Mathematical Statistics I
Recitation Session 6

Definition 1. Let X be a sample from an unknown population $P \in \mathcal{P}$ and ϑ be a real-valued parameter related to P. An unbiased estimator $T(X)$ of ϑ is called the uniformly minimum variance unbiased estimator (UMVUE) if and only if $\operatorname{Var}(T(X)) \leq \operatorname{Var}(U(X))$ for any unbiased estimator $U(X)$ of ϑ.

Theorem. *(Rao-Blackwell). Let $X_1, X_2, \ldots, X_n, n$ a fixed positive integer, denote a random sample from a distribution (continuous or discrete) that has pdf or pmf $f(x;\theta), \theta \in \Omega$. Let $Y_1 = u_1(X_1, X_2, \ldots, X_n)$ be a sufficient statistic for θ, and let $Y_2 = u_2(X_1, X_2, \ldots, X_n)$, not a function of Y_1 alone, be an unbiased estimator of θ. Then $E(Y_2 \mid y_1) = \varphi(y_1)$ defines a statistic $\varphi(Y_1)$. This statistic $\varphi(Y_1)$ is a function of the sufficient statistic for θ; it is an unbiased estimator of θ; and its variance is less than or equal to that of Y_2.*

Theorem. *(Lehmann and Scheffé). Let $X_1, X_2, \ldots, X_n, n$ a fixed positive integer, denote a random sample from a distribution that has pdf or pmf $f(x;\theta), \theta \in \Omega$, let $Y_1 = u_1(X_1, X_2, \ldots, X_n)$ be a sufficient statistic for θ, and let the family $\{f_{Y_1}(y_1;\theta) : \theta \in \Omega\}$ be complete. If there is a function of Y_1 that is an unbiased estimator of θ, then this function of Y_1 is the unique MVUE of θ.*

Exercise. Let $X_1, \ldots, X_n$ be iid according to the Poisson distribution $P(\lambda)$. Find the UMVU estimator of (a) λ^k for any positive integer k and (b) $e^{-\lambda}$.

Exercise. Let $X_1, \ldots, X_n$ be iid according to the uniform distribution $U(0,\theta)$. Find the UMVU estimator of θ^k for any integer $k > -n$.

Exercise. Let $(X_1, \ldots, X_n), n > 2$, be a random sample from the uniform distribution on the interval $(\theta_1 - \theta_2, \theta_1 + \theta_2)$, where $\theta_1 \in \mathcal{R}$ and $\theta_2 > 0$. Find the UMVUE of θ_1/θ_2.

Exercise. Let $X_1, \ldots, X_n$ be a random sample from poisson distribution with parameter $\theta > 0$. Using completeness of $\sum_i X_i$, find $E(X_1^2 | \sum_i X_i)$.

Theorem. *Let $\mathcal{U}$ be the set of all unbiased estimators of 0 with finite variances and T be an unbiased estimator of ϑ with $E(T^2) < \infty$. A necessary and sufficient condition for $T(X)$ to be a UMVUE of ϑ is that $E[T(X)U(X)] = 0$ for any $U \in \mathcal{U}$ and any $P \in \mathcal{P}$.*

As a consequence, we have the following useful result. Let T_j be a UMVUE of $\vartheta_j, j = 1, \ldots, k$, where k is a fixed positive integer. Then $\sum_{j=1}^k c_j T_j$ is a UMVUE of $\vartheta = \sum_{j=1}^k c_j \vartheta_j$ for any constants $c_1, \ldots, c_k$.

Exercise. Let $(X_1, \ldots, X_n)$ be a sample of binary random variables with $P(X_i = 1) = p \in (0,1)$.
(i) Find the UMVUE of p^m, where m is a positive integer and $m \leq n$.
(ii) Find the UMVUE of $P(X_1 + \cdots + X_m = k)$, where m and k are positive integers and $k \leq m \leq n$.
(iii) Find the UMVUE of $P(X_1 + \cdots + X_{n-1} > X_n)$.