Stochastic Processes
Assignment 1 - Solutions

1. A quadratic equation has real roots when the discriminant is non-negative, hence the probability that we're looking for is $\mathrm{P}(U_2^2 - 4U_1U_3 \geq 0)$. We have,

$$\mathrm{P}(U_2^2 - 4U_1U_3 \geq 0) = \mathrm{P}((U_1, U_2, U_3) \in V) = \int_V f_{(U_1,U_2,U_3)}(u_1, u_2, u_3)\ dV$$

where $V = \{(u_1, u_2, u_3)|u_2^2 - 4u_1u_3 \geq 0, 0 \leq u_i \leq 1\}$. Due to the independence assumption,

$$f_{(U_1,U_2,U_3)}(u_1, u_2, u_3) = f_{U_1}(u_1)f_{U_2}(u_2)f_{U_3}(u_3) = 1, \qquad 0 \leq u_i \leq 1, i = 1, 2, 3$$

So we have,

$$\int_V f_{(U_1,U_2,U_3)}(u_1, u_2, u_3)\ dV = \int_0^1 \int_0^1 \int_{min(1,2\sqrt{u_1u_3})}^1 du_2du_3du_1 = \frac{1}{36}(5 + 6\ln(2))$$

2. a. If $f_{(X,Y)}(x, y)$ is a joint density function, we know that f is non-negative and $\int_{S^\star_{(X,Y)}} f = 1$ where $S^\star_{(X,Y)} = \{(x, y)|f_{(X,Y)}(x, y) > 0\}$. Hence we have,

$$\int_0^\infty \int_{-x}^x f_{(X,Y)}(x, y)\ dydx = \int_0^\infty \int_{-x}^x c\left(x^2 - y^2\right)e^{-x}\ dydx = 1$$

After solving the above equation for c, we find that $c = 1/8$.
b. We can compute the marginal densities as follows, note that the domain $-x \leq y \leq x$ can be rewritten as $|y| \leq x$.

$$f_X(x) = \int_{-x}^x f_{(X,Y)}(x, y)\ dy = \int_{-x}^x \frac{1}{8}(x^2 - y^2)e^{-x}\ dy = \frac{1}{6}x^3e^{-x}$$

$$f_Y(y) = \int_{|y|}^\infty f_{(X,Y)}(x, y)\ dx = \int_{|y|}^\infty \frac{1}{8}(x^2 - y^2)e^{-x}\ dx = \frac{e^{-|y|}(|y| + 1)}{4}$$

c. We can compute the conditional densities as follows,

$$f_{Y|X}(y) = \frac{f_{(X,Y)}(x, y)}{f_X(x)} = \frac{1/8\left(x^2 - y^2\right)e^{-x}}{1/6(x^3e^{-x})} = \frac{3}{4}\left[\frac{1}{x} - \frac{y^2}{x^3}\right]$$

$$f_{X|Y}(x) = \frac{f_{(X,Y)}(x, y)}{f_Y(y)} = \frac{\frac{1}{8}\left(x^2 - y^2\right)e^{-x}}{\frac{e^{-|y|}(|y|+1)}{4}} = \frac{\left(x^2 - y^2\right)e^{-x}}{2e^{-|y|}(|y| + 1)}$$

3. We know that $T \sim E(\lambda)$ and $U|T = t \sim U[0, t]$. Therefore $f_T(t) = \lambda e^{-\lambda t}$ and $f_{U|T=t}(u) = 1/t$. Also note that if X be uniform on $[0, a]$ then $\mathrm{E}(X) = a/2$ and $\mathrm{V}(X) = a^2/12$.
a.

$$\mathrm{E}(U) = \mathrm{E}\left[\mathrm{E(U|T)}\right] = \mathrm{E}\left(\frac{T}{2}\right) = \int_0^\infty \frac{t}{2}\lambda e^{-\lambda t}\ dt = \frac{1}{2\lambda}$$

b. First, note that

$$\mathrm{E}\left(T^2/k\right) = \int_0^{\infty} \frac{t^2}{k} \lambda e^{-\lambda t}\, dt = \frac{2}{k\lambda^2}, \tag{1}$$

now we can compute the unconditional variance as follows,

$$\begin{aligned}
\mathrm{V}(U) &= \mathrm{V}[\mathrm{E}(U|T)] + \mathrm{E}[\mathrm{V}(U|T)] \\
&= \mathrm{V}\left(\frac{T}{2}\right) + \mathrm{E}\left(\frac{T^2}{12}\right) \\
&= \left(\mathrm{E}\left(\frac{T^2}{4}\right) - \mathrm{E}^2\left(\frac{T}{2}\right)\right) + \mathrm{E}\left(\frac{T^2}{12}\right).
\end{aligned}$$

Using (1), finally we have,

$$\mathrm{V}(U) = \left(\frac{1}{2\lambda^2} - \frac{1}{4\lambda^2}\right) + \frac{1}{6\lambda^2} = \frac{5}{12\lambda^2}$$

4. a. For every positive integer n and every $s \in (-1, 1)$ we have,

$$\begin{aligned}
\frac{d}{ds} G(s) &= \sum_{k=0}^{\infty} k s^{k-1}\ \mathrm{P}(X = k), \\
\frac{d^2}{ds^2} G(s) &= \sum_{k=0}^{\infty} k(k-1) s^{k-2}\ \mathrm{P}(X = k), \\
&\ \vdots \\
\frac{d^n}{ds^n} G(s) &= \sum_{k=0}^{\infty} \frac{k!}{(k-n)!} s^{k-n}\ \mathrm{P}(X = k).
\end{aligned}$$

Hence,

$$\left.\frac{d^n}{ds^n} G(s)\right|_{s=0} = n!\ \mathrm{P}(X = n), \qquad n = 1, 2, \ldots$$

and finally,

$$\left.\frac{1}{n!}\frac{d^n}{ds^n} G(s)\right|_{s=0} = \mathrm{P}(X = n), \qquad n = 1, 2, \ldots$$

b.

$$\begin{aligned}
\left.\frac{dG}{ds}\right|_{s=1} &= \sum_{k=0}^{\infty} k\ \mathrm{P}(X = k) = E(X) \\
\left.\frac{d^2G}{ds^2}\right|_{s=1} &= \sum_{k=0}^{\infty} k(k-1)\ \mathrm{P}(X = k) = E[X(X-1)]
\end{aligned}$$

c. By our definition of probability-generating function we have $G_X(s) = \mathrm{E}\left(s^X\right)$. Define $t := \ln s$ so that $s = e^t$, then

$$G_X(s) = \mathrm{E}\left(s^X\right) = \mathrm{E}\left((e^t)^X\right) = \mathrm{E}\left(e^{tX}\right) = M_X(t) = M_X(\ln\ s)$$

d. Let X be a discrete random variable with a Poisson distribution with parameter λ for some $\lambda \in \mathbb{R}$. We know that $M_X(t) = e^{\lambda(e^t - 1)}$, then from part (c) we have,

$$G_X(t) = M_X(\ln\ t) = \exp\left[\lambda\left(e^{\ln\ t} - 1\right)\right] = e^{\lambda(t-1)}$$

5. a. Letting X denote the number of men that select their own hats, we can compute $\mathrm{E}(X)$ by noting that $X=\sum_{i=1}^{n} X_i$ where,

$$X_i = \begin{cases} 1, & \text{if the } i\text{th man selects his own hat} \\ 0, & \text{otherwise} \end{cases}$$

Now, because the ith man is equally likely to select any of the n hats, it follows that

$$\mathrm{E}(X_i) = \mathrm{P}\left(X_i = 1\right) = P\left(i\text{th man selects his own hat}\right) = \frac{1}{n}.$$

Hence we obtain,

$$\mathrm{E}(X) = \sum_{i=1}^{n} \mathrm{E}(X_i) = n\left(\frac{1}{n}\right) = 1$$

b. Letting $X=\sum_{i=1}^{n} X_i$, where

$$X_i = \begin{cases} 1, & \text{if } i\text{th man selects his own hat} \\ 0, & \text{otherwise} \end{cases}$$

we obtain

$$\mathrm{Var}(X) = \sum_{i=1}^{n} \mathrm{Var}\left(X_i\right) + \sum_{i \neq j} \mathrm{Cov}\left(X_i, X_j\right)$$

Since $X_i \sim B(1/n)$ for $i = 1, \ldots n$, we see

$$\mathrm{Var}\left(X_i\right) = \frac{1}{n}\left(1 - \frac{1}{n}\right) = \frac{n-1}{n^2}$$

Also,

$$\mathrm{Cov}\left(X_i, X_j\right) = E\left(X_i X_j\right) - E\left(X_i\right) E\left(X_j\right)$$

Now,

$$X_i X_j = \begin{cases} 1, & \text{if the } i\text{th and } j\text{th men both select their own hats} \\ 0, & \text{otherwise} \end{cases}$$

and thus

$$\begin{aligned} E\left[X_i X_j\right] &= P\left(X_i = 1, X_j = 1\right) \\ &= P\left(X_i = 1\right) P\left(X_j = 1 \mid X_i = 1\right) \\ &= \frac{1}{n} \frac{1}{n-1} \end{aligned}$$

Hence,

$$\mathrm{Cov}\left(X_i, X_j\right) = \frac{1}{n(n-1)} - \left(\frac{1}{n}\right)^2 = \frac{1}{n^2(n-1)}$$

and

$$\begin{aligned} \mathrm{Var}(X) &= \frac{n-1}{n} + 2\binom{n}{2} \frac{1}{n^2(n-1)} \\ &= \frac{n-1}{n} + \frac{1}{n} \\ &= 1 \end{aligned}$$

c. Let X denote the number of matches, and let X_1 equal 1 if the first person has a match and 0 otherwise. Then,

$$\begin{aligned} E[X] &= E\left[X \mid X_1=0\right] P\left(X_1=0\right)+E\left[X \mid X_1=1\right] P\left(X_1=1\right) \\ &= E\left[X \mid X_1=0\right] \frac{n-1}{n}+E\left[X \mid X_1=1\right] \frac{1}{n} \end{aligned}$$

But, by (a)

$$E[X]=1$$

Moreover, given that the first person has a match, the expected number of matches is equal to 1 plus the expected number of matches when $n-1$ people select among their own $n-1$ hats, showing that

$$E\left[X \mid X_1=1\right]=2$$

Therefore, we obtain the result

$$E\left[X \mid X_1=0\right]=\frac{n-2}{n-1}$$

6. a. To prove independence, we can equivalently show that

$$M_{(X,Y,Z)}(t,u,v)=M_X(t)M_Y(u)M_Z(v). \tag{2}$$

We have,

$$\begin{aligned} M_X(t) &= M_{(X,Y,Z)}(t,0,0)=e^{t+t^2} \\ M_Y(u) &= M_{(X,Y,Z)}(0,u,0)=e^{2u^2} \\ M_Z(v) &= M_{(X,Y,Z)}(0,0,v)=\frac{1}{1-v}. \end{aligned}$$

Therefore, equality (2) holds.

b. According to the assumption of independence that we proved in (a), we can rewrite the required expected value as below,

$$\mathrm{E}\left[e^{2X}\left(Y^2+Z\right)\right]=\mathrm{E}\left(e^{2X}\right)\mathrm{E}\left(Y^2+Z\right)$$

Hence we have,

$$\begin{aligned} \mathrm{E}\left(e^{2X}\right)\mathrm{E}\left(Y^2+Z\right) &= \mathrm{E}\left(e^{2X}\right)\left[\mathrm{E}(Y^2)+\mathrm{E}(Z)\right] \\ &= M_X(2)\left[\left.\frac{d^2}{du^2}M_Y(u)\right|_{u=0}+\left.\frac{d}{dv}M_Z(v)\right|_{v=0}\right] \\ &= e^6\left[\left.4e^{2u^2}(1+4u^2)\right|_{u=0}+\left.\frac{1}{(1-v)^2}\right|_{v=0}\right]=5e^6 \end{aligned}$$