Basics of Dynamical and Control System (CS—)

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Starting with some Examples

Cruise Control System

- Cruise control is a standard example found in many modern vehicles.
- It maintains a constant vehicle speed despite of external disturbances, such as changes in wind or road grade.
- It controls the speed by measuring the vehicle speed, comparing it to the desired or reference speed, and automatically adjusting the throttle.

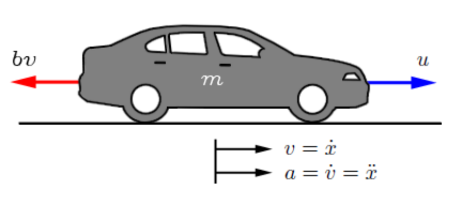

Figure: Free-Body Diagram of Cruise Control

- m is the mass of the vehicle and u is the control force acting on it.
- The force u represents the force generated at the road/tire interface.
- The resistive forces, bv, due to rolling resistance and wind drag, are assumed to vary linearly with the vehicle velocity, v, and act in the direction opposite the vehicle's motion.

System Equations

Applying Newton's 2nd law over the summing forces in the x-direction, we arrive at the following system equation:

$$m\dot{v} + bv = u$$

Since we are interested in controlling the speed of the vehicle, the output equation is chosen as follows:

$$y = v$$

Source: http://ctms.engin.umich.edu/CTMS

DC Motor Speed System

- The basic principle of the DC motor is to convert DC energy into mechanical energy.
- It provides rotary motion as well as translational motion coupled with wheels or drums and cables.
- Since the torque generated by a DC motor is proportional to the armature current and the strength of the magnetic field, its speed control can be done either by controlling armature or field.

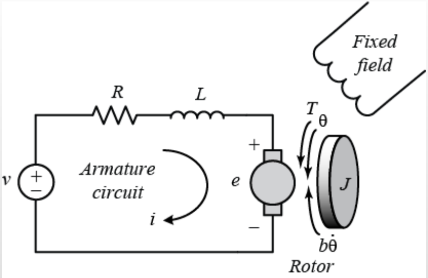

Figure: Free-Body Diagram of DC Motor Speed System

- V is the voltage source applied to the motor's armature.
- $J \rightarrow$ moment of inertia of the rotor, $b \rightarrow$ motor viscous friction constant, $T \rightarrow$ torque, $R \rightarrow$ the electric resistance and $L \rightarrow$ electric inductance.
- The rotational speed of the shaft is $d(\theta)/dt$ where θ is the angular displacement of rotor.

System Equations

1. Since we are assuming that the magnetic field is constant, the motor torque (T) is proportional to only the armature current i by a constant factor Kt as shown below:

$$T = K_t i$$

2. The back emf, e, is proportional to the angular velocity ($\dot{\theta}$) of the shaft by a constant factor Ke.

$$e = K_e \dot{\theta}$$

3. Following Newton's 2nd law and Kirchhoff's voltage law we can derive the system equations as given below:

$$J\ddot{\theta} + b\dot{\theta} = Ki \quad and \quad L\frac{di}{dt} + Ri = V - K\dot{\theta}$$

where K represents both the constants Kt and Ke (in SI units $Kt = Ke$).

Formal Definition

Dynamical System

- It is a system that changes over time according to a set of fixed rules that determine how one state of the system moves to another state.
- Comprising of two components,
  - a state vector, which is a minimum set of variables that fully describe the system and its response to any given set of inputs.
  - a function, which tells us, given the current state, what the state of the system will be in the next instant of time- i.e., rate of change of the state vector

System State

- A state vector can be described by,

$$x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]$$

- For any state-determined system, from the knowledge of statw variables at an initial time t_0 and the system inputs for time $t \geq t_0$, it is possible to predict the future system state and outputs for all time $t > t_0$.
- The state variables are an internal description of the system which completely characterize the system state at any time t, and from which any output variables $y_i(t)$ may be computed.
- The number of state variables, n, is equal to the number of independent energy storage elements in the system.

System Function - the state equation

- A function can be described by a single function or by a set of functions

$$f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots, f_n(x_1, x_2, \ldots, x_n)$$

- Entire system can be then described by a set of differential equations

$$\dot{x}_1 = \frac{dx_1}{dt} = f_1(x_1, x_2, \ldots, x_n)$$

$$\dot{x}_2 = \frac{dx_2}{dt} = f_2(x_1, x_2, \ldots, x_n)$$

$$\vdots$$

$$\dot{x}_n = \frac{dx_n}{dt} = f_n(x_1, x_2, \ldots, x_n)$$

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Classification of Dynamical System

Linear	Non-linear
Autonomous	Non-autonomous
Conservative	Non-conservative
Discrete	Continous
One-dimensional	Multidimensional

Linear system - the respective state equation(s) describing the system behavior must satisfy two basic properties:

- Additivity: $f(x+y) = f(x) + f(y)$
- Homogeneity: $f(\alpha y) = \alpha f(y)$

Example: $f(x) = 3x$ and $f(y) = 3y$

- Additivity $f(x+y) = 3(x+y) = 3x+3y = f(x)+f(y)$
- Homogeneity: $5 * f(x) = 5 * 3x = 15x = f(5x)$

Non-Linear system - the respective state equation(s) are described by a nonlinear function. It does not satisfy previous basic properties.
Example: $f(x) = x^2; f(y) = y^2$

Autonomous system- is a system of ordinary differential equations, which do not depend on the independent variable. If the independent variable is time, we call it time-invariant system.

Property: By definition, a time-invariant systems output will shift in time if its input shifts in time, otherwise will remain exactly the same.

Example: Let $y(t) = 10x(t)$ be the output of a system.
Consider a delay of the input: $x_d(t) = x(t+\delta)$ and
$y_1(t) = y(t) = 10x_d(t) = 10x(t+\delta)$.
Now delay the output by δ: $y_2(t) = y(t+\delta) = 10x(t+\delta)$.
Clearly $y_1(t) = y_2(t)$. Hence the system is time-invariant or autonomous.
Whereas, if $y(t) = tx(t)$, for a delay of the input we get:
$y_1(t) = y(t) = tx_d(t) = tx(t+\delta)$ and for delay in output we get: $y_2(t) = y(t+\delta) = (t+\delta)x(t+\delta)$.
Clearly $y_1(t) \neq y_2(t)$. Hence this system is non-autonomous.

Conservative system - the total mechanical energy remains constant, there are no dissipations present, e.g. simple harmonic oscillator

Nonconservative (dissipative) system -the total mechanical energy changes due to dissipations like friction or damping, e.g. damped harmonic oscillator

Discrete system - is described by a difference equation or set of difference equations. For this system we denote time by **k**, and the system is typically specified by the equations:

$$x(0) = x_0, \quad and \quad x(k+1) = f(x(k))$$

It thus follows that $x(k) = f^k(x_0)$, where f^k denotes a k-fold application of f to x_0.

Example: Typical example is annual progress of a bank account. If the initial deposit is 10 Lakh and annual interest is 3%, then we can describe the system by,

$$x(0) = 10, \quad x(k+1) = 1.03x(k), \quad and \quad x(k) = 1.03^k \times 10$$

Continuous system -is described by a differential equation or a set of differential equations. For a continuous time dynamical system, we denote time by t, and the following equations specify the system:

$$x(0) = x_0, \quad and \quad \dot{x(t)} = f(x(t))$$

Example: Vertical throw of a ball is described by,

$$\dot{h(t)} = v(t), \quad and \quad \dot{v(t)} = -g$$

with initial condition $h(0) = 0$ (height) and $v(0) = 0$ (velocity of the ball).

One-dimensional system- is described by a single function like

$$x(k+1) = ax(k) + b \quad \textit{discrete system}$$

$$\dot{x}(t) = ax(t) + b \quad \textit{continuous system}$$

where a,b are constants.

Multidimensional system-is described by a vector of functions like

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B} \quad \textit{discrete system}$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B} \quad \textit{continuous system}$$

where $\mathbf{x}$ is a vector with n components, $\mathbf{A}$ is $n \times n$ matrix and $\mathbf{B}$ is a constant vector.

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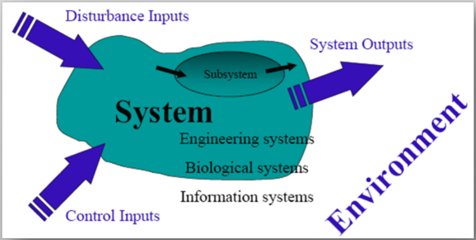
Plant/Process

Plant is the device or system under control. The input and output interaction represents the cause-and-effect relationship of the plants.

The interaction is defined in terms of variables.

1. System input
2. System output
3. Environmental disturbances

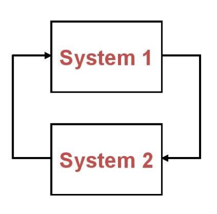


What is Feedback?

- Its a signal that return to the input as a part of the output of the plant (as for producing changes in an electronic circuit that improve performance or in an automatic control device that provide self-corrective action)

- In other words, feedback is a mutual interconnection of two (or more) systems

    - System 1 affects System 2
    - System 2 affects System 1
    - Systems are mutually dependent

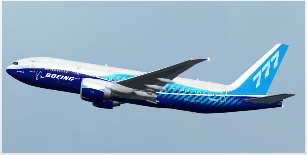


- Feedback is ubiquitous in natural and engineered systems

Source: Merriam-Webster

What do these two have in common?

Tornado



Boeing 777

Similarities

- Highly nonlinear, complicated dynamics!
- Both are capable of transporting goods and people over long distances

Difference

- One is controlled, but the other is not.

- Control is "the hidden technology that you meet every day"
- It heavily relies on the notion of "feedback"

Ref: https://www.seas.upenn.edu/

Control System

- Control is a process of causing a system variable such as temperature or position to follow some desired value, called reference value.
  - Manual control (driving the car yourself)
  - Automatic control (involving machines or computers only)

A control system is one which can control any quantity of interest in a machine, mechanism or other equipment in order to achieve the desired performance or output.

OR

A control system is an interconnection of components connected or related in such a way so that it can command, direct, or regulate itself or another system.

Example

Let us consider the driving system of an automobile. Speed of the automobile is a function of the position of its accelerator. The desired speed can be maintained (or a desired change in speed can be achieved) by controlling pressure on the accelerator pedal. This automobile driving system (accelerator, carburetor and engine-vehicle) constitutes a control system.

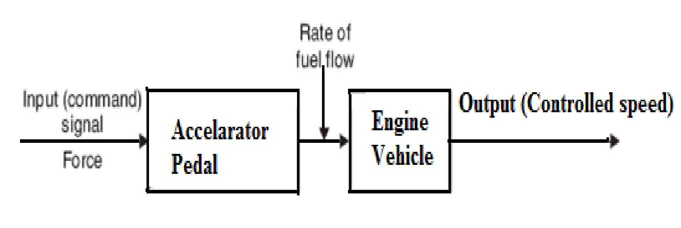
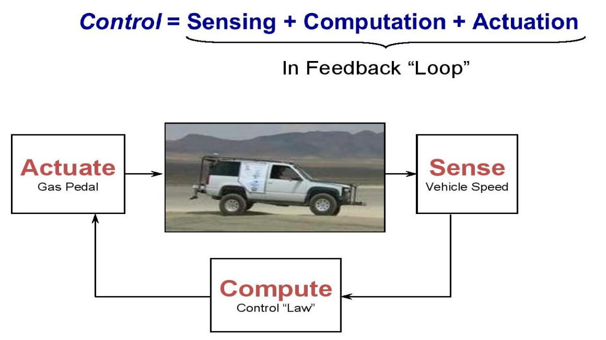


Figure: Block diagram of a control system.



Goals

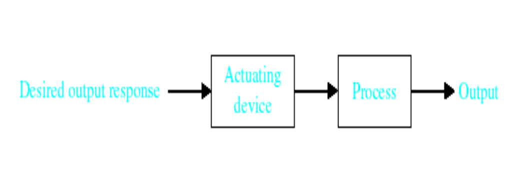
- **Stability**: system maintains desired operating point (hold steady speed)
- **Performance**: system responds rapidly to changes (accelerate to high speed)
- **Robustness**: system tolerates perturbations in dynamics (mass, drag etc)

Ref: https://www.seas.upenn.edu/

Control System Classification

Open-Loop Control Systems

- A system which does not possess any feedback network, and contains only the input and output relationship.
- utilize a controller or control actuator to obtain the desired response.
- Examples are light switches, gas ovens etc.
- The drawback of an open loop control system is that it is incapable of making automatic adjustments.



Missile Launcher System

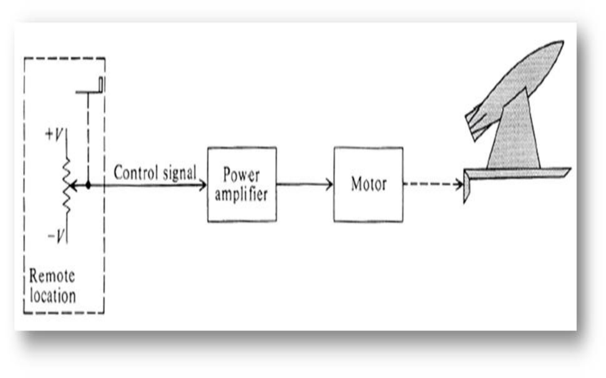


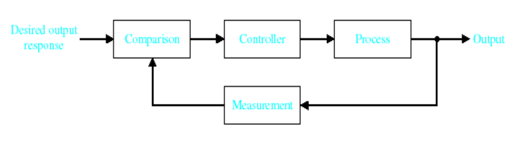
Figure: Open-Loop Control System.

Ref: ee.yeditepe.edu.tr

Control System Classification

Closed-Loop Control Systems

- A closed loop system is one which uses a feedback control between input and output.
- Compares the output with the expected result or command status, then it takes appropriate control actions to adjust the input signal.
- Examples are air conditioners, refrigerators, automatic rice cookers, automatic ticketing machines etc.
- One advantage of using the closed loop control system is that it is able to adjust its output automatically by feeding the output signal back to the input.



Missile Launcher System

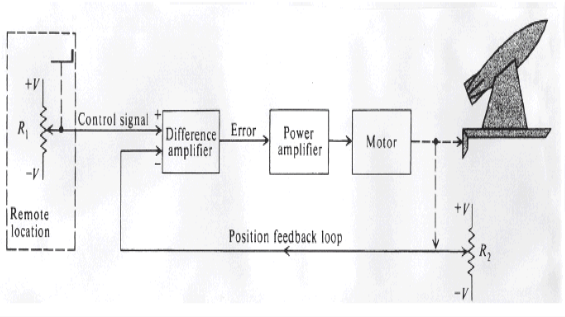


Figure: Closed-Loop Feedback Control System

Ref: ee.yeditepe.edu.tr

Control System Classification

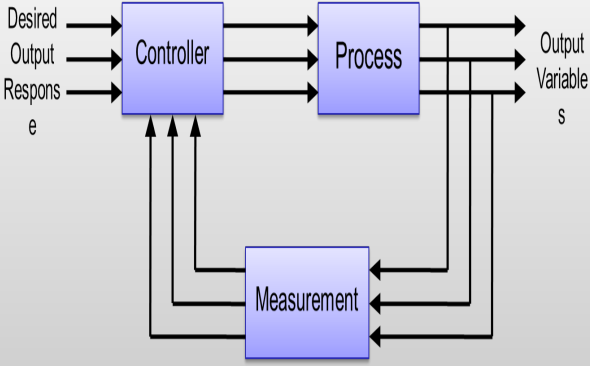


Figure: Multi Input Multi Output (MIMO) System

Control System Components

1. System, plant or process
   - To be controlled
2. Actuators
   - Converts the control signal to a power signal
3. Sensors
   - Provides measurement of the system output
4. Reference input
   - Represents the desired output

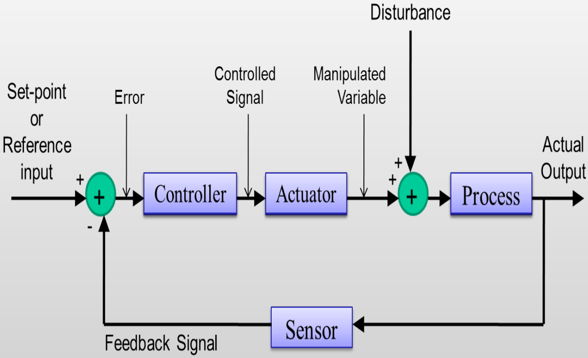


Figure: General block diagram of an automatic control system.

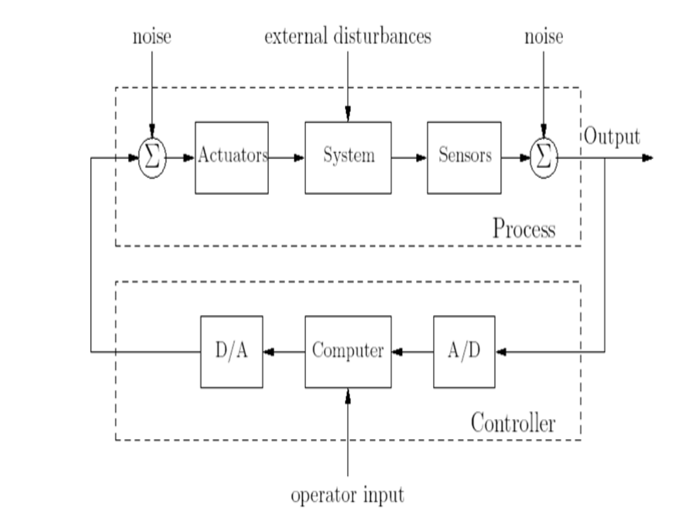


Figure: A modern Feedback Control System.

Ref: https://www.seas.upenn.edu/

Purpose of Control Systems

1. Power Amplification (Gain)
   - Positioning of a large radar antenna by low-power rotation of a knob
2. Remote Control
   - Using robotic arm to pick up radioactive materials
3. Convenience of Input Form
   - Changing room temperature by thermostat position
4. Compensation for Disturbances
   - Controlling antenna position in the presence of large wind disturbance torque

Ref: ee.yeditepe.edu.tr

A Practical Control System

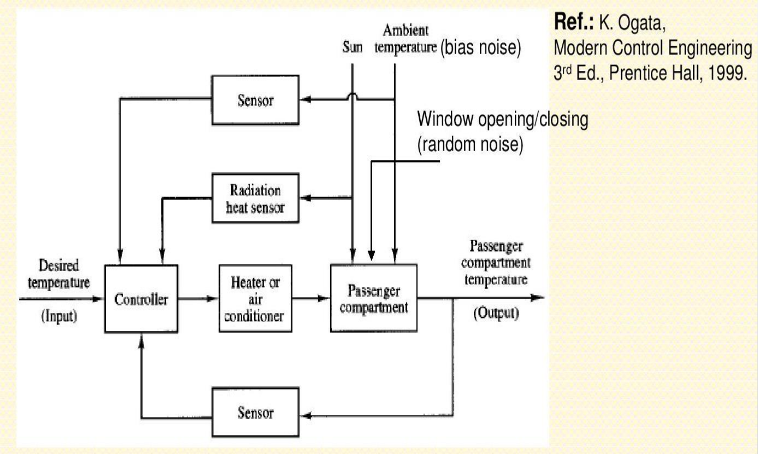
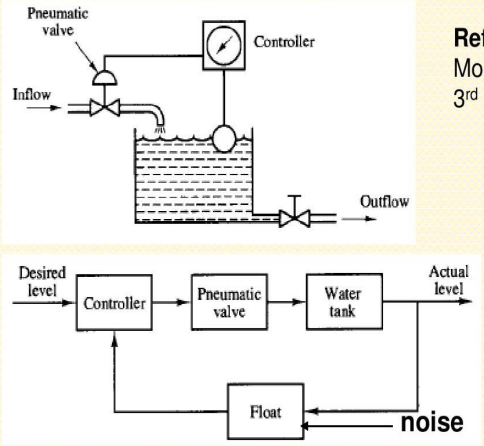


Figure: Temperature control system in a car.

Another Practical Control System



Ref.: K. Ogata,
Modern Control Engineering
3rd Ed., Prentice Hall, 1999.

Figure: Water level control in an overhead tank.

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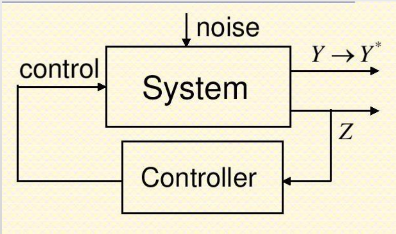
State Space Representation

- **Input variable:**
  - Manipulative (control)
  - Non-manipulative (noise)



- **Output variable:** Variables of interest which can be measured or calculated.
- **State variable:** Minimum set of parameters which entirely summarize the system's status.

Ref: nptel.ac.in

Definitions

- **State:** The state of a dynamic system is the smallest number of variables (called **state variables**) such that the knowledge of these variables at $t = t_0$, together with the knowledge of the input for $t = t_0$, completely determine the behaviour of the system for any time $t >= t_0$.

Note: State variables need not be physically measurable or observable quantities. This gives extra flexibility.

Ref: nptel.ac.in

Definitions

- **State Vector:** A n - dimensional vector whose components are n state variables that describe the system completely.
- **State Space:** The n - dimensional space whose co-ordinate axes consist of the x_1 axis, x_2 axis, ..., x_n axis is called a state space.

Note: For any dynamical system, the state space remains unique, but the *state variables are not unique*.

Ref: nptel.ac.in

Critical Considerations while Selecting State Variables

- **Minimum number of variables**
  1. Minimum number of first-order differential equations needed to describe the system dynamics completely
  2. *Lesser number of variables*: wont be possible to describe the system dynamics
  3. *Larger number of variables*:
     i. Computational complexity
     ii. Loss of either controllability, or observability or both.

- **Linear independence. If not, it may result in:**
  1. *Bad*: May not be possible to solve for all other system variables
  2. *Worst*: May not be possible to write the complete state equations

Ref: nptel.ac.in

State Variable Selection

- Typically, the number of state variables (i.e. the order of the system) is equal to the number of independent energy storage elements. However, there are exceptions!

- Is there a restriction on the selection of the state variables ?
  YES! All state variables should be linearly independent and they must collectively describe the system completely.

Ref: nptel.ac.in

State-space model

Finite-dimensional linear systems can always be modeled using a set of differential (or difference) equations as follows:

Definition (Continuous-time State-Space Models)

$$\frac{d}{dt}x(t) = A(t)x(t) + B(t)u(t);$$
$$y(t) = C(t)x(t) + D(t)u(t);$$

Definition (Discrete-time State-Space Models)

$$x(k+1) = A(k)x(k) + B(k)u(k);$$
$$y(k) = C(k)x(k) + D(k)u(k);$$

The matrices appearing in the above formulas are in general functions of time, and have the correct dimensions to make the equations meaningful.

Ref: https://ocw.mit.edu/

Linear Time-Invariant (LTI) Systems

- If a continuous-time system is both linear and time-invariant, then the output $y(t)$ is related to the input $x(t)$ by a convolution integral.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

where $h(t)$ is the impulse response of the system.

LTI State-space model

If the system is Linear Time-Invariant (LTI), the equations simplify to:

Definition (Continuous-time State-Space Models)

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t);$$

$$y(t) = Cx(t) + Du(t);$$

Definition (Discrete-time State-Space Models)

$$x(k+1) = Ax(k) + Bu(k);$$

$$y(k) = Cx(k) + Du(k);$$

In the above formulas, $A \in \Re^{n\times n}$, $B \in \Re^{n\times 1}$, $C \in \Re^{1\times n}$, $D \in \Re$, and n is the dimension of the state vector.

Ref: https://ocw.mit.edu/

Advantages of State Space Representation

- Systematic analysis and synthesis of higher order systems without truncation of system dynamics
- Convenient tool for MIMO systems
- Uniform platform for representing time-invariant systems, time-varying systems, linear systems as well as nonlinear systems
- Can describe the dynamics in almost all systems (mechanical systems, electrical systems, biological systems, economic systems, social systems etc.)

Note: Transfer function representations are valid for only for linear time invariant (LTI) systems

Ref: nptel.ac.in

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Introduction

- If for a specific input the output of a control system varies with respect to time, then it is called the time response of the control system.
- Time response analysis means subjecting the control system to inputs that are function of time and their outputs which are also function of time.
- It is possible to compute the time response of a system if the nature of input and the mathematical model of the system are known.

- The time response of a control system consists of two parts:
    i. Transient response
    ii. Steady state response

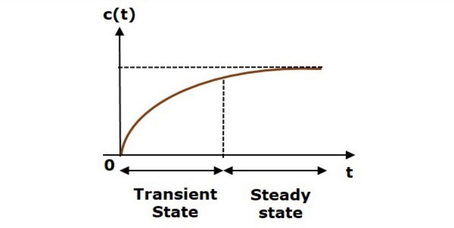


Figure: response of control system in time domain.

Mathematically, from the figure the time response $c(t)$ can be represented as

$$c(t) = c_{ts}(t) + c_{ss}(t)$$

Where,

- $c_{ts}(t) \rightarrow$ the transient response
- $c_{ss}(t) \rightarrow$ the steady state response

Ref: https://www.tutorialspoint.com

Transient Response

The time required to achieve the final value is called transient period. The transient response is represented as the part of the time response which goes from the initial state to the final state and reduces to zero as time becomes very large.
Mathematically, it can be defined as

$$\lim_{x \to \infty} c_{ts}(t) = 0$$

Steady state Response

The steady-state response is defined as the behaviour of the system or the part of the total response as t approaches infinity after the transients have died out.

Example

Let us assume the time response of a control system: $c(t) = 1 + 10e^{-2t}$
Here, the second term $10e^{-2t}$ will be zero as t goes to infinity. Hence, this is the transient term. And the first term 1 remains constant as t approaches infinity. So, this is the steady state term.

Standard Test Signals

- The characteristics of actual input signals are a *sudden shock*, a *sudden change*, a *constant velocity*, and a *constant acceleration*.
- Therefore, the dynamic behavior of a system is judged and compared under application of standard test signals an *impulse*, a *step*, a *constant velocity*, and a *constant acceleration*.
- The other important standard signal in the area of control system is a *sinusoidal signal*.

Ref: www.ece.mtu.edu

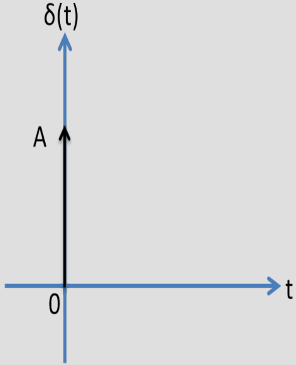
Standard Test Signals

Impulse signal

- The impulse signal represents the sudden shock characteristics of actual input signal.

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

- If $A = 1$, the impulse signal is called unit impulse signal.



Ref: www.ece.mtu.edu

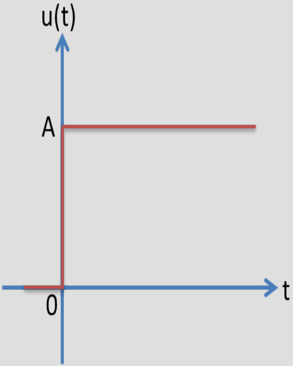
Standard Test Signals

Step signal

- The step signal represents the sudden change characteristics of actual input signal.

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

- If $A = 1$, the step signal is called unit step signal.



Ref: www.ece.mtu.edu

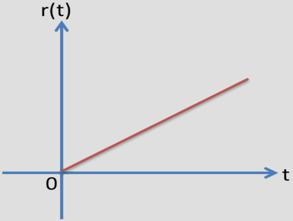
Standard Test Signals

Ramp signal

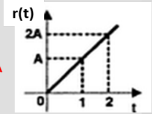
- The ramp signal represents the constant velocity characteristics of actual input signal.

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

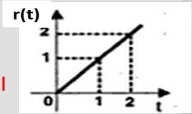
- If $A = 1$, the ramp signal is called unit ramp signal.



ramp signal with slope A



unit ramp signal



Ref: www.ece.mtu.edu

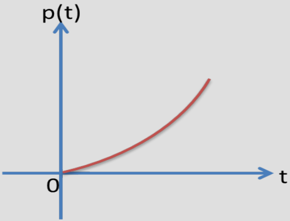
Standard Test Signals

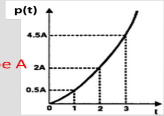
Parabolic signal

- The parabolic signal represents the constant acceleration characteristics of actual input signal.

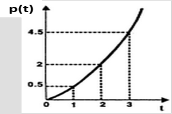
$$p(t) = \begin{cases} \frac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

- If $A = 1$, the parabolic signal is called unit parabolic signal.





parabolic signal with slope A



Unit parabolic signal

Ref: www.ece.mtu.edu

Relation between standard Test Signals

- Impulse $\quad \delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$
- Step $\quad u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$
- Ramp $\quad r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$
- Parabolic $\quad p(t) = \begin{cases} \frac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$

$\int \quad \int \quad \int$

$\frac{d}{dt} \quad \frac{d}{dt} \quad \frac{d}{dt}$

Ref: www.ece.mtu.edu

Laplace Transform of Test Signals

Impulse

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$$L\{\delta(t)\} = \delta(s) = A$$

Step

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{u(t)\} = U(s) = \frac{A}{S}$$

Ramp

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{r(t)\} = R(s) = \frac{A}{S^2}$$

Parabolic

$$p(t) = \begin{cases} \frac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

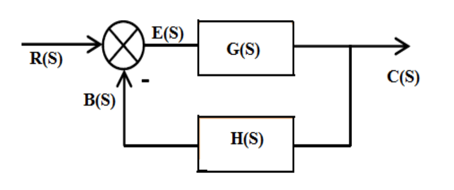
$$L\{p(t)\} = P(s) = \frac{A}{S^3}$$

Ref: www.ece.mtu.edu

Transfer Function

The input- output relationship in a linear time invariant system is generally represented by the transfer function. For a time invariant system, it is defined as the ratio of Laplace transform of the output to the Laplace transform of the input.

Let us consider the following block diagram is representing a feedback control system:



In the diagram, $B(S) \rightarrow$ feedback signal, $C(S) \rightarrow$ output, $R(S) \rightarrow$ input function, $G(S) \rightarrow$ open loop gain and $H(S) \rightarrow$ feedback loop gain of the system.

Transfer Function

Hence, the feedback signal $B(S)$ is given by

$$B(S) = H(S).C(S)$$

Now, from the block diagram

$$G(S) = \frac{C(S)}{E(S)}$$

$$E(S) = R(S) – B(S)$$

So,

$$C(S) = G(S).E(S) = G(S)[R(S) - B(S)] = G(S)[R(S) - H(S).C(S)]$$

Therefore,

$$\frac{C(S)}{R(S)} = \frac{G(S)}{1 + G(S)H(S)}$$

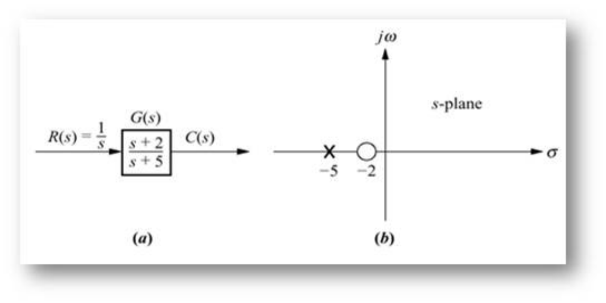
This is the transfer function of the closed loop control system.

Poles and Zeros

- The fundamental concept of poles and zeros in design of control system simplifies the evaluation of system response.
- The poles of a transfer function are:
  1. Values of the Laplace Transform variable s, that cause the transfer function to become infinite.
  2. The roots of the denominator of the transfer function.
- The zeros of a transfer function are:
  1. Values of the Laplace Transform variable s, that cause the transfer function to become zero.
  2. The roots of the denominator of the transfer function.

Influence of Poles on Time Response

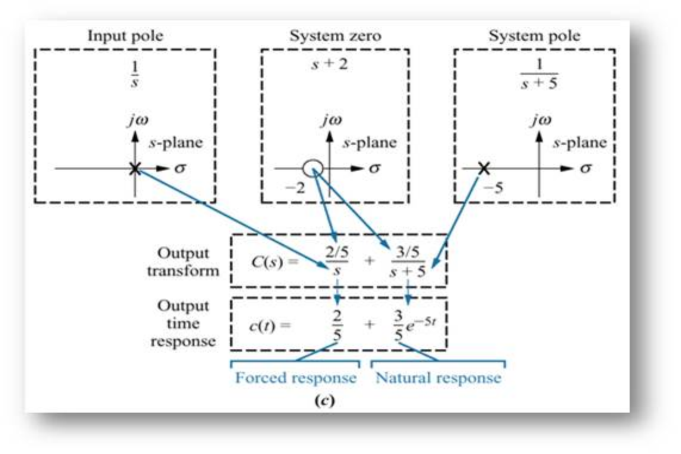
- The output response of a system has two components:
  - i. Forced response
  - ii. Natural response



(a) A system having an input and an output
(b) Pole-zero plot of the system

Ref: www.portal.unimap.edu.my

Influence of Poles on Time Response

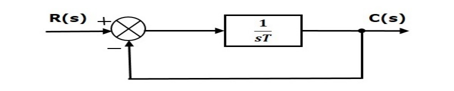


(c) Evolution of the system response. Follow the blue arrows to see the evolution of system component generated by the pole or zero

Ref: www.portal.unimap.edu.my

First Order System

Let us consider the following block diagram of the closed loop control system, where, a unity negative feedback is connected with an open loop transfer function, $\frac{1}{sT}$.



We can write the transfer function of the closed loop control system as,

$$\frac{C(s)}{R(s)} = \frac{1}{sT+1} \Rightarrow C(s) = (\frac{1}{sT+1})R(s)$$

Where,

- $C(s) \rightarrow$ the Laplace transform of the output signal $c(t)$,
- $R(s) \rightarrow$ the Laplace transform of the input signal $r(t)$, and
- T is the time constant.

The power of s is one in the denominator term. Therefore, the above transfer function is of the first order and the system is called the first order system.

Ref: www.tutorialspoint.com

Impulse Response of First Order System

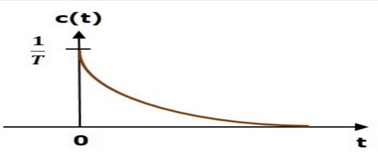
Let us consider the unit impulse signal as an input to the first order system described earlier. So,

$$r(t) = \delta(t)$$

Hence, Impulse Response of the First Order System is defined by the following equation:

$$c(t) = \frac{1}{T} e^{\left(-\frac{t}{T}\right)} u(t)$$

The unit impulse response is shown in the following figure:



The unit impulse response, $c(t)$ is an exponential decaying signal for positive values of 't' and it is zero for negative values of 't'.

Ref: www.tutorialspoint.com

Step Response of First Order System

Let us consider the unit step signal as an input to the same first order system. So,

$$r(t) = u(t)$$

Hence, unit step response of the First Order System is defined by the following equation:

$$c(t) = (1 - e^{(-\frac{t}{T})})u(t)$$

The transient term in the unit step response is -

$$c_{ts}(t) = -e^{(-\frac{t}{T})}u(t)$$

The steady state term in the unit step response is -

$$c_{ss}(t) = u(t)$$

The value of the unit step response, $c(t)$ is zero when t = 0 and (-)ve. It is gradually increasing from zero value and finally reaches to one in steady state. Therefore, the steady state value depends on the magnitude of the input.

Ref: www.tutorialspoint.com

Step Response of First Order System

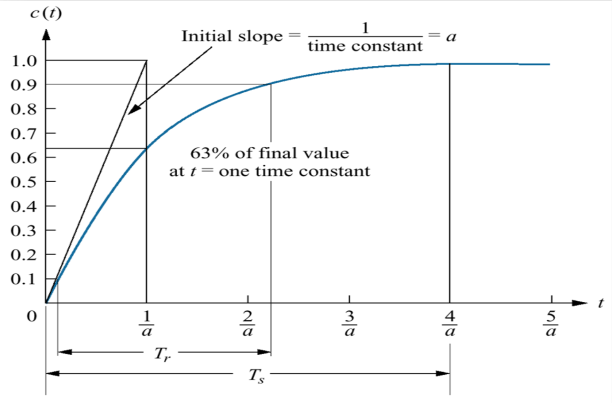


Figure: First Order System response to a unit step.

Ref: www.portal.unimap.edu.my

Transient unit step response specifications

Time constant, τ

The time required for e^{-at} to decay 37% of its initial value.

$$\tau = \frac{1}{a}$$

Rise time, t_r

The time required by the waveform to go from 0.1 to 0.9 of its final value.

$$t_r = \frac{2.2}{a}$$

Settling time, t_s

The time required by the response to reach, and stay within 2% of its final value.

$$t_s = \frac{4}{a}$$

Ref: www.portal.unimap.edu.my

Ramp Response of First Order System

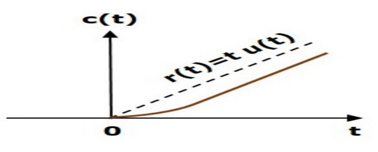
Let us consider the unit ramp signal as an input to the same first order system. So,

$$r(t) = tu(t)$$

Hence, unit ramp response of the First Order System is defined by the following equation:

$$c(t) = (t - T + Te^{(-\frac{t}{T})})u(t)$$

The unit ramp response is shown in the following figure:



The unit ramp response, $c(t)$ follows the unit ramp input signal for all (+)ve values of t. But, there is a deviation of T units from the input signal.

Ref: www.tutorialspoint.com

Parabolic Response of First Order System

Let us consider the unit parabolic signal as an input to the same first order system. So,

$$r(t) = \frac{t^2}{2}u(t)$$

Hence, unit parabolic response of the First Order System is defined by the following equation:

$$c(t) = (\frac{t^2}{2} - Tt + T^2 - T^2 e^{(-\frac{t}{T})})u(t)$$

The transient term in the unit parabolic response is -

$$c_{ts}(t) = -T^2 e^{(-\frac{t}{T})} u(t)$$

The steady state term in the unit parabolic response is -

$$c_{ss}(t) = (\frac{t^2}{2} - Tt + T^2)u(t)$$

Ref: www.tutorialspoint.com

First Order System: Conclusion

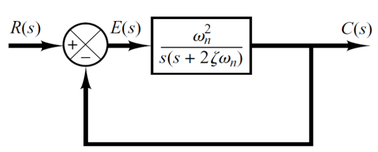
- The first order control systems are not stable with the ramp and parabolic inputs because these responses go on increasing even at infinite amount of time.
- The first order control systems are stable with impulse and step inputs because these responses have bounded output. But, the impulse response does not have steady state term. Hence, the step signal is widely used in the time domain for analyzing the control systems from their responses.

Ref: www.tutorialspoint.com

Second Order System

A general second-order system is characterized by the following transfer function:

$$\frac{C(S)}{R(S)} = \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$



- $\omega_n \rightarrow$ un-damped natural frequency of the second order system, which is the frequency of oscillation of the system without damping.
- $\zeta \rightarrow$ damping ratio of the second order system, which is a measure of the degree of resistance to change in the system output.

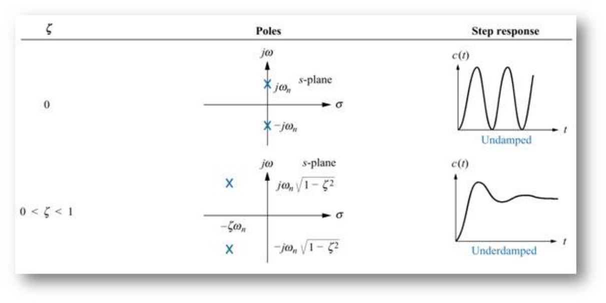
Ref: www.ece.mtu.edu

Response as a function of damping ratio

Roots of denominator

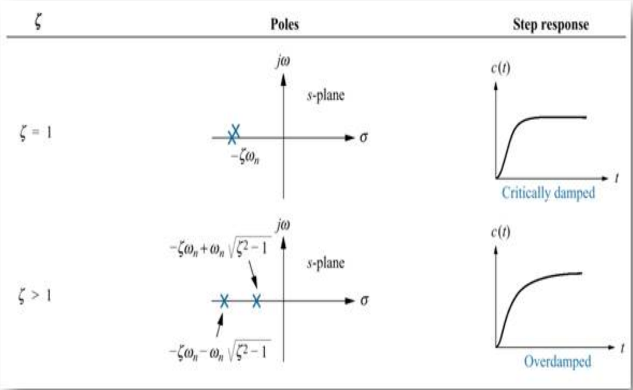
$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$
$$\Rightarrow S_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

From the equation above, we see that the various cases of second-order response are a function of ζ; they are summarized in the table below.



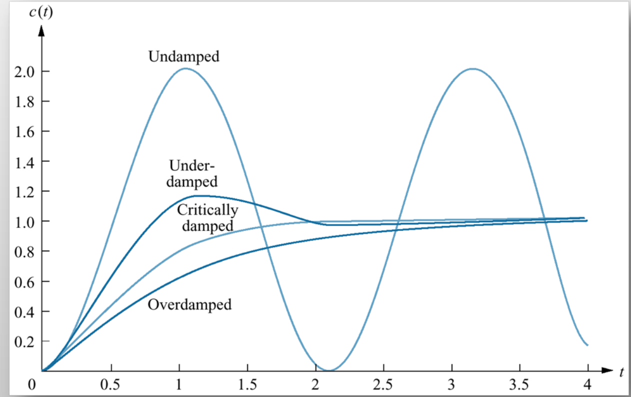
Ref: web.iku.edu.tr

Response as a function of damping ratio



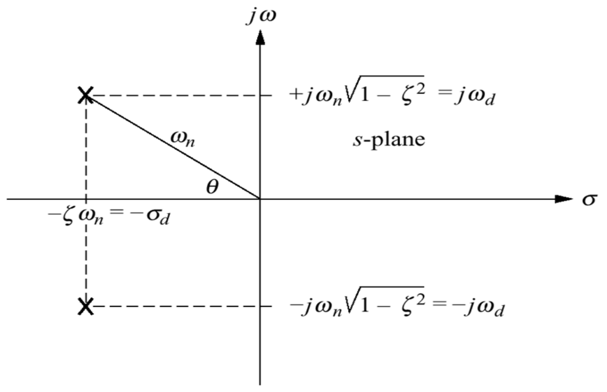
Ref: www.portal.unimap.edu.my

Step responses for second-order system damping cases



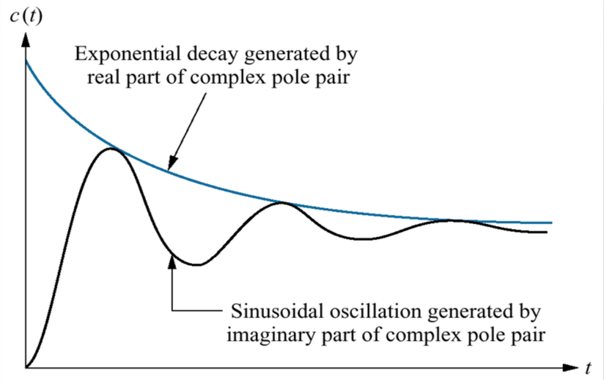
Ref: www.portal.unimap.edu.my

Pole plot for the underdamped second-order system



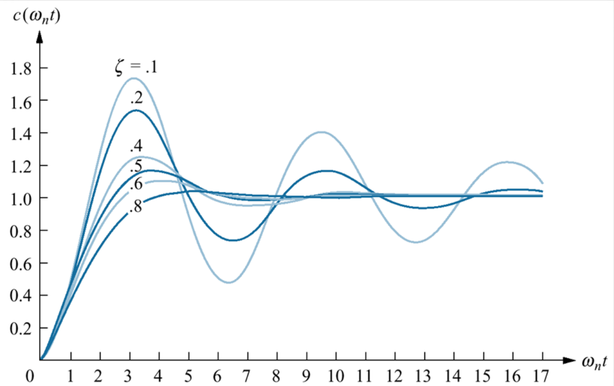
Ref: www.portal.unimap.edu.my

Second-order response components generated by complex poles



Ref: www.portal.unimap.edu.my

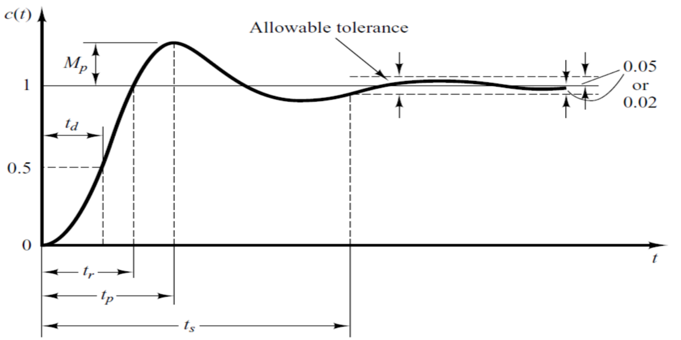
Second-order underdamped responses for damping ratio value



Ref: www.portal.unimap.edu.my

Second-order underdamped system response specifications

Important timing characteristics: delay time, rise time, peak time, maximum overshoot, and settling time.



Ref: www.ece.mtu.edu

Second-order underdamped system response specifications

Delay Time, t_d

The delay (t_d) time is the time required for the response to reach half the final value the very first time.

Rise time, t_r

The time required by the waveform to go from 10% to 90% of its final value for the over-damped systems and 0% to 100% of its final value for the under-damped systems.

Peak time, t_p

The peak time is the time required for the response to reach the first peak of the overshoot.

$$t_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}$$

Ref: www.ece.mtu.edu

Second-order underdamped system response specifications

Settling time, t_s

The settling time is the time required for the response curve to reach and stay within 2% or 5% of the steady-state value.

$$t_s = \frac{4}{\zeta\omega_n} \quad (2\%) \quad \text{OR} \quad t_s = \frac{3}{\zeta\omega_n} \quad (5\%)$$

Maximum Overshoot, M_p

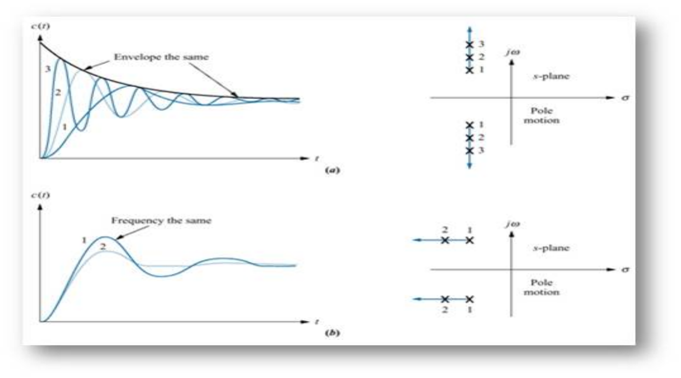
The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is generally used as the maximum percent overshoot. It is defined by

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

Ref: www.ece.mtu.edu

System Performance

Step responses of second-order underdamped systems as poles move

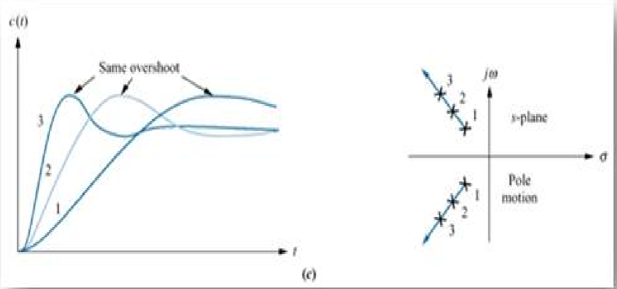


(a) With constant real part
(b) With constant imaginary part

Ref: www.portal.unimap.edu.my

System Performance

Step responses of second-order underdamped systems as poles move



(c) With constant damping ratio

Ref: www.portal.unimap.edu.my