Mathematical Statistics I
Assignment 4 - Questions

Due by May 8, 2024

1. Let X/Y and Y be independent random variables,
(i) Show that
$$\mathrm{E}\left(\frac{X}{Y}\right)^k=\frac{\mathrm{E}\left(X^k\right)}{\mathrm{E}\left(Y^k\right)}.$$
(ii) Use (i) result and Basu's Theorem to show that if $X_1,\ldots,X_n$ denote a random sample from a distribution that is Gamma(α,β), where α is known, then for $T=\sum_i X_i$
$$\mathrm{E}\left(X_{(i)}\mid T\right)=\mathrm{E}\left(\left.\frac{X_{(i)}}{T}T\right| T\right)=T\frac{\mathrm{E}\left(X_{(i)}\right)}{\mathrm{E}(T)}.$$

2. Let $\left(Y_{(1)},Y_{(2)},\ldots,Y_{(n)}\right)$ be the order statistics of a random sample of size n from a distribution that has pdf $f(x;\theta)=(1/\theta)e^{-x/\theta}, 0<x<\infty, 0<\theta<\infty$, zero elsewhere.
(i) Show that the ratio $R=nY_{(1)}/\sum_1^n Y_i$ and its denominator are independent.
(ii) Determine $E\left(R^k\right)$, for $k\in\mathbb{N}$.

3. The random variable X takes the values $0,1,2$ according to one of the following distributions:

	$P(X=0)$	$P(X=1)$	$P(X=2)$	
Distribution 1	p	$3p$	$1-4p$	$0<p<1/4$
Distribution 2	p	p^2	$1-p-p^2$	$0<p<1/2$

In each case determine whether the family of distributions of X is complete.

4. Let $X_1,X_2,\ldots,X_n$ be a sample from the inverse Gaussian pdf,
$$f(x;\mu,\lambda)=\left(\frac{\lambda}{2\pi x^3}\right)^{1/2}\exp\left\{-\lambda(x-\mu)^2/\left(2\mu^2x\right)\right\},\quad x>0.$$
Show that the MLEs of μ and λ are
$$\hat{\mu}_n=\overline{X}\quad\text{and}\quad\hat{\lambda}_n=n\left(\sum_i\frac{1}{X_i}-\frac{1}{\overline{X}}\right)^{-1}.$$

5. Let $X_1,\ldots,X_n$ be a random sample from a distribution that has pdf,
$$f(x;\theta)=\theta x^{\theta-1},\quad 0\leq x\leq 1,\quad 0<\theta<\infty.$$
(i) Find the MLE of θ, and show that its variance $\to 0$ as $n\to\infty$.
(ii) Find the method of moments estimator of θ.