Stochastic Processes
Assignment 6 - Questions

Due on May 28, 2023

1. Consider a pure death process on $\{0, 1, 2, \ldots\}$.
   (a) Write the forward equation.
   (b) Find $P_{xx}(t)$
   (c) Solve for $P_{xy}(t)$ in terms of $P_{x,y+1}(t)$.
   (d) Find $P_{x,x-1}(t)$.
   (e) Show that if $\mu_x = x\mu, x \geq 0$, for some constant μ, then

$$P_{xy}(t) = \binom{x}{y} \left(e^{-\mu t}\right)^y \left(1 - e^{-\mu t}\right)^{x-y}, \quad 0 \leq y \leq x.$$

2. Consider a general birth and death process with birth rates $\{\lambda_n\}$ and death rates $\{\mu_n\}$, where $\mu_0 = 0$, and let T_i denote the time, starting from state i, it takes for the process to enter state $i+1$, $i \geq 0$. Compute $E(T_i)$ and $Var(T_i)$. *Hint:* For i > 0, condition whether the first transition takes the process into state $i-1$ or $i+1$.

3. Let $\{X(t), t \geq 0\}$ be a continuous-time Markov chain with state space $S = \{0, 1, 2\}$ and the following generator

$$q = \begin{bmatrix} -2 & 1 & 1 \\ 2 & -3 & 1 \\ 1 & 2 & -3 \end{bmatrix}$$

   Determine the following quantities:
   (a) the distribution of the holding times for state 1.
   (b) the transition matrix Q of the embedded chain.
   (c) $P\left(\tau_2 > 1 \mid X\left(\tau_1\right) = 1\right)$ where τ_i is the ith transition time and $X(0) = 0$.

4. There are two servers available to process n jobs. Initially, each server begins work on a job. Whenever a server completes work on a job, that job leaves the system and the server begins processing a new job (provided there are still jobs waiting to be processed). Let T denote the time until all jobs have been processed. If the time that it takes server i to process a job is exponentially distributed with rate $\mu_i, i = 1, 2$. Find $E(T)$ and $Var(T)$.

5. Customers arrive at a two-server service station according to a Poisson process with rate λ. Whenever a new customer arrives, any customer that is in the system immediately departs. A new arrival enters service, first with server 1 and then with server 2. If the service times at the servers are independent exponentials with respective rates μ_1 and μ_2, what proportion of entering customers completes their service with server 2? (i.e. what proportion of entering customers completes both services without being interrupted.)

6. Teams 1 and 2 are playing a match. The teams score points according to independent Poisson processes with respective rates λ_1 and λ_2. The match ends when one of the teams has scored k more points than the other. Find the probability that team 1 wins. *Hint:* Relate this to the gambler's ruin problem.