Stochastic Processes
Assignment 3 - Solutions

1. (a)

$$\begin{aligned}
E\left(X_{n+1}\right) &= \sum_y yP\left(X_{n+1}=y\right) \\
&= \sum_x \sum_y yP\left(X_{n+1}=y \mid X_n=x\right) P\left(X_n=x\right) \\
&= \sum_x P\left(X_n=x\right) \sum_y yP(x,y) \\
&= \sum_x (Ax+b) P\left(X_n=x\right) \\
&= E\left(AX_n+B\right) = AE\left(X_n\right)+B
\end{aligned}$$

Note that in the second equation we interchanged the order of summation, this is possible because the summands are non-negative. Also note that the whole process can be repeated for $E_x(X_{n+1})$ in a similar way without any problem.

(b) Using the recurrence relation that we established in (a) we have,

$$\begin{aligned}
E\left(X_n\right) &= AE\left(X_{n-1}\right)+B \\
&= A\left(AE\left(X_{n-2}\right)+B\right)+B \\
&= A^2E\left(X_{n-2}\right)+B(A+1)
\end{aligned}$$

By repeating the above argument we arrive at the equation,

$$E\left(X_n\right) = A^nE\left(X_0\right)+B\sum_{k=0}^{n-1}A^k$$

note that $\sum_{k=0}^{n} A^k$ is a geometric series, hence we have,

$$\begin{aligned}
E\left(X_n\right) &= A^nE\left(X_0\right)+B\left(\frac{1-A^n}{1-A}\right) \\
&= A^nE\left(X_0\right)-A^n\left(\frac{B}{1-A}\right)+\frac{B}{1-A} \\
&= \frac{B}{1-A}+A^n\left(E\left(X_0\right)-\frac{B}{1-A}\right)
\end{aligned}$$

(c) The transition function of the Ehrenfest chain is given by,

$$P(x,y) = \begin{cases} \dfrac{x}{d}, & y=x-1, \\ 1-\dfrac{x}{d}, & y=x+1, \\ 0, & \text{elsewhere.} \end{cases}$$

Hence we have,

$$\sum_y yP(x,y) = (x-1)\frac{x}{d} + (x+1)\left(1-\frac{x}{d}\right) = \left(1-\frac{2}{d}\right)x + 1.$$

So the assumption of the Exercise holds for $A = (1-2/d)$ and $B = 1$, using (b) we have,

$$\begin{aligned} E_x\left(X_n\right) &= \frac{1}{1-(1-2/d)} + \left(1-\frac{2}{d}\right)^n \left(E_x\left(X_0\right) - \frac{1}{1-(1-2/d)}\right) \\ &= \frac{d}{2} + \left(1-\frac{2}{d}\right)^n \left(x - \frac{d}{2}\right) \end{aligned}$$

2. (a) We say that x leads to y if $\rho_{xy} > 0$ or equivalently if $p^n(x,y) > 0$ for some positive integer n. An irreducible Markov chain is a chain whose state space $\mathscr{S}$ is irreducible, that is if x leads to y for all choices of x and y in $\mathscr{S}$. Let x and y be members of the set of non-negative integers, it suffice to show that x leads to y. if $x < y$, starting at state x, we can get to the state y with positive probability p^{y-x} and if $x \geq y$, starting at state x, we can get to the state y with positive probability $(1-p)p^y$.

   (b) The desired probability indicates the probability that, starting from the state 0, we will return to this state for the first time in the nth step, that is $p^{n-1}(1-p)$.

   (c) A Markov chain is called a transient chain if all of its states are transient and a recurrent chain if all of its states are recurrent and an irreducible Markov chain is necessarily either a transient chain or a recurrent chain, so it suffice to show the recurrence of a single state, we'll pick state 0.

$$\begin{aligned} \rho_{00} &= P_0(T_0 < \infty) \\ &= \lim_{n\to\infty} P_0(T_0 < n) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n-1} P_0(T_0 = k) \\ &= \sum_{k=1}^{\infty} p^{k-1}(1-p) \qquad \text{by (b)} \\ &= (1-p)\sum_{k=1}^{\infty} p^{k-1} = \frac{1-p}{1-p} = 1 \end{aligned}$$

3. (a) A nonempty set C of states is said to be closed if no state inside of C leads to any state outside of C, i.e., if

$$\rho_{xy} = 0, \qquad x \in C \text{ and } y \notin C,$$

   and if C be a finite irreducible closed set of states, then every state in C is recurrent. Note that $C_1 = \{1,2,3\}$ and $C_2 = \{4,5,6\}$ are irreducible closed sets, therefore the above argument implies that $1,2,3,4,5$ and 6 are recurrent states. Also note that state 0 leads to state 2, but can't be reached from 2, hence state 0 is transient. Let $\mathscr{S}_T$ denote the collection of transient states in $\mathscr{S}$, and let $\mathscr{S}_R$ denote the collection of recurrent states in $\mathscr{S}$, so we have $\mathscr{S}_T = \{0\}$ and $\mathscr{S}_R = \{1,2,3,4,5,6\}$.

   (b) Let C be one of the irreducible closed sets of recurrent states, and let $P_C(x) = P_x(T_C < \infty)$ be the probability that a Markov chain starting at x eventually hits C. Observe that if $x \in \mathscr{S}_T$, a chain starting at x can enter C only by entering C at time 1 or by being in $\mathscr{S}_T$ at time 1 and entering C at some future time. The former event has probability $\sum_{y\in C} P(x,y)$ and the latter event has probability $\sum_{y\in\mathscr{S}_T} P(x,y)\rho_C(y)$. Thus

$$\rho_C(x) = \sum_{y\in C} P(x,y) + \sum_{y\in\mathscr{S}_T} P(x,y)\rho_C(y), \quad x\in\mathscr{S}_T.$$

From the above equation and the transition matrix of the Exercise,

$$\begin{aligned} P_{C_1}(0) &= P(0,2) + P(0,3) + P(0,0)P_{C_1}(0) = \frac{3}{8} + \frac{1}{2}P_{C_1}(0), \\ P_{C_2}(0) &= P(0,4) + P(0,0)P_{C_2}(0) = \frac{1}{8} + \frac{1}{2}P_{C_2}(0). \end{aligned}$$

Hence, $P_{C_1}(0) = 3/4$ and $P_{C_2}(0) = 1/4$. Note that a Markov chain starting at a transient state x enters an irreducible closed set C of recurrent states, it visits every state in C. Thus

$$\rho_{xy} = \rho_C(x), \quad x\in\mathscr{S}_T \text{ and } y\in C.$$

hence it follows that in our example

$$\begin{aligned} \rho_{01} = \rho_{02} = \rho_{03} = \rho_{C_1}(0) = \frac{3}{4} \\ \rho_{04} = \rho_{05} = \rho_{06} = \rho_{C_2}(0) = \frac{1}{4}. \end{aligned}$$

Finally, ρ_{00}, the probability that the Markov chain starting at 0 will ever return to 0 is $1/2$.

4. If the birth and death chain starts at y, then in one step it goes to $y-1, y$, or $y+1$ with respective probabilities q_y, r_y, or p_y. It follows that

$$u(y) = q_y u(y-1) + r_y u(y) + p_y u(y+1), \quad a < y < b. \tag{1}$$

Since $r_y = 1 - p_y - q_y$, we can rewrite (1) as

$$u(y+1) - u(y) = \frac{q_y}{p_y}(u(y) - u(y-1)), \quad a < y < b \tag{2}$$

From (2) we see that

$$u(y+1) - u(y) = \frac{\gamma_y}{\gamma_{y-1}}(u(y) - u(y-1)), \quad a < y < b,$$

from which it follows that

$$\begin{aligned} u(y+1) - u(y) &= \frac{\gamma_{a+1}}{\gamma_a}\cdots\frac{\gamma_y}{\gamma_{y-1}}(u(a+1) - u(a)) \\ &= \frac{\gamma_y}{\gamma_a}(u(a+1) - u(a)). \end{aligned}$$

Consequently,

$$u(y) - u(y+1) = \frac{\gamma_y}{\gamma_a}(u(a) - u(a+1)), \quad a \le y < b. \tag{3}$$

Summing (3) on $y = a, \ldots, b-1$ and recalling that $u(a) = 1$ and $u(b) = 0$, we conclude that

$$\frac{u(a) - u(a+1)}{\gamma_a} = \frac{1}{\sum_{y=a}^{b-1}\gamma_y}.$$

Thus (3) becomes

$$u(y) - u(y+1) = \frac{\gamma_y}{\sum_{y=a}^{b-1} \gamma_y}, \quad a \le y < b.$$

Summing this equation on $y = x, \dots, b-1$ and again using the formula $u(b) = 0$, we obtain

$$u(x) = \frac{\sum_{y=x}^{b-1} \gamma_y}{\sum_{y=a}^{b-1} \gamma_y}, \quad a < x < b$$

It now follows from the definition of $u(x)$ that

$$P_x\left(T_a < T_b\right) = \frac{\sum_{y=x}^{b-1} \gamma_y}{\sum_{y=a}^{b-1} \gamma_y}, \quad a < x < b$$

5. (a) Let X_n denote the capital of the gambler at time n with $X_0 = 1000$. Then $X_n, n \ge 0$, forms a birth and death chain on $\{0, 1, \dots, 1001\}$ with birth and death rates and

$$\begin{aligned} p_x &= 9/19, \quad 0 < x < 1001, \\ q_x &= 10/19, \quad 0 < x < 1001. \end{aligned}$$

States 0 and 1001 are absorbing states. Formula that we established in the previous Exercise is applicable with $a = 0, x = 1000$, and $b = 1001$. We conclude that

$$\gamma_y = (10/9)^y, \quad 0 \le y \le 1000$$

and hence that

$$P_{1000}\left(T_0 < T_{1001}\right) = \frac{(10/9)^{1000}}{\sum_{y=0}^{1000} (10/9)^y} \approx 0.1.$$

Thus the gambler has probability 0.1 of losing his initial capital.

(b) Let X denote the amount of loss after the result is determined, hence X takes two possible values namely -1 and 1000 corresponding to wining a dollar more than the initial capital or losing the initial capital respectively. Also note that X takes 1000 with probability 0.1, therefore the expected loss can be calculated as follows,

$$E(X) = (-1)(1 - 0.1) + (1000)(0.1) = 99.1.$$

6. Suppose that each individual gives rise to ξ individuals in the next generation, where ξ is a non-negative integer-valued random variable having Poisson distribution with parameter λ. We suppose that the number of offspring of the various individuals in the various generations are independent.

(a) We'll solve the question for the more general form $P_x(X_1 = y)$. We have for $x \ge 1$,

$$P(x, y) = P\left(\xi_1 + \dots + \xi_x = y\right),$$

hence we have,

$$\begin{aligned} P_x(X_1 = y) &= P(X_1 = y | X_0 = x) \\ &= P(x, y) = P\left(\xi_1 + \dots + \xi_x = y\right). \end{aligned}$$

Note that $\xi_1, \dots, \xi_x$ are independent Poisson random variables having common density $f(k) = e^{-\lambda}\lambda^k/k!$, therefore $\xi_1 + \dots + \xi_x$ is a Poisson random variable having density $f(k) = e^{-(\lambda x)}(\lambda x)^k/k!$, finally we have,

$$P_x(X_1 = y) = P\left(\xi_1 + \dots + \xi_x = y\right) = e^{-(\lambda x)}(\lambda x)^y/y!$$

Hence $P_1(X_1 = 0) = e^{-\lambda}$.

(b) We'll solve the question for the more general form $E_x(X_n)$. By conditioning on X_{n-1}, we obtain,

$$\begin{aligned} E\left(X_n\right) &= E\left[E\left(X_n \mid X_{n-1}\right)\right] \\ &= E\left[E\left(\sum_{i=1}^{X_{n-1}} \xi_i \mid X_{n-1}\right)\right] \\ &= E\left(X_{n-1} E\left(\xi_i\right)\right) \\ &= E\left(\xi_i\right) E\left(X_{n-1}\right) \end{aligned}$$

By repeating the above argument we arrive at the equation,

$$E\left(X_n\right) = E^n(\xi_i) E\left(X_0\right),$$

using the fact that $E(\xi_i) = \lambda$ finally we have,

$$E_x(X_n) = \lambda^n x.$$

Hence $E_1(X_1) = \lambda$.

(c) We'll solve the question for the more general form $E_x(X_n X_{n+1})$. We begin by calculating $E_x(X_n^2)$, note that,

$$\begin{aligned} E\left(X_n^2 | X_{n-1} = x\right) &= E^2(X_n | X_{n-1} = x) + \mathrm{Var}(X_n | X_{n-1} = x) \\ &= \left(E\left(\xi_i\right) x\right)^2 + \mathrm{Var}\left(\xi_i\right) x = \lambda^2 x^2 + \lambda x, \end{aligned}$$

hence we have,

$$\begin{aligned} E_x\left(X_n^2\right) &= \sum_y E_x\left(X_n^2 | X_{n-1} = y\right) P_x(X_{n-1} = y) \\ &= \lambda \sum_y y P_x(X_{n-1} = y) + \lambda^2 \sum_y y^2 P_x(X_{n-1} = y) \\ &= \lambda E_x(X_{n-1}) + \lambda^2 E_x\left(X_{n-1}^2\right) \\ &= x\lambda^n + \lambda^2 E_x\left(X_{n-1}^2\right), \end{aligned}$$

iterating this gives,

$$E_x\left(X_n^2\right) = x\left(\lambda^n + \cdots + \lambda^{2n-1}\right) + x^2 \lambda^{2n}, \quad n \geq 1.$$

Finally, we calculate the desired expected value $E_x(X_n X_{n+1})$, by conditioning on X_n we have,

$$\begin{aligned} E_x(X_n X_{n+1}) &= E_x[E_x(X_n X_{n+1} | X_n)] \\ &= E_x[X_n E_x(X_{n+1} | X_n)] \\ &= E_x\left[X_n^2 E\left(\xi_i\right)\right] \\ &= \lambda E_x\left[X_n^2\right] \\ &= x\left(\lambda^{n+1} + \cdots + \lambda^{2n}\right) + x^2 \lambda^{2n+1}. \end{aligned}$$

Hence $E_1(X_1 X_2) = \lambda^2 + \lambda^3$.