Mathematical Statistics I
Assignment 4 - Solutions

1. (i)

$$\mathrm{E}\left(X^k\right)=\mathrm{E}\left(\frac{X}{Y}Y\right)^k=\mathrm{E}\left[\left(\frac{X}{Y}\right)^k\left(Y^k\right)\right]=\mathrm{E}\left(\frac{X}{Y}\right)^k\mathrm{E}\left(Y^k\right).$$

where the last equality results from the independence assumption of X/Y and Y. Divide both sides by $\mathrm{E}\left(Y^k\right)$ to obtain the desired equality.
(ii) If α is fixed, $T=\sum_i X_i$ is a complete sufficient statistic for β by the theorem on completeness in exponential family. Because β is a scale parameter, if $Z_1,\ldots,Z_n$ is a random sample from a $\Gamma(\alpha,1)$ distribution, then $X_{(i)}/T$ has the same distribution as $\left(\beta Z_{(i)}\right)/\left(\beta\sum_i Z_i\right)=Z_{(i)}/\left(\sum_i Z_i\right)$, and this distribution does not depend on β. Thus, $X_{(i)}/T$ is ancillary, and by Basu's Theorem, it is independent of T. We have

$$\mathrm{E}\left(X_{(i)}\mid T\right)=\mathrm{E}\left(\left.\frac{X_{(i)}}{T}T\right|T\right)=T\mathrm{E}\left(\left.\frac{X_{(i)}}{T}\right|T\right)=T\mathrm{E}\left(\frac{X_{(i)}}{T}\right)=T\frac{\mathrm{E}\left(X_{(i)}\right)}{\mathrm{E}\left(T\right)},$$

where the last two equalities result from the independence of $X_{(i)}/T$ and T that we've just proved. Note, this expression is correct for each fixed value of (α,β), regardless whether α is "known" or not.

2. $T=\sum_i Y_i$ is a complete sufficient statistic for θ by the theorem on completeness in exponential family. Because θ is a scale parameter, if $Z_1,\ldots,Z_n$ is a random sample from a $E(1)$ distribution, then $R=nY_{(1)}/\sum_i Y_i$ has the same distribution as $\left(n\theta Z_{(i)}\right)/\left(\theta\sum_i Z_i\right)=nZ_{(i)}/\left(\sum_i Z_i\right)$, and this distribution does not depend on θ. Thus, R is ancillary, and by Basu's Theorem, it is independent of its denominator.
(ii) Note that $nY_{(1)}\sim E(\theta)$ and $\sum_i Y_i\sim\Gamma(n,\theta)$, hence $M_1(t)=E\left[\exp\left(tnY_{(1)}\right)\right]=(1-\theta t)^{-1}$ for $t<1/\theta$, and $M_2(t)=E\left[\exp\left(t\sum_i Y_i\right)\right]=(1-\theta t)^{-n}$ for $t<1/\theta$, so we have $M_1^{(k)}(0)=\theta^k\Gamma(k+1)$ and $M_2^{(k)}(0)=\theta^k\Gamma(n+k)/\Gamma(n)$. According to the result of the part (i) of previous question we now have $E\left(R^k\right)=M_1^{(k)}(0)/M_2^{(k)}(0)=\Gamma(k+1)\Gamma(n)/\Gamma(n+k)$.

3. To check if the family of distributions of X is complete, we check if $\mathrm{E}_p\left[g(X)\right]=0$ for all p, implies that $g(X)$ is identically zero. For Distribution 1

$$\mathrm{E}_p\left[g(X)\right]=\sum_{x=0}^{2}g(x)P(X=x)=pg(0)+3pg(1)+(1-4p)g(2).$$

Note that if $g(0)=-3g(1)$ and $g(2)=0$, then the expectation is zero for all p, but $g(x)$ need not be identically zero. Hence the family is not complete. For Distribution 2 calculate

$$\mathrm{E}_p\left[g(X)\right]=g(0)p+g(1)p^2+g(2)\left(1-p-p^2\right)=[g(1)-g(2)]p^2+[g(0)-g(2)]p+g(2).$$

This is a polynomial of degree 2 in p. To make it zero for all p each coefficient must be zero. Thus, $g(0)=g(1)=g(2)=0$, so the family of distributions is complete.

4. Let $x = (x_1, \ldots, x_n)$. The likelihood is

$$L(\mu, \lambda; x) = \frac{\lambda^{n/2}}{(2\pi)^n \prod_i x_i} \exp\left\{ -\frac{\lambda}{2} \sum_i \frac{(x_i - \mu)^2}{\mu^2 x_i} \right\}.$$

For fixed λ, maximizing with respect to μ is equivalent to minimizing the sum in the exponential.

$$\frac{d}{d\mu} \sum_i \frac{(x_i - \mu)^2}{\mu^2 x_i} = \frac{d}{d\mu} \sum_i \frac{((x_i/\mu) - 1)^2}{x_i} = -\sum_i \frac{2\left((x_i/\mu) - 1\right)}{x_i} \frac{x_i}{\mu^2} = \frac{-2}{\mu^2} \sum_i \left(\frac{x_i}{\mu} - 1 \right).$$

Setting this equal to zero is equivalent to setting

$$\sum_i \left(\frac{x_i}{\mu} - 1 \right) = 0,$$

and solving for μ yields $\hat{\mu}_n = \overline{X}$. Plugging in this $\hat{\mu}_n$ and maximizing with respect to λ amounts to maximizing an expression of the form $\lambda^{n/2} e^{-\lambda b}$, where $b = \sum_i \frac{(x_i - \bar{x})^2}{2\bar{x}^2 x_i}$.

$$\frac{d}{d\lambda} \lambda^{n/2} e^{-\lambda b} = \frac{n}{2} \lambda^{n/2-1} e^{-\lambda b} - \lambda^{n/2} b e^{-\lambda b} = e^{-\lambda b} \lambda^{n/2-1} \left(n/2 - \lambda b\right).$$

Setting this equal to zero is equivalent to setting

$$n/2 - \lambda b = 0,$$

and solving for λ yields $\lambda = \frac{n}{2b}$. Finally,

$$2b = \sum_i \frac{x_i}{\bar{x}^2} - 2 \sum_i \frac{1}{\bar{x}} + \sum_i \frac{1}{x_i} = -\frac{n}{\bar{x}} + \sum_i \frac{1}{x_i} = \sum_i \left(\frac{1}{x_i} - \frac{1}{\bar{x}} \right),$$

therefore,

$$\hat{\lambda}_n = n \left(\sum_i \frac{1}{X_i} - \frac{1}{\overline{X}} \right)^{-1}.$$

5. (i) Let $x = (x_1, \ldots, x_n)$. The likelihood is

$$L(\theta; x) = f(x; \theta) = \prod_i \theta x_i^{\theta-1} = \theta^n \left(\prod_i x_i \right)^{\theta - 1}$$

$$\frac{d}{d\theta} \log L = \frac{d}{d\theta} \left[n \log \theta + (\theta - 1) \sum_i \log x_i \right] = \frac{n}{\theta} + \sum_i \log x_i$$

Set the derivative equal to zero and solve for θ to obtain $\hat{\theta} = \left(-\frac{1}{n} \sum_i \log x_i \right)^{-1}$. The second derivative is $-n/\theta^2 < 0$, so this is the MLE. To calculate the variance of $\hat{\theta}$, note that $Y_i = -\log X_i \sim \mathrm{E}(1/\theta)$, so $-\sum_i \log X_i \sim \Gamma(n, 1/\theta)$. Thus $\hat{\theta} = n/T$, where $T \sim \Gamma(n, 1/\theta)$. We can either calculate the first and second moments directly, or use the fact that $\hat{\theta}$ is inverted gamma. We have

$$\mathrm{E}\left(\frac{1}{T}\right) = \frac{\theta^n}{\Gamma(n)} \int_0^\infty \frac{1}{t} t^{n-1} e^{-\theta t} dt = \frac{\theta^n}{\Gamma(n)} \frac{\Gamma(n-1)}{\theta^{n-1}} = \frac{\theta}{n-1}.$$

$$\mathrm{E}\left(\frac{1}{T^2}\right) = \frac{\theta^n}{\Gamma(n)} \int_0^\infty \frac{1}{t^2} t^{n-1} e^{-\theta t} dt = \frac{\theta^n}{\Gamma(n)} \frac{\Gamma(n-2)}{\theta^{n-2}} = \frac{\theta^2}{(n-1)(n-2)},$$

and thus

$$\mathrm{E}(\hat{\theta}) = \frac{n}{n-1}\theta \quad \text{and} \quad \mathrm{Var}(\hat{\theta}) = \frac{n^2}{(n-1)^2(n-2)}\theta^2 \to 0 \text{ as } n \to \infty.$$

(ii) Because $X \sim \text{beta}(\theta, 1)$, $\mathrm{E}(X) = \theta/(\theta+1)$ and the method of moments estimator is the solution to

$$\frac{1}{n}\sum_i X_i = \frac{\theta}{\theta+1}$$

by solving for θ we obtain $\tilde{\theta} = \sum_i X_i/(n - \sum_i X_i)$.