Dynamical Systems

September 4, 2021

1 State-space Model to Transfer Function

Consider the following state-space model in continuous time domain,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{1}$$

Applying Laplac transform on Eq. 1 we get

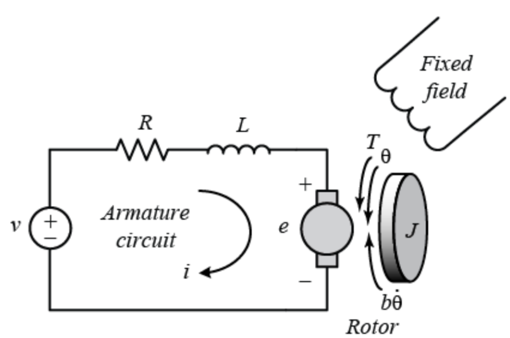
$$\begin{aligned} sX(s) &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s) \end{aligned} \tag{2}$$

From the above equations, the transfer function is computed as,

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D \tag{3}$$

Here, I is the identity matrix of order of the matrix A. The denominator of Eq. 3 is the characteristic equation whose roots are the poles of the system.

2 Example System: DC Motor Speed



The dynamical equations of the DC motor are given below,

$$J\ddot{\theta} + b\dot{\theta} = Ki$$
$$L\frac{di}{dt} + Ri = V - K\dot{\theta} \tag{4}$$

The above system has 2 states : the rotational speed of the motor θ and current i. θ is to be controlled by input voltage V, and also measured as output. The other notations are described in Tab. 1. From Eq. 4, we derive the system matrices as,

$$A = \begin{bmatrix} -\frac{b}{J} & \frac{K}{J} \\ -\frac{K}{L} & -\frac{R}{L} \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ \frac{1}{L}; \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0; \end{bmatrix} \quad D = 0; \tag{5}$$

Table 1: Notations

Symbol	Description	Value
R	armature resistance	1
L	armature inductance	0.5
K	electromotive force constant	0.01
J	moment of inertia	0.01
b	friction co efficient	0.1

Considering the parameters given in Tab. 1, we can compute the system matrices A and B. And, subesequently, the transfer function can be derived following the Eq. 3 as,

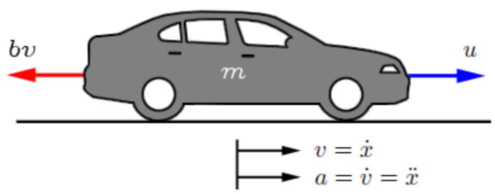
$$\frac{2}{s^2 + 12s + 20.02}$$

The characteristic equation is thus, $s^2 + 12s + 20.02$. The solutions of this equation are $s \simeq -2$ and $s \simeq -10$. Note that, these are nothing but the eigen values of A. Also, since both the poles have negative real part, the system is stable.

3 Practice Problems

Derive i) state space model, ii) transfer function, iii) poles (and thus, comment on their open-loop stability), iv) un-damped natural frequency, v) damping ratio, vi) peak-time, and vii) time required for the response curve to reach and stay within 2% or 5% of the steady-state value of the following systems.

3.1 Cruise Control



Purpose of the cruise control is to maintain constant speed. The dynamical equation of the system is,

$$m\dot{v} + bv = u \tag{6}$$

Here, m is the mass, b is the damping co-eeficient, and v is the velocity. The control force is u. Consider, v is measurable state. Given: $m = 100kg$ and $b = 10N.s/m$.

3.2 Power Generator

Consider the following system equations:

$$\begin{aligned} w(t) &= \dot{\theta(t)} \\ M\dot{w}(t) &= -Dw(t) - b\theta(t) + u(t) \end{aligned} \tag{7}$$

θ and w are the phase angle and frequency deviation of the power generator. Both the states are measurable. $u(t)$ is the normalized mechanical power provided to the generator. M and D are the inertia and damping coeffiecient. And, b is the susceptance parameter of the transmission line. Given, $M = D = b = 1$.