Stochastic Processes
Assignment 4 - Solutions

1. (a) We say that x leads to y if $\rho_{xy} > 0$ or equivalently if $p^n(x,y) > 0$ for some positive integer n. An irreducible Markov chain is a chain whose state space $\mathscr{S}$ is irreducible, that is if x leads to y for all choices of x and y in $\mathscr{S}$. Let x and y be members of the set $\{1, 2, \ldots, c+d\}$, it suffice to show that x leads to y. Define $\mathscr{S}_1 := \{1, 2, \ldots, c\}$ and $\mathscr{S}_2 := \{c+1, c+2, \ldots, c+d\}$. Notice that if $x \in \mathscr{S}_1$ and $y \in \mathscr{S}_2$ then $P(x,y) = 1/d > 0$ and if $x \in \mathscr{S}_2$ and $y \in \mathscr{S}_1$ then $P(x,y) = 1/c > 0$. Hence if $x, y \in \mathscr{S}_1$ or $x, y \in \mathscr{S}_2$ then $P^2(x,y) = 1/(cd) > 0$.

   (b) Let X_n, be a Markov chain having state space $\mathscr{S}$ and transition function P, then π is called a stationary distribution if $\pi(x), x \in \mathscr{S}$, are non-negative numbers summing to one, and if

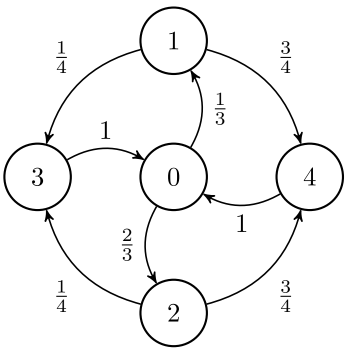
$$\sum_x \pi(x)P(x,y) = \pi(y), \quad y \in \mathscr{S}. \tag{1}$$

   Using (1) we have

$$\begin{aligned}
\pi(y) &= \sum_{x \in \mathscr{S}_1} \frac{1}{d}\pi(x) \qquad y \in \mathscr{S}_2 \\
\pi(y) &= \sum_{x \in \mathscr{S}_2} \frac{1}{c}\pi(x) \qquad y \in \mathscr{S}_1,
\end{aligned}$$

   solving above equations along with $\sum_x \pi(x) = 1$ for $\pi(y)$ we get, $\pi(y) = 1/2c$ for $y \in \mathscr{S}_1$ and $\pi(y) = 1/2d$ for $y \in \mathscr{S}_2$.

2. (a) The irreducibility of the chain can be easily checked by drawing the chain's one-step transition diagram.



   We can clearly see that x leads to y for all choices of x and y in $\{0, 1, 2, 3, 4\}$.

   (b) Using (1), by solving the following system of equations

$$\begin{aligned}
\pi(0) &= \pi(3) + \pi(4) \\
\pi(1) &= \frac{1}{3}\pi(0) \\
\pi(2) &= \frac{2}{3}\pi(0) \\
\pi(3) &= \frac{1}{4}\pi(1) + \frac{1}{4}\pi(2) \\
\pi(4) &= \frac{3}{4}\pi(1) + \frac{3}{4}\pi(2),
\end{aligned}$$

   along with $\sum_x \pi(x) = 1$ we get $\pi = (\frac{1}{3}, \frac{1}{9}, \frac{2}{9}, \frac{1}{12}, \frac{1}{4})$.

3. (a) Suppose that there are x black balls in box 1 at time n. Then with probability x/d the ball drawn from box 1 will be black and with probability x/d the ball drawn from box 2 will be white. In this case there will be $x-1$ black balls in box 1 at time $n+1$ with probability $(x/d)^2$. Similarly, with probability $(1-x/d)^2$ there will be $x+1$ black balls in box 1 at time $n+1$. Finally, if we take two black balls or two white balls out of both boxes, the number of black balls in box 1 will not change. This event will occur with probability $2(x/d)(1-x/d)$. Thus the transition function of this Markov chain is given by

$$P(x,y)=\begin{cases}(\frac{x}{d})^2, & y=x-1,\\ 2(\frac{x}{d})(1-\frac{x}{d}), & y=x,\\ (1-\frac{x}{d})^2, & y=x+1,\end{cases}$$

(b) We know [1] that the birth and death chain on $\{0,1,\dots,d\}$ has a unique stationary distribution, given by

$$\pi(x)=\frac{\pi_x}{\sum_{y=0}^{d}\pi_y},\quad 0\le x\le d \tag{2}$$

where

$$\pi_x=\begin{cases}1, & x=0,\\ \dfrac{p_0\dots p_{x-1}}{q_1\dots q_x}, & 0<x\le d.\end{cases}$$

Hence we have,

$$\pi_x=\frac{\displaystyle\prod_{k=0}^{x-1}(1-k/d)^2}{\displaystyle\prod_{k=1}^{x}(k/d)^2}=\left(\frac{d!}{x!\,(x-d)!}\right)^2=\binom{d}{x}^2,$$

now using (2) we have,

$$\pi(x)=\frac{\binom{d}{x}^2}{\binom{d}{0}^2+\dots+\binom{d}{d}^2}=\frac{\binom{d}{x}^2}{\binom{2d}{d}}.$$

4. (a) The Ehrenfest chain is an example of the birth and death chain, therefore we can use (2) in order to achieve the stationary distribution of the chain. Using the chain's transition function,

$$P(x,y)=\begin{cases}\frac{x}{d}, & y=x-1,\\ 1-\frac{x}{d}, & y=x+1,\\ 0, & \text{elsewhere.}\end{cases}$$

We have,

$$\pi_x=\frac{\displaystyle\prod_{k=0}^{x-1}(1-k/d)}{\displaystyle\prod_{k=1}^{x}(k/d)}=\frac{d!}{x!\,(x-d)!}=\binom{d}{x},$$

now using (2) we have,

$$\pi(x)=\frac{\binom{d}{x}}{\binom{d}{0}+\dots+\binom{d}{d}}=\binom{d}{x}\frac{1}{2^d}\qquad x=0,1,\dots,d.$$

Notice that $\pi(x)$ is the probability mass function of a binomial distribution with parameters d and 1/2.

[1] For the proof refer to Section 2.2.1 of *Introduction to Stochastic Processes (1972) - Paul G. Hoel.*

(b) We know that if the random variable X follows the binomial distribution with parameters $n \in \mathbb{N}$ and $p \in [0,1]$ then $E(X) = np$ and $Var(X) = np(1-p)$, hence the stationary distribution have the mean $d/2$ and the variance $d/4$.

(c) We know that[2] an irreducible positive recurrent Markov chain has a unique stationary distribution π, given by

$$\pi(x) = \frac{1}{m_x}, \quad x \in \mathscr{S},$$

where $m_x = E_x(T_x)$ denote the mean return time to x. Note that initially all of the balls are in the second box i.e. $X_0 = 0$, therefore we have,

$$E(T_0|X_0 = 0) = E_0(T_0) = m_0 = 1/\pi(0) = 2^d.$$

5. (a) The density f of the number of male children of a given man is the binomial density with parameters $n = 3$ and $p = \frac{1}{2}$. Thus $f(0) = \frac{1}{8}$, $f(1) = \frac{3}{8}, f(2) = \frac{3}{8}, f(3) = \frac{1}{8}$, and $f(x) = 0$ for $x \geq 4$. The mean number of male offspring is $\mu = \frac{3}{2}$. Since $\mu > 1$, the extinction probability ρ is the root of the equation

$$\frac{1}{8} + \frac{3}{8}t + \frac{3}{8}t^2 + \frac{1}{8}t^3 = t$$

lying in $[0,1)$. We can rewrite this equation as

$$t^3 + 3t^2 - 5t + 1 = 0$$

or equivalently as

$$(t-1)\left(t^2 + 4t - 1\right) = 0.$$

This equation has three roots, namely, $1, -\sqrt{5} - 2$, and $\sqrt{5} - 2$. Consequently, $\rho = \sqrt{5} - 2$.

(b) Since the numbers of male offspring of these mans in the various generations are chosen independently of each other, the probability ρ_{x0} that the male descendants of each of the x mans eventually become extinct is just the xth power of the probability that the male descendants of any one man eventually become extinct. In other words,

$$\rho_{x0} = \rho^x, \qquad x = 1, 2, \ldots.$$

Hence, the probability that the male line of a man who has two boys and one girl will continue forever is just,

$$1 - \rho_{20} = 1 - \rho^2 = 1 - (\sqrt{5} - 2)^2 = 4(\sqrt{5} - 2).$$

6. Let Φ be the probability generating function of f and let μ denote the expected number of offspring of any given particle. Suppose $\mu \leq 1$. Then the equation $\Phi(t) = t$ has no roots in $[0,1)$ (under the assumption that $f(1) < 1$), and hence $\rho = 1$. Thus ultimate extinction is certain if $\mu \leq 1$ and $f(1) < 1$. Suppose instead that $\mu > 1$. Then the equation $\Phi(t) = t$ has a unique root ρ_0 in $[0,1)$, and hence ρ equals either ρ_0 or 1. Actually ρ always equals ρ_0. Consequently, if $\mu > 1$ the probability of ultimate extinction is less than one[3]

According to the above explanations, we first calculate the expected value of ξ, then we decide on the probability of extinction, thus we have,

$$\begin{aligned}
\mu = E(\xi) &= \sum_{x=0}^{\infty} xp(1-p)^x \\
&= p(1-p)\sum_{x=0}^{\infty} x(1-p)^{x-1} \\
&= -p(1-p)\left(\sum_{x=0}^{\infty}(1-p)^x\right)' \\
&= -p(1-p)\,(1/p)' \\
&= p(1-p)/p^2 = (1-p)/p.
\end{aligned}$$

[2]For the proof refer to Section 2.5, Theorem 5 of *Introduction to Stochastic Processes (1972) - Paul G. Hoel.*

[3]The proof of these results are provided in Chapter 1 appendix of *Introduction to Stochastic Processes (1972) - Paul G. Hoel.*

Note that if $p \geq 1/2$ then $\mu \leq 1$ and the extinction is certain i.e. $\rho = 1$. For $p < 1/2$, $\mu > 1$ therefore the equation $\Phi(t) = t$ has a unique root in $[0, 1)$. For $0 \leq t \leq 1$ we have,

$$\begin{aligned}
\Phi(t) = E(t^{\xi}) &= \sum_{x=0}^{\infty} t^x p(1-p)^x \\
&= p \sum_{x=0}^{\infty} [t(1-p)]^x \\
&= \frac{p}{1 - t(1-p)}.
\end{aligned}$$

For $p < 1/2$, since $\mu > 1$, the extinction probability ρ is the root of the equation,

$$\frac{p}{1 - t(1-p)} = t.$$

We can rewrite this equation as,

$$\left(t - \frac{p}{1-p}\right)(t-1) = 0.$$

This equation has two roots, namely, 1 and $p/(1-p)$. Since $p < 1/2$, $p/(1-p) < 1$, Consequently $\rho = p/(1-p)$.