# Linear Quadratic Regulator and Kalman Filter

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## 1 Discrete Time System

In this section we discuss how a continuous time system can be transformed into a discrete time system by considering the behaviour of the signals at the sampling instants.

### 1.1 Sampling Continuous Time Signal

Sampling a continuous time system means to replace the signals by its values at a discrete set of points. Let $\{t_k : k \in \mathbb{Z}\}$ be a subset of real numbers called sampling instants. In periodic sampling, the sampling instants are equally spaced in time i.e., $t_k = k.h$ where $h$ is the sampling period.

### 1.2 Sampling a continuous Time System

Assume that the continuous time system is given in the following state-space form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ \dot{x}(t) - Ax(t) &= Bu(t) \end{aligned} \tag{1}$$

Let us consider any differential equation which is of the form as given below.

$$\dot{y} + P(x)y = Q(x)$$

Both side of such equation can be multiplied with $e^{\int P(x)dx} = e^{-At}$ to facilitate subsequent integration process. Using this in Eq. 1 we get,

$$\begin{aligned} e^{-At}\dot{x}(t) - e^{-At}Ax(t) &= e^{-At}Bu(t) \\ \Rightarrow \frac{d}{dt}(e^{-At}x(t)) &= e^{-At}Bu(t) \\ \Rightarrow \left| e^{-At}x(t) \right|_{t_k}^{t} &= \int_{t_k}^{t} e^{-At}Bu(t)dt \\ \Rightarrow e^{-At}x(t) - e^{-At_k}x(t_k) &= \int_{t_k}^{t} e^{-At}Bu(t)dt \end{aligned} \tag{2}$$

The value of control signal $u(t)$ changes only at sampling instants. Therefore, $u(t) = u(t_k)$ in $(t_k,\ t_{k+1})$ and

$$\int_{t_k}^{t} e^{-At} Bu(t)dt = Bu(t_k)\left|\frac{e^{-At}}{-A}\right|_{t_k}^{t}$$

Using the above result in Eq. 2,

$$\begin{aligned} e^{-At}x(t) &= e^{-At_k}x(t_k) + \frac{Bu(t_k)}{A}(e^{-At_k} - e^{-At}) \\ x(t) &= e^{A(t-t_k)}x(t_k) + \frac{Bu(t_k)}{A}(e^{A(t-t_k)} - 1) \end{aligned} \tag{3}$$

Let, $t_k = kh$ and $t = (k+1)h$. Therefore,

$$\begin{aligned} x((k+1)h) &= e^{Ah}x(kh) + \frac{Bu(kh)}{A}(e^{Ah} - 1) \\ &= \Phi x(kh) + \Gamma u(kh) \end{aligned} \tag{4}$$

where $\Phi = e^{Ah}$ and $\Gamma = \frac{B}{A}(e^{Ah} - 1)$

## 1.3 Sampling a System with Time Delay

Sampled systems often have delays, when the controller output is available after some delay. Let the system be described by,

$$\dot{x}(t) = Ax(t) + Bu(t-\tau) \tag{5}$$

It is assumed that the $\tau \leq h$. Integrating Eq. 5 over one sampling period gives,

$$x(kh+h) = e^{Ah}x(kh) + \int_{kh}^{kh+h} e^{A(kh+h-s)}Bu(s-\tau)ds \tag{6}$$

Because, the signal $u(t)$ is constant over sampling interval, the delayed signal $u(t\text{-}\tau)$ is also constant over sampling interval. The delayed signal will, however, change between the sampling instants.

$$\begin{aligned} \int_{kh}^{kh+h} e^{A(kh+h-s)}Bu(s-\tau)ds &= u(kh-h)\int_{kh}^{kh+\tau} e^{A(kh+h-s)}Bds + u(kh)\int_{kh+\tau}^{kh+h} e^{A(kh+h-s)}Bds \\ &= \Gamma_1 u(kh-h) + \Gamma_0 u(kh) \end{aligned}$$

Sampling the continuous time system in Eq. 5 thus gives,

$$x(kh+h) = \Phi x(kh) + \Gamma_0 u(kh) + \Gamma_1 u(kh-h) \tag{7}$$

where

$$\begin{aligned} \Phi &= e^{Ah} \\ \Gamma_0 &= B\int_0^{h-\tau} e^{As}ds \\ \Gamma_1 &= Be^{A(h-\tau)}\int_0^{\tau} e^{As}ds \end{aligned} \tag{8}$$

## 2 Linear Quadratic Regulator

### 2.1 The Process

The process to be controlled is described by the following continuous time model

$$dx = Axdt + Budt + dv_c \tag{9}$$

Here $A$, $B$ may be time varying matrices and $v_c$ has mean value of zero. The above model can be sampled. The input $u(t)$ is constant over the sampling period. For noise free case, the solution of Eq. 9 can be written as,

$$x(t) = \Phi(t, kh)x(kh) + \Gamma(t, kh)u(kh) \tag{10}$$

where $\Phi$(t) is fundamental matrix of Eq. 10 satisfying

$$\frac{d}{dt}\Phi(t, kh) = A(t)\Phi(t, kh), \qquad \Phi(kh, kh) = I, \qquad \Gamma(t, kh) = \int_{kh}^{t} \Phi(t, s)B(s)ds$$

Omitting the time arguments of the matrices, the sampled model can be written as,

$$\begin{aligned} x(kh + h) &= \Phi x(kh) + \Gamma u(kh) + v(kh) \\ y(kh) &= Cx(kh) + e(kh) \end{aligned} \tag{11}$$

where $v$ is process noise and $e$ is measurement noise. Both $v$ and $e$ are discrete time Gaussian white noise processes with zero mean value and

$$\begin{aligned} E[v(kh)v^T(kh)] &= R_1 = \int_0^h e^{A\Gamma} R_{1C} e^{A^T\Gamma} d\Gamma \\ E[v(kh)e^T(kh)] &= R_{12} \\ E[e(kh)e^T(kh)] &= R_2 \end{aligned}$$

### 2.2 The Criterion

We consider the output of the process to be controlled be representative of the state variable $x(k)$. Let the operating point be origin. Ideally, for the process to be stable, we want the $x(k)$ values to be near to zero and restrict the control input $u(k)$ not to change drastically. The design criteria we will use is a way of weighting the magnitude of the state and the control signals. One way to look at the power of the state is,

$$J = \int_0^{Nh} |x(t)|^2 dt = \int_0^{Nh} x^T(t)x(t)dt$$

The components of the state may have different dimensions and we can instead use a more general weighting

$$J = \int_0^{Nh} x^T(t)Q_{1C}x(t)dt$$

where $Q_{1C}$ is a symmetric positive semi-definite matrix. This leads to a control problem where we want to minimize the loss function,

$$\begin{aligned} J &= E[\int_0^{Nh} (x^T(t)Q_{1C}x(t) + 2x^T(t)Q_{12C}u(t)+ \\ &u^T(t)Q_{2C}u(t))dt + x^T(Nh)Q_{0C}x(Nh)] \\ &= E[\int_0^{Nh} \left[ \begin{array}{cc} x^T(t) & u^T(t) \end{array} \right] Q_c \left[ \begin{array}{c} x(t) \\ u(t) \end{array} \right] dt + x^T(Nh)Q_{0C}x(Nh)] \end{aligned} \tag{12}$$

with

$$Q_c = \left[ \begin{array}{cc} Q_{1c} & Q_{12c} \\ Q_{12c}^T & Q_{2c} \end{array} \right]$$

where $Q_{0c}$, $Q_{1c}$, and $Q_{2c}$ are symmetric.

## 2.3 The Problem

The optimal control problem is defined to be finding the admissible control signal that minimizes the loss function of Eq. 12 when the process is described by the model of Eq. 9 or equivalent model of Eq. 11. The design parameters are the matrices in the loss function and the sampling period.

## 2.4 Sampling the Loss Function

We assume that periodic sampling is used and the control signal is constant over sampling periods.The continuous time loss function of Eq. 12 can be transformed into discrete time loss function as

$$J = E[\sum_{k=0}^{N-1} J(k) + x^T(Nh)Q_{0C}x(Nh)]$$

where

$$J(k) = \int_{kh}^{kh+h} (x^T(t)Q_{1C}x(t) + 2x^T(t)Q_{12C}u(t) + u^T(t)Q_{2C}u(t))dt \tag{13}$$

Using Eq. 10 in Eq. 13 and the fact that $u(t)$ is constant over the sampling period gives

$$J(k) = x^T(kh)Q_1x(kh) + 2x^T(kh)Q_{12}u(kh) + u^T(kh)Q_2u(kh)$$

where

$$Q_1 = \int_{kh}^{kh+h} \Phi^T(s,kh)Q_{1c}\Phi(s,kh)ds \tag{14}$$

$$Q_{12} = \int_{kh}^{kh+h} \Phi^T(s,kh)(Q_{1c}\Gamma(s,kh) + Q_{12c})ds \tag{15}$$

$$Q_2 = \int_{kh}^{kh+h} (\Gamma^T(s,kh)Q_{1c}\Gamma(s,kh) + 2\Gamma^T(s,kh)Q_{12c} + Q_{2c})ds \tag{16}$$

Minimizing the loss function of Eq. 12 when $u(t)$ is constant over sampling period is thus the same as minimizing the discrete time loss function

$$\begin{aligned} J &= E[\sum_{k=0}^{N-1}(x^T(kh)Q_1x(kh) + 2x^T(kh)Q_{12}u(kh)+ \\ &u^T(kh)Q_2u(kh)) + x^T(Nh)Q_0x(Nh)] \\ &= E[\sum_{k=0}^{N-1} \begin{bmatrix} x^T(kh) & u^T(kh) \end{bmatrix} Q \begin{bmatrix} x(kh) \\ u(kh) \end{bmatrix} + x^T(Nh)Q_0x(Nh)] \end{aligned} \tag{17}$$

where

$$Q = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix} \tag{18}$$

## 2.5 Linear Quadratic Control for Deterministic System

The deterministic case with $v(k) = e(k) = 0$ is considered. We assume that the sampling period is used as time unit i.e., $h = 1$. Thus, the system in Eq. 11 is described by,

$$x(k+1) = \Phi x(k) + \Gamma u(k) \tag{19}$$

where $x(0)$ is given. The problem is now to determine the control sequence $u(0)$, $u(1)$, ..., $u(N - 1)$ such that the loss function in Eq. 17 is minimized.

The idea behind the derivation of the control law is to use the *principle of optimality* and *dynamic programming*. The principle of optimality states that an optimal policy has the property that whatever the initial state and the initial decision are the remaining decisions must be optimal with respect to the state resulting from the first decision. By using this idea and starting from the end time $N$ and going backwards in time, it is possible to determine the best control law for the last step independent of how the state at time $N$ - 1 was reached.

Let $S(N) = Q_0 = W_x$. Let the minimum value of the loss (Eq. 17) is,

$$minJ = V_0 = x^T(0)S(0)x(0)$$

We introduce,

$$\begin{aligned} V_k = \min_{u(k),\ldots,u(N-1)} \sum_{i=k}^{N-1} (x^T(i)Q_1x(i) + u^TQ_2u(i) + \\ 2x^T(i)Q_{12}u(i)) + x^T(N)Q_0x(N) \end{aligned} \tag{20}$$

$V_k$ can be interpreted as loss from $k$ to $N$. For $k = N$ we have,

$$V_N = x^T(N)S(N)x(N)$$

Using Eq. 19 we get,

$$V_N = [\Phi x(N-1) + \Gamma u(N-1)]^T S(N)[\Phi x(N-1) + \Gamma u(N-1)] \tag{21}$$

Now, for $k = N-1$ we have,

$$V_{N-1} = \min_{u(N-1)} \{x^T(N-1)W_x x(N-1) + u^T(N-1)W_u u(N-1) + V_N\}$$

Since we want to minimize $|x(t)|^2$ and $|u(t)|^2$, we ignore the covariance term of $x(t)$ and $u(t)$ in the subsequent derivations (considering for some distribution, covariance of $x(t)$ and $u(t)$ is zero). Using Eq. 21 we get,

$$V_{N-1} = \min_{u(N-1)} \{x^T(N-1)W_x x(N-1) + u^T(N-1)W_u u(N-1) + x^T(N-1)\Phi^T S(N)\Phi x(N-1) +$$
$$u^T(N-1)\Gamma^T S(N)\Phi x(N-1) + x^T(N-1)\Phi^T s(N)\Gamma u(N-1) + u^T\Gamma^T S(N)\Gamma u(N-1)\}$$

$$\begin{aligned} V_{N-1} = \min_{u(N-1} \{ & x^T(N-1)(W_x + \Phi^T S(N)\Phi)x(N-1) \\ & + x^T(N-1)\Phi^T S(N)\Gamma u(N-1) \\ & + u^T(N-1)\Gamma^T S(N)\Phi x(N-1) \\ & + u^T(N-1)(\Gamma^T S(N)\Gamma + W_u)u(N-1)\} \end{aligned} \tag{22}$$

First term in Eq. 22 has no role in the minimization process since it is independent of $u(t)$. Eq. 22 can be written in the following form,

$$F(U) = U^T AU + Z^T U + U^T Z$$

Where $A = W_u + \Gamma^T S(N)\Gamma$ and $Z = \Gamma^T S(N)\Phi x(N-1)$.Now,

$$\begin{aligned} F(U) &= U^T AU + Z^T U + U^T Z + Z^T A^{-1} Z - Z^T A^{-1} Z \\ &= (U + A^{-1}Z)^T A(U + A^{-1}Z) - Z^T A^{-1} Z \end{aligned}$$

Each term in the above equation is non-negative. Hence, F($U$) is minimum if $(U + A^{-1}Z)^T A(U + A^{-1}Z) = 0$ i.e., $U = -A^{-1}Z$. Thus, optimal value of $u$($N$ - 1) is,

$$\begin{aligned} u(N-1) &= -(W_u + \Gamma^T S(N)\Gamma)^{-1}\Gamma^T S(N)\Phi x(N-1) \\ &= -G(N-1)x(N-1) \end{aligned} \tag{23}$$

where gain matrix $G$($N$ - 1),

$$G(N-1) = (W_u + \Gamma^T S(N)\Gamma)^{-1}\Gamma^T S(N)\Phi \tag{24}$$

Using Eq. 23 in Eq. 22 we get,

$$
\begin{aligned}
V_{N-1} &= x^T(N-1)(W_x + \Phi^T S(N)\Phi)x(N-1) \\
&- x^T(N-1)\Phi^T S(N)\Gamma G(N-1)x(N-1) \\
&- x^T(N-1)G^T(N-1)\Gamma^T S(N)\Phi x(N-1) \\
&+ x^T(N-1)G^T(N-1)(\Gamma^T S(N)\Gamma + W_u)G(N-1)x(N-1) \\
&= x^T(N-1)\{(W_x + \Phi^T S(N)\Phi) - \Phi^T S(N)\Gamma G(N-1) - G^T(N-1)\Gamma^T S(N)\Phi \\
&+ G^T(N-1)(\Gamma^T S(N)\Gamma + W_u)G(N-1)\}x(N-1) \\
&= x^T(N-1)\{(W_x + \Phi^T S(N)\Phi) - \Phi^T S(N)\Gamma G(N-1) - G^T(N-1)\Gamma^T S(N)\Phi \\
&+ G^T(N-1)\Gamma^T S(N)\Gamma G(N-1) + G^T(N-1)W_u G(N-1)\}x(N-1) \\
&= x^T(N-1)\{[\Phi - \Gamma G(N-1)]^T S(N)[\Phi - \Gamma G(N-1)] \\
&+ W_x + G^T(N-1)W_u G(N-1)\}x(N-1) \\
&= x^T(N-1)S(N-1)x(N-1)
\end{aligned} \tag{25}
$$

where,
$S(N) = [\Phi - \Gamma G(N)]^T S(N+1)[\Phi - \Gamma G(N)] + W_x + G^T(N)W_u G(N)$.

Note that we can thus estimate $S(N) \to G(N-1) \to S(N-1) \to \cdots$ The equations of $G$ and $S$ give stable solutions for $N \to \infty$ (when the system matrices obey certain regularity conditions) and are known as the Algebraic Riccati Equations (ARE). If $(\Phi, \Gamma)$ is reachable and if $(\Phi, \Sigma)$ is observable pair, where $W_u = \Sigma^T\Sigma$, then there exists a unique, symmetric, non-negative definite solution to the ARE.

# 3 Kalman Filter

It is unrealistic to assume that all states of a system can be measured, particularly if disturbances are part of the state. It is therefore of interest to determine the states of a system from available measurements and a model.It is assumed that the system is described by the sampled model in Eq. 11 with $h = 1$. The problem is thus to calculate or reconstruct the state $x(k)$ from input and output sequences $y(k)$, $y(k\text{-}1)$, ... $u(k)$, $u(k\text{-}1)$,... is considered in this section.

Postulate an one step ahead estimator of the form,

$$\hat{x}(k+1|k) = \Phi\hat{x}(k|k-1) + \Gamma u(k) + K(k)(y(k) - C\hat{x}(k|k-1)) \tag{26}$$

The reconstruction error $\widetilde{x} = x - \hat{x}$ is governed by,

$$\begin{aligned}
\widetilde{x}(k+1) &= x(k+1) - \hat{x}(k+1) \\
&= \Phi x(k) + \Gamma u(k) + v(k) - [\Phi \hat{x}(k|k-1) + \Gamma u(k) + K(k)(y(k) - C\hat{x}(k|k-1))] \\
&= \Phi(x(k) - \hat{x}(k|k-1)) + v(k) - K(k)(y(k) - C\hat{x}(k|k-1)) \\
&= \Phi \widetilde{x}(k) + v(k) - K(k)(y(k) - C\hat{x}(k|k-1)) \\
&= \Phi \widetilde{x}(k) + K(k)C\hat{x}(k) + v(k) - K(k)y(k) \\
&= \Phi \widetilde{x}(k) - K(k)C(x(k) - \hat{x}(k)) + K(k)Cx(k) + v(k) - K(k)y(k) \\
&= (\Phi - K(k)C)\widetilde{x}(k) + v(k) - K(k)(y(k) - Cx(k))
\end{aligned} \tag{27}$$

Measurement noise $e(k)$ is given by,

$$e(k) = y(k) - Cx(k) \tag{28}$$

Using Eq. 28 in Eq. 27,

$$\begin{aligned}
\widetilde{x}(k+1) &= (\Phi - K(k)C)\widetilde{x}(k) + v(k) - K(k)e(k) \\
&= \begin{bmatrix} I & -K(k) \end{bmatrix} \left( \begin{bmatrix} \Phi \\ C \end{bmatrix} \widetilde{x}(k) + \begin{bmatrix} v(k) \\ e(k) \end{bmatrix} \right)
\end{aligned} \tag{29}$$

The criterion is to minimize the variance of the estimation error, which is denoted by $P(k)$.

$$P(k) = E[(\widetilde{x}(k) - E[\widetilde{x}(k)])(\widetilde{x}(k) - E[\widetilde{x}(k)])^T]$$

The mean value of $\widetilde{x}$ is obtained from Eq. 29,

$$E[\widetilde{x}(k+1)] = (\Phi - K(k)C)E[\widetilde{x}(k)]$$

We assume that the initial state $x(0)$ is Gaussian distributed with mean, i.e. $E(x(0)) = m_0$. Given that $x(k)$ is the linear transformation sequence of $x(0)$, the reconstruction error $\widetilde{x}(k)$ will have zero mean, i.e. $E(\widetilde{x}(k)) = 0$. Hence,

$$P(k) = E[\widetilde{x}(k)\widetilde{x}^T(k)]$$

$$
\begin{aligned}
P(k+1) &= E[\widetilde{x}(k+1)\widetilde{x}^T(k+1)] \\
&= E[\begin{bmatrix} I & -K(k) \end{bmatrix}(\begin{bmatrix} \Phi \\ C \end{bmatrix}\widetilde{x}(k) + \begin{bmatrix} v(k) \\ e(k) \end{bmatrix}) \\
&(\begin{bmatrix} \Phi \\ C \end{bmatrix}\widetilde{x}(k) + \begin{bmatrix} v(k) \\ e(k) \end{bmatrix})^T \begin{bmatrix} I \\ -K^T(k) \end{bmatrix}] \\
&= E[\begin{bmatrix} I & -K(k) \end{bmatrix}(\begin{bmatrix} \Phi \\ C \end{bmatrix}\widetilde{x}(k)\widetilde{x}^T(k)\begin{bmatrix} \Phi \\ C \end{bmatrix}^T + \begin{bmatrix} v(k) \\ e(k) \end{bmatrix}\widetilde{x}^T(k)\begin{bmatrix} \Phi \\ C \end{bmatrix}^T \\
&\begin{bmatrix} \Phi \\ C \end{bmatrix}\widetilde{x}(k)\begin{bmatrix} v(k) \\ e(k) \end{bmatrix}^T + \begin{bmatrix} v(k) \\ e(k) \end{bmatrix}\begin{bmatrix} v(k) \\ e(k) \end{bmatrix}^T)\begin{bmatrix} I \\ -K^T(k) \end{bmatrix}] \\
&= \begin{bmatrix} I & -K(k) \end{bmatrix}(\begin{bmatrix} \Phi \\ C \end{bmatrix}P(k)\begin{bmatrix} \Phi \\ C \end{bmatrix}^T + \\
&\begin{bmatrix} E[v(k)v^T(k)] & E[v(k)e^T(k)] \\ E[e(k)v^T(k)] & E[e(k)e^T(k)] \end{bmatrix})\begin{bmatrix} I \\ -K^T(k) \end{bmatrix}
\end{aligned} \tag{30}
$$

We have two terms becoming zero as $E(\widetilde{x}(k))$ is zero. Let,

$$
\begin{aligned}
E[v(k)v^T(k)] &= R_1 \\
E[v(k)e^T(k)] &= R_{12} \\
E[e(k)e^T(k)] &= R_2
\end{aligned} \tag{31}
$$

Using Eq. 31 in Eq 30 we get,

$$
P(k+1) = \begin{bmatrix} I & -K(k) \end{bmatrix}\begin{bmatrix} \Phi P(k)\Phi^T + R_1 & \Phi P(k)C^T + R_{12} \\ CP(k)\Phi^T + R_{12}^T & CP(k)C^T + R_2 \end{bmatrix}\begin{bmatrix} I \\ -K^T(k) \end{bmatrix} \tag{32}
$$

The above is a quadratic equation of the below form,

$$
F(x,u) = \begin{bmatrix} x^T & u^T \end{bmatrix}\begin{bmatrix} Q_x & Q_{xu} \\ Q_{xu}^T & Q_u \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} \tag{33}
$$

and we want to find the minimum with respect to $u$. If there exists an $L$ satisfying,

$$
Q_u L = Q_{xu}^T \tag{34}
$$

then the function becomes ("completing the square technique"),

$$
F(x,u) = x^T(Q_x - L^T Q_u L)x + (u+Lx)^T Q_u (u+Lx)
$$

Since this expression is quadratic in $u$ and both terms are $\geq 0$, it is minimum with $u = -Lx$. Note that Eq. 32 has similar form as Eq. 33 in the sense that $u = (-K(k))^T$, $Q_{xu} = \Phi P(k)C^T + R_{12}$, $Q_u = CP(k)C^T + R_2, x = I^T$. So we have,

$$
\begin{aligned}
&Q_u L = Q_{xu}^T \\
\Rightarrow &(CP(k)C^T + R_2)L = (\Phi P(k)C^T + R_{12})^T \\
\Rightarrow &L = (CP(k)C^T + R_2)^{-1}(\Phi P(k)C^T + R_{12})^T
\end{aligned}
$$

Hence, the expression of $P$ in Eq. 32, is minimized with,

$$
\begin{aligned}
u &= -Lx \\
\Rightarrow (-K(k))^T &= -(CP(k)C^T + R_2)^{-1}(\Phi P(k)C^T + R_{12})^T I \\
\Rightarrow K(k) &= (\Phi P(k)C^T + R_{12})((CP(k)C^T + R_2)^{-1})^T \\
\Rightarrow K(k) &= (\Phi P(k)C^T + R_{12})((CP(k)C^T + R_2)^{-1})^T \\
\Rightarrow K(k) &= (\Phi P(k)C^T + R_{12})(R_2 + CP(k)C^T)^{-1}
\end{aligned}
\tag{35}
$$

since both $R_2$ and $CP(k)C^T$ are symmetric. Now, inserting Eq. 35 into Eq. 32,

$$
\begin{aligned}
P(k+1) = {} & (\Phi P(k)\Phi^T + R_1) \\
& - (\Phi P(k)C^T + R_{12})(R_2 + CP(k)C^T)^{-1}(CP(k)\Phi^T + R_{12}^T)
\end{aligned}
\tag{36}
$$

where $P(0) = R_0$. The reconstruction defined by Eq. 26, Eq. 35, and Eq. 36 is called the *Kalman Filter*. After a few iterations, the gain value $K$ converges.