Mathematical Statistics I
Recitation Session 1

Consider a family $\{f(x;\theta) : \theta \in \Omega\}$ of probability density or mass functions, where Ω is the interval set $\Omega = \{\theta : \gamma < \theta < \delta\}$, where γ and δ are known constants (they may be $\pm\infty$), and where

$$f(x;\theta) = \begin{cases} \exp[p(\theta)K(x) + H(x) + q(\theta)] & x \in \mathcal{S} \\ 0 & \text{elsewhere,} \end{cases} \tag{1}$$

where $\mathcal{S}$ is the support of X.

Definition. A pdf of the form (1) is said to be a member of the regular exponential class of probability density or mass functions if
1. $\mathcal{S}$, the support of X, does not depend upon θ
2. $p(\theta)$ is a nontrivial continuous function of $\theta \in \Omega$
3. Finally,
(a) if X is a continuous random variable, then each of $K'(x) \not\equiv 0$ and $H(x)$ is a continuous function of $x \in \mathcal{S}$,
(b) if X is a discrete random variable, then $K(x)$ is a nontrivial function of $x \in \mathcal{S}$.

Theorem. *Let $X_1, X_2, \ldots, X_n$ denote a random sample from a distribution that represents a regular case of the exponential class, with pdf or pmf given by (1). Consider the statistic $Y = \sum_{i=1}^{n} K(X_i)$. Then*
1. The pdf or pmf of Y has the form

$$f_Y(y;\theta) = R(y) \exp[p(\theta)y + nq(\theta)]$$

for $y \in \mathcal{S}_Y$ and some function $R(y)$. Neither $\mathcal{S}_Y$ nor $R(y)$ depends on θ.
2. $E(Y) = -n\frac{q'(\theta)}{p'(\theta)}$.
3. $\operatorname{Var}(Y) = n\frac{1}{p'(\theta)^3}\{p''(\theta)q'(\theta) - q''(\theta)p'(\theta)\}$.

Exercise. Given that $f(x;\theta) = \exp[\theta K(x) + H(x) + q(\theta)], a < x < b, \gamma < \theta < \delta$, represents a regular case of the exponential class, show that the moment-generating function $M(t)$ of $Y = K(X)$ is $M(t) = \exp[q(\theta) - q(\theta + t)], \gamma < \theta + t < \delta$.

Exercise. Let X and Y be random variables having the bivariate normal distribution with $\mathrm{E}(X) = \mathrm{E}(Y) = 0, \operatorname{Var}(X) = \operatorname{Var}(Y) = 1$, and $\operatorname{Cov}(X,Y) = \rho$. Show that $E(\max\{X,Y\}) = \sqrt{(1-\rho)/\pi}$.