Tutorial-4

LQR, Kalman Filter, Lyapunov Stability

LQR Controller

LQR Controller computes the control inputs ***u_1, u_2, …,*** and there by the closed –loop feedback gain ***K*** such that the following cost function is minimized:

$$J = \min_{u_1, u_2, \ldots} E \left(\sum_{i=1}^{\infty} [(x_i - x_{ref}) Q (x_i - x_{ref})^T + u_i R u_i^T] \right)$$

This optimization problem returns a constant gain ***K*** for which ***u_i = -Kx_i***

Is minimum.

LQR Controller

Q. Consider the following first order system: $\dot{x}(t) = x(t) + u(t)$. Desired equilibrium condition is x(t) = 0 as $t \rightarrow \infty$. The performance of the system is defined as:

$$J = \int_0^\infty [(x(t) - 2)^2 + u^2(t)]dt$$

Design a feedback controller gain ***K*** such that ***u(t) =−Kx(t)*** and ***J*** is minimized. Given that x(0) = 2.

LQR Controller

Sol. We have x(0) = 2 and

$J = \int_0^\infty [(x(t) - 2)^2 + u^2(t)]dt$ ---- (1)

$\dot{x}(t) = x(t) + u(t) = \mathrm{x(t)} - \mathrm{Kx(t)} = (1 - \mathrm{K})\mathrm{x(t)}$ ---(2)

Taking Laplace transform of (2),

$SX(S) - X(0) = (1 - K)X(S)$
$\Longrightarrow [S - (1 - K)]X(S) = X(0)$

$\Longrightarrow X(S) = \frac{X(0)}{S - (1-K)}$ --- (3)

Taking Inverse Laplace of (3)

$\Longrightarrow x(t) = e^{(1-K)t}x(0) = 2e^{(1-K)t}$ -------(4)

From (4) we can state: ***for the system to be stable, K must be greater than 1***

LQR Controller: Solution of Problem 1

Replacing (4) in (1)

$$J = 4(1 + K^2)\int_0^{\infty} e^{2(1-K)t}dt \; - 8\int_0^{\infty} e^{(1-K)t}dt + c(constant)$$

$$\Longrightarrow J \; = \frac{2(1+K^2)}{(K-1)} - \frac{8}{(K-1)} + C$$ (since K>1)

Since, we want to find a K such that J is minimized,

$$\frac{dJ}{dK} = \frac{2K^2 \; - 4K + 6}{(K-1)^2} = 0$$

$$\Longrightarrow K = 1{,}3$$

Since, K must be grater than 1 to ensure stability, $\boldsymbol{K = 3}$

LQR Controller: Homework

A first-order system is represented by the following differential equation: $\dot{x}(t) = x(t) + u(t)$

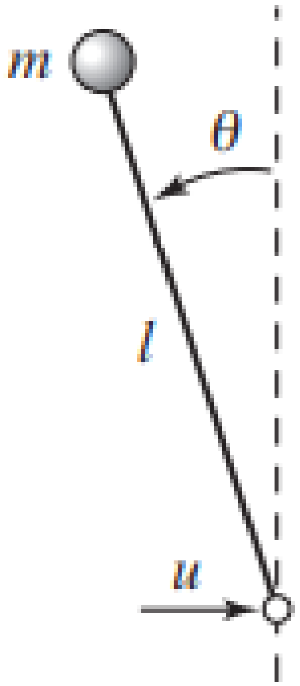
We want to design a feedback controller such that u(t) = -Kx(t) and desired equilibrium condition is: x(t) = 0 as $t \to \infty$. The performance of the system is measured by:

$$J = \int_0^\infty [x^2(t) + \alpha u^2(t)]dt$$

Initial state x(0) = 2. Compute K such that J is minimum.

Lyapunov Function/Stability

An Inverted Pendulum



Q. For the described inverted pendulum, where x1 is the angular deviation from the upright position and u is the (scaled) acceleration of the pivot, as shown in Figure. The system has an equilibrium at $x_1 = x_2$ = 0, which corresponds to the pendulum standing upright. This equilibrium is unstable. Design a stabilizing controller for the system using the Lyapunov Function $\boldsymbol{V(x)} = (\boldsymbol{\cos x_1} - \mathbf{1}) + \boldsymbol{a}(\mathbf{1} - \mathbf{cos^2}\, \boldsymbol{x_1}) + \mathbf{0.5}\, \boldsymbol{x_2^2}$. How would you characterize this stability?

$$\dot{x}_1 = x_2,\ \dot{x}_2 = \sin(x_1) + u\cos(x_1)$$

Lyapunov Function/Stability

Sol. Since the angular deviation is a small quantity, we can approximate to make the computation easier, i.e.

$$V(x) = (co\,s\,x_1 - 1) + a\left(1 - \cos^2 x_1\right) + 0.5\,x_2^2 \approx (a - 0.5)x_1^2 + 0.5x_2^2.$$

The Taylor series expansion shows that the function is positive definite near the origin if a > 0.5.

The time derivative of V (x)is ,

$$\dot{V} = -\dot{x_1}\sin x_1 + 2a\ \dot{x_1}\sin x_1 \cos x_1 + \dot{x_2}x_2 = x_2(u + 2a\sin x_1)\cos x_1.$$

Choosing the feedback law $\boldsymbol{u = -2a\,sin\,x_1 - x_2\,cos\,x_1}$ gives

$$\dot{V} = -x_2^2\cos^2 x_1 \leq 0.$$

It follows from Lyapunov's theorem that Since the function is only negative semidefinite, we cannot conclude asymptotic stability the equilibrium is **locally stable**.

Now $\dot{V} = 0$ implies that $x_2 = 0\ or\ x_1 = \pi/2 \pm n\pi$. So in order to keep $\dot{V}$ negative definite, if we can manage to keep the system within a region $(x_1, x_2) \in B_r$ s.t. $x_2 = 0 \Rightarrow \dot{x_2}(t) = 0 \Rightarrow x_1(t) = 0$, meaning it comes back to the equilibrium point (0,0), we can call the system **locally asymptotically stable.**

Lyapunov Function/Stability

Q. Consider the non-linear system

$$\dot{x} = f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} -x_1 + 2{x_1}^2 x_2 \\ -x_2 \end{bmatrix}$$

The candidate Lyapunov function is : $v(x) = \alpha_1 {x_1}^2 + \alpha_2 {x_2}^2$

For $\alpha_1 = \alpha_2 = 1$, check if the above one is the good candidate of Lyapunov function for stability at origin.

Lyapunov Function/Stability

sol. According to Lyapunov stability theorem, a function $v: D \rightarrow R$ is candidate for Lypunov stability at the equilibrium as origin if

(i) $\mathrm{v}(0) = 0$

(ii) $v(x) > 0, \ \forall \mathrm{x} \in \mathrm{D} - \{0\}$

(iii) $\dot{\mathrm{v}}(\mathrm{x}) \leq 0, \ \forall \mathrm{x} \in \mathrm{D}$

For this problem, we assume $D \in R^2$. Therefore,

(i) $\mathrm{v}(0) = \alpha_1 * 0 + \alpha_2 * 0 = 0$

(ii) $\mathrm{v}(\mathrm{x}_1, \mathrm{x}_2) = \mathrm{x}_1{}^2 + \mathrm{x}_2{}^2 > 0 \ \forall x \in D - \{(0,0)\}$

Lyapunov Function/Stability

(iii) $\dot{v}(x_1, x_2) = 2\alpha_1 x_1(-x_1 + 2{x_1}^2 x_2) + 2\alpha_2 x_2(-x_2)$

Replacing with $\alpha_1 = \alpha_2 = 1,$

$\dot{v}(x_1, x_2) = -2{x_2}^2 - 2{x_1}^2(1 - 2x_1 x_2)$

Now, $\dot{v}(x_1, x_2)$ is guaranteed to be negative whenever $(1 - 2x_1 x_2) > 0 \Longrightarrow x_1 x_2 < \frac{1}{2}$.

We can conclude that $v(x_1, x_2)$ can be a good candidate for Lyapunov function whenever $x_1 x_2 < ½$.

Lyapunov Function/Stability: Homework

Q. Comment on the stability of the following system.

System : $\dot{x} = \frac{2}{1+x} - x$

Points of equilibrium: 1

Candidate Lyapunov Function: $V(x) = 0.5x^2 - x + 0.5$

Matlab Code Snippets

LQR

```
p = 50000;
Q = p*(C'*C);
R = 0.1;
[K] = dlqr(A,B,Q,R);
% Q and R are weight
matrix for x and u
respectively

x = [0.1;1];
u = -K*x
for i=1:time
    x = A*x + B*u;
    u = -K*x;
    plot_vectorx1(i) = x(1);
    plot_vectorx2(i) = x(2);
end
```

Kalman Filter

```
QN = 50;
RN = 0.01*eye(1);
[kalmf,L,P,M] =
kalmd(sys_ss,QN,RN,Ts);
% QN, RN are process error
and measurement error
covariance matrix

time = 20;
x = [0.1;1];
z = [0;0];
u = -K*z;

for i=1:time
    r = C*x-C*z;
    z = A*z + B*u + L*r;
    x = A*x + B*u;
    u = -K*z;
end
```

Lyapunov Function

```
% V (x) = x'P x, dV/dt (x) < 0, A'P + P A = − Q, Q > 0
Q=25;
setlmis([]);
P=lmivar(1,[size(A,1),1]);
lmi_id=1;
lmiterm([lmi_id 1 1 P],-Q,1);           % LMI # 1, element @ (1,1): -P
lmiterm([lmi_id 1 2 -P],A',1);          % LMI # 1, element @ (1,2) : (P*A)'
= (A' P'
lmiterm([lmi_id 2 1 P],1,A);            % LMI # 1, element @ (2,1) : P*A
lmiterm([lmi_id 2 2 P],-1,1);                 % LMI # 1, element @ (2,2): -P
lmiterm([-lmi_id 1 1 P],1,1);                 % LMI # 2  P       for P > 0
lmisys = getlmis;                       % Create the LMI system
[tmin,rP_feas] = feasp(lmisys) ;           % Solve LMI

P = dec2mat(lmisys,rP_feas,P)              % display P matrix
plot_vectorx = [plot_vectorx1;plot_vectorx2];

for i= 1:time
    V(i) = plot_vectorx(:,i)'*P*plot_vectorx(:,i);
end
```