0.9x_2) = 7.8x_1 + 7.1x_2$.

3. **the constraints are:**

- associated with the assembly time: $\frac{1}{4}x_1 + \frac{1}{3}x_2 \leqslant 90$;
- associated with the testing time: $\frac{1}{8}x_1 + \frac{1}{3}x_2 \leqslant 80$.
- non-negativity restrictions: $x_1 \geqslant 0$, $x_2 \geqslant 0$.

The complete model will be:

$$\begin{array}{ll} \text{maximize} & z = 7.8x_1 + 7.1x_2 \\ \text{subject to} & \\ & \begin{aligned} 3x_1 + 4x_2 &\leqslant 1,080 \\ 3x_1 + 8x_2 &\leqslant 1,920 \\ x_1, x_2 &\geqslant 0 \end{aligned} \end{array}$$

The first modification can be easily integrated in the linear model:

- the costs increase with \$350, therefore the objective becomes: maximize $z = 7.8x_1 + 7.1x_2 - 350$.
- the constraint associated with the assembly time becomes $\frac{1}{4}x_1 + \frac{1}{3}x_2 \leqslant 140$;

The complete model becomes:

$$
\begin{aligned}
\text{maximize} \quad & z = 7.8x_1 + 7.1x_2 - 350 \\
\text{subject to} \quad & \\
& \begin{aligned} 3x_1 + 4x_2 &\leqslant 1,680 \\ 3x_1 + 8x_2 &\leqslant 1,920 \\ x_1, x_2 &\geqslant 0 \end{aligned}
\end{aligned}
$$

Company Products - The Model

Considerations for the second modification of the original problem:

- if the bill for the raw material exceeds \$300, i.e., if $1.2x_1 + 0.9x_2 \geqslant 300$, then the costs with raw material will be $0.9(1.2x_1 + 0.9x_2) = 1.08x_1 + 0.81x_2$;
- a possible[1] solution may be to solve the original problem, and, if the raw material bill exceeds \$ 300 for the optimal solution, solve the problem with a new objective: maximize $z = 7.92x_1 + 7.19x_2$.

The complete model becomes:

$$
\begin{array}{ll}
\text{maximize} & z = 7.92x_1 + 7.19x_2 \\
\text{subject to} & \\
& \begin{aligned} 3x_1 + 4x_2 &\leqslant 1,080 \\ 3x_1 + 8x_2 &\leqslant 1,920 \\ x_1, x_2 &\geqslant 0 \end{aligned}
\end{array}
$$

[1] Is this an optimal solving of the modified problem?

Diet Problem [Taha07]

Ozarks Farms uses at least 800 lb of special feed daily. The special feed is a mixture of corn and soybean meal with the following compositions:

	lb per lb of feed-stuff		
Feed-stuff	Protein	Fiber	Cost ($/lb)
Corn	0.09	0.02	0.30
Soybean meal	0.60	0.06	0.90

The dietary requirements of the special feed are at least 30% protein and at most 5% fiber. Ozark Farms wishes to determine the daily minimum cost feed mix.

1. **the decision variables of the model are:**
   - x_1 = lb of corn in the daily mix;
   - x_2 = lb of soybean meal in the daily mix.
2. **the objective:**
   - letting z represent the cost for lb of special feed, the objective of the company is
   - to minimize $z = 0.3x_1 + 0.9x_2$.

3. **the constraints are:**

- the dietary requirements: $x_1 + x_2 \geqslant 800$;
- the protein dietary requirement: $0.09x_1 + 0.6x_2 \geqslant 0.3(x_1 + x_2)$;
- the fiber dietary requirement: $0.02x_1 + 0.06x_1 \leqslant 0.05(x_1 + x_2)$.
- non-negativity restrictions: $x_1 \geqslant 0$, $x_2 \geqslant 0$.

The complete model is:

$$\begin{aligned} \text{minimize} \quad & z = 0.3x_1 + 0.9x_2 \\ \text{subject to} \quad & \\ & x_1 + x_2 \geqslant 800 \\ & 0.21x_1 - 0.3x_2 \leqslant 0 \\ & 0.03x_1 - 0.01x_2 \geqslant 0 \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

Reddy Mikks Company ([Taha07]) produces both interior and exterior paints from two raw materials, M_1 and M_2

	Tons of raw material/ton of exterior paint	interior paint	Maximum daily availability (tons)
Raw material M_1	6	4	24
Raw material M_2	1	2	6
Profit per ton ($1000)	5	4	

A market survey indicates that the daily demand for interior paint cannot exceed that for exterior paint by more than one ton. Also, the maximum daily demand for interior paint is 2 tons. Reddy Mikks wants to determine the optimal (best) product mix of interior and exterior paints that maximizes the total daily profit.

The correspondingly LP (OR) model has three basic components:

1. the decision variables that we see to determine:
   - x_1 = tons produced daily of exterior paint;
   - x_2 = tons produced daily of interior paint.
2. the objective (goal) that we need to optimize:
   - if z represent the total daily profit, the objective of the company is
   - to maximize $z = 5x_1 + 4x_2$[2].

[2] Thousands of $.

3. the constraints that the solution must satisfy:
   - restrictions associated to daily availabilities of M_1 and M_2:
     $6x_1 + 4x_2 \leqslant 24$ (Raw material M_1);
     $x_1 + 2x_2 \leqslant 6$ (Raw material M_2).
   - market restrictions:
     $x_2 - x_1 \leqslant 1$ (market limit);
     $x_2 \leqslant 2$ (demand limit).
   - non-negativity restrictions:
     $x_1 \geqslant 0, x_2 \geqslant 0.$

The complete linear model:

$$
\begin{aligned}
\text{maximize} \quad & z = 5x_1 + 4x_2 \\
\text{subject to} \quad & \\
& 6x_1 + 4x_2 \leqslant 24 \\
& x_1 + 2x_2 \leqslant 6 \\
& x_2 - x_1 \leqslant 1 \\
& x_2 \leqslant 2 \\
& x_1, x_2 \geqslant 0
\end{aligned}
$$

- Linear programs can be solved **geometrically** or **algebraically**, both approaches being equivalent.
- Geometrically solving is based on the geometry of the possible solutions set (feasible region) - this set is a convex one in a certain euclidean space.
- One of the advantages of this approach is that the form of the constraints does not influence the process of solving; on the other hand the algebraic approaches is heavily based on specific form of the constraints.
- Using geometry, many of the central concepts in linear programming become easier to understand.
- The only clear disadvantage is that the geometric method successfully applies only in *two dimensions*.

Consider the following linear program

$$
\begin{aligned}
\text{maximize} \quad & z = x_1 + 2x_2 \\
\text{subject to} \quad & \\
& 2x_1 + x_2 \leqslant 12 \\
& x_1 + x_2 \geqslant 5 \\
& -x_1 + 3x_2 \geqslant 3 \\
& 6x_1 - x_2 \geqslant 12 \\
& x_1, x_2 \geqslant 0
\end{aligned}
$$

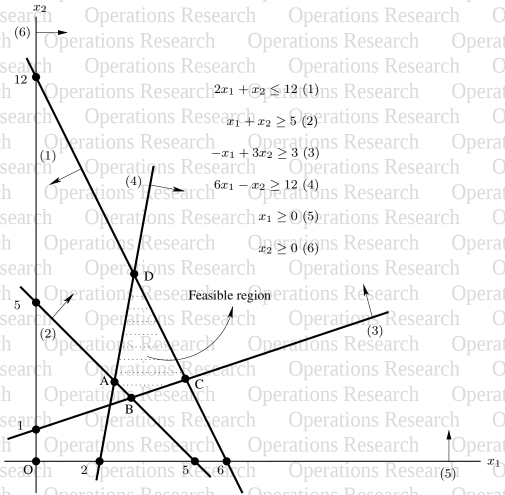

Figure: Feasible Region

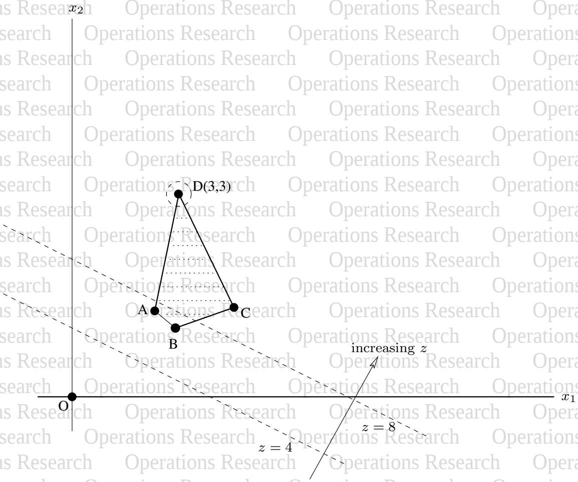

Figure: Optimal Solution

Consider now one of the previous examples (Diet Problem):

$$\begin{aligned} &\text{minimize} \quad z = 0.3x_1 + 0.9x_2 \\ &\text{subject to} \\ &\qquad x_1 + x_2 \geqslant 800 \\ &\qquad 0.21x_1 - 0.3x_2 \leqslant 0 \\ &\qquad 0.03x_1 - 0.01x_2 \geqslant 0 \\ &\qquad x_1, x_2 \geqslant 0 \end{aligned}$$

Or, equivalently,

$$\begin{aligned} &\text{minimize} \quad z = 3x_1 + 9x_2 \\ &\text{subject to} \\ &\qquad x_1 + x_2 \geqslant 800 \\ &\qquad 7x_1 - 10x_2 \leqslant 0 \\ &\qquad 3x_1 - x_2 \geqslant 0 \\ &\qquad x_1, x_2 \geqslant 0 \end{aligned}$$

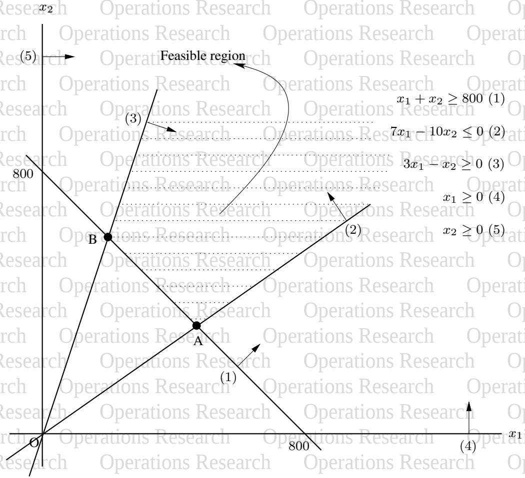

Figure: Feasible Region

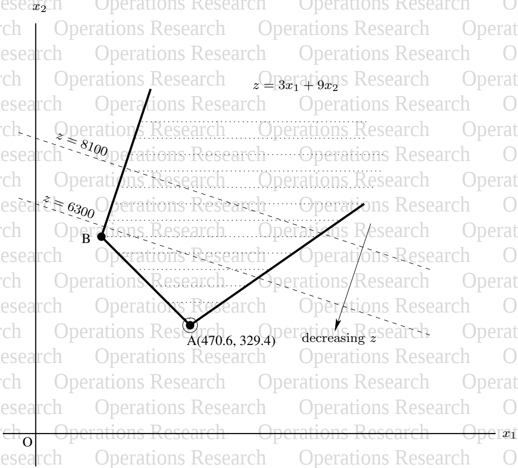

Figure: Optimal Solution

Consider now the Reddy Mikks example:

$$
\begin{aligned}
\text{maximize} \quad & z = 5x_1 + 4x_2 \\
\text{subject to} \quad & \\
& 6x_1 + 4x_2 \leqslant 24 \\
& x_1 + 2x_2 \leqslant 6 \\
& x_2 - x_1 \leqslant 1 \\
& x_2 \leqslant 2 \\
& x_1, x_2 \geqslant 0
\end{aligned}
$$

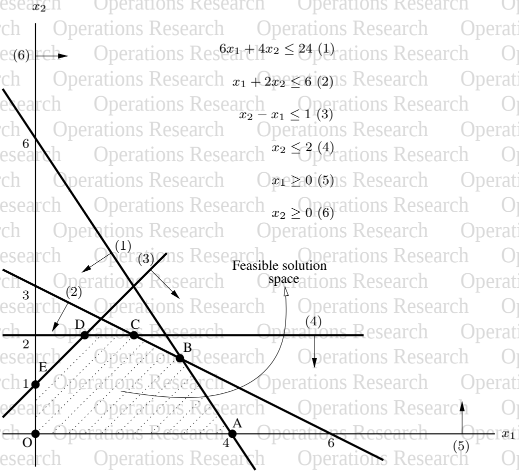

Figure: Feasible Region

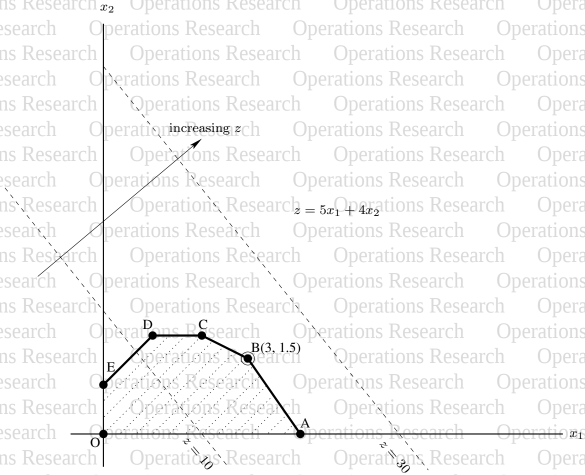

Figure: Optimal Solution

The graphical procedure includes two steps:

1. Determination of the **feasible region** or **feasible solution space**.
   - the non-negativity of the variables restricts the solution-space area to the *first quadrant* (the so-called positive orthant);
   - for the remaining constraints:
     - first replace each inequality with an equation and then graph the resulting line;
     - next, consider the effect of the inequality: the line divides the plane into two *half-planes* or *semi-planes* and only one of these two halves satisfies the inequality (we use an arrow to point to the feasible semi-plane);
     - the intersection of all these semi-planes gives the feasible region.

2. Finding of the optimal solution from among all the feasible points in the solution space.

- first, identify the *direction* in which the profit function *improves* (increases for maximizing z, or decreases for minimizing z);
- we can do so by assigning two arbitrary values to z, which is equivalent to graphing two lines;
- the optimal solution occurs at a *corner* (a point in the feasible region) beyond which any further improvement will put z outside the feasible region.

- An important characteristic of the optimum LP solution (if any) is that it is always associated with a *corner* or *extreme point* of the feasible region (where two or more lines intersect).
- This is true even if the objective function happens to be parallel to a constraint-line (in which case it is possible that any point on that line segment will be an alternative optimum, but the important observation here is that the line segment is totally defined by its corner points).

- We will prove that the geometric approach is equivalent to the algebraic one. In order to do this we will describe first particular (*standard* and *canonical*) forms of the constraints.
- Standard forms will be used to define a *basic feasible solution*; the algebraic notion of a basic feasible solution is equivalent to the geometric notion of an extreme point.
- This is of great value because, in higher dimensions, basic feasible solution are easier to generate than extreme points. In this way we can see why the algebraic method is more practical than the geometric one.

- It will be shown that any feasible solution (or feasible point) can be represented in terms of basic feasible solutions (extreme points). This leads to show that any linear program with a finite optimal solution has an optimal extreme point.
- This last result will greatly motivate the introduction of the *simplex algorithm*: a method that solves a linear program by examining basic feasible solutions (that is, extreme points) one by one, until an optimal one is found.

Algebra background

- A *matrix* of dimensions $m \times n$ is an array of numbers a_{ij}:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Its (i, j)-th entry is a_{ij} or $[\mathbf{A}]_{ij}$. $\mathbf{A}_j$ it is the jth *column* of matrix $\mathbf{A}$, and $\mathbf{a}'_i$ is its ith *row*:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \cdots & \mathbf{A}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}'_1 \\ \mathbf{a}'_2 \\ \vdots \\ \mathbf{a}'_m \end{bmatrix}$$

- The *transpose* of a $m \times n$matrix $\mathbf{A}$ is the following $n \times m$ matrix

$$\mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

- A vector $\mathrm{x} \in \mathbb{R}^n$ is a column and has components $x_1, x_2, \ldots x_n$.

$$\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ its transpose is } x^T = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}.$$

The euclidean norm of x is $||\mathrm{x}||_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$.

- When $\mathbf{x}$ is a vector, $\mathbf{x} \geqslant 0$ ($\mathbf{x} > 0$) means that every component of $\mathbf{x}$ is non-negative (positive). Similar meaning have the notations $\mathbf{A} \geqslant 0$, $\mathbf{A} > 0$, for a matrix $\mathbf{A}$.
- The *inner product* of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is

$$\mathbf{x}^T\mathbf{y} = \mathbf{y}^T\mathbf{x} = \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=0}^{n} x_i \cdot y_i$$

- If $\mathbf{A}$ is a $m \times n$ matrix and $\mathbf{x} \in \mathbb{R}^n$, then

$$\mathbf{A}\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{A}_i = \begin{bmatrix} \langle \mathbf{a}_1', \mathbf{x} \rangle \\ \langle \mathbf{a}_2', \mathbf{x} \rangle \\ \vdots \\ \langle \mathbf{a}_m', \mathbf{x} \rangle \end{bmatrix}$$

- The vectors $\mathbf{x}^1, \mathbf{x}^2, \ldots \mathbf{x}^p \in \mathbb{R}^n$ are called *linearly independent* if, for every $\alpha_1, \alpha_2, \ldots, \alpha_p \in \mathbb{R}$, we have

$$\alpha_1 \mathbf{x}^1 + \alpha_2 \mathbf{x}^2 + \cdots \alpha_p \mathbf{x}^p = \mathbf{0} \Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_p = 0.$$

  Otherwise $\mathbf{x}^1, \mathbf{x}^2, \ldots \mathbf{x}^p \in \mathbb{R}^n$ are called *linearly dependent*.
- If $\mathbf{x}^1, \mathbf{x}^2, \ldots \mathbf{x}^p \in \mathbb{R}^n$ are linearly independent, then $p \leqslant n$. $\mathbb{R}^n$ has n linearly independent vectors - which a *base* in $\mathbb{R}^n$.
- If $\mathbf{A} \in \mathbb{R}^{m \times n}$, the *rank* of $\mathbf{A}$, $rank(\mathbf{A})$, is the maximum number of linearly independent rows of $\mathbf{A}$ (which equals the maximum number of linearly independent columns of $\mathbf{A}$).
- Thus, $rank(\mathbf{A}) \leqslant \max\{m, n\}$. $\mathbf{A}$ is said to have *full row rank* if $rank(\mathbf{A}) = m$ and $\mathbf{A}$ have *full column rank* if $rank(\mathbf{A}) = n$.

Satisfiability problem

- Consider a set of boolean variables $X = \{x_1, x_2, \dots, x_n\}$, and a *conjuctive normal form (CNF)* formula, like, for e. g.,

$$F = (x_3 \vee \overline{x}_4) \wedge (x_1 \vee \overline{x}_2 \vee x_3) \wedge (\overline{x}_1 \vee \overline{x}_3 \vee x_4).$$

- The *SAT problem* is to find an assignment of truth over X such that a given CNF formula is true.
- The integer LP model for the SAT problem (e. g., [Wang01]):
  - each boolean variable becomes a LP variable x_i, and each negative literal $\overline{x}_i$ is transformed in $1 - x_i$;
  - the logical operator or ($\vee$) is replaced by addition operator ($+$);
  - each disjunction of literals (clause) is replaced by a an inequality of the following type: the sum of the LP variables of the corresponding boolean variables is greater or equal with 1.

- For the CNF formula F from above:

$$\begin{aligned} x_3 \vee \overline{x}_4 \quad &\text{becomes} \quad x_3 + (1 - x_4) \geqslant 1 \\ x_1 \vee \overline{x}_2 \vee x_3 \quad &\text{becomes} \quad x_1 + (1 - x_2) + x_3 \geqslant 1 \\ \overline{x}_1 \vee \overline{x}_3 \vee x_4 \quad &\text{becomes} \quad (1 - x_1) + (1 - x_3) + x_4 \geqslant 1 \end{aligned}$$

- To solve this SAT instance is to find a solution to the following set of inequalities:

$$\begin{aligned} x_3 - x_4 &\geqslant 0 \\ x_1 - x_2 + x_3 &\geqslant 0 \\ -x_1 - x_3 + x_4 &\geqslant -1 \\ x_i \in \{0, 1\}, i &= \overline{1, 3} \end{aligned}$$

- An LP problem can be written like follows:

$$
\begin{aligned}
\min z = y_1 + y_2 + y_3 & \\
x_3 - x_4 + y_1 &\geqslant 0 \\
x_1 - x_2 + x_3 + y_2 &\geqslant 0 \\
-x_1 - x_3 + x_4 + y_3 &\geqslant -1 \\
\mathbf{x} \in \{0,1\}^4, \mathbf{y} \in \{0,1\}^3 &
\end{aligned}
$$

- More general, to a SAT instance corresponds an ILP problem

$$
\begin{aligned}
\min z = \mathbf{1}^T \mathbf{y} & \\
A\mathbf{x} + I_m \mathbf{y} &\geqslant \mathbf{1} - \mathbf{b} \\
\mathbf{x} \in \{0,1\}^n, \mathbf{y} \in \{0,1\}^m &
\end{aligned}
$$

where A is the coefficient $m \times n$ matrix of $\mathbf{x}$, I_m is the $m \times m$ identity matrix, and $\mathbf{b}$ is the vector containing the number of negative literals in each clause.

- This is one of the simplest of all integer LP problems.
- The problem is to select a maximum value collection of n objects subject to restriction on some consumed resources (like weight or volume).
- Suppose that item j has weight b_j and value c_j; we add a boolean variable for each item j: $x_j = 1$ iff item j is included in our collection.
- The problem becomes:

$$\max z = \mathbf{c}^T\mathbf{x}$$
$$\mathbf{b}^T\mathbf{x} \leqslant b$$
$$\mathbf{x} \in \{0,1\}^n$$

  where b is the maximum weight of the knapsack.
- Among the posible variants: replace $\mathbf{x} \in \{0,1\}^n$ with $\mathbf{x} \in \mathbb{Z}_+^n$ and you will get the *unbounded knapsack problem*.

- We have a graph $G = (V, E)$ and we want to *color its vertices* such that adjacent vertices have different colors - using as few colors as possible.
- The integer LP model for the vertex coloring problem (e. g., [Wang01])
  - to each possible color (no more than $n = |V|$), j, we associate a boolean variable: $w_j = 1$ iff color j is used by some vertex;
  - to each pair (vertex, color), (u, j), we introduce a boolean variable: $x_{uj} = 1$ iff the vertex u is colored with j;
  - to each edge $uv \in E$ and each color j we add an inequality which constrains at most one of the end points to receive the color j: $x_{uj} + x_{vj} \leqslant w_j$.
  - each vertex must be in the end colored: exactly one of the variables $(x_{uj})_{1 \leqslant j \leqslant n}$ must be 1.

- Hence the integer LP model for coloring the vertices of a given graph becomes ([Mendez09])

$$
\begin{aligned}
\min z = \mathbf{1}^T \mathbf{w} & & & \\
\sum_{j=1}^{n} x_{uj} &= 1, & & \forall u \in V \\
x_{uj} + x_{vj} &\leqslant w_j, & & \forall uv \in E, 1 \leqslant j \leqslant n \\
x_{uj}, w_j \in \{0, 1\} & & &
\end{aligned}
$$

- Sometimes the contraints $w_{j+1} \leqslant w_j$, $j = \overline{1, n}$ are added in order to inforce that color $j+1$ not to be used if color j is not used.

- Consider a salesman traveling from city to city (from a given list). The salesman starts in a city and has to visit all cities on a business trip before returning home. The problem then consists of *finding the shortest tour which visits every city once*.
- TSP can be modelled using a simple (and undirected) weighted graph $G = (V, E)$, $w : E \to \mathbb{R}_+$ being a distance or a cost function defined on the edges.
- The integer LP model for TSP:
  - to each edge, uv, we associate a boolean variable: $x_{uv} = 1$ the tour uses edge uv - note that $x_{uv} = x_{vu}$, this is the symmetrical version of the problem;
  - since the traveler must enter and leave each city (vertex) exactly once, we must have

$$\sum_{v \in V} x_{uv} = 2$$

- However the last restriction doesn't eliminate the subtours (instead of just one tour we can get a list with disjoint subtours which covers all edges).
- In order to have only one tour each subgraph induced by sets travelled edges must be a forest except when all vertices (cities) belong to that subgraph.
- Therefore the integer LP model for TSP is

$$
\begin{aligned}
\min z = \sum_{uv \in E} w_{uv} x_{uv} & & & \\
\sum_{v \in V} x_{uv} & = & 2, & \quad \forall u \in V \\
\sum_{u,v \in U, u \neq v} x_{uv} & \leqslant & |U| - 1, & \quad \forall \varnothing \neq U \subsetneq V \\
x_{uv} \in \{0,1\} & & &
\end{aligned}
$$

- A variant of TSP: using a directed graph (i. e., x_{uv} doesn't necessary equal x_{vu}).

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Operations Research - Lecture 2

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Table of contents

Definition

An LP in **canonical form** *will be written as*

$$\begin{aligned} \textit{minimize} \quad & z = \sum_{j=1}^{n} c_j x_j + d, \\ \textit{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \geqslant b_i, i = \overline{1, m}, \\ & x_j \geqslant 0, j = \overline{1, n} \end{aligned} \tag{1}$$

Sometimes the constant from the objective function is dropped (being considered 0 does not modify the optimal solution, but only the value of the objective).

Canonical Form

In matricial notations (1) becomes

$$\begin{aligned} \text{minimize} \quad & z = \mathbf{c}^T\mathbf{x} + d, \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \geqslant \mathbf{b}, \\ & \mathbf{x} \geqslant \mathbf{0}. \end{aligned} \tag{2}$$

where $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$, and $\mathbf{A} \in \mathbb{R}^{m \times n}$ is the *constraint matrix*. A LP problem in canonical form has the following features:

- is a minimization problem;
- all the variables are restricted to be non-negative;
- all other constraints are "$\geqslant$" inequations.

Standard Form

Definition

An LP in **standard form** *will be written as*

$$\begin{aligned} \text{minimize} \quad & z = \sum_{j=1}^{n} c_j x_j + d, \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j = b_i,\ i = \overline{1,m}, \\ & x_j \geqslant 0, j = \overline{1,n} \end{aligned} \tag{3}$$

where $b_j \geqslant 0$*, for* $j = \overline{1,m}$.

In matricial notations

$$\begin{aligned} \text{minimize} \quad & z = \mathbf{c}^T \mathbf{x} + d, \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geqslant \mathbf{0}. \end{aligned} \tag{4}$$

where $\mathbf{b} \geqslant \mathbf{0}$. An LP problem in standard form has the following features:

- is a minimization problem;
- all the variables are restricted to be non-negative;
- all other constraints are equations;
- the components of the right-hand side vector $\mathbf{b}$ are non-negative.

Any LP problem can be converted to standard form; we illustrate the techniques of converting using some examples.

- If we have a maximization problem:

$$\text{maximize } z = \mathbf{c}^T \mathbf{x} + d$$

we multiply the objective by (-1):

$$\text{minimize } z' = -\mathbf{c}^T \mathbf{x} - d$$

After we solve the problem, the optimal objective value must be multiplied by (-1): $z_* = -z'_*$.
However, the optimal solutions (that is, the values of the variables after solving the problem) are the same.

- A (non-zero) lower bound on a variable:

$$x_h \geqslant b_h, \text{ with } b_h \neq 0.$$

is treated by replacing the variable with:

$$x'_h = x_h - b_h \text{ and } x'_h \geqslant 0.$$

- Upper bounds of a variable can be treated in a similar manner or as a general constraint.

- An inequation having a negative right-hand side:

$$a_{j1}x_1 + a_{j2}x_2 + \ldots + a_{jn}x_n \leqslant b_j \text{ (or } \geqslant b_j), \text{ with } b_j < 0.$$

must be multiplied by (-1):

$$-a_{j1}x_1 - a_{j2}x_2 - \ldots - a_{jn}x_n \geqslant -b_j \text{ (or } \leqslant -b_j).$$

- An unrestricted variable:

$$x_h \in \mathbb{R}$$

can be replaced by a pair of non-negative variables like this

$$x_h = x_h' - x_h''.$$

Converting to Standard Form - Techniques

- An "$\leqslant$" inequation with a non-negative right-hand side:

$$a_{j1}x_1 + a_{j2}x_2 + \ldots + a_{jn}x_n \leqslant b_j, \text{ with } b_j \geqslant 0.$$

is converted to an equation by including a *slack variable* $s_j \geqslant 0$

$$a_{j1}x_1 + a_{j2}x_2 + \ldots + a_{jn}x_n + s_j = b_j.$$

- An "$\geqslant$" inequation with a non-negative right-hand side:

$$a_{h1}x_1 + a_{h2}x_2 + \ldots + a_{hn}x_n \geqslant b_h, \text{ with } b_h \geqslant 0.$$

is converted to an equation by including an *excess variable* $e_h \geqslant 0$

$$a_{h1}x_1 + a_{h2}x_2 + \ldots + a_{hn}x_n - e_h = b_h.$$

$$
\begin{array}{ll}
\text{maximize} & z = 3x_1 - 2x_2 + 4x_3 + 2 \\
\text{subject to} & \\
& \begin{aligned}
x_1 + 4x_2 - x_3 &\leqslant 10 \\
-x_1 + 8x_2 + 2x_3 + x_4 &\geqslant 1 \\
x_1 - 3x_2 - x_4 &= 4 \\
x_1 \geqslant 3, 0 \leqslant x_3 &\leqslant 5
\end{aligned}
\end{array}
$$

First we convert the variables and the objective function:

$$
\begin{array}{ll}
\text{minimize} & z' = -3x_1' + 2x_2' - 2x_2'' - 4x_3 + 11 \\
\text{subject to} & \\
& \begin{aligned}
x_1' + 4x_2' - 4x_2'' - x_3 &\leqslant 7 \\
-x_1' + 8x_2' - 8x_2'' + 2x_3 + x_4' - x_4'' &\geqslant 4 \\
x_1' - 3x_2' + 3x_2'' - x_4' + x_4'' &= 1 \\
x_3 &\leqslant 5 \\
x_1', x_2', x_2'', x_3, x_4', x_4'' &\geqslant 0
\end{aligned}
\end{array}
$$

Then we convert the constraints:

$$
\begin{array}{ll}
\text{minimize} & z' = -3x_1' + 2x_2' - 2x_2'' - 4x_3 + 11 \\
\text{subject to} & \\
& \begin{aligned}
x_1' + 4x_2' - 4x_2'' - x_3 + s_1 &= 7 \\
-x_1' + 8x_2' - 8x_2'' + 2x_3 + x_4' - x_4'' - e_2 &= 4 \\
x_1' - 3x_2' + 3x_2'' - x_4' + x_4'' &= 1 \\
x_3 + s_3 &= 5
\end{aligned} \\
& x_1', x_2', x_2'', x_3, x_4', x_4'', s_1, e_2, s_3 \geqslant 0
\end{array}
$$

In matricial notations the problem is

$$\begin{aligned} \text{minimize} \quad & z = \mathbf{c}^T\mathbf{x} + d, \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geqslant 0. \end{aligned} \tag{5}$$

where $d = 11, \mathbf{c} = (-3, 2, -2, -4, 0, 0, 0, 0, 0)^T, \mathbf{b} = (7, 4, 1, 5)^T$, and

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & -4 & -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 8 & -8 & 2 & 1 & -1 & 0 & -1 & 0 \\ 1 & -3 & 3 & 0 & -1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Hyperplanes and Halfspaces

Definition

Let $\mathbf{a} \in \mathbb{R}^n$ *be a non-zero vector and* $b \in \mathbb{R}$.

(i) *The set* $\mathcal{H}_s(\mathbf{a}, b) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{a} \geqslant b\}$ *is called a* *halfspace*.

(ii) *The set* $H_p(\mathbf{a}, b) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{a} = b\}$ *is called a* *hyperplane*.

- Obviously, a hyperplane $H_p(\mathbf{a}, b)$ is the boundary of two halfspaces: $\mathcal{H}_s(-\mathbf{a}, -b)$ and $\mathcal{H}_s(\mathbf{a}, b)$.
- Geometrically, the vector $\mathbf{a}$ is orthogonal on the hyperplane $H_p(\mathbf{a}, b)$; it is called the *normal* vector to $H_p(\mathbf{a}, b)$.

Definition

Let $\mathbf{A}$ be a $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^n$ be a vector. The set $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \geqslant \mathbf{b}\}$ is called a polyhedron.

- The feasible region of any LP problem is a polyhedron: such a problem can be described by inequality constraints of the form $\mathbf{A}\mathbf{x} \geqslant \mathbf{b}$.
- Obviously a polyhedron is a finite intersection of halfspaces, and can be a bounded or an unbounded set in $\mathbb{R}^n$.

Definition

A set $\mathcal{M} \in \mathbb{R}^n$ is bounded if it exists a constant $K \in \mathbb{R}_+^*$, such that $||\mathbf{x}||_2 < K$, for every $\mathbf{x} \in \mathcal{M}$, otherwise $\mathcal{M}$ is unbounded.

Definition

A set $\mathcal{M} \in \mathbb{R}^n$ is *convex* *if for any $\mathbf{x}, \mathbf{y} \in \mathcal{M}$, and any $\lambda \in [0, 1]$, we have $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in \mathcal{M}$.*

The *line segment* joining $\mathbf{x}$ and $\mathbf{y}$ is $\{\mathbf{z} = \lambda\mathbf{x} + (1-\lambda)\mathbf{y} \,:\, \lambda \in [0, 1]\}$.

Definition

(i) *A* *convex combination* *of vectors $\mathbf{x}^1, \mathbf{x}^2, \ldots \mathbf{x}^p \in \mathbb{R}^n$ is a vector $\sum_{i=1}^{p} \lambda_i \mathbf{x}^i$, where $\lambda_1, \lambda_2, \ldots, \lambda_p \geqslant 0$ and $\sum_{i=1}^{p} \lambda_i = 1$.*

(ii) *The* *convex hull*, *conv$(\mathcal{M})$, of a set $\mathcal{M} \subseteq \mathbb{R}^n$ is the smallest convex set of all convex sets containing $\mathcal{M}$.*

Convexity

Theorem

(i) *An intersection of convex sets is a convex set.*

(ii) *Any of the following sets are convex: a polyhedron, a halfspace, and a hyperplane.*

(iii) *Any convex combination of a finite number of elements of a convex set belongs to that set.*

(iv) *The convex hull of a set* $\mathcal{M} \subseteq \mathbb{R}^n$ *is*

$$conv(\mathcal{M}) = \Big\{ \mathbf{z} \in \mathbb{R}^n \ : \ \exists p \in \mathbb{N}^*, \exists \mathbf{x}^1, \ldots, \mathbf{x}^p \in \mathcal{M},$$

$$and\ \exists \lambda^1, \ldots, \lambda^p \in [0,1], \sum_{i=1}^{p} \lambda_i = 1, \ s.t. \ \mathbf{z} = \sum_{i=1}^{p} \lambda_i \mathbf{x}^i \Big\}$$

- As we already saw from examples an optimal solution of a LP problem is a "corner" of the polyhedron obtained from all constraints of the problem.
- This is an *extreme point* of a polyhedron: a point that cannot be expressed as a convex combination of other points from that polyhedron.
- The geometric definition follows:

Definition

A vector $\mathbf{x}$ *of a set* $\mathcal{M} \subseteq \mathbb{R}^n$ *is an* **extreme point** *of* $\mathcal{M}$ *if we cannot find two vectors* $\mathbf{y}, \mathbf{z} \in \mathcal{M}$ *and* $\lambda \in (0, 1)$ *such that* $\mathbf{x} = \lambda \mathbf{y} + (1 - \lambda)\mathbf{z}$. *The set of extreme points of* $\mathcal{M}$ *is denoted by* $\mathbb{E}_{\mathcal{M}}$.

The importance of extreme points in optimization over a polyhedron is revealed by the next result - given here without proof.

Theorem

(Krein-Milman) Any convex, compact subset of $\mathbb{R}^n$ coincides with the convex hull of its extreme points.

Consider now the problem (2) with its subjacent polyhedron $\mathcal{P} = \{\mathrm{x} \in \mathbb{R}^n_+ : \mathbf{A}\mathrm{x} \geqslant \mathrm{b}\}$. If $\mathcal{P}$ is bounded, it is compact (since is obviously closed), hence

Corollary

If $\mathcal{P}$ is bounded, then $\mathcal{P} = conv(\mathbb{E}_{\mathcal{P}})$.

Now consider a LP problem in standard form

$$\begin{aligned} \text{maximize} \quad & z = \mathbf{c}^T\mathbf{x} + d, \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geqslant 0. \end{aligned} \tag{6}$$

where $m \leqslant n$ and matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ is full rank, that is, its rows are linearly independent. We will see that this condition is not restrictive (see seminar 2).

Definition

A **basic solution** *is a vector* $\mathbf{x} \in \mathbb{R}^n$ *such that*

(i) $\mathbf{x}$ *satisfies the constraints of the linear program:* $\mathbf{A}\mathbf{x} = \mathbf{b}$.

(ii) *The columns of* $\mathbf{A}$ *corresponding to non-zero components of* $\mathbf{x}$ *are linearly independent.*

- The components of a basic solution x can always be separated in two classes: a sub-vector x_N of $(n-m)$ zero components, and x_B of m (possible non-zero) components.
- This separation is possible because $\mathbf{A}$ having full rank we can find an $m \times m$ invertible sub matrix $\mathbf{B}$ of $\mathbf{A}$; the columns of $\mathbf{B}$ corresponds to the variables from x_B also named *basic variables*.
- The set of basic variables is called the *basis* (corresponding to x).
- The variables from x_N are called *non-basic variables*.
- If some of the basic variables are also zero, the above separation could be not unique.

Definition

1. *A basic solution* $\mathbf{x}$ *is a* **basic feasible solution** *if, in addition, it satisfies the non-negativity restrictions, that is,* $\mathbf{x} \geqslant \mathbf{0}$.
2. *A basic feasible solution is called* **optimal basic feasible solution** *if it is also optimal for the linear program.*

Definition

A ***feasible solution*** *is a vector* $\mathbf{x} \in \mathbb{R}^n$ *which satisfies the constraints of the linear program:* $\mathbf{A}\mathbf{x} = \mathbf{b}$ *and* $\mathbf{x} \geqslant \mathbf{0}$.

Consider the Reddy Mikks problem from last course (but as a minimization one)

$$
\begin{aligned}
\text{minimize} \quad & z = -5x_1 - 4x_2 \\
\text{subject to} \quad & \\
& 6x_1 + 4x_2 \leqslant 24 \\
& x_1 + 2x_2 \leqslant 6 \\
& x_2 - x_1 \leqslant 1 \\
& x_2 \leqslant 2 \\
& x_1, x_2 \geqslant 0
\end{aligned}
$$

We already solved this problem: optimal value $z_* = -21$ is reached at point $\mathrm{x}_* = (3, 1.5)$ (B). The boundaries of the feasible region are the lines

$$
\begin{array}{rclclrcll}
6x_1 + 4x_2 & = & 24 & (1) & \quad & x_2 & = & 2 & (4) \\
x_1 + 2x_2 & = & 6 & (2) & & x_1 & = & 0 & (5) \\
x_2 - x_1 & = & 1 & (3) & & x_2 & = & 0 & (6)
\end{array}
$$

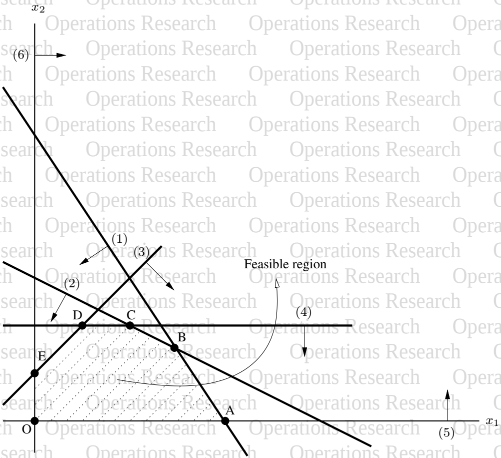

Figure: Feasible Region

- Each corner of the feasible region corresponds to the intersection of two of these lines: theoretically there are $\binom{6}{2} = 15$ such intersections.
- Only six of them are corners of the feasible region.
- In standard form this problem becomes (it has six variables)

$$
\begin{array}{ll}
\text{minimize} & z = -5x_1 - 4x_2 \\
\text{subject to} & \\
& \begin{aligned}
6x_1 + 4x_2 + s_1 &= 24 \\
x_1 + 2x_2 + s_2 &= 6 \\
-x_1 + x_2 + s_3 &= 1 \\
x_2 + s_4 &= 2 \\
x_1, x_2, s_1, s_2, s_3, s_4 &\geqslant 0
\end{aligned}
\end{array}
$$

- The basis $\{x_2, s_2, s_3, s_4\}$ gives the basic (infeasible) solution

$$(x_1\ x_2\ s_1\ s_2\ s_3\ s_4)^T = (0\ 6\ 0\ -6\ 1\ 2)^T$$

which corresponds to an infeasible corner (intersection of (1) with (6)).

- The basis $\{x_1, x_2, s_1, s_3\}$ gives the basic feasible solution

$$(x_1\ x_2\ s_1\ s_2\ s_3\ s_4)^T = (2\ 2\ 4\ 0\ 1\ 0)^T$$

which corresponds to a feasible corner (intersection of (2) with (4)).

- The basis $\{x_1, x_2, s_2, s_4\}$ gives the basic infeasible solution

$$(x_1\ x_2\ s_1\ s_2\ s_3\ s_4)^T = (2\ 3\ 0\ -2\ 0\ -2)^T$$

which corresponds to an infeasible corner (intersection of (1) with (3)).

- The basis $\{x_1, x_2, s_3, s_4\}$ gives the optimal basic feasible solution

$$(x_1\ x_2\ s_1\ s_2\ s_3\ s_4)^T = (3\ 1.5\ 0\ 0\ 1.5\ 0.5)^T$$

which corresponds to an optimal feasible corner (intersection of (1) with (2)).

- If $\mathbf{x}$ is a basic feasible solution, once a set of basic variables has been selected, we can reorder the variables such that the basic variables are listed first:

$$\mathbf{x}^T = (\mathbf{x}_B^T \ \mathbf{x}_N^T).$$

- The constraint matrix can be written (by rearranging the columns)

$$\mathbf{A} = (\mathbf{B} \ \mathbf{N}),$$

where $\mathbf{B}$ has the columns corresponding to $\mathbf{x}_B$, and those of $\mathbf{N}$ correspond to $\mathbf{x}_N$.

- For a basic solution $\mathbf{x}$ we have $\mathbf{x}_N = \mathbf{0}$ ($\in \mathbb{R}^{n-m}$), therefore the constraints $\mathbf{A}\mathbf{x} = \mathbf{b}$ become

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{B} \cdot \mathbf{x}_B + \mathbf{N} \cdot \mathbf{x}_N = \mathbf{B} \cdot \mathbf{x}_B = \mathbf{b}.$$

- The number of basic feasible solutions is upper bounded by the number of ways we can select m (basic) variables from the n existing variables:
$$\binom{n}{m} = \frac{n!}{m!(n-m)!}.$$
- This bound can be very large, but not all choices of basic variables may correspond to basic feasible solutions (see the above example).

In this section we will prove that the notion of basic feasible solution and that of extreme point coincide.

Theorem

Let $\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n \; : \; \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geqslant \mathbf{0}\}$ *be a non-empty polyhedron and* $\mathbf{x} \in \mathcal{P}$*. Then, the following are equivalent*

(i) $\mathbf{x}$ *is an extreme point of* $\mathcal{P}$*.*

(ii) $\mathbf{x}$ *is a basic feasible solution.*

Proof. "$(i) \Longrightarrow (ii)$" We prove by contradiction; first, we reorder the variables so that the non-zero variables come first: $\mathbf{x}^T = (\mathbf{x}_B^T \mathbf{x}_N^T)$, where $\mathbf{x}_N = \mathbf{0}$, $\mathbf{x}_B > \mathbf{0}$. Correspondingly, we can write $\mathbf{A} = (\mathbf{B} \; \mathbf{N})$ (but $\mathbf{B}$ may not be a square matrix). Obviously $\mathbf{B}\mathbf{x}_B = \mathbf{b}$.
If the columns of $\mathbf{B}$ are linearly independent, then the proof is done.

In what follows we suppose that the columns of $\mathbf{B}$ are not independent. Let $\mathbf{B}_j$ be the jth column of $\mathbf{B}$; it must exist real numbers $y_1, \ldots, y_t$, not all of which are zero, such that $\sum_{j=1}^{t} y_j \mathbf{B}_j = \mathbf{0}$. If we put $\mathbf{y} = (y_1 \ldots y_t)^T$, then we have $\mathbf{B}\mathbf{y} = \mathbf{0}$. For every $\varepsilon \in \mathbb{R}_+^*$ we can write

$$B(\mathbf{x}_B \pm \varepsilon \mathbf{y}) = \mathbf{B}\mathbf{x}_b \pm \varepsilon \mathbf{B}\mathbf{y} = \mathbf{B}\mathbf{x}_B = \mathbf{b}. \tag{7}$$

Now, if ε is small enough, we must have $\mathbf{x}_B \pm \varepsilon \mathbf{y} > \mathbf{0}$; we define

$$\mathbf{x}^1 = \begin{pmatrix} \mathbf{x}_B + \varepsilon \mathbf{y} \\ \mathbf{x}_N \end{pmatrix}, x^2 = \begin{pmatrix} \mathbf{x}_B - \varepsilon \mathbf{y} \\ \mathbf{x}_N \end{pmatrix}.$$

From (7) it follows that $\mathbf{x}^{1,2} \in \mathcal{P}$; as $\mathbf{x}^1 \neq \mathbf{x}^2$ and $\mathbf{x} = 0.5\mathbf{x}^1 + 0.5\mathbf{x}^2$, $\mathbf{x}$ cannot be an extreme point - which is a contradiction.

"$(ii) \Longrightarrow (i)$" We may assume that the first variables are basic: $\mathbf{x}^T = (\mathbf{x}_B^T \ \mathbf{x}_N^T)$, where $\mathbf{x}_N = \mathbf{0}$, $\mathbf{x}_B \geqslant \mathbf{0}$, and $\mathbf{A} = (\mathbf{B} \ \mathbf{N})$ ($\mathbf{B}$ being an $m \times m$ matrix), with $\mathbf{B}\mathbf{x}_B = \mathbf{b}$, the columns of B being linearly independent, matrix B is non-singular.

The proof is also by contradiction: suppose that $\mathbf{x}$ is not an extreme point, then there exist two distinct points $\mathbf{x}^1, \mathbf{x}^2 \in \mathcal{P}$, and $\alpha \in (0,1)$, such that $\mathbf{x} = \alpha\mathbf{x}^1 + (1-\alpha)\mathbf{x}^2$. We write ${\mathbf{x}^1}^T = ({\mathbf{x}_B^1}^T \ {\mathbf{x}_N^1}^T)$ and ${\mathbf{x}^2}^T = ({\mathbf{x}_B^2}^T \ {\mathbf{x}_N^2}^T)$, with $\mathbf{x}_B^i, \mathbf{x}_N^i \geqslant \mathbf{0}$, $i = \overline{1,2}$.

Obviously, we have $\mathbf{0} = \mathbf{x}_N = \alpha\mathbf{x}_N^1 + (1-\alpha)\mathbf{x}_N^2$, therefore, all terms being non-negative, $\mathbf{x}_N^1 = \mathbf{x}_N^2 = \mathbf{0}$. From here it follows that

$$\mathbf{B}\mathbf{x}_B = \mathbf{B}\mathbf{x}_B^1 = \mathbf{B}\mathbf{x}_B^2 = \mathbf{b} \Longrightarrow \mathbf{x}_B = \mathbf{x}_B^1 = \mathbf{x}_B^2 = \mathbf{B}^{-1}\mathbf{b},$$

which is again a contradiction. This completes the proof. □

A procedure for constructing basic solutions, for a linear program with a full-rank matrix is the following:

- Choose m linearly independent columns $\mathbf{A}_{j_1}, \mathbf{A}_{j_2}, \ldots, \mathbf{A}_{j_m}$.
- Let $x_j = 0$ for all $j \in \{1, 2, \ldots, m\} \setminus \{j_1, j_2, \ldots, j_m\}$.
- Solve the system of m equations and m variables $\mathbf{Ax} = \mathbf{b}$.

If, in addition, the basic variables have non-negative values, then the basic solution will be feasible. Conversely, since every basic feasible solution is a basic solution, it can be obtained with this procedure.

Degeneracy and Adjacency

- When one or more basic variables of a basic feasible solution are zero, the corresponding solution (or vertex) is called *degenerate*.
- At a degenerate vertex, several different bases correspond to the same basic feasible solution. *Degeneracy* can occur when the problem has a redundant constraint.
- Two extreme points are adjacent if they are connected by an "edge" of the feasible region. Two bases are *adjacent* if they share $(m-1)$ variables.
- Adjacent bases give *adjacent basic feasible solutions* - which may or may not be distinct.

- Let $\mathcal{M} \subseteq \mathbb{R}^n$ be a convex set (say a polyhedron), $\mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ is a *direction of unboundedness* if

$$\mathbf{x} + \alpha\mathbf{y} \in \mathcal{M}, \forall\, \mathbf{x} \in \mathcal{M}, \forall\, \alpha \in \mathbb{R}_+.$$

- Let $\mathcal{M} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geqslant \mathbf{0}\}$ (that is, a linear program in standard form) and $\mathbf{y} \neq \mathbf{0}$ a direction of unboundedness for $\mathcal{M}$. It follows that $\mathbf{y} \geqslant \mathbf{0}$ (exercise), and $\mathbf{x}, (\mathbf{x} + \alpha\mathbf{y})$ must be feasible points ($\forall\, \alpha \in \mathbb{R}_+$) hence

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{A}(\mathbf{x} + \alpha\mathbf{y}) = \mathbf{b} \Longrightarrow \mathbf{A}\mathbf{y} = \mathbf{0}.$$

- The reverse is also true: if $\mathbf{y} \geqslant \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$, and $\mathbf{A}\mathbf{y} = \mathbf{0}$, then $\mathbf{y}$ is a direction of unboundedness.

Consider the linear program

$$
\begin{aligned}
\text{minimize} \quad & z = -x_1 - 2x_2 \\
\text{subject to} \quad & \\
& -2x_1 + x_2 \leqslant 2 \\
& -x_1 + x_2 \leqslant 3 \\
& x_1, x_2 \geqslant 0
\end{aligned}
$$

- This problem has three extreme points $A(0,0)$, $A(0,2)$, and $B(1,4)$;
- Point $\mathbf{y} = (1,0)^T$ is a direction of unboundedness, because $\mathbf{A}\mathbf{y} = \mathbf{0}$ and $\mathbf{y} \geqslant \mathbf{0}$.

Theorem

Consider the polyhedron $\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n_+ \ : \ \mathbf{A}\mathbf{x} = \mathbf{b}\}$ *representing the feasible region for problem (4), and* $\mathbb{E}_{\mathcal{P}} = \{\mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^t\}$*. If* $\mathcal{P}$ *is nonempty, then* $\mathbb{E}_{\mathcal{P}}$ *is nonempty, and every feasible solution* $\mathbf{x} \in \mathcal{P}$ *can be written as*

$$\mathbf{x} = \mathbf{y} + \sum_{i=1}^{t} \lambda_i \mathbf{v}^i, \tag{8}$$

where $\mathbf{A}\mathbf{y} = \mathbf{0}$ *(*$\mathbf{y}$ *is zero or is a direction of unboundedness), and* $\sum_{i=1}^{t} \lambda_i = 1, \lambda_i \geqslant 0, \forall i = \overline{1, i}$*.*

Proof. First we analyze the case when $\mathcal{P}$ is bounded: using corollary 2.1 we get that $\mathcal{P} = conv(\mathbb{E}_{\mathcal{P}})$, therefore (8) holds with $\mathbf{y} = \mathbf{0}$.

For the unbounded case, let $\mathbf{x}$ be a feasible solution of $\mathcal{P}$. We proceed by induction on the number of non-zero components of $\mathbf{x}$. If $\mathbf{x}$ is a basic feasible solution, then it is an extreme point $\mathbf{v}^i$ of $\mathcal{P}$ and (8) holds with $\mathbf{y} = \mathbf{0}, \lambda_i = 1$, and $\lambda_j = 0$, for $j \neq i$.

If $\mathbf{x}$ is not an extreme point (i.e., it is not a basic feasible solution), the columns of $\mathbf{A}$ corresponding to the non-zero components of $\mathbf{x}$ are linearly dependent: there are n real numbers $y_1, y_2, \ldots, y_n$, not all of which are zero, such that

$$\sum_{i=1}^{n} y_i \mathbf{A}_i = \mathbf{0} \text{ and } y_i = 0, \text{ if } x_i = 0.$$

Let $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$, we have $\mathbf{y} \neq \mathbf{0}$ and $\mathbf{A}\mathbf{y} = \mathbf{0}$. Hence $\mathbf{A}(\mathbf{x} + \varepsilon \mathbf{y}) = \mathbf{A}\mathbf{x} = \mathbf{b}$, that is, for small enough ε, $(\mathbf{x} + \varepsilon \mathbf{y}) \in \mathcal{P}$.

Proof of Representation Theorem

- Observe first that $\mathbf{y}$ and $-\mathbf{y}$ cannot be both directions of unboundedness (why?).
- If $\mathbf{y}$ is not a direction of unboundedness, for some $\varepsilon > 0$, $(\mathbf{x} + \varepsilon \mathbf{y})$ will meet the boundary of $\mathcal{P}$, then we can choose

$$\varepsilon_1 = \max\{\varepsilon > 0 \; : \; \mathbf{x} + \varepsilon \mathbf{y} \geqslant \mathbf{0}\}.$$

Obviously, $\mathbf{x}^1 = (\mathbf{x} + \varepsilon_1 \mathbf{y})$ has less non-zero components than $\mathbf{x}$.
- In the same way, if $-\mathbf{y}$ is not a direction of unboundedness, then we can find out an $\varepsilon_2 > 0$, such that $\mathbf{x}^2 = (\mathbf{x} - \varepsilon_2 \mathbf{y})$ has less non-zero components than $\mathbf{x}$.

- If both $\mathbf{y}$ and $-\mathbf{y}$ are not directions of unboundedness, then

$$\mathbf{x} = \lambda \mathbf{x}^1 + (1-\lambda)\mathbf{x}^2, \text{ where } \lambda = \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2},$$

and we can apply the induction hypothesis for $\mathbf{x}^1$ and $\mathbf{x}^2$.

- If only $\mathbf{y}$ $(-\mathbf{y})$ is a direction of unboundedness, then $\mathbf{x} = \mathbf{y} + \mathbf{x}^1$ (respectively $\mathbf{x} = \mathbf{y} + \mathbf{x}^2$), and, for $\mathbf{x}^i$, we can apply the induction hypothesis.

□

Bounded and Unbounded LP Problems.

Definition

An LP problem is called **bounded** *if it has a finite optimum, and* **unbounded** *when it has an infinite optimum.*

- Bertsimas, D., J. N. Tsitsiklis, *Introduction to Linear Optimization*, Athena Scientific, Belmont, Massachusetts, 1997.
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- Kolman, B., R. E. Deck, *Elementary Linear Programming with Applications*, Elsevier Science and Technology Books, 1995.
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Operations Research - Lecture 3

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Consider an LP problem in standard form

$$\begin{aligned} \text{minimize} \quad & z = \mathbf{c}^T\mathbf{x} + d, \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geqslant 0. \end{aligned} \tag{1}$$

Theorem

If an LP problem in standard form has a finite optimal solution (i. e., the problem is bounded), then it has an optimal basic feasible solution.

Proof. From the Representation Theorem, we know that, for some $\mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^t \in \mathbb{E}_{\mathcal{P}}$, and $\mathbf{y}$ (a direction of unboundedness or a zero vector), we have

$$\mathbf{x} = \mathbf{y} + \sum_{i=1}^{t} \lambda_i \mathbf{v}^i, \text{ where } \sum_{i=1}^{t} \lambda_i = 1, \lambda_i \geqslant 0, \forall i = \overline{1, t}.$$

Let $\mathbf{x}$ be an optimal solution. Since we know that $\mathbf{x} + \alpha\mathbf{y} \in \mathcal{P}$, for all $\alpha > 0$[1], we must have

$$\mathbf{c}^T\mathbf{x} \leqslant \mathbf{c}^T(\mathbf{x} + \alpha\mathbf{y}) \Longrightarrow \alpha\mathbf{c}^T\mathbf{y} \geqslant 0 \Longrightarrow \mathbf{c}^T\mathbf{y} \geqslant 0.$$

Suppose that $\mathbf{c}^T\mathbf{y} > 0$; if we denote $\widetilde{\mathbf{x}} = \mathbf{x} - \mathbf{y} \in \mathcal{P}$, then $\mathbf{c}^T\mathbf{x} > \mathbf{c}^T\widetilde{\mathbf{x}}$, which implies that $\mathbf{x}$ would not be optimal. Therefore, $\mathbf{c}^T\mathbf{y} = 0$, and $\mathbf{c}^T\mathbf{x} = \mathbf{c}^T\widetilde{\mathbf{x}}$, that is, $\widetilde{\mathbf{x}}$ is also an optimal solution. On the other hand

$$\mathbf{c}^T\widetilde{\mathbf{x}} = \sum_{i=1}^{t}\lambda_i\mathbf{c}^T\mathbf{v}^i \geqslant \sum_{i=1}^{t}\lambda_i\mathbf{c}^T\widetilde{\mathbf{x}} = \mathbf{c}^T\widetilde{\mathbf{x}} \Longrightarrow \mathbf{c}^T\mathbf{v}^i = \mathbf{c}^T\widetilde{\mathbf{x}}, \forall i = \overline{1,t},$$

Hence any $\mathbf{v}^j$, with $\lambda_j > 0$, is an optimal basic feasible solution for problem (1). ($\mathbf{v}^j$ being an extreme point of $\mathcal{P}$ it corresponds to a basic feasible solution.) □

[1] Recall the definition of a direction of unboundedness.

- A *feasible direction* for $\mathbf{x}$ is a vector $\mathbf{y} \in \mathbb{R}^n$ satisfying

$$\mathbf{A}\mathbf{y} = \mathbf{0} \text{ and } x_i = 0 \Rightarrow y_i = 0.$$

- Suppose now that $\mathbf{x}$ is a finite optimal basic feasible solution and $\mathbf{y}$ is a feasible direction of the same problem. For small enough $\varepsilon > 0$, $(\mathbf{x} + \varepsilon \mathbf{y})$ must be a feasible solution too.
- Under these conditions $\mathbf{y}$ must satisfy

$$\mathbf{c}^T\mathbf{y} \geqslant 0, \mathbf{A}\mathbf{y} = 0, \text{ and } x_i = 0 \Rightarrow y_i = 0.$$

- Simplex method was developed in the 1940's, in the research department of US Air Force, when the linear programming models show their appeal for military and later economic planning (as a matter of fact linear programming means linear planning).
- This method benefited from the almost simultaneous development of digital programmable computers, which gave tools for automated solving of large scale linear problems.
- Over the years simplex algorithm has proved its efficiency and was the main LP method until the 1980's when another LP technique was discovered (namely, the interior path method).
- Even today simplex remains a prominent method, partly because of its simplicity (sic!) and because of its theoretical applications.

- We already know from the previous section that, if an LP problem in standard form has a finite optimal solution, then it has an optimal basic feasible one.
- In other words, if an LP problem has a finite optimum, then an optimum feasible solution may be found among the extreme points of the subjacent polyhedra.
- *Simplex algorithm* is based on these observations and searches for an optimal feasible solution by moving from one basic feasible solution to another (adjacent one), along the boundary of the feasible region, improving the objective function.
- Eventually a basic feasible solution is reached at which no improvments of the objective function are possible. Such a basic feasible solution is an optimal one.

- We consider here the LP problem (1) in standard form (remember that $\mathbf{b} \geqslant 0$). Let $\mathbf{x}$ be a basic feasible solution with the variables ordered so that

$$\mathbf{x}^T = (\mathbf{x}_B^T\ \mathbf{x}_N^T),$$

where $\mathbf{x}_B$ is the vector of basic variables and $\mathbf{x}_N$ is the vector of non-basic ones ($\mathbf{x}_N = \mathbf{0}$).

- Correspondingly we split $\mathbf{c}$ and $\mathbf{A}$:

$$\mathbf{c}^T = (\mathbf{c}_B^T\ \mathbf{c}_N^T), \mathbf{A} = (\mathbf{B}\ \mathbf{N}).$$

- The objective function and the constraints become

$$z = \mathbf{c}_B^T\mathbf{x}_B + \mathbf{c}_N^T\mathbf{x}_N,\quad \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b}.$$

- From the last equalities we get

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \Rightarrow z = \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} + (\mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N})\mathbf{x}_N.$$

- If we take $\mathbf{y} = (\mathbf{c}_B^T \mathbf{B}^{-1})^T$, the objective function become

$$z = \mathbf{y}^T \mathbf{b} + (\mathbf{c}_N^T - \mathbf{y}^T \mathbf{N})\mathbf{x}_N.$$

- $\mathbf{y}$ is the vector of *simplex multipliers*.
- The current values of basic variables and objective function are

$$\mathbf{x}_B = \widehat{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b}, \ \ \widehat{z} = \mathbf{y}^T \mathbf{b}.$$

We consider an example already analyzed

$$
\begin{array}{ll}
\text{minimize} & z = -5x_1 - 4x_2 \\
\text{subject to} & \\
& \begin{aligned} 6x_1 + 4x_2 + x_3 &= 24 \\ x_1 + 2x_2 + x_4 &= 6 \\ -x_1 + x_2 + x_5 &= 1 \\ x_2 + x_6 &= 2 \\ x_1, x_2, \ldots, x_6 &\geqslant 0 \end{aligned}
\end{array}
$$

We have $\mathbf{c} = (-5, -4, 0, 0, 0, 0)^T$, $\mathbf{b} = (24, 6, 1, 2)^T$, and

$$
\mathbf{A} = \begin{bmatrix} 6 & 4 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.
$$

- Consider the feasible base $\{x_1, x_4, x_5, x_6\}$, $\mathbf{x}_B = (x_1\ x_4\ x_5\ x_6)^T$, $\mathbf{x}_N = (x_2\ x_3)^T$, and

$$\mathbf{B} = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{B}^{-1} = \begin{bmatrix} 1/6 & 0 & 0 & 0 \\ -1/6 & 1 & 0 & 0 \\ 1/6 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{N} = \begin{bmatrix} 4 & 1 \\ 2 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix},$$

$\mathbf{c}_B = (-5\ 0\ 0\ 0)^T$, $\mathbf{c}_N = (-4\ 0)^T$.

- We may further compute

$$\mathbf{y} = (-5/6\ 0\ 0\ 0)^T, \mathbf{x}_B = \widehat{\mathbf{b}} = (4\ 2\ 5\ 2)^T, \widehat{z} = -28.$$

- The general formula for the objective value is

$$z = \mathbf{y}^T\mathbf{b} + (\mathbf{c}_N^T - \mathbf{y}^T\mathbf{N})\mathbf{x}_N.$$

Definition

The coefficient corresponding to a non-basic variable x_j in the vector $\hat{\mathbf{c}}_N^T = (\mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N})$ is called the reduced cost of variable x_j*:*

$$\hat{c}_j = c_j - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{A}_j \tag{2}$$

- If we want to test for optimality we examine, in (2), the variation of the objective function if a non-basic variable x_j is increased from zero: if $\hat{c}_j < 0$, then the objective will decrease, if $\hat{c}_j > 0$, then the objective will increase, and if $\hat{c}_j = 0$, then the objective will not change.

Theorem

Let $\mathbf{x}$ be a basic feasible solution to the LP problem (1), having the basis matrix $\mathbf{B}$, and $\widehat{\mathbf{c}}_N$ the vector of reduced costs for non-basic variables. The following are true

(i) *If $\widehat{\mathbf{c}}_N \geqslant \mathbf{0}$, then $\mathbf{x}$ is an optimal solution.*

(ii) *If $\mathbf{x}$ is an optimal and nondegenerate solution, then $\widehat{\mathbf{c}}_N \geqslant \mathbf{0}$.*

Proof. (i) Let $\widetilde{\mathbf{x}}$ be an arbitrary feasible solution of (1), and $\overline{\mathbf{x}} = \widetilde{\mathbf{x}} - \mathbf{x}$. We have

$$\mathbf{A}\mathbf{x} = \mathbf{A}\widetilde{\mathbf{x}} = \mathbf{b} \Rightarrow \mathbf{A}\overline{\mathbf{x}} = \mathbf{0} \Rightarrow \mathbf{B}\overline{\mathbf{x}}_B + \sum_{j \in N} \mathbf{A}_j \overline{x}_j = \mathbf{0}.$$

From here we get $\overline{\mathbf{x}}_B = -\sum_{j \in N} \mathbf{B}^{-1} \mathbf{A}_j \overline{x}_j$, and

$$\mathbf{c}^T\overline{\mathbf{x}} = \mathbf{c}_B^T\overline{\mathbf{x}}_B + \sum_{j\in N} c_j\overline{x}_j = \sum_{j\in N}(c_j - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{A}_j)\overline{x}_j = \sum_{j\in N}\widehat{c}_j\overline{x}_j =$$

$$\sum_{j\in N}(\widehat{c}_j\widetilde{x}_j - \widehat{c}_j x_j) = \sum_{j\in N}\widehat{c}_j\widetilde{x}_j \geqslant 0.$$

Thus, $\mathbf{c}^T\mathbf{x} \leqslant \mathbf{c}^T\widetilde{\mathbf{x}}$, for any feasible solution $\widetilde{\mathbf{x}}$, which means that $\mathbf{x}$ is an optimal solution.

(ii) Let $\mathbf{x}$ be a nondegenerate (that is, $x_i > 0, \forall i \in B$) optimal solution and suppose that we have $\widehat{c}_j < 0$, for some $j \in N$.

We can build a feasible solution $\mathbf{x}+\alpha\mathbf{y}$ $(\alpha > 0)$, such that x_j is increased, and all other non-basic variables remain zero: $y_j = 1$, and $y_h = 0, \forall h \in N \setminus \{j\}$.

$$\mathbf{A}(\mathbf{x}+\alpha\mathbf{y}) = \mathbf{b} \Rightarrow \mathbf{A}\mathbf{y} = \mathbf{0} \Rightarrow 0 = \sum_{i=1}^{n}\mathbf{A}_i y_i = \sum_{i\in B}\mathbf{A}_i y_i + \mathbf{A}_j = \mathbf{B}\mathbf{y}_B + \mathbf{A}_j.$$

From the above relations we get $\mathbf{y}_B = -\mathbf{B}^{-1}\mathbf{A}_j$; with these values $\mathbf{y}$ is the *j-th basic direction*.

Obviously, for small enough $\alpha > 0$, we will have $(\mathbf{x} + \alpha\mathbf{y}) > \mathbf{0}$. Now,

$$\mathbf{c}^T\mathbf{y} = \mathbf{c}_B^T\mathbf{y}_B + c_j y_j = -\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{A}_j + c_j y_j = \widehat{c}_j.$$

Since $\widehat{c}_j < 0$, we have $\mathbf{c}^T(\mathbf{x} + \alpha\mathbf{y}) = \mathbf{c}^T\mathbf{x} + \alpha\widehat{c}_j < \mathbf{c}^T\mathbf{x}$, hence $\mathbf{x}$ cannot be an optimal solution - a contradiction. We must have $\widehat{\mathbf{c}}_N \geqslant 0$. $\square$

- Using Theorem 3.1, we acknowledge that, if $\mathbf{x}$ is a basic feasible solution, and $\widehat{c}_j < 0$, we can eventually increase x_j (this is the *entering variable*) until a nonnegativity constraint is violated (this will give the *leaving variable*).
- The basic variables are $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N$ and all components of $\mathbf{x}_N$ are zero, except x_j. Therefore,

$$\mathbf{x}_B = \widehat{\mathbf{b}} - \widehat{\mathbf{A}}_j x_j, \text{ where } \widehat{\mathbf{A}}_j = \mathbf{B}^{-1}\mathbf{A}_j. \tag{3}$$

- We examine equation (3) componentwise: $x_i = \widehat{b}_i - \widehat{a}_{ij} x_j$, $i \in B$:
  - if $\widehat{a}_{ij} > 0$, then x_i will decrease as x_j increases and will become zero when $x_j = \dfrac{\widehat{b}_i}{\widehat{a}_{ij}}$;
  - if $\widehat{a}_{ij} < 0$, then x_i will increase as x_j increases;
  - if $\widehat{a}_{ij} = 0$, then x_i will remain the same.
- The variable x_j can be increased as long as all the variables have nonnegative values:

$$\widehat{x}_j = \min\left\{\frac{\widehat{b}_i}{\widehat{a}_{ij}} \; : \; \widehat{a}_{ij} > 0\right\}. \tag{4}$$

- What happens if $\widehat{a}_{ij} \leqslant 0, \forall i \in B$? The answer is given by the following result.

Theorem

Let $\mathbf{x}$ *be a basic feasible solution to the LP problem (1), having the basis matrix* $\mathbf{B}$, *and* $\widehat{\mathbf{c}}_N$ *the vector of reduced costs for non-basic variables. Suppose that* $\widehat{c}_j < 0$, *for some* $j \in N$*; if* $\widehat{a}_{ij} \leqslant 0$, *for all* $i \in B$, *then problem (1) has an infinite optimum (it is unbounded).*

Proof. Obviously, all the basic variables will not decrease, and x_j can be made arbitrary large. The new values of the basic variables and of the objective function are

$$\mathbf{x}_B \leftarrow \mathbf{x}_B - \mathbf{A}_j \widehat{x}_j,\ z = \widehat{z} + \widehat{c}_j \widehat{x}_j \tag{5}$$

We have $\lim\limits_{\widehat{x}_j \to +\infty} \widehat{z} = -\infty$ - this is the optimum of the problem. □

The algorithm starts with with a basis matrix $\mathbf{B}$, corresponding to the basic feasible solution $\mathbf{x}_B = \widehat{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} \geqslant \mathbf{0}$. The algorithm follows:

The Optimality Test. Compute $\mathbf{y}^T = \mathbf{c}_B^T \mathbf{B}^{-1}$ and $\widehat{\mathbf{c}}_N^T = \mathbf{c}_N^T - \mathbf{y}^T \mathbf{N}$; if $\widehat{\mathbf{c}}_N^T \geqslant \mathbf{0}$, then the current base is optimal, if not, select an index $j \in N$, such that $\widehat{c}_j < 0$. x_j will be the *entering variable*.

The Main Step. Compute $\widehat{\mathbf{A}}_j = \mathbf{B}^{-1}\mathbf{A}_j$. If $\widehat{a}_{hj} \leqslant 0$, for all $h \in B$, then Stop - the problem has infinite optimum. Otherwise find an $i \in B$ such that

$$\frac{\widehat{b}_i}{\widehat{a}_{ij}} = \min\left\{\frac{\widehat{b}_h}{\widehat{a}_{hj}} : \widehat{a}_{hj} > 0\right\}.$$

x_i will be the *leaving variable* and $\widehat{a}_{ij}$ will be the *pivot* entry.

The Update. Compute the new basis matrix $\mathbf{B}$, the new vector of basic variables $\mathbf{x}_B$, and the new reduced costs $\widehat{c}$. Go to the *optimality test*.

- We consider again our example

$$\mathbf{A} = \begin{bmatrix} 6 & 4 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 24 \\ 6 \\ 1 \\ 2 \end{bmatrix}, \text{ and } \mathbf{c} = \begin{bmatrix} -5 \\ -4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

- But now we will use the slack variables as an initial basic feasible solution: $\{x_3, x_4, x_5, x_6\}$. Hence $\mathrm{x}_B = (x_3\ x_4\ x_5\ x_6)^T$, $\mathrm{x}_N = (x_1\ x_2)^T$, $\mathbf{B} = I_4 = \mathbf{B}^{-1}$, $\mathbf{c}_B^T = (0\ 0\ 0\ 0)$, $\mathbf{c}_N^T = (-5\ -4)$, and

$$N = \begin{bmatrix} 6 & 4 \\ 1 & 2 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

- We compute the basis: $\mathbf{x}_B = \widehat{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} = (24\ \ 6\ \ 1\ \ 2)^T$; then

$$\mathbf{y}^T = \mathbf{c}_B^T\mathbf{B}^{-1} = (0\ \ 0\ \ 0\ \ 0),\ \widehat{\mathbf{c}}_N^T = \mathbf{c}_N^T - \mathbf{y}^T\mathbf{N} = (-5\ \ -4).$$

- The current basis is not optimal because $\widehat{\mathbf{c}}_N$ has negative components; we choose $\widehat{c}_1 < 0$ - hence, x_1 will be the *entering varable*. We also compute

$$\widehat{\mathbf{A}}_1 = \mathbf{B}^{-1}\mathbf{A}_1 = (6\ \ 1\ \ -1\ \ 0)^T$$

- In order to find out the *leaving variable* we apply the ratio test:

$$\frac{\widehat{b}_3}{\widehat{a}_{31}} = 4, \frac{\widehat{b}_4}{\widehat{a}_{41}} = 6 \;\; (\widehat{a}_{51} < 0, \widehat{a}_{61} = 0).$$

- x_3 will be the leaving variable;
- In the next iteration x_1 will replace x_3 in the new basis: $\mathbf{x}_B = (x_1 \; x_4 \; x_5 \; x_6)^T, \mathbf{x}_N = (x_3 \; x_2)^T$,

$$\mathbf{B} = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{B}^{-1} = \begin{bmatrix} 1/6 & 0 & 0 & 0 \\ -1/6 & 1 & 0 & 0 \\ 1/6 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{N} = \begin{bmatrix} 1 & 4 \\ 0 & 2 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$\mathbf{c}_B = (-5 \; 0 \; 0 \; 0)^T, \mathbf{c}_N = (0 \; -4)^T$.

- Thus

$$\mathbf{x}_B = \widehat{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} = (4\ \ 2\ \ 5\ \ 2)^T,$$

$$\mathbf{y}^T = \mathbf{c}_B^T\mathbf{B}^{-1} = (-5/6\ \ 0\ \ 0\ \ 0),$$

$$\widehat{\mathbf{c}}_N^T = \mathbf{c}_N^T - \mathbf{y}^T\mathbf{N} = (5/6\ \ -4/6).$$

- The second reduced cost is negative, hence the current base is not optimal, x_2 will be the next the entering variable ...

Tableau Implementation

- The tableaux are a convenient and compact form to present the simplex algorithm; they are just a notational tool.
- In this format the inverse of basis matrix are updated at every iteration and not computed anew, increasing the speed of the method.
- The original LP problem correspond to the tableau

	$\mathbf{x}_B$	$\mathbf{x}_N$	RHS
$\mathbf{x}_B$	$\mathbf{B}$	$\mathbf{N}$	$\mathbf{b}$
z	$\mathbf{c}_B^T$	$\mathbf{c}_N^T$	0

- The tableau for the problem in the current basis is

	$\mathbf{x}_B$	$\mathbf{x}_N$	RHS
$\mathbf{x}_B$	I_n	$\mathbf{B}^{-1}\mathbf{N}$	$\mathbf{B}^{-1}\mathbf{b}$
z	$\mathbf{0}$	$\mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N}$	$\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b}$

- We consider the following problem

$$
\begin{aligned}
\text{minimize} \quad & z = -10x_1 - 12x_2 - 12x_3 \\
\text{subject to} \quad & \\
& x_1 + 2x_2 + 2x_3 \leqslant 20 \\
& 2x_1 + x_2 + 2x_3 \leqslant 20 \\
& 2x_1 + 2x_2 + x_3 \leqslant 20 \\
& x_1, x_2, \ldots, x_3 \geqslant 0
\end{aligned}
$$

- In standard form, the problem becomes

$$
\begin{array}{ll}
\text{minimize} & z = -10x_1 - 12x_2 - 12x_3 \\
\text{subject to} & \\
& \begin{aligned}
x_1 + 2x_2 + 2x_3 + x_4 &= 20 \\
2x_1 + x_2 + 2x_3 + x_5 &= 20 \\
2x_1 + 2x_2 + x_3 + x_6 &= 20 \\
x_1, x_2, \ldots, x_6 &\geqslant 0
\end{aligned}
\end{array}
$$

- Observe that $\mathrm{x} = (0\ 0\ 0\ 20\ 20\ 20)^T$ is a basic feasible solution and it can start the algorithm.

Table: First Simplex tableau (note that $\mathbf{B} = I_3$).

	x_1	x_2	x_3	x_4	x_5	x_6	RHS		
x_4	1	2	2	1	0	0	20	20/1	
x_5	$\boxed{2}$	1	2	0	1	0	20	20/2	$\leftarrow$ min
x_6	2	2	1	0	0	1	20	20/2	$\leftarrow$ min
z	-10	-12	-12	0	0	0	0		

- The reduced cost of x_1 is negative: we let this variable to *enter the basis*; the *pivot column* is that labeled by x_1.
- The smallest ratio corresponds to either the row labeled by x_5 or by x_6; we choose the former. That will be the *pivot row*; x_5 will *leave the basis*.

- The process of updating the tableau is called *pivoting*: add to each row of the tableau a constant multiple of the pivot row, such that the pivot element becomes 1 and all other entries of the pivot column become 0.
- We apply this rule of transformation to our tableau
  - multiply the pivot row by -0.5 and add it to the first row;
  - substract the pivot row from the third row;
  - multiply the pivot row by 5 and add it to the last row;
  - finally we divide the pivot row by 2.
- The tableau becomes (x_1 will now label the second row):

Table: Second Simplex tableau.

	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_4	0	1.5	1	1	-0.5	0	10
x_1	1	0.5	1	0	0.5	0	10
x_6	0	1	-1	0	-1	1	0
z	0	-7	-2	0	5	0	100

- The current basis is $\{x_1, x_4, x_6\}$ - not an optimal one, since the non-basic variables x_2 and x_3 have negative reduced costs. We choose x_3 to be the *entering variable*.
- Minimum ratio corresponds to both first and second rows; we choose x_4 to be the *leaving variable*.
- The position of the *pivot* is at the intersection of the first row with the third column.

Table: Second Simplex tableau decorated.

	x_1	x_2	x_3	x_4	x_5	x_6	RHS		
x_4	0	1.5	$\boxed{1}$	1	-0.5	0	10	10/1	← min
x_1	1	0.5	1	0	0.5	0	10	10/1	← min
x_6	0	1	-1	0	-1	1	0		
z	0	-7	-2	0	5	0	100		

- We pivot again:
  - substract the pivot row from the second row;
  - add the pivot row to the third row;
  - multiply by 2 the pivot row and add it to the last row;
  - the pivot row remains unchanged (since the pivot has value 1).

Table: Third Simplex tableau.

	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_3	0	1.5	1	1	-0.5	0	10
x_1	1	-1	0	-1	1	0	0
x_6	0	2.5	0	1	-1.5	1	10
z	0	-4	0	2	4	0	120

- The current basis is $\{x_1, x_3, x_6\}$ - not an optimal one, since the non-basic variable x_2 has negative reduced cost; x_2 will be the *entering variable*.
- Minimum ratio corresponds to both third row; we choose x_6 to be the *leaving variable*.
- The position of the *pivot* is at the intersection of the third row with the second column.

Table: Third Simplex tableau decorated.

	x_1	x_2	x_3	x_4	x_5	x_6	RHS		
x_3	0	1.5	1	1	-0.5	0	10	10/1.5	
x_1	1	-1	0	-1	1	0	0		
x_6	0	$\boxed{2.5}$	0	1	-1.5	1	10	10/2.5	$\leftarrow$ min
z	0	-4	0	2	4	0	120		

- We pivot again:
  - multiply by -0.6 the pivot row and add it to the first row;
  - multiply by 0.4 the pivot row and add it to the second row;
  - multiply by 1.6 the pivot row and add it to the last row;
  - divide the pivot row by 2.5.

Table: Fourth Simplex tableau.

	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_3	0	0	1	0.4	0.4	-0.6	4
x_1	1	0	0	-0.6	0.4	0.4	4
x_2	0	1	0	0.4	-0.6	0.4	4
z	0	0	0	3.6	1.6	1.6	136

- The current basis - $\{x_1, x_2, x_3\}$ - is an optimal one, since all non-basic variables have nonnegative reduced costs.
- We find out an optimal solution: $x_1 = x_2 = x_3 = 4$, the optimal value of the objective function is -136 (don't forget to change the sign).

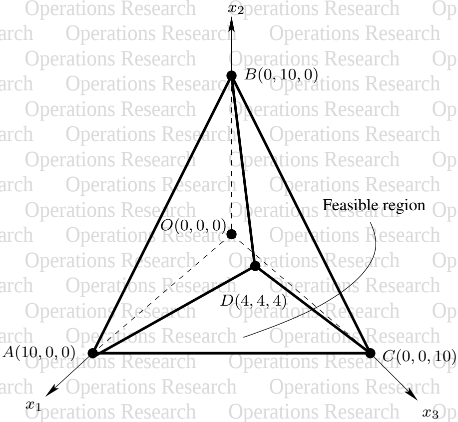

Figure: Feasible Region of our example

- If compare the walk of our simplex tableaus with the graphic of the feasible region we see that this corresponds to the path O, A, C, D.
- Obviously, if we choose other variables to enter or to leave (when this is possible), we may find another path through the extreme points of the feasible region.
- However, some paths are not eligible for the simplex algorithm: path O, A, D could not be traced, since the initial and the final bases differ by three variables (at least three basis changes are required).

- Obviously, an LP problem can have more than one optimal solution: such a problem can have one, none or an infinite set of optimal solutions.
- We can detect such situations: when, for an optimal basic feasible solution x, one of the non-basic variables, $x_j, j \in N$, has zero reduced cost: $\widehat{c}_j = 0$.
- If we let x_j enter the current base, the new base will give the same value for the objective function. Hence, we have another optimal solution; by, say, geometric reasons, this imply that we will have an infinite number of solutions.
- It is easy to check that, if $\mathrm{x}^1, \mathrm{x}^2$ are optimal solution for a LP problem, then any vector of the line segment joining x^1 and x^2 is an optimal solution too.

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Operations Research - Lecture 4

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Table of contents

- In the last lecture we reviewed some of the issues related with the Simplex method. Part of them are globally linked to the framework of solving an LP problem, but some of them are strictly related to the algorithm.
- The special situations we will discuss here are:
  - Degeneracy.
  - Unboundedness.
  - Multiple optimal solutions.
  - Cycling.
  - Initial basic feasible solution.

Throught this section we will consider a LP problem in standard form

$$\begin{aligned} \text{minimize} \quad & z = \mathbf{c}^T\mathbf{x}, \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geqslant 0. \end{aligned} \tag{1}$$

Definition

Let $\mathbf{x}$ be a basic feasible solution to problem (1). $\mathbf{x}$ is said to be degenerate if $x_i = \widehat{b}_i = 0$, for some $i \in B$.

- That is *degeneracy* occurs when the current basic feasible solution has a basic variable having zero value.

- If the current basis is degenerate, it is possible that a zero value basic variable to be chosen to leave the basis.
- Degeneracy is a sign of redundancy in information; a side effect is that the value of the objective function doesn't change, hence the algorithm doesn't progress.
- If this issue occurs we may find ourselves in a more difficult situation: cycling, which is enabled by the existence of degenerate bases.

Table: An Example of Degeneracy.

	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_4	0	1.5	1	1	-0.5	0	10
x_1	1	0.5	1	0	0.5	0	10
x_6	0	1	-1	0	-1	1	0
z	0	-7	-2	0	5	0	100

Definition

Problem (1) is said to be unbounded if doesn't have a finite optimal feasible solution.

- Unboundedness means that the "optimal" value of objective is $-\infty$.
- In Simplex this situation is revealed when we can't find a leaving variable.

Table: An Example of Unboundedness.

	x_1	x_2	x_3	x_4	x_5	RHS
x_1	1	0	-2	3	0	1
x_5	0	0	0	-1	1	5
x_2	0	1	-1	0	0	3
z	0	2	-2	0	0	12

Definition

Problem (1) has multiple optimal solutions if there exist $\mathrm{x}^1 \neq \mathrm{x}^2$, *both optimal feasible solutions of it.*

- We already know from the last lecture that, if we have two different optimal feasible solutions, then we have an infinite number of optimal feasible solutions.
- In Simplex framework: when we have an optimal basic feasible solution with a non-basic solution having a zero reduced cost.
- In this situation a non-basic variable can be introduced in the current basis; the next basis will be optimal too.

Table: Simplex Example of Alternate Optimal Solutions.

	x_1	x_2	x_3	x_4	x_5	RHS		
x_1	1	3	0	$\boxed{3}$	0	1	1/3	← min
x_5	0	1	0	-1	1	5		
x_3	0	2	1	0	0	3		
z	0	2	0	0	0	12		

	x_1	x_2	x_3	x_4	x_5	RHS
x_4	1/3	1	0	1	0	1/3
x_5	1/3	2	0	0	1	16/3
x_3	0	2	1	0	0	3
z	0	2	0	0	0	12

$(1\ 0\ 3\ 0\ 5)^T$ and $(0\ 0\ 3\ 1\ 16/3)^T$ are both optimal basic feasible solutions.

Definition

A cycle occurs in the execution of the Simplex algorithm if, after a finite number of iterations, we meet an already computed tableau.

- Consider the following LP problem

$$
\begin{array}{ll}
\text{minimize} & z = -3/4x_1 + 20x_2 - 1/2x_3 + 6x_4 + 3 \\
\text{subject to} & \\
& \begin{array}{rcl}
1/4x_1 - 8x_2 - x_3 + 9x_4 & \leqslant & 0 \\
1/2x_1 - 12x_2 - 1/2x_3 + 3x_4 & \leqslant & 0 \\
x_3 & \leqslant & 1 \\
x_1, x_2, \ldots, x_4 & \geqslant & 0
\end{array}
\end{array}
$$

Cycling

- In standard form, the problem becomes

$$
\begin{array}{ll}
\text{minimize} & z = -3/4x_1 + 20x_2 - 1/2x_3 + 6x_4 + 3 \\
\text{subject to} & \\
& \begin{aligned}
1/4x_1 - 8x_2 - x_3 + 9x_4 + x_5 &= 0 \\
1/2x_1 - 12x_2 - 1/2x_3 + 3x_4 + x_6 &= 0 \\
x_3 + x_7 &= 1 \\
x_1, x_2, \ldots, x_7 &\geqslant 0
\end{aligned}
\end{array}
$$

- We will use the following rules for finding a pivot:
  - the entering variable will be that with the most negative reduced cost (sometimes called the Dantzig rule);
  - the leaving variable will be that with the smallest index among those that are eligible (for leaving).

Table: First Simplex Tableau.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS		
x_5	$\boxed{1/4}$	-8	-1	9	1	0	0	0	0/0.25	← min
x_6	1/2	-12	-1/2	3	0	1	0	0	0/0.5	← min
x_7	0	0	1	0	0	0	0	1		
z	$-3/4$	20	-1/2	6	0	0	0	3		

Table: Second Simplex Tableau.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS		
x_1	1	-32	-4	36	4	0	0	0		
x_6	0	$\boxed{4}$	3/2	-15	-2	1	0	0	0/4	← min
x_7	0	0	1	0	0	0	0	1		
z	0	-4	$-7/2$	33	3	0	0	3		

Cycling

Table: Third Simplex Tableau.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS		
x_1	1	0	$\boxed{8}$	-84	-12	8	0	0	0/8	$\leftarrow$ min
x_2	0	1	3/8	-30/8	-1/2	1/4	0	0	0/0.375	$\leftarrow$ min
x_7	0	0	1	0	0	0	0	1	1/1	
z	0	0	-2	18	1	1	0	3		

Table: Fourth Simplex Tableau.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS		
x_3	1/8	0	1	-21/2	-3/2	1	0	0		
x_2	-3/64	1	0	$\boxed{3/16}$	1/16	-1/8	0	0	0/0.1875	$\leftarrow$ min
x_7	-1/8	0	0	21/2	3/2	-1	1	1	2/21	
z	1/4	0	0	-3	-2	3	0	3		

Cycling

Table: Fifth Simplex Tableau.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS		
x_3	-5/2	56	1	0	$\boxed{2}$	-6	0	0	0/2	← min
x_4	-1/4	16/3	0	1	1/3	-2/3	0	0	0/0.33	← min
x_7	5/2	-56	0	0	-2	6	1	1		
z	-1/2	16	0	0	-1	1	0	3		

Table: Sixth Simplex Tableau.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS		
x_5	-5/4	28	1/2	0	1	-3	0	0		
x_4	1/6	-4	-1/6	1	0	$\boxed{1/3}$	0	0	0/0.33	← min
x_7	0	0	1	0	0	0	1	1		
z	-7/4	44	1/2	0	0	-2	0	3		

Cycling

Table: Seventh Simplex Tableau.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_5	1/4	-8	-1	9	1	0	0	0
x_6	1/2	-12	-1/2	3	0	1	0	0
x_7	0	0	1	0	0	0	0	1
z	-3/4	20	-1/2	6	0	0	0	3

- After six pivots we are again in the initial situation, with the same tableau, and same basis.
- This sequence of pivots can be repeated over and over, and the algorithm never ends.

- Obviously, this situation is induced by the degeneracy - all the intermediate bases are (and must be) degenerate (why?).
- Although the degeneracy doesn't always imply cycling, without degeneracy we cannot have cycles.
- The solution to this issue stands in choosing a certain pivoting rule, degeneracy being sometimes unavoidable - as in our example.
- We will describe below two anticycling rules: lexicographic and Bland's rule.

Definition

Let $\mathbf{u} \neq \mathbf{v} \in \mathbb{R}^n$; $\mathbf{u}$ *is* *lexicographically larger* *than* $\mathbf{v}$, *and write* $\mathbf{u} >_L \mathbf{v}$, *if the first non-zero component of* $\mathbf{u} - \mathbf{v}$ *is positive.*

- *Lexicographic Pivoting Rule*:
  - Choose an entering variable x_j as long as its reduced cost is negative; let $\mathbf{u}$ be the column corresponding to x_j (i.e., the jth column).
  - For each $u_i > 0$, divide the ith row of the table by u_i, and choose the lexicographically smalest row - this will be the row of the leaving variable.
- *Bland's Rule* (smallest index pivoting rule):
  - Find the smallest index j such that the reduced cost $\widehat{c}_j$ is negative.
  - Among all the indexes k for which $\dfrac{\widehat{b}_k}{\widehat{a}_{kl}} = \min\left\{\dfrac{\widehat{b}_h}{\widehat{a}_{hl}} : \widehat{a}_{hl} > 0\right\}$, choose the minimum one - the variable which labels the kth row will be the leaving variable.

- Simplex algorithm iterates from one basic feasible solution to another until an optimal solution is found or until unboundedness is proved.
- In our examples the initial basic feasible solution is the set formed with all slack variables. This was possible because the original problem has all constraints of the form $\mathbf{Ax} \leqslant \mathbf{b}$ and $\mathbf{b} \geqslant \mathbf{0}$.
- By introducing slack variables the constraints become $\mathbf{Ax} + \mathbf{s} = \mathbf{b}$. The vector $(\mathbf{x}, \mathbf{s})$ with $\mathbf{s} = \mathbf{b}$ and $\mathbf{x} = \mathbf{0}$ is a basic feasible solution (with $\mathbf{B} = I$).

- Usually, problems in standard form may have constraints which doesn't contain any slack variable. In this way occurs the following question: *how to choose an initial basic feasible solution for a problem in general form?*
- This section give two answers to the above question: the **Two Phase Method** and the **Big M Method**.
- Both these methods rely on solving an auxiliary LP problem; after that we can know if the original problem has or has not an initial basic feasible solution
- That is, our methods will tell us if the original problem has or has not feasible solutions at all, since having a feasible solution means having a basic feasible solution also.

- Consider a problem in standard form ($\mathbf{b} \geqslant \mathbf{0}$)

$$\begin{aligned} \text{minimize} \quad & z = \mathbf{c}^T\mathbf{x}, \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geqslant 0. \end{aligned} \tag{2}$$

- We introduce a vector of artificial variables $\mathbf{y} \in \mathbb{R}^m$, that will play the role of slack variables vector, and replace the constraints with

$$\begin{aligned} & \mathbf{A}\mathbf{x} + \mathbf{y} = \mathbf{b}, \\ & \mathbf{x}, \mathbf{y} \geqslant 0. \end{aligned} \tag{3}$$

- Obviously, this will be a distinct problem, and the objective function will be modified in different ways by the above methods.
- Sometimes it is not necessary to add m artificial variables, since some of the original variables can play the roles of slack variables.

Finding Initial Basic Feasible Solutions - Example

- We will use the following example

$$\begin{array}{ll} \text{minimize} & z = 2x_1 + 3x_2 \\ \text{subject to} & \end{array}$$

$$\begin{aligned} 3x_1 + 2x_2 &= 14 \\ 2x_1 - 4x_2 &\geqslant 2 \\ 4x_1 + 3x_2 &\leqslant 19 \\ x_1, x_2 &\geqslant 0 \end{aligned}$$

- In standard form the problem becomes

$$\begin{array}{ll} \text{minimize} & z = 2x_1 + 3x_2 \\ \text{subject to} & \end{array}$$

$$\begin{aligned} 3x_1 + 2x_2 &= 14 \\ 2x_1 - 4x_2 - x_3 &= 2 \\ 4x_1 + 3x_2 + x_4 &= 19 \\ x_1, x_2, x_3, x_4 &\geqslant 0 \end{aligned}$$

- We add artificial variables

$$
\begin{array}{ll}
\text{minimize} & z = 2x_1 + 3x_2 \\
\text{subject to} & \\
& \begin{aligned} 3x_1 + 2x_2 + y_1 &= 14 \\ 2x_1 - 4x_2 - x_3 + y_2 &= 2 \\ 4x_1 + 3x_2 + x_4 &= 19 \\ x_1, x_2, x_3, x_4, y_1, y_2 &\geqslant 0 \end{aligned}
\end{array}
$$

- Now, we can start the Simplex with the initial base $\{y_1, y_2, x_4\}$; note that x_4 can play the role of a slack variable, hence, two artificial variables are enough.
- But this basis doesn't correspond to a basic feasible solution of the original problem, since the artificial variables doesn't belong to the original problem.

- In the Two Phase Method, the artificial variables are used to create an auxiliary LP problem - the *phase I problem*.
- This new problem aims only to find a basic feasible solution to the original problem.
- The objective for the phase I problem is

$$\text{minimize} \quad z' = \sum_j y_j .$$

- The phase I problem is

$$\begin{array}{ll} \text{minimize} & z' = \sum_j y_j, \\ \text{subject to} & \mathbf{A}\mathbf{x} + \mathbf{y} = \mathbf{b}, \\ & \mathbf{x}, \mathbf{y} \geqslant 0. \end{array} \tag{4}$$

- Let z'_* be the optimal value of the objective function for the phase I problem; note that this problem has a finite optimum, since it cannot be unbounded.
- If the original problem is feasible, then $z'_* = 0$, otherwise $z'_* > 0$. Hence, the original problem is feasible if and only if $z'_* = 0$.
- The phase I problem for our example is

$$\begin{array}{ll}
\text{minimize} & z = y_1 + y_2 \\
\text{subject to} & \\
& \begin{aligned} 3x_1 + 2x_2 + y_1 &= 14 \\ 2x_1 - 4x_2 - x_3 + y_2 &= 2 \\ 4x_1 + 3x_2 + x_4 &= 19 \\ x_1, x_2, x_3, x_4, y_1, y_2 &\geqslant 0 \end{aligned}
\end{array}$$

	x_1	x_2	x_3	x_4	y_1	y_2	
y_1	3	2	0	0	1	0	14
y_2	2	-4	-1	0	0	1	2
x_4	4	3	0	1	0	0	19
z'	0	0	0	0	1	1	0

- Obviously this tableau is not in a proper simplex form: we must express z' only in terms of non-basic variables, by eliminating basic (i.e., artificial) variables from their constraints:

$$y_1 = 14 - 3x_1 - 2x_2, y_2 = 2 - 2x_1 + 4x_2 + x_3,$$

$$z' = y_1 + y_2 = -5x_1 + 2x_2 + x_3 + 16.$$

Table: First Simplex Tableau - Phase I.

	x_1	x_2	x_3	x_4	y_1	y_2	RHS		
y_1	3	2	0	0	1	0	14	14/3	
y_2	$\boxed{2}$	-4	-1	0	0	1	2	2/2	← min
x_4	4	3	0	1	0	0	19	19/4	
z'	-5	2	1	0	0	0	-16		

Table: Second Simplex Tableau - Phase I.

	x_1	x_2	x_3	x_4	y_1	y_2	RHS		
y_1	0	8	3/2	0	1	-3/2	11	11/8	
x_1	1	-2	-1/2	0	0	1/2	1		
x_4	0	$\boxed{11}$	2	1	0	-2	15	15/11	← min
z'	0	-8	-3/2	0	0	5/2	-11		

Table: Third Simplex Tableau - Phase I.

	x_1	x_2	x_3	x_4	y_1	y_2	RHS	
y_1	0	0	$\boxed{1/22}$	-8/11	1	-1/22	1/11	2 $\leftarrow$ min
x_1	1	0	-3/22	2/11	0	3/22	41/11	
x_2	0	1	2/11	1/11	0	-2/11	15/11	15/2
z'	0	0	$-1/22$	8/11	0	23/22	-1/11	

Table: Fourth Simplex Tableau - Phase I.

	x_1	x_2	x_3	x_4	y_1	y_2	RHS
x_3	0	0	1	-16	22	-1	2
x_1	1	0	0	-2	3	0	4
x_2	0	1	0	3	-4	0	1
z'	0	0	0	0	1	1	0

- After three iterations, the curent basis doesn't contain any artificial variable and the objective value is zero, hence we have a basic feasible solution for the original problem.
- We can remove the columns corresponding to artificial variables and restate the original objective function:

	x_1	x_2	x_3	x_4	RHS
x_3	0	0	1	-16	2
x_1	1	0	0	-2	4
x_2	0	1	0	3	1
z	2	3	0	0	0

- Obviously, this is not a proper form Simplex tableau, since there are some non-zero reduced costs of basic variables; we must replace these variables from their equations:

$$x_1 = 2x_4 + 4, x_2 = -3x_4 + 1,$$

$$z = 2x_1 + 3x_2 = -5x_4 + 11.$$

Table: First Simplex Tableau - Phase II.

	x_1	x_2	x_3	x_4	RHS
x_3	0	0	1	-16	2
x_1	1	0	0	-2	4
x_2	0	1	0	3	1
z	0	0	0	-5	-11

- From this point we can use Simplex to solve the original problem - this is *phase II* (left as an exercise).

- After solving Phase I problem, it may happen that the optimal value is zero, but some artificial variables are basic ones, in this case we proceed like this:
  - Let the ith basic variable (from the optimal basic feasible solution) be an artificial one, x_h.
  - We choose an $\widehat{a}_{ij} \neq 0$, where x_j is a non-basic variable from the original problem and pivot such that x_h leaves and x_j enters the basis.
  - If we can't find such a variable x_j, then we can remove the ith line (it is not relevant for the original problem) and the hth column.
  - Repeat this steps until there are no more artificial basic variables.
  - After all that, transform the Simplex tableau to proper form and apply the second phase.

- Consider the problem

$$\begin{aligned} \text{minimize} \quad & z = x_1 + x_2 \\ \text{subject to} \quad & \\ & 2x_1 + x_2 + x_3 = 4 \\ & x_1 + x_2 + 2x_3 = 2 \\ & x_1, x_2, x_3 \geqslant 0 \end{aligned}$$

- We add artificial variables and modify the objective function

$$\begin{aligned} \text{minimize} \quad & z = y_1 + y_2 \\ \text{subject to} \quad & \\ & x_1 + x_2 + 2x_3 + y_1 = 2 \\ & 2x_1 + x_2 + x_3 + y_2 = 4 \\ & x_1, x_2, x_3, y_1, y_2 \geqslant 0 \end{aligned}$$

Table: First Simplex Tableau - Phase I.

	x_1	x_2	x_3	y_1	y_2	RHS		
y_1	$\boxed{1}$	1	2	1	0	2	2/1	← min
y_2	2	1	1	0	1	4	4/2	
z'	-3	-2	-3	0	0	-6		

Table: Second Simplex Tableau - Phase I.

	x_1	x_2	x_3	y_1	y_2	RHS
x_1	1	1	2	1	1	2
y_2	0	$\boxed{-1}$	-3	-2	1	0
z'	0	1	3	3	0	0

- The second tableau is already optimal, but the artificial y_2 remains in the basis; we eliminate it and introduce the (original) non-basic variable x_2 (the pivot is $-1 \neq 0$).

Table: Third Simplex Tableau - Phase I.

	x_1	x_2	x_3	y_1	y_2	RHS
x_1	1	0	-1	-2	2	2
x_2	0	1	3	2	-1	0
z'	0	0	0	1	1	0

- We remove the artificial variables and restate the original objective function in terms of the nonbasic variables:

Table: First Simplex Tableau - Phase II.

	x_1	x_2	x_3	RHS
x_1	1	0	-1	2
x_2	0	1	3	0
z'	0	0	-2	-2

- Now, one can proceed with the phase II (left as an exercise).

- We consider another example:

$$\begin{array}{ll} \text{minimize} & z = x_1 + 2x_2 \\ \text{subject to} & \\ & \begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 + 2x_2 &= 4 \\ x_1, x_2 &\geqslant 0 \end{aligned} \end{array}$$

- We add artificial variables and modify the objective function

$$\begin{array}{ll} \text{minimize} & z = y_1 + y_2 \\ \text{subject to} & \\ & \begin{aligned} x_1 + x_2 + y_1 &= 2 \\ 2x_1 + 2x_2 + y_2 &= 4 \\ x_1, x_2, y_1, y_2 &\geqslant 0 \end{aligned} \end{array}$$

Table: First Simplex Tableau - Phase I.

	x_1	x_2	y_1	y_2	RHS		
y_1	$\boxed{1}$	1	1	0	2	2/1	← min
y_2	2	2	0	1	4	4/2	
z'	-3	-3	0	0	-6		

Table: Second Simplex Tableau - Phase I.

	x_1	x_2	y_1	y_2	RHS
x_1	1	1	1	0	2
y_2	0	0	-2	1	0
z'	0	0	3	0	0

- The second tableau is already optimal, but the artificial y_2 remains in the basis.

- We cannot pivot again in order to eliminate y_2, since in the second row all the coefficients corresponding to non-basic variables from the original problem (namely, y_2) are zero.
- In this case we simply remove the row corresponding to variable y_2.
- Then, we remove the artificial variables and restate the original objective function.

Table: First Simplex Tableau - Phase II.

	x_1	x_2	RHS
x_1	1	1	2
z'	0	1	-2

- From now on we can start the phase II (left as an exercise).

- An example which shows the infeasability of the original problem:

$$\begin{array}{ll} \text{minimize} & z = -x_1 \\ \text{subject to} & \\ & \begin{aligned} x_1 + x_2 &\geqslant 6 \\ 2x_1 + 3x_2 &\leqslant 4 \\ x_1, x_2 &\geqslant 0 \end{aligned} \end{array}$$

- We convert to the standard form, add artificial variables and modify the objective function

$$\begin{array}{ll} \text{minimize} & z = y_1 \\ \text{subject to} & \\ & \begin{aligned} x_1 + x_2 - x_3 + y_1 &= 6 \\ 2x_1 + 3x_2 + x_4 &= 4 \\ x_1, x_2, x_3, x_4, y_1 &\geqslant 0 \end{aligned} \end{array}$$

Table: First Simplex Tableau - Phase I.

	x_1	x_2	x_3	x_4	y_1	RHS		
y_1	1	1	-1	0	1	6	6/1	
x_4	$\boxed{2}$	3	0	1	0	4	4/2	← min
z'	-1	-1	1	0	0	-6		

Table: Second Simplex Tableau - Phase I.

	x_1	x_2	x_3	x_4	y_1	
y_1	0	-1/2	-1	-1/2	1	4
x_1	1	3/2	0	1/2	0	2
z'	0	1/2	1	1/2	0	-4

- The Phase I problem has a non-zero optimum value, hence the original problem in infeasible. We must stop here - there is no Phase II problem.

- Historically, the big M method precedes the two phase method; it has been replaced due to the grater practical efficiency of the former.
- The big M method ensures that the artificial variables are zero in an (or, equivalently, in any) optimal feasible solution.
- That is, it pushes the artificial variables out of the optimal basis, by assigning a penalty cost M to each such variable in the objective function, where $M >$ is a big real number.
- Hence, instead of the original problem, we will solve

$$\begin{aligned} \text{minimize} \quad & z' = \mathbf{c}^t \mathbf{x} + \sum_j M y_j, \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} + \mathbf{y} = \mathbf{b}, \\ & \mathbf{x}, \mathbf{y} \geqslant 0. \end{aligned} \tag{5}$$

- Problem (2) has feasible solutions if and only if there is an optimal feasible solution of (5) having $\mathbf{y} = \mathbf{0}$.
- A basic feasible solution for (2) can be derived from an optimal solution to (5) in a similar manner to that of two phase method.
- Obviously, in order to solve (5), having the artificial variables as the initial basis, we must eliminate all of them from the objective function:

$$y_j = b_j - \sum_i a_{ji} x_i, \forall j$$

$$z' = \sum_i \left(c_i - M \sum_j a_{ji} \right) x_i + M \sum_j b_j.$$

- As for the Two Phase method, we don't add artificial variables to those constraints who already have slack variables.

- We will use again the following example

$$\begin{aligned} &\text{minimize} \quad z = 2x_1 + 3x_2 \\ &\text{subject to} \\ &\qquad 3x_1 + 2x_2 = 14 \\ &\qquad 2x_1 - 4x_2 \geqslant 2 \\ &\qquad 4x_1 + 3x_2 \leqslant 19 \\ &\qquad x_1, x_2 \geqslant 0 \end{aligned}$$

- In standard form the problem becomes

$$\begin{aligned} &\text{minimize} \quad z = 2x_1 + 3x_2 \\ &\text{subject to} \\ &\qquad 3x_1 + 2x_2 = 14 \\ &\qquad 2x_1 - 4x_2 - x_3 = 2 \\ &\qquad 4x_1 + 3x_2 + x_4 = 19 \\ &\qquad x_1, x_2, x_3, x_4 \geqslant 0 \end{aligned}$$

- We add artificial variables

$$\begin{aligned} \text{minimize} \quad & z = 2x_1 + 3x_2 + My_1 + My_2 \\ \text{subject to} \quad & \\ & 3x_1 + 2x_2 + y_1 = 14 \\ & 2x_1 - 4x_2 - x_3 + y_2 = 2 \\ & 4x_1 + 3x_2 + x_4 = 19 \\ & x_1, x_2, x_3, x_4, y_1, y_2 \geqslant 0 \end{aligned}$$

- Or, after we eliminate y_1 and y_2

$$\begin{aligned} \text{minimize} \quad & z = (2 - 5M)x_1 + (3 + 2M)x_2 + Mx_3 + 16M \\ \text{subject to} \quad & \\ & 3x_1 + 2x_2 + y_1 = 14 \\ & 2x_1 - 4x_2 - x_3 + y_2 = 2 \\ & 4x_1 + 3x_2 + x_4 = 19 \\ & x_1, x_2, x_3, x_4, y_1, y_2 \geqslant 0 \end{aligned}$$

Table: First Simplex Tableau.

	x_1	x_2	x_3	x_4	y_1	y_2	RHS	
y_1	3	2	0	0	1	0	14	14/3
y_2	$\boxed{2}$	-4	-1	0	0	1	2	1/2 ← min
x_4	4	3	0	1	0	0	19	19/2
z'	$2-5M$	3+2M	M	0	0	0	-16M	

Table: Second Simplex Tableau.

	x_1	x_2	x_3	x_4	y_1	y_2	RHS	
y_1	0	8	3/2	0	1	-3/2	11	
x_1	1	-2	-1/2	0	0	1/2	1	
x_4	0	$\boxed{11}$	2	1	0	-2	15	15/11 ←
z'	0	$7-8M$	1 - 3/2M	0	0	-1 + 5/2M	-2 - 11M	

Table: Third Simplex Tableau.

	x_1	x_2	x_3	x_4	y_1	y_2	RHS		
y_1	0	0	$\boxed{1/22}$	-8/11	1	-1/22	1/11	2	← min
x_1	1	0	-3/22	2/11	0	3/22	41/11		
x_2	0	1	2/11	1/11	0	-2/11	15/11	15/2	
z'	0	0	$-\frac{M+6}{22}$	$\frac{8M-7}{11}$	0	$\frac{6-23M}{22}$	$-\frac{127+M}{11}$		

Table: Fourth Simplex Tableau.

	x_1	x_2	x_3	x_4	y_1	y_2	RHS
x_3	0	0	1	-16	22	-1	2
x_1	1	0	0	-2	3	0	4
x_2	0	1	0	3	-4	0	1
z'	0	0	0	-5	M + 6	-12M/11	-11

- Although not optimal, the current basis doesn't contain any artificial variable, so this is a basic feasible solution for the original problem.
- We remove the artificial variable and restate the original objective function: $z = 2x_1 + 3x_2 = 11 - 5x_4$.

Table: Modified Simplex Tableau.

	x_1	x_2	x_3	x_4	RHS		
x_3	0	0	1	-16	2		
x_1	1	0	0	-2	4		
x_2	0	1	0	$\boxed{3}$	1	1/3	← min
z'	0	0	0	-5	-11		

- From this point we can use the simplex algorithm for the original problem ...

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Operations Research - Lecture 5

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Table of contents

Naïve motivation

- Consider the following canonical form LP problem

$$\begin{array}{lrcl} \text{minimize} & z = 4x_1 + x_2 + 3x_3 & & \\ \text{subject to} & x_1 - 2x_2 + x_3 & \geqslant & 2 \\ & 2x_1 + x_2 + x_3 & \geqslant & 3 \\ & x_1, x_2, x_3 & \geqslant & 0 \end{array} \tag{1}$$

- Every feasible solution gives an upper bound to the optimal objective function value: the solution $(x_1, x_2, x_3) = (2, 0, 0)$ says that $z_* \leqslant 8$.
- Now, if we multiply the first constraint by 2 and add it up the second constraint we get

$$\begin{array}{rrcl} & 2 \cdot (x_1 - 2x_2 + x_3) & \geqslant & 2 \cdot 2 \\ + & 2x_1 + x_2 + x_3 & \geqslant & 3 \\ \hline & 4x_1 - 3x_2 + 3x_3 & \geqslant & 7 \end{array}$$

- Comparing the last expression with the objective function we get

$$4x_1 + x_2 + 3x_3 \geqslant 4x_1 - 3x_2 + 3x_3 \geqslant 7,$$

for all $\mathrm{x} \in \mathbb{R}^3_+$. Therefore, $7 \leqslant z_* \leqslant 8$.

- A more systematic motivation will lead us to multiply the constraints not by specific numbers, but by variables, say y_1 and y_2.
- After that we can try to find values for this variables that gives us the best (largest) lower bound for optimal objective function value.
- Following this procedure we get ($y_1, y_2 \geqslant 0$):

$$\begin{array}{rcr} y_1 \cdot (x_1 - 2x_2 + x_3) & \geqslant & y_1 \cdot 2 \\ + \qquad\qquad y_2 \cdot (2x_1 + x_2 + x_3) & \geqslant & y_2 \cdot 3 \\ \hline (y_1 + 2y_2)x_1 + (-2y_1 + y_2)x_2 + (y_1 + y_2)x_3 & \geqslant & 2y_1 + 3y_2 \end{array}$$

- Now, we compare this sum (by seeing it as a lower bound) with the objective function:

$$z = 4x_1 + x_2 + 3x_3 \geqslant (y_1 + 2y_2)x_1 + (-2y_1 + y_2)x_2 + (y_1 + y_2)x_3 \geqslant 2y_1 + 3y_2$$

- Furthermore, we impose that every coefficient of x_i to be as small as the corresponding coefficient of x_i in the objective function:

$$\begin{aligned} y_1 + 2y_2 &\leqslant 4 \\ -2y_1 + y_2 &\leqslant 1 \\ y_1 + y_2 &\leqslant 3 \end{aligned}$$

- We found ourselves in the face of a new optimization (maximization) problem:

$$\begin{array}{lrcl} \text{maximize} & w = 2y_1 + 3y_2 & & \\ \text{subject to} & y_1 + 2y_2 & \leqslant & 4 \\ & -2y_1 + y_2 & \leqslant & 1 \\ & y_1 + y_2 & \leqslant & 3 \\ & y_1, y_2 & \geqslant & 0 \end{array} \quad (2)$$

- This problem is called the *dual LP problem* associated with problem (1).
- The above procedure is called *Lagrange multiplier method* - a more general method used for mimimize a function under some equation constraints.

Consider the problem in canonical form, called the primal problem:

$$\begin{array}{rl} \text{minimize} & z = \mathrm{c}^T \mathrm{x} \\ \text{subject to} & \mathrm{A}\mathrm{x} \geqslant \mathrm{b} \\ & \mathrm{x} \geqslant 0. \end{array} \tag{3}$$

By definition the associated dual problem of (3) is

$$\begin{array}{rl} \text{maximize} & w = \mathrm{b}^T \mathrm{y} \\ \text{subject to} & \mathrm{A}^T \mathrm{y} \leqslant \mathrm{c} \\ & \mathrm{y} \geqslant 0. \end{array} \tag{4}$$

Definition

A LP maximization problem is said to be in canonical form if it is presented like problem (4). That is, all the constraints are "$\leqslant$", and all variables are nonnegative.

- We know that every LP problem can be converted to a minimization problem and then converted in canonical form.
- Hence, if we see the dual problem as a minimization problem in its canonical form, we can dualize it again. It is no surprize that, by doing so, we get the primal problem.

Lemma

The dual of the dual problem is the primal problem.

Properties Related to Duality

Proof. First, let us convert problem (4) to a minimization problem in canonical form

$$\begin{aligned} \text{minimize} \quad & w' = -\mathrm{b}^T \mathrm{y} \\ \text{subject to} \quad & -\mathrm{A}^T \mathrm{y} \geqslant -\mathrm{c} \\ & \mathrm{y} \geqslant 0. \end{aligned}$$

The dual of this problem is

$$\begin{aligned} \text{maximize} \quad & z' = -\mathrm{c}^T \mathrm{x} \\ \text{subject to} \quad & (-\mathrm{A}^T)^T \mathrm{x} \leqslant -\mathrm{b} \\ & \mathrm{x} \geqslant 0. \end{aligned}$$

Which is equivalent to

$$\begin{aligned} \text{minimize} \quad & z = \mathrm{c}^T \mathrm{x} \\ \text{subject to} \quad & \mathrm{A}\mathrm{x} \geqslant \mathrm{b} \\ & \mathrm{x} \geqslant 0, \end{aligned}$$

that is, the primal problem (3). □

- As we already pointed out, every LP problem can be transformed into a minimization problem and then converted in canonical form.
- Therefore, any LP problem has a dual. The rules of general duality will be deduced after we will apply these transformations.
- Let us consider an LP problem with non-negative variables, that is

$$
\begin{aligned}
\text{minimize} \quad & z = \mathrm{c}^T \mathrm{x} \\
\text{subject to} \quad & \mathrm{A}_1 \mathrm{x} \geqslant \mathrm{b}_1 \\
& \mathrm{A}_2 \mathrm{x} \leqslant \mathrm{b}_2 \\
& \mathrm{A}_3 \mathrm{x} = \mathrm{b}_3 \\
& \mathrm{x} \geqslant 0
\end{aligned}
\tag{5}
$$

- We convert it to a canonical form

$$\begin{array}{lrcl} \text{minimize} & z = \mathrm{c}^T \mathrm{x} & & \\ \text{subject to} & \mathrm{A}_1 \mathrm{x} & \geqslant & \mathrm{b}_1 \\ & -\mathrm{A}_2 \mathrm{x} & \geqslant & -\mathrm{b}_2 \\ & \mathrm{A}_3 \mathrm{x} & \geqslant & \mathrm{b}_3 \\ & -\mathrm{A}_3 \mathrm{x} & \geqslant & -\mathrm{b}_3 \\ & \mathrm{x} \geqslant 0 & & \end{array}$$

- We must define four groups of dual variables: one for every group of constraints: $\mathrm{y}_1, \mathrm{y}_2', \mathrm{y}_3', \mathrm{y}_3''$. The dual problem is

$$\begin{array}{lrcl} \text{maximize} & w = \mathrm{b}_1^T \mathrm{y}_1 - \mathrm{b}_2^T \mathrm{y}_2' + \mathrm{b}_3^T \mathrm{y}_3' - \mathrm{b}_3^T \mathrm{y}_3'' & & \\ \text{subject to} & \mathrm{A}_1^T \mathrm{y}_1 - \mathrm{A}_2^T \mathrm{y}_2' + \mathrm{A}_3^T \mathrm{y}_3' - \mathrm{A}_3^T \mathrm{y}_3'' & \leqslant & \mathrm{c} \\ & \mathrm{y}_1, \mathrm{y}_2', \mathrm{y}_3', \mathrm{y}_3'' \geqslant 0 & & \end{array} \tag{6}$$

- If we change the notations, by putting $y_2 = y_2'$ and $y_3 = y_3' - y_3''$, then we have an equivalent form of the dual

$$\begin{array}{lrcl} \text{maximize} & w = b_1^T y_1 + b_2^T y_2 + b_3^T y_3 & & \\ \text{subject to} & A_1^T y_1 + A_2^T y_2 + A_3^T y_3 & \leqslant & c \\ & y_1 \geqslant 0, y_2 \leqslant 0, y_3 \text{ unrestricted} & & \end{array}$$

- The directions of the constraints in the primal problem are not canonical, this implies that the signs of the variables in the dual problem are not canonical. The rules are
  - the dual variables associated with "$\geqslant$" constraints are nonnegative;
  - the dual variables associated with "$\leqslant$" constraints are nonpositive;
  - the dual variables associated with equations are unrestricted.

- Now, let us onsider a LP problem having only the constraints in canonical form, i. e.,

$$\begin{array}{lrcl} \text{minimize} & z = c_1^T x_1 + c_2^T x_2 + c_3^T x_3 & & \\ \text{subject to} & A_1 x_1 + A_2 x_2 + A_3 x_3 & \geqslant & b \\ & x_1 \geqslant 0, x_2 \leqslant 0, x_3 \text{ unrestricted} & & \end{array} \tag{7}$$

- We put this problem in canonical form by converting the variables, $x_2' = -x_2$ and $x_3 = x_3' - x_3''$:

$$\begin{array}{lrcl} \text{minimize} & z = c_1^T x_1 - c_2^T x_2' + c_3^T x_3' - c_3^T x_3' & & \\ \text{subject to} & A_1 x_1 - A_2 x_2' + A_3 x_3' - A_3 x_3'' & \geqslant & b \\ & x_1, x_2', x_3', x_3'' \geqslant 0 & & \end{array}$$

- The dual of the later problem is

$$\begin{array}{lrcl} \text{maximize} & w = \mathbf{b}^T\mathbf{y} & & \\ \text{subject to} & \mathrm{A}_1^T\mathbf{y} & \leqslant & \mathbf{c}_1 \\ & -\mathrm{A}_2^T\mathbf{y} & \leqslant & -\mathbf{c}_2 \\ & \mathrm{A}_3^T\mathbf{y} & \leqslant & \mathbf{c}_3 \\ & -\mathrm{A}_3^T\mathbf{y} & \leqslant & -\mathbf{c}_3 \\ & \mathbf{y} \geqslant 0 & & \end{array}$$

- Which is equivalent with

$$\begin{array}{lrcl} \text{maximize} & w = \mathbf{b}^T\mathbf{y} & & \\ \text{subject to} & \mathrm{A}_1^T\mathbf{y} & \leqslant & \mathbf{c}_1 \\ & \mathrm{A}_2^T\mathbf{y} & \geqslant & \mathbf{c}_2 \\ & \mathrm{A}_3^T\mathbf{y} & = & \mathbf{c}_3 \\ & \mathbf{y} \geqslant 0 & & \end{array} \tag{8}$$

- We see that the signs of the variables in the primal problem influence the types of constraints in the dual problem. The rules are:
  - the dual constraints associated with nonnegative variables are "$\leqslant$", i.e., consistent with canonical form;
  - the dual constraints associated with nonpositive variables are "$\geqslant$"i.e., reversed from canonical form;
  - the dual constraints associated with unrestricted variables are equations.

Table: Duality Rules

primal/dual constraint		dual/primal variable
consistent with canonical form	$\Leftrightarrow$	variable nonnegative ($\geqslant$)
reversed from canonical form	$\Leftrightarrow$	variable nonpositive ($\leqslant$)
equation	$\Leftrightarrow$	variable unrestricted

- Consider the primal problem

$$\begin{array}{lrcr} \text{maximize} & w = 2x_1 - 3x_2 + 3x_3 + 2x_4 & & \\ \text{subject to} & 3x_1 + x_2 - 2x_3 & \geqslant & 5 \\ & 2x_1 - x_2 + 2x_3 + 2x_4 & = & 4 \\ & x_1 + x_2 + x_3 - x_4 & \leqslant & 2 \\ & \multicolumn{3}{r}{x_1 \geqslant 0, x_2, x_4 \leqslant 0, x_3 \text{ unrestricted}} \end{array} \tag{9}$$

- Its dual is

$$\begin{array}{lrcr} \text{minimize} & z = 5y_1 + 4y_2 + 2y_3 & & \\ \text{subject to} & 3y_1 + 2y_2 + y_3 & \geqslant & 2 \\ & y_1 - y_2 + y_3 & \leqslant & -3 \\ & -2y_1 + 2y_2 + y_3 & = & 3 \\ & 2y_2 - y_3 & \leqslant & 2 \\ & \multicolumn{3}{r}{y_1 \leqslant 0, y_2 \text{ unrestricted}, y_3 \geqslant 0} \end{array} \tag{10}$$

Weak Duality

- Consider the following pair of primal/dual problems

$$\left\{\begin{array}{rl} \text{minimize} & z = c^T x \\ \text{subject to} & Ax = b \\ & x \geqslant 0. \end{array}\right. \qquad \left\{\begin{array}{rl} \text{maximize} & w = b^T y \\ \text{subject to} & A^T y \leqslant c \\ & y \text{ unrestricted.} \end{array}\right. \tag{11}$$

Theorem

(Weak Duality) *Let* x *be a feasible solution for the primal problem and* y *a feasible solution for the dual problem. Then*

$$z = c^T x \geqslant b^T y = w.$$

Proof. Using the inequalities from both, the primal and the dual problems, and the nonnegativity of all variables $z = \mathrm{c}^T\mathrm{x} \geqslant (\mathrm{A}^T\mathrm{y})^T\mathrm{x} = \mathrm{y}^T\mathrm{A}\mathrm{x} = \mathrm{y}^T\mathrm{b} = \mathrm{b}^T\mathrm{y} = w$. □

- The importance of the Weak Duality Theorem comes from the following consequence.

Corollary

(i) *Unboundedness of the primal problem implies the infeasibility of the dual.*

(ii) *Let* x *be a feasible solution for the primal problem and* y *a feasible solution for the dual problem, such that* $\mathrm{c}^T\mathrm{x} = \mathrm{b}^T\mathrm{y}$*. Then* x *and* y *are optimal for their respective problems.*

Proof. (i) If the primal problem is unbounded, then it exists a sequence of feasible solutions $\left(\mathrm{x}^k\right)_{k\geqslant 0}$, such that $\lim_{k\to\infty} \mathrm{c}^T\mathrm{x}^k = -\infty$.

Suppose, on the contrary, that the dual problem has a feasible solution y. From Weak Duality Theorem we have $\mathrm{c}^T\mathrm{x}^k \geqslant \mathrm{b}^T\mathrm{y}$, for all $k \geqslant 0$. Hence, $-\infty \geqslant \mathrm{b}^T\mathrm{y}$ - which is a contradiction: the dual problem is infeasible.

(ii) If both problems are feasible, then they are both bounded; let x_* and y_* two optimal solutions for their respective problems, primal and dual. From the following sequence or relations

$$\mathrm{c}^T\mathrm{x} \geqslant \mathrm{c}^T\mathrm{x}_* \geqslant \mathrm{b}^T\mathrm{y}_* \geqslant \mathrm{b}^T\mathrm{y} = \mathrm{c}^T\mathrm{x},$$

follows that $\mathrm{c}^T\mathrm{x} = \mathrm{c}^T\mathrm{x}_* = \mathrm{b}^T\mathrm{y}_* = \mathrm{b}^T\mathrm{y}$. □

- Consider the following pair of primal/dual problems

$$(P)\begin{cases} \max & z = 2x_1 + 3x_2 \\ \text{s. t.} & x_1 - 2x_2 \leqslant 2 \\ & x_1 + x_2 \geqslant 1 \\ & x_1, x_2 \geqslant 0 \end{cases} \qquad (D)\begin{cases} \min & w = 2y_1 + y_2 \\ \text{s. t.} & y_1 + y_2 \geqslant 2 \\ & -2y_1 + y_2 \geqslant 3 \\ & y_1 \geqslant 0, y_2 \leqslant 0 \end{cases}$$

Since (P) is unbounded, (D) must be infeasible (*verify!*).

- Now, consider another pair of primal/dual problems

$$(P')\begin{cases} \min & z = -x_1 + 2x_2 \\ \text{s. t.} & x_1 - x_2 \geqslant 2 \\ & 2x_1 - 2x_2 \leqslant 1 \\ & x_1, x_2 \geqslant 0 \end{cases} \qquad (D')\begin{cases} \max & w = 2y_1 + y_2 \\ \text{s. t.} & y_1 + 2y_2 \leqslant -1 \\ & -y_1 - 2y_2 \leqslant 2 \\ & y_1 \geqslant 0, y_2 \leqslant 0 \end{cases}$$

Here (P) is infeasible, hence (D) can be infeasible or unbounded (*test the two possibilities!*).

- If x and y are feasible solutions for primal and dual problems, respectively, the *duality gap between x and y* is the difference between the objective functions values corresponding to these solutions:
$$c^T x - b^T y \geqslant 0.$$
- The *duality gap between the two problems* is the duality gap between two optimal solutions for the pair of primal/dual problems provided that the two problems are bounded.
- Corollary 3.1 says that if the duality gap between two solutions is zero, then the two solutions are optimal for their respective problems.
- The next result will show that in most of the cases the duality gap between problems is zero, because this is equivalent to the fact that one of the problems has a (finite) optimal feasible solution.

Theorem

(Strong Duality) *If the primal problem or the dual problem has an optimal solution then so does the other, and the optimal objective values are equal.*

Proof. Without restrain the generality we can assume that the primal problem has an optimal solution and that this problem is in standard form.

Let x_* be an optimal basic feasible solution for the primal (obtained using the simplex algorithm). By reordering the variables we can suppose that $x_*^T = (x_B^T \ x_N^T)$. In line with this separation, we write $A = (B \ N)$ and $c^T = (c_B^T \ c_N^T)$.

Then, $x_B = B^{-1}b$ and, since x_* is optimal, the reduced costs of non-basic variables are nonnegative: $c_N^T - c_B^T B^{-1} N \geqslant 0$.

Let $y_* = (B^{-1})^T c_B$. y_* is feasible solution for the dual:

$$A^T y_* = \begin{pmatrix} B^T \\ N^T \end{pmatrix} y_* = \begin{pmatrix} B^T y_* \\ N^T y_* \end{pmatrix} = \begin{pmatrix} B^T (B^{-1})^T c_B \\ N^T (B^{-1})^T c_B \end{pmatrix} =$$

$$\begin{pmatrix} (B^{-1}B)^T c_B \\ N^T (B^{-1})^T c_B \end{pmatrix} = \begin{pmatrix} c_B \\ (c_B^T B^{-1} N)^T \end{pmatrix} \leqslant \begin{pmatrix} c_B \\ c_N \end{pmatrix} = c.$$

On the other hand

$$z = c^T x_* = c_B^T x_B = c_B^T B^{-1} b = y_*^T b = b^T y_* = w.$$

Using corollary (3.1) (ii) we get the desired conclusion. □

- The Strong Duality Theorem holds for every pair of primal/dual problems without regarding their forms. This is true because any problem can be equivalently converted to one in standard form.
- There is a deeper relation between the nonnegativity variables in the primal problem and the constraints in the dual problem.
- This relation is called *complementary slackness* and says that its impossible to have both $x_i > 0$ and $(A^T y)_i < c_i$, where x and y are optimal solutions for their respective problems.
- This property may help to recover an optimal solution for the dual problem from an optimal solution for the primal problem (and viceversa).

Theorem

(Complementary Slackness) If x *is a feasible solution for the primal problem and* y *is a feasible solution for the dual problem, then the following are equivalent*

(i) $\mathrm{x}^T(\mathrm{c}-\mathrm{A}^T\mathrm{y})=0.$

(ii) x *and* y *are optimal solutions for their respective problems.*

Proof. (i) $\Rightarrow$ (ii) If $\mathrm{x}^T(\mathrm{c}-\mathrm{A}^T\mathrm{y})=0$, then $\mathrm{c}^T\mathrm{x}=\mathrm{b}^T\mathrm{y}$ and the conclusion follows from Corrolary (3.1).

(ii) $\Rightarrow$ (i). We know that $z=\mathrm{c}^T\mathrm{x}\geqslant \mathrm{y}^T\mathrm{A}\mathrm{x}=\mathrm{y}^T\mathrm{b}=\mathrm{b}^T\mathrm{y}$; from the Strong Duality Theorem, $\mathrm{c}^T\mathrm{x}=\mathrm{b}^T\mathrm{y}$, therefore, $\mathrm{c}^T\mathrm{x}=\mathrm{y}^T\mathrm{A}\mathrm{x}$ which yields $\mathrm{x}^T(\mathrm{c}-\mathrm{A}^T\mathrm{y})=0.$ $\square$

- Consider the following primal LP problem and its dual

$$(P)\begin{cases} \min & z = 13x_1 + 10x_2 + 6x_3 \\ \text{s. t.} & 5x_1 + x_2 + 3x_3 = 8 \\ & 3x_1 + x_2 = 3 \\ & x_1, x_2, x_3 \geqslant 0 \end{cases} \qquad (D)\begin{cases} \max & w = 8y_1 + 3y_2 \\ \text{s. t.} & 5y_1 + 3y_2 \leqslant 13 \\ & y_1 + y_2 \leqslant 10 \\ & 3y_1 \leqslant 6 \end{cases}$$

- We will verify if $\mathrm{x} = (1\ 0\ 1)^T$ is an optimal feasible solution for the primal problem.
- Assume that, indeed, x is optimal; as $x_1, x_3 > 0$, we must have $5y_1 + 3y_2 = 13$ and $3y_1 = 6$ which yields $y_1 = 2, y_2 = 1$.
- Now, we compute the objective functions values: $\mathrm{c}^T\mathrm{x} = 19 = \mathrm{b}^T\mathrm{y}$ - this equality says that x is optimal for the primal problem (and y is optimal for the dual problem).

- The dual problem can be used to improve the interpetation of the model for the original problem at hand.
- This interpretation may vary from problem to problem; we will review this approach by using an example.
- Example: A bakery makes and sells two types of cakes, one simple and a more fancy one. These products require basic ingredients (flour, sugar, eggs etc), and some decorations and flavors (fruits, nuts etc) with the fancier cake using more of the decorations, but also more labor force. The baker would like to maximize profit.
- An LP problem associated to this situation follows.

$$
(P)\begin{cases}
\max & z = 24x_1 + 14x_2 \\
\text{s. t.} & 3x_1 + 2x_2 \leqslant 120 \\
& 4x_1 + x_2 \leqslant 100 \\
& 2x_1 + x_2 \leqslant 70 \\
& x_1, x_2 \geqslant 0
\end{cases}
$$

- x_1 and x_2 represent the number of batches of the fancier and simple cakes produced per day.
- The first constraint is associated with the daily limits on the avalaibility of basic ingredients (a batch of simple cake requires 2 pounds, a batch of fancier cake requires 3 pounds).
- In a similar manner, the second constraint represents the limits on decorations and the third constraint records the work force avalaibility (1 hour/batch of simple cake vs. 2 hours/batch of fancier cake).

$$(D)\begin{cases} \min & w = 120y_1 + 100y_2 + 70y_3 \\ \text{s. t.} & 3y_1 + 4y_2 + 2y_3 \geqslant 24 \\ & 2y_1 + y_2 + y_3 \geqslant 14 \\ & y_1, y_2, y_3 \geqslant 0 \end{cases}$$

- The optimal feasible solution for (P) is $\mathrm{x}_* = (1636)^T$, with $z_* = 888$. The optimal feasible solution for (D) is $\mathrm{y}_* = (6.4\ \ 1.2\ \ 0)^T$, with $w_* = 888$. (Complementary slackness conditions are satisfied and optimal objective values are equal.)
- The limiting factors in this problem are the resources (ingredients, labor force). The bakery might want to hire some new employers or to buy additional quantities of ingredients.
- In this case how much the bakery should pay?

- Each extra pound of ingredients will be worth $y_1 = 6.4\$$ in profit and each extra pound of decorations will be worth $y_2 = 1.2\$$ in profit.
- Additional work force is of no value, since there the labor force is in excess. This argument fails if the bakery produces too much fancy cake batches because it can drain up the labor force.

- Another interpretation of the dual problem: if someone wants to takeover the bakery, what price should be offered?
- First the potential buyer records the values of the bakery's assets: ingredients (y_1), decorations (y_2), workforce (y_3).
- The buyer wants to minimize this total value

$$\text{minimize } w = 120y_1 + 100y_2 + 70y_3$$

- Such a price would be fair if the bakery gives a profit greater or equal to the profit obtained by producing cakes

$$\begin{aligned} 3y_1 + 4y_2 + 2y_3 &\geqslant 24 \\ 2y_1 + y_2 + y_3 &\geqslant 14 \end{aligned}$$

- The dual problem helps us to determine the daily value of the bakery's bussiness.

- The Simplex method we already studied will be referred as primal simplex algorithm.
- This algorithm starts with a basic feasible solution to the primal problem and iterates until the *primal optimality conditions* are fulfilled.
- It is possible to apply the simplex algorithm to the dual problem, starting with a feasible solution to the dual problem and iterating until the *dual optimality conditions* are satisfied.

- The proof of the Strong Duality Theorem 3.2 shows that primal optimality conditions for the primal problem ($c_N^T - c_B^T B^{-1} N \geqslant 0$) are equivalent with the *dual feasibility conditions* ($A^T y \leqslant c$).
- The primal simplex algorithm goes through a sequence of primal feasible but dual infeasible bases, trying at each iteration to "reduce" the dual infeasibility, until the dual feasibility conditions are satisfied.
- The dual simplex algorithm works in a dual fashion: it goes through a sequence of dual feasible but primal infeasible bases, trying at each iteration to "reduce" the primal infeasibility, until the primal feasibility conditions are satisfied.
- When these conditions are satisfied, then the duality theorems ensure us that an optimal dual basis was reached.

- Suppose we have an initial dual basic feasible solution $\mathrm{x}_B = \mathrm{B}^{-1}\mathrm{b}$ (that is, all reduced costs are nonnegative: $\widehat{c}_j \geqslant 0$, for all $j \in N$);
- For the primal point of view such a solution is optimal or *super-optimal*, since it gives an objective function value greater than optimum, without being necessarily primal feasible.
- The dual algorithm ends when the current basis becomes dual optimal, or, eqiuvalently, primal feasible.
- In a common iteration, suppose that the current basis is not primal feasible, this corresponds to a negative right-hand side entry $\widehat{b}_k$; the k constraint is

$$(\mathrm{x}_B)_k + \sum_{i \in N} \widehat{a}_{ki} x_i = \widehat{b}_k < 0.$$

- If a non-basic variable x_i $(i \in N)$ were to replace $(\mathrm{x}_B)_k$ in the new basis, then the value for x_i will be $\widehat{b}_k / \widehat{a}_{ki} > 0$ - which makes sense as long as, in this way, we reduce the infeasibility.
- The new reduced costs will be

$$\widehat{\widehat{c}}_h = \widehat{c}_h - \widehat{c}_i \frac{\widehat{a}_{kh}}{\widehat{a}_{ki}}, \forall h = \overline{1, n},$$

since we must preserve the primal optimality, we impose

$$\widehat{\widehat{c}}_h \geqslant 0 \Leftrightarrow \frac{\widehat{c}_h}{\widehat{a}_{kh}} \leqslant \frac{\widehat{c}_i}{\widehat{a}_{ki}} \text{ (for } \widehat{a}_{kh} < 0, \text{ otherwise } \widehat{\widehat{c}}_h \geqslant 0)$$

- Therefore, if x_i is the entering variable, then it must have the smallest ratio $\left|\frac{\widehat{c}_i}{\widehat{a}_{ki}}\right|$ with $\widehat{a}_{ki} < 0$.

The algorithm starts with with a basis matrix B, corresponding to the dual basic feasible solution, that is, $\widehat{c}_j \geqslant 0$. The algorithm follows:

The Feasibility Test. Compute $\mathrm{x}_B = \widehat{\mathrm{b}} = \mathrm{B}^{-1}\mathrm{b}$, if $\mathrm{x}_B \geqslant 0$, then the current basis is a dual optimal feasible solution. Otherwise choose $(\mathrm{x}_B)_k$, as the leaving variable, such that $\widehat{b}_k < 0$. The kth row is the pivot row.

The Main Step. Compute $\widehat{\mathrm{A}}_j = \mathrm{B}^{-1}\mathrm{A}_j$. Find an $i \in N$ such that

$$\left|\frac{\widehat{c}_i}{\widehat{a}_{ki}}\right| = \min\left\{\left|\frac{\widehat{c}_h}{\widehat{a}_{kh}}\right| : \widehat{a}_{kh} < 0, h \in N\right\}.$$

x_i will be the entering variable and $\widehat{a}_{ij}$ the *pivot* entry. If $\widehat{a}_{hj} \geqslant 0$, for all $h \in N$, then Stop - the problem has infinite optimum.

The Update Compute the new basis matrix B, the new vector of basic variables x_B, and the new reduced costs $\widehat{c}$.

- Consider the following LP problem

$$\begin{array}{ll} \text{minimize} & z = 2x_1 + 3x_2 \\ \text{subject to} & -3x_1 + 2x_2 \leqslant -4 \\ & -x_1 - 2x_2 \leqslant -3 \\ & x_1, x_2 \geqslant 0 \end{array}$$

- In standard form the problem becomes

$$\begin{array}{ll} \text{minimize} & z = 2x_1 + 3x_2 \\ \text{subject to} & -3x_1 + 2x_2 + x_3 = -4 \\ & -x_1 - 2x_2 + x_4 = -3 \\ & x_1, x_2, x_3, x_4 \geqslant 0 \end{array}$$

- The basis $\{x_3, x_4\}$ is infeasible but the primal optimality conditions are satisfied (this is a super-optimal solution)

Table: First Dual Simplex tableau (note that $B = I_2$).

	x_1	x_2	x_3	x_4	RHS
x_3	$\boxed{-3}$	2	1	0	-4
x_4	-1	-2	0	1	-3
z	2	3	0	0	0
	$\left\lvert \frac{2}{-3} \right\rvert$ min				

Table: Second Dual Simplex tableau.

	x_1	x_2	x_3	x_4	RHS
x_1	1	-2/3	-1/3	0	4/3
x_4	0	$\boxed{-8/3}$	-1/3	1	$-5/3$
z	0	13/3	2/3	0	-8/3
		$\left\lvert\frac{13/3}{-8/3}\right\rvert$	$\left\lvert\frac{2/3}{-1/3}\right\rvert$		
		min			

Table: Third Dual Simplex tableau.

	x_1	x_2	x_3	x_4	RHS
x_1	1	0	-1/4	-1/4	7/4
x_2	0	1	1/8	-3/8	5/8
z	0	0	1/8	13/8	-43/8

- The current basis is primal feasible, hence is optimal. The dual optimal feasible solution is $y = (1/8\ 13/8)^T$.

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Operations Research - Lecture 6

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- Many discrete optimization problems can be modeled as an **Integer Linear Programming** problem (*ILP* or simply *IP*).
- A pure ILP problem is an LP problem with the additional restriction that all variables are integer

$$\boxed{\begin{aligned} \text{maximize} \quad & z = \mathbf{c}^T \mathbf{x}, \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leqslant \mathbf{b}, \\ & \mathbf{x} \in \mathbb{Z}^n. \end{aligned} \qquad (1)}$$

- From a (more) geometric point of view an ILP problem is maximizing (or minimizing) a linear function $\mathbf{c}^T\mathbf{x}$ over the integer vectors of the polyhedron $\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leqslant \mathbf{b}\}$.
- The problem (1) can be written as

$$\max\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \in \mathcal{P}, \mathbf{x} \in \mathbb{Z}^n\}. \qquad (2)$$

- Sometimes such problems are *mixed*: some, but not all, of the variables are constrained to be integer and the rest of them are unrestricted.
- In this way we get the **Mixed Integer Linear Programming** (*MILP* or *MIP*) problem:

$$\boxed{\begin{aligned} \text{maximize} \quad & z = \mathbf{c}^T\mathbf{x}, \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leqslant \mathbf{b}, \\ & x_i \in \mathbb{Z}, \forall i \in \mathcal{I}. \end{aligned} \qquad (3)}$$

where $\varnothing \neq \mathcal{I} \subsetneq \{1, 2, \ldots, n\}$.

- We present next some discrete optimization problems written as integer linear programs.

- Suppose we have a knapsack that can carry a maximum weight b and there are n types of items that we could take: an item of type i has weight $a_i > 0$.
- We want to load the knapsack with items without exceeding the knapsack capacity.
- On the other hand, suppose that an item of type i has value $c_i \geqslant 0$.
- The problem of loading the knapsack so as to maximize the value of all loaded items is the **knapsack problem**.

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{n} c_i x_i, \\ \text{subject to} \quad & \sum_{i=1}^{n} a_i x_i \leqslant b, \\ & \mathbf{x} \in \mathbb{Z}^n, \mathbf{x} \geqslant 0. \end{aligned} \tag{4}$$

- If only one item of each type is allowed to be loaded, then we can use binary variables instead of general integers.

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{n} c_i x_i, \\ \text{subject to} \quad & \sum_{i=1}^{n} a_i x_i \leqslant b, \\ & x_i \in \{0,1\}, \forall i = \overline{1,n}. \end{aligned}$$

- In this way we get the **Binary Integer Linear Programming** (*BILP* or *BIP*) problem:

$$\boxed{\begin{aligned} \text{maximize} \quad & z = \mathbf{c}^T \mathbf{x}, \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leqslant \mathbf{b}, \\ & \mathbf{x} \in \{0,1\}^n. \end{aligned} \qquad (5)}$$

- Suppose we have a finite set X, and a family of subsets $\mathcal{F} \subseteq 2^X$.
- A subfamily $\mathcal{F}' \subseteq \mathcal{F}$ is said to be a *packing* of $\mathcal{F}$ if $\mathcal{F}'$ contains only pairwise disjoint sets.
- The *Set Packing* problem is: given X, $\mathcal{F} \subseteq 2^X$, and $k \in \mathbb{N}$, there exists a packing of cardinality of least k?
- The optimization version of this problem is the **Maximum Set Packing** problem: what is the maximum number of pairwise disjoint sets in $\mathcal{F}$?

$$\begin{aligned} \text{maximize} \quad & \sum_{F \in \mathcal{F}} x_F, \\ \text{subject to} \quad & \sum_{F \in \mathcal{F}, x \in F} x_F \leqslant 1, \forall x \in X, \\ & x_F \in \{0, 1\}, \forall F \in \mathcal{F}. \end{aligned}$$

(The solution x is the characteristic vector of $\mathcal{F}'$.)

- A subfamily $\mathcal{F}' \subseteq \mathcal{F}$ is said to be a *covering* of $\mathcal{F}$ if every element of X is covered by some set in $\mathcal{F}'$ (that is, $\bigcup F_{F \in \mathcal{F}'} = X$).
- The *Set Cover* problem is: given X, $\mathcal{F} \subseteq 2^X$, and $k \in \mathbb{N}$, there exists a covering of cardinality of most k?
- The optimization version of this problem is the **Minimum Set Cover** problem: what is the minimum cardinality covering of $\mathcal{F}$?

$$
\begin{aligned}
\text{minimize} \quad & \sum_{F \in \mathcal{F}} x_F, \\
\text{subject to} \quad & \sum_{F \in \mathcal{F}, x \in F} x_F \geqslant 1, \forall x \in X, \\
& x_F \in \{0, 1\}, \forall F \in \mathcal{F}.
\end{aligned}
$$

(The solution x is the characteristic vector of $\mathcal{F}'$.)

- Suppose we have a finite set X of boolean variables, U the set of (positive or negative) literals over X, and C a formula in disjunctive normal form (that is, a disjunction of conjunctions of literals) where each clause is limited to at most three literals from U.
- The *3-SAT* problem is: given X and C like above, there exists an assignment of truth on X that satisfies all the clauses in C?
- The optimization version of this problem is the **Max 3-SAT** problem: what is the maximum number of satisfiable clauses in C?
- In order to give the LP description of this problem we define $X = \{x_1, x_2, \dots, x_k\}$, $u_i = x_i, v_i = \overline{x}_i$ for all $i = \overline{1, k}$, $L = \{u_i, v_i : i = \overline{1, k}\}$, and and $C = \{C_1, C_2, \dots, C_p\}$, where $C_j = w_1^j \vee w_2^j \vee w_3^j$, with $w_i^j \in L$.

3-SAT Problem

- The LP formulation of Max 3-SAT is

$$
\begin{aligned}
\text{maximize} \quad & \sum_{C \in \mathcal{C}} x_C, \\
\text{subject to} \quad & w_1^j + w_2^j + w_3^j - x_{C_j} \geqslant 0, \forall j = \overline{1,p}, \\
& u_i + v_i = 1, \forall i = \overline{1,k} \\
& x_C \in \{0,1\}, \forall C \in \mathcal{C}, \\
& u_i, v_i \in \{0,1\}, \forall i = \overline{1,k}.
\end{aligned}
$$

- In the corresponding truth assignment the boolean variable x_i is true iff $u_i = 1$.
- Constraint $u_i + v_i = 1$ was introduced to insure that x_i is true iff $\overline{x}_i$ is false.
- The construction of the linear program was made such that, for an optimal solution, $x_C = 1$ iff the clause C is satisfied by the current truth assignment.

- Suppose we have n people and n tasks; every pair person/task has a certain value c_{ij}.
- The **assignment problem**: allocate exactly one person to each task so that the total value is maximized.

$$\begin{aligned}
\text{maximize} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}, \\
\text{subject to} \quad & \sum_{i=1}^{n} x_{ij} = 1, \forall j = \overline{1,n} \\
& \sum_{j=1}^{n} x_{ij} = 1, \forall i = \overline{1,n} \\
& x_{ij} \in \{0,1\}, \forall i,j = \overline{1,n}.
\end{aligned}$$

- All the problems from above, except for the last, are **NP**-hard problems.
- This observation implies that a typical ILP problem is hard to solve.
- The assignment problem is also called *maximum weight bipartite matching problem* and can be solved in polynomial time complexity.

- There are some classes of ILP problems which can be solved in polynomial time; this is possible for particular combinatorial problems.
- Generally we don't expect to find a polynomial time algorithm for solving an ILP problem, thus we are interested in finding broad methods to solve such a problem.
- We will study two approaches for solving ILP problems:
  - One approach is based on the particular structure of the problem (namely, particular properties of matrix $\mathbf{A}$, such as *total unimodularity*);
  - The second approach is a more general one and is based on solving basic LP problems (using Simplex algorithm or other tools) and consists in several strategies: *branch-and-bound*, *cutting plane*, *branch-and cut methods* etc.

Definition

An matrix $\mathbf{A}$ *is called* **totally unimodular** *if every square submatrix of* A *has its determinant in the set* $\{-1, 0, 1\}$.

- Obviously, a totally unimodular matrix has entries from $\{-1, 0, 1\}$.
- The following result underlines the importance of totally unimodular matrices in connection with ILP.

Theorem

Let $\mathbf{A}$ *be a totally unimodular* $m \times n$ *matrix, and* $\mathbf{b} \in \mathbb{Z}^m$. *Then all the extreme points of the polyhedron* $\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n \,:\, \mathbf{A}\mathbf{x} \leqslant \mathbf{b}\}$ *are integral vectors (i. e., from* $\mathbb{Z}^n$*).*

Proof. Let $\mathbf{x}$ be an extreme point of $\mathcal{P}$ and $\mathbf{A_x}$ be the submatrix which contains only those rows $\mathbf{a}_j'$, for which $\langle \mathbf{a}_j', \mathbf{x}\rangle = b_j$.

Lemma

Matrix $\mathbf{A_x}$ has rank n.

Proof (for lemma). If $rank(\mathbf{A_x}) < n$, then the n columns of $\mathbf{A_x}$ are linearly dependent: there exists $\mathbf{y} \in \mathbb{R}^n, \mathbf{y} \neq \mathbf{0}$ with $\mathbf{A_x}\mathbf{y} = \mathbf{0}$. Now, we can find an $\varepsilon > 0$ such that for every row $\mathbf{a}_j'$ which doesn't occur in $\mathbf{A_x}$ ($\mathbf{a}_j'\mathbf{x} < b_j$) we have

$$\langle \mathbf{a}_j', \mathbf{x} + \varepsilon\mathbf{y}\rangle \leqslant b_j \text{ and } \langle \mathbf{a}_j', \mathbf{x} - \varepsilon\mathbf{y}\rangle \leqslant b_j.$$

Since $\mathbf{A_x}\mathbf{y} = \mathbf{0}$ and $\mathbf{A}\mathbf{x} \leqslant \mathbf{b}$, we get

$$\mathbf{A}(\mathbf{x} + \varepsilon\mathbf{y}) \leqslant \mathbf{b} \text{ and } \mathbf{A}(\mathbf{x} - \varepsilon\mathbf{y}) \leqslant \mathbf{b},$$

which means that $(\mathbf{x} + \varepsilon\mathbf{y}), (\mathbf{x} - \varepsilon\mathbf{y}) \in \mathcal{P}$ - a contradiction (why?). $\square$

Proof (cont'd for theorem). Let $\mathbf{x}$ be an extreme vector of $\mathcal{P}$ and $\mathbf{A_x}$ defined as above. Since $rank(\mathbf{A_x}) = n$, there exist a square submatrix of $\mathbf{A_x}$, $\mathbf{A}_1$, of rank n. Let $\mathbf{b}_1$ be a vector formed with elements of $\mathbf{b}$ that corresponds to $\mathbf{A}_1$; we must have $\mathbf{A}_1\mathbf{x} = \mathbf{b}_1$. Therefore, $\mathbf{x} = \mathbf{A}_1^{-1}\mathbf{b}_1$; but, since $det(\mathbf{A}_1) = \pm 1$, $\mathbf{A}_1^{-1} \in \mathbb{Z}^{n\times n}$, hence $\mathbf{x}$ is an integral vector. □

Proposition

Let $(\mathbf{x}, \mathbf{y})$ *be an extreme point of the polyhedron* $\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^m_+ \;:\; \mathbf{A}\mathbf{x} + \mathbf{y} = \mathbf{b}\}$, *then* $\mathbf{x}$ *is an extreme point of the polyhedron* $\mathcal{P}' = \{\mathbf{x} \in \mathbb{R}^n \;:\; \mathbf{A}\mathbf{x} \leqslant \mathbf{b}\}$.

Proof. Let $(\mathbf{x}, \mathbf{y})$ be an extreme point of $\mathcal{P}$. Suppose, on the contrary, that there exist two points $\mathbf{x}^1 \neq \mathbf{x}^2$ in $\mathcal{P}'$ such that

$$\mathbf{x} = \frac{1}{2}\mathbf{x}^1 + \frac{1}{2}\mathbf{x}^2.$$

Let $i \in \{1, 2, \ldots, m\}$; we have three cases

(i) $y_i = 0$. We define $y_i^1 = y_i^2 = 0$.

(ii) $y_i > 0$, $\mathbf{a}_i'\mathbf{x}^1 = b_i$, and $\mathbf{a}_i'\mathbf{x}^2 < b_i$. We define $y_i^1 = 0$ and $y_i^2 = b_i - \mathbf{a}_i'\mathbf{x}^2$; since $\mathbf{a}_i'\mathbf{x}^1 + \mathbf{a}_i'\mathbf{x}^2 = 2\mathbf{a}_i^T\mathbf{x} = 2b_i - 2y_i$, we must have $y_i^2 = 2y_i$.

(iii) $y_i > 0$, $\mathbf{a}_i'\mathbf{x}^1 < b_i$, and $\mathbf{a}_i'\mathbf{x}^2 < b_i$. We define $y_i^1 = b_i - \mathbf{a}_i'\mathbf{x}^1$ and $y_i^2 = b_i - \mathbf{a}_i'\mathbf{x}^2$; since $\mathbf{a}_i'\mathbf{x}^1 + \mathbf{a}_i'\mathbf{x}^2 = 2\mathbf{a}_i'\mathbf{x} = 2b_i - 2y_i$, we must have $y_i^1 + y_i^2 = 2y_i$.

In all of the above situations we have

$$\mathbf{a}_i^T \mathbf{x}^1 + y_i^1 = \mathbf{a}_i^T \mathbf{x}^2 + y_i^2 = b_i, \text{ and } \frac{1}{2}(x_i^1, y_i^1) + \frac{1}{2}(x_i^2, y_i^2) = (x_i, y_i).$$

Obviously $(\mathbf{x}^1, \mathbf{y}^1) \neq (\mathbf{x}^2, \mathbf{y}^2)$ are points from $\mathcal{P}$. Since

$$\frac{1}{2}(\mathbf{x}^1, \mathbf{y}^1) + \frac{1}{2}(\mathbf{x}^2, \mathbf{y}^2) = (\mathbf{x}, \mathbf{y}),$$

we come to the conclusion that $(\mathbf{x}, \mathbf{y})$ cannot be an extreme point in $\mathcal{P}$, which is a contradiction. □

- An immediate consequence of Theorem 3.1 and Proposition 3.1 is the following: when we have to optimize over a polyhedron defined by a totally unimodular matrix, we can use the Simplex Algorithm.
- Suppose that we have to optimize over the polyhedron $\mathcal{P}'$. We add the slack variables y and optimize over the new polyhedron $\mathcal{P}$; using simplex algorithm we eventually find an optimal basic feasible solution $(\mathrm{x}, \mathrm{y}) \in \mathcal{P}$.
- We already know that such a solution corresponds to an extreme point (x, y) of $\mathcal{P}$. Therefore, from Proposition 3.1, x is an extreme point and, also, an optimal solution in $\mathcal{P}'$.
- From Theorem 3.1, x must be an integral vector. Hence, we have found an optimal solution in $\mathcal{P}'$ which is an integral vector.

Definition

A polyhedron $\mathcal{P}$ *is* integral *if, for every* $\mathbf{c} \in \mathbb{R}^n$*, for which* $\sup\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \in \mathcal{P}\} \in \mathbb{R}$*, the supremum is attained at an integral vector.*

- A simple consequence is

Corollary

If $\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leqslant \mathbf{b}\}$*,* $\mathbf{A}$ *is an* $m \times n$ *totally unimodular matrix, and* $\mathbf{b} \in \mathbb{Z}^m$*, then* $\mathcal{P}$ *is an integral polyhedron.*

Proof. Let $\mathbf{c} \in \mathbb{R}^n$ and $\mathbf{x}_*$ an optimal solution to the problem

$$\max\{\mathbf{c}^T\mathbf{x} \,:\, \mathbf{x} \in \mathcal{P}\}.$$

Let $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{Z}^n$, such that $\mathbf{b}_1 \leqslant \mathbf{x}_* \leqslant \mathbf{b}_2$, and define

$$\mathcal{P}' = \{\mathbf{x} \in \mathcal{P} \,:\, \mathbf{b}_1 \leqslant \mathbf{x} \leqslant \mathbf{b}_2\}.$$

Obviously,

$$\max\{\mathbf{c}^T\mathbf{x} \,:\, \mathbf{x} \in \mathcal{P}\} = \max\{\mathbf{c}^T\mathbf{x} \,:\, \mathbf{x} \in \mathcal{P}'\} = \mathbf{c}^T\mathbf{x}_*.$$

On the other hand $\mathcal{P}'$ is bounded, therefore an optimal solution to $\max\{\mathbf{c}^T\mathbf{x} \,:\, \mathbf{x} \in \mathcal{P}'\}$ can be found at an extreme point of $\mathcal{P}'$, $\mathbf{x}'_*$; thus, $\mathbf{c}^T\mathbf{x}_* = \mathbf{c}^T\mathbf{x}'_*$. Let us define

$$\mathbf{A}' = \begin{bmatrix} \mathbf{A} \\ -I_n \\ I_n \end{bmatrix}, \mathbf{b}' = \begin{bmatrix} \mathbf{b} \\ -\mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}.$$

Since, matrix $\mathbf{A}'$ is totally unimodular (why?) and $\mathcal{P}' = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}'\mathbf{x} \leqslant \mathbf{b}'\}$, $\mathbf{x}'_*$ must be an integral vector (by Theorem 3.1). □

- If $\mathcal{P}$ would be a polytope (that is, a bounded polyhedron), the above proof would be shorter.
- It will follow that every LP problem with integral data ($\mathbf{b}$ and $\mathbf{c}$) and totally unimodular matrix has integral optimal primal and dual solutions.

Corollary

If $\mathbf{A}$ is an $m \times n$ totally unimodular matrix, $\mathbf{b} \in \mathbb{Z}^m$, and $\mathbf{c} \in \mathbb{Z}^n$, then the following primal/dual pair of problems have integral optimum solutions (if the optima are finite)

$$\max\{\mathbf{c}^T\mathbf{x} \;:\; \mathbf{A}\mathbf{x} \leqslant \mathbf{b}\} = \min\{\mathbf{b}^T\mathbf{y} \;:\; \mathbf{y} \geqslant \mathbf{0}, \mathbf{A}^T\mathbf{y} = \mathbf{c}\}$$

Proof. Using Corollary 3.1 and the (easy to prove) fact that the following matrix is also totally unimodular

$$\mathbf{A}' = \begin{bmatrix} -I_m \\ \mathbf{A}^T \\ -\mathbf{A}^T \end{bmatrix}. \square$$

- It can be proved that the property listed in Corollary 3.1 is a characterization of total unimodularity.

Theorem

(Hoffman-Kruskal Theorem) Let $\mathbf{A}$ *be an integral* $m \times n$ *matrix. Then* $\mathbf{A}$ *is totally unimodular iff, for each* $\mathbf{b} \in \mathbb{Z}^m$*, the polyhedron* $\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n_+ \ : \ \mathbf{A}\mathbf{x} \leqslant \mathbf{b}\}$ *is integral.*

Totally Unimodular Matrices from Bipartite Graphs

- Let $G = (V, E)$ be a graph, with $V = \{v_1, v_2, \ldots, v_m\}$ and $E = \{e_1, e_2, \ldots, e_n\}$.
- The *incidency matrix* of G is $A \in \{0, 1\}^{m \times n}$, such that $a_{ij} = 1$ iff vertex v_i is adjacent with edge e_j.
- The incidency matrix can or cannot be totally unimodular, what we can prove is

Theorem

A graph G is bipartite iff its incidency matrix is totally unimodular.

Proof.

"$\Longrightarrow$" Suppose $\mathbf{A}'$ is a square submatrix of $\mathbf{A}$, of order k - we prove that $det(\mathbf{A}') \in \{-1, 0, 1\}$ by induction on $k \geqslant 2$ (for $k = 1$ the result obviously holds). We have three possible situations

(i) $\mathbf{A}'$ has a null column, then $det(\mathbf{A}') = 0$.

(ii) $\mathbf{A}'$ has a column with exactly one 1. Then, using the Laplace expansion, $det(\mathbf{A}') = \pm 1 \cdot det(\mathbf{A}'')$, where $\mathbf{A}''$ is a square submatrix of $\mathbf{A}'$ (hence, of $\mathbf{A}$) of order $(k-1)$. Therefore $det(\mathbf{A}'') \in \{-1, 0, 1\}$ and $det(\mathbf{A}') \in \{-1, 0, 1\}$.

(iii) Every column of $\mathbf{A}'$ has exactly two 1's. $\mathbf{A}'$ is the incidence matrix of a subgraph G' of G. Since G' is also bipartite, with bipartition (V_1', V_2'), if we add the rows corresponding to vertices from V_1' we get a row full of 1's, and the same result if we add the rows corresponding to vertices from V_2'. Therefore $det(A') = 0$.

"$\Longleftarrow$" We will use the well-known characterization: a graph is bipartite iff it doesn't contain odd cycles. If G is not bipartite, then it contains an odd cycle through vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$, and edges $e_{j_1}, e_{j_2}, \ldots, e_{j_k}$. The submatrix of $\mathbf{A}$ having rows $i_1, i_2, \ldots, i_k$ and columns $j_1, j_2, \ldots, j_k$ has the determinant equal with 2 (verify!) - which is a contradiction.

$\square$

- Using the last result and the Corollary 3.2, we give some combinatorial results concerning bipartite graphs.
- One of these results links minimum cardinality edge covers to maximum cardinality stable sets in G; another one relates maximum cardinality matchings to minimum vertex covers (both König's theorems).
- Both these results are generalized in the following theorem.

Theorem

Let G be bipartite graph and $w : E(G) \to \mathbb{N}$ be a weight function defined on its edges. Then

(i) *The maximum weight of a matching in G is equal to the minimum value of $\sum_{v \in V(G)} f(v)$, where f ranges over all functions $f : V(G) \to \mathbb{N}$, such that $f(u) + f(v) \geqslant w(uv), \forall uv \in E(G)$.*

(ii) *The minimum weight of an edge cover in G is equal to the maximum value of $\sum_{v \in V(G)} f(v)$, where f ranges over all functions $f : V(G) \to \mathbb{N}$, such that $f(u) + f(v) \leqslant w(uv), \forall uv \in E(G)$.*

Proof. The properties are equivalent with

$$\max\{w^T\mathbf{x} : \mathbf{A}\mathbf{x} \leqslant \mathbf{1}, \mathbf{x} \geqslant 0\} = \min\{\mathbf{1}^T\mathbf{y} : \mathbf{A}^T\mathbf{y} \geqslant w, \mathbf{y} \geqslant \mathbf{0}\},$$

$$\min\{w^T\mathbf{x} : \mathbf{A}\mathbf{x} \geqslant \mathbf{1}, \mathbf{x} \geqslant 0\} = \max\{\mathbf{1}^T\mathbf{y} : \mathbf{A}^T\mathbf{y} \leqslant w, \mathbf{y} \geqslant \mathbf{0}\},$$

respectively, where $\mathbf{A}$ is the incidence matrix of G. Now, using Theorem 3.3 and Corollary 3.2 we conclude that both these relations are true. □

- Let $D = (V, A)$ be a directed graph with $V = \{v_1, v_2, \ldots, v_m\}$ and $A = \{e_1, e_2, \ldots, e_n\}$. Let us define the incidency matrix of D, the $m \times n$ matrix $\mathbf{M}$ with entries

$$m_{ij} = \begin{cases} 1, & \text{if } e_j \text{ leaves } v_i \\ -1, & if \text{ if } e_j \text{ enters } v_i \\ 0, & \text{otherwise} \end{cases}$$

Theorem

The incidency matrix of a digraph is totally unimodular.

Proof. Let $\mathbf{M}'$ be a square submatrix of $\mathbf{M}$ of order k. We proceed by induction on k. Suppose $k > 2$; we have three possible situation:

(i) $\mathbf{M}'$ has a null column, then $det(\mathbf{M}') = 0$.

(ii) $\mathbf{M}'$ has a column with exactly one non-null value. Then, using the Laplace expansion, $det(\mathbf{M}') = \pm 1 \cdot det(\mathbf{M}'')$, where $\mathbf{M}''$ is a square submatrix of $\mathbf{M}'$ (hence, of $\mathbf{M}$) of order $(k-1)$. $det(\mathbf{M}'') \in \{-1, 0, 1\}$ by our induction hypothesis; therefore, $det(\mathbf{A}') \in \{-1, 0, 1\}$.

(iii) Every column of $\mathbf{M}'$ has exactly two nonzero values (an 1 and a -1). By adding up all the rows of $\mathbf{M}'$ we get a null row, hence $det(\mathbf{M}') = 0$. $\square$

- The incidence matrix of a digraph relates with flows and circulations in D, because the homogeneous system of linear equations $\mathbf{M}\mathbf{x} = \mathbf{0}$ is equivalent with

$$\sum_{e_j \in \delta^+(v)} x_j = \sum_{e_j \in \delta^-(v)} x_j, \forall v \in V$$

which is the *conservation law*.

Definition

Let $D = (V, A)$ be a digraph; a circulation of D is a function $\mathbf{x} : A \to \mathbb{R}$ which obeys the conservation law.

Corollary

Let $D = (V, A)$ be a digraph and $\mathbf{c}_1, \mathbf{c}_2 : A \to \mathbb{Z}$. There exists a circulation $\mathbf{x}$ of D with $\mathbf{c}_1 \leqslant \mathbf{x} \leqslant \mathbf{c}_2$ iff there exists an integral circulation $\mathbf{x}$ of D with $\mathbf{c}_1 \leqslant \mathbf{x} \leqslant \mathbf{c}_2$.

Proof. Obviously we need to prove only the if part.

If there exists a circulation $\mathbf{x}$ of D with $\mathbf{c}_1 \leqslant \mathbf{x} \leqslant \mathbf{c}_2$, then the following polytope (that is, a bounded polyhedron) is nonempty

$$\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{M}\mathbf{x} = \mathbf{0}, \mathbf{c}_1 \leqslant \mathbf{x} \leqslant \mathbf{c}_2\}.$$

Any extreme point, $\mathbf{x}_*$, of $\mathcal{P}$ is an integral circulation of D, with $\mathbf{c}_1 \leqslant \mathbf{x}_* \leqslant \mathbf{c}_2$. □

- Another interesting application is the *max-flow min-cut theorem*:

Corollary

Let $D = (V, A)$ be a digraph, $s \neq t \in V$, and $\mathbf{c} : A \to \mathbb{R}_+$ be a capacity function. The maximum value of an st-flow is equal with the minimum capacity of an st-cut.

Proof. It is obvious that the value of any st-flow cannot exceed the capacity of any st-cut. It suffices to prove that there exists a flow $\mathbf{x}$ and cut (S, T) such that $v(\mathbf{x}) \geqslant \mathbf{c}(S, T)$.

If we delete from $\mathbf{M}$ the rows corresponding to s and t, we get a matrix $\mathbf{M}'$. The flow conservation law is equivalent with $\mathbf{M}'\mathbf{x} = \mathbf{0}$. Now, if $\mathbf{a}_{i_0}$ is the row corresponding to s, then $\mathbf{a}_{i_0}\mathbf{x}$ is the value of the flow $\mathbf{x}$.

The maximum value of an st-flow is

$$\max\{\mathbf{a}_{i_0}^T\mathbf{x} \,:\, \mathbf{0} \leqslant \mathbf{x} \leqslant \mathbf{c}, \mathbf{M}'\mathbf{x} = \mathbf{0}\}$$

The dual problem is

$$\min\{\mathbf{c}^T\mathbf{y}'' \,:\, \mathbf{y}'' \geqslant \mathbf{0}, \mathbf{M}'^T\mathbf{y}' + \mathbf{y}'' \geqslant \mathbf{a}_{i_0}\},$$

or

$$\min \{\mathbf{c}^T \mathbf{y}'' : \mathbf{y}'' \geqslant \mathbf{0}, \mathbf{M}_0^T (\mathbf{y}'^T \ \mathbf{y}''^T)^T \geqslant \mathbf{a}_0\},$$

where

$$\mathbf{M}_0 = \begin{bmatrix} \mathbf{M}' \\ I_n \end{bmatrix}, \mathbf{a}_0 = \begin{bmatrix} \mathbf{a}_{i_0}'^T \\ \mathbf{0} \end{bmatrix}.$$

Obviously, $\mathbf{M}_0$ is a totally unimodular matrix (why?). Since $\mathbf{a}_{i_0}$ is an integral vector, the optimal solution of the dual $(\mathbf{y}', \mathbf{y}'')$ is an integral vector. We can build a st-cut like this:

$$S = \{v \in V \ : v \neq s, t, y''_v \leqslant -1\} \cup \{s\}, T = V \setminus S.$$

The proof is completed by observing that $v(\mathrm{x}) \leqslant \mathbf{c}(S, T)$ (exercise). □

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