Stochastic Processes
Assignment 5 - Questions

Due on May 13, 2023

1. Let $\{X(t) : \text{t} \geq 0\}$ be a Poisson process with rate λ :
   a) If T_1 , $T_2, \cdots$, T_n are arrival times for the n events, show that $\mathbb{P}(T_1 \leq t_1, \cdots, T_n \leq t_n | X(t) = n)$ is free of λ.
   b) Let T be a random variable that is independent of the times when events occur. Suppose that T has an exponential density with parameter ν. Find the distribution of $X(T)$,which is the number of events occurring by time T.

2. Let $X(t)$ and $Y(t)$ be two independent homogeneous Poisson processes with intensities $\lambda_X > 0$ and $\lambda_Y > 0$.
   a) Verify that $m_{X(t)}(\text{s}) = \mathbb{E}[e^{sX(t)}] = e^{t\lambda_X(e^s-1)}$ for all t ≥ 0, s $\in \mathbb{R}$.
   b) For $0 < s < t$, characterize the conditional distribution of $X(s)$ given $X(t)$ in terms of a common distribution,find $\mathbb{P}\{X(s) = k | X(t) = n\}$
   c) Let T be a random variable having an exponential distribution with parameter ν.Suppose that for all times $t \in [0, 1)$, T and $X(t)$ are independent. Find the distribution of the random variable $X(T)$.find $\mathbb{P}\{X(T) = n\}$.
   d) Find the probability that $X(t)$ jumps to 1 before $Y(t)$ does.

3. Let $\{X_t\}_{t\geq 0}$ be a Markov chain with state space S $= \{0, 1, \cdots, 5\}$ and transition matrix:

$$\text{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{8} & 0 & \frac{7}{8} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{5}{7} & 0 & 0 & \frac{2}{7} \end{bmatrix}$$

   a) Classify each state as either transient, null recurrent, or positive recurrent.
   b) Find the period of each state. Is the chain regular?(Aperiodic + irreducible)
   c) Show that there is a unique stationary distribution. Find it.
   d) For each state, determine the proportion of time the chain spend in it, in the long run.

4. You are given the following information:
   - Taxis arrive according to a Poisson process at a rate of 12 per hour.
   - Buses arrive according to a Poisson process at a rate of 6 per hour.
   - The arrival of buses and taxis are independent.
   - You get a ride to university from either a bus or a taxi, whichever arrives first.

   a) Find the distribution of your waiting time.
   b) Find the probability you will have to wait more than 10 minutes for a ride to university.

5. **a)** Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate λ and waiting times $S_1 < S_2 <$... independent of N(t), find $\mathbb{P}(N(3) = 5 | N(7) = 2)$ and $\mathbb{E}[S_k]$ for each $k \in \mathbb{N}$.
   b) Suppose that passengers arrive at a train station as a Poisson process with rate λ. The only train departs after a deterministic time T. Let W be the combined waiting time for all passengers. Compute $\mathbb{E}(W)$.

6. Consider a birth and. death process on the non-negative integers ,whose death rates are $\mu_x = x, x > 0$. Determine whether the process is transient, null recurrent, or positive recurrent if the birth rates are:
   a) $\lambda_x = x + 1, \quad x \geq 0$
   b) $\lambda_x = x + 2, \quad x \geq 0$