Mathematical Statistics I
Recitation Session 4

Theorem. *(Basu's Theorem) If $T(\mathbf{X})$ is a complete and minimal sufficient statistic, then $T(\mathbf{X})$ is independent of every ancillary statistic.*

Theorem. *If a minimal sufficient statistic exists, then any complete statistic is also a minimal sufficient statistic.*

So even though the word "minimal" is redundant in the statement of Basu's Theorem, it was stated in this way as a reminder that the statistic $T(\mathbf{X})$ in the theorem is a minimal sufficient statistic.

Definition. A statistic $S(X)$ whose distribution does not depend on the parameter θ is called an ancillary statistic.

Exercise. Suppose X_1 and X_2 are iid observations from the pdf $f(x;\alpha) = \alpha x^{\alpha-1}e^{-x^{\alpha}}, x > 0, \alpha > 0$. Show that $(\log X_1) / (\log X_2)$ is an ancillary statistic.

Exercise. Let $X_1, \ldots, X_n$ be a random sample from a location family. Show that $M - \bar{X}$ is an ancillary statistic, where M is the sample median.

Exercise. Let $X_1, \ldots, X_n$ i.i.d. random variables having the exponential distribution with parameter θ. Determine the expected value of $g(\mathbf{X})$ where,

$$g(\mathbf{X}) = \frac{X_n}{X1 + \cdots + X_n}$$

Exercise. Let $X_1, \ldots, X_n$ be a random sample from the pdf $f(x;\mu) = e^{-(x-\mu)}$, where $\mu < x$.
(a) Show that $X_{(1)}$ is a complete sufficient statistic.
(b) Use Basu's Theorem to show that $X_{(1)}$ and S^2 are independent.

Exercise. Let $X_1, \ldots, X_n$ be i.i.d. random variables having the uniform distribution on the interval (a, b), where $-\infty < a < b < \infty$. Show that $\left(X_{(i)} - X_{(1)}\right) / \left(X_{(n)} - X_{(1)}\right), i = 2, \ldots, n-1$, are independent of $\left(X_{(1)}, X_{(n)}\right)$ for any a and b.

Exercise. Let $X_1, \ldots, X_n$ be i.i.d. random variables having the gamma distribution $\Gamma(\alpha, \gamma)$. Show that $\sum_{i=1}^{n} X_i$ and $\sum_{i=1}^{n} \left[\log X_i - \log X_{(1)}\right]$ are independent for any (α, γ).