Stochastic Processes
Assignment 6 - Solutions

1. (a) Consider a pure death process on $\{0, 1, 2, \dots\}$ with death rate $\mu_x = q_{x,x-1}$. By forward equation formula,

$$P'_{xy}(t) = \sum_z P_{xz}(t) q_{zy}$$

we have,

$$P'_{xy}(t) = -\mu_y P_{xy}(t) + \mu_{y+1} P_{x,y+1}(t), \quad y \le x - 1; \tag{1}$$
$$P'_{xx}(t) = -\mu_x P_{xx}(t). \tag{2}$$

(b) By equation (2) we have,

$$\frac{P'_{xx}(t)}{P_{xx}(t)} = -\mu_x$$

which can be written as,

$$\frac{d}{dt} \ln(P_{xx}(t)) = -\mu_x.$$

Integrating both sides yields

$$\ln(P_{xx}(t)) = -\mu_x t + C,$$

hence we have, $P_{xx}(t) = e^{-\mu_x t + C}$. Using the initial condition $P_{xx}(0) = 1$, we find that $C = 0$, hence $P_{xx}(t) = e^{-\mu_x t}$.

(c) By equation (1) we have,

$$P'_{xy}(t) + \mu_y P_{xy}(t) = \mu_{y+1} P_{x,y+1}(t), \tag{3}$$

to solve this linear differential equation, first we compute the integrating factor $\xi(t)$ as follows,

$$\xi(t) = \exp\left[\int \mu_y dt\right] = e^{\mu_y t}$$

now we can rewrite (3) as follows,

$$\frac{d}{dt}\left[\xi(t) P_{xy}(t)\right] = \xi(t) \mu_{y+1} P_{x,y+1}(t).$$

hence by fundamental theorem of calculus we have,

$$\mu_{y+1} \int_0^t \xi(t) P_{x,y+1}(t) ds = \xi(t) P_{xy}(t) - \xi(0) P_{xy}(0) = \xi(t) P_{xy}(t)$$

where the last equality follows from the fact that for $x \neq y$, $P_{xy}(0) = 0$. By substituting $\xi(t)$ and rearranging we obtain,

$$P_{xy}(t) = \mu_{y+1} \int_0^t e^{-\mu_y (t-s)} P_{x,y+1}(s) ds$$

(d) Using part (c), by setting $y = x - 1$ we have,

$$P_{x,x-1}(t) = \mu_x \int_0^t e^{-\mu_{x-1}(t-s)} P_{xx}(s) ds = \mu_x \int_0^t e^{-\mu_{x-1}(t-s)} e^{-\mu_x s} ds,$$

hence we have,

$$P_{x,x-1}(t) = \frac{\mu_x}{\mu_{x-1} - \mu_x}\left(e^{-\mu_x t} - e^{-\mu_{x-1} t}\right),$$

(e) Here we use backward induction. Note that for $y=x$, $P_{xy}(t)=e^{-x\mu t}=e^{-\mu_x t}$, which is established according to part (b). Now suppose the equation is established for $y+1\leq x$, by part (c) we have,

$$
\begin{aligned}
P_{xy}(t) &= \mu_{y+1}\int_0^t e^{-\mu_y(t-s)}P_{x,y+1}(s)ds \\
&= (y+1)\mu\int_0^t e^{-y\mu(t-s)}\binom{x}{y+1}\left(e^{-\mu s}\right)^{y+1}\left(1-e^{-\mu s}\right)^{x-(y+1)}ds \\
&= \binom{x}{y+1}(y+1)\mu e^{-y\mu t}\int_0^t e^{-\mu s}\left(1-e^{-\mu s}\right)^{x-(y+1)}ds \\
&= \binom{x}{y+1}(y+1)\mu e^{-y\mu t}\left.\frac{\left(1-\mathrm{e}^{-\mu s}\right)^{x-y}}{\mu(x-y)}\right|_0^t \\
&= \binom{x}{y+1}\frac{y+1}{x-y}\left(e^{-\mu t}\right)^y\left(1-e^{-\mu t}\right)^{x-y} \\
&= \binom{x}{y}\left(e^{-\mu t}\right)^y\left(1-e^{-\mu t}\right)^{x-y}.
\end{aligned}
$$

2. We will recursively compute $E(T_i), i\geq 0$, by starting with $i=0$. Since T_0 is exponential with rate λ_0, we have

$$E(T_0)=\frac{1}{\lambda_0}$$

For $i>0$, we condition whether the first transition takes the process into state $i-1$ or $i+1$. That is, let

$$I_i=\begin{cases}1, & \text{if the first transition from } i \text{ is to } i+1\\ 0, & \text{if the first transition from } i \text{ is to } i-1\end{cases}$$

and note that

$$
\begin{aligned}
E(T_i \mid I_i=1) &= \frac{1}{\lambda_i+\mu_i} \\
E(T_i \mid I_i=0) &= \frac{1}{\lambda_i+\mu_i}+E(T_{i-1})+E(T_i)
\end{aligned}
\tag{4}
$$

This follows since, independent of whether the first transition is from a birth or death, the time until it occurs is exponential with rate $\lambda_i+\mu_i$; if this first transition is a birth, then the population size is at $i+1$, so no additional time is needed; whereas if it is death, then the population size becomes $i-1$ and the additional time needed to reach $i+1$ is equal to the time it takes to return to state i (this has mean $E(T_{i-1})$) plus the additional time it then takes to reach $i+1$ (this has mean $E[T_i]$). Hence, since the probability that the first transition is a birth is $\lambda_i/(\lambda_i+\mu_i)$, we see that

$$E(T_i)=\frac{1}{\lambda_i+\mu_i}+\frac{\mu_i}{\lambda_i+\mu_i}\left[E(T_{i-1})+E(T_i)\right]$$

or, equivalently,

$$E[T_i]=\frac{1}{\lambda_i}+\frac{\mu_i}{\lambda_i}E[T_{i-1}], \quad i\geq 1$$

Starting with $E(T_0)=1/\lambda_0$, the preceding yields an efficient method to successively compute $E(T_1), E(T_2)$, and so on.

We can also compute the variance of the time to go from i to $i+1$ by utilizing the conditional variance formula. First note that equation (4) can be written as

$$E[T_i \mid I_i]=\frac{1}{\lambda_i+\mu_i}+(1-I_i)\left(E[T_{i-1}]+E[T_i]\right)$$

Thus,

$$
\begin{aligned}
\operatorname{Var}(E[T_i \mid I_i]) &= \left(E[T_{i-1}]+E[T_i]\right)^2\operatorname{Var}(I_i) \\
&= \left(E[T_{i-1}]+E[T_i]\right)^2\frac{\mu_i\lambda_i}{(\mu_i+\lambda_i)^2}
\end{aligned}
\tag{5}
$$

where $\mathrm{Var}(I_i)$ is as shown since I_i is a Bernoulli random variable with parameter $p = \lambda_i/(\lambda_i + \mu_i)$. Also, note that if we let X_i denote the time until the transition from i occurs, then

$$\begin{aligned} \mathrm{Var}(T_i \mid I_i = 1) &= \mathrm{Var}(X_i \mid I_i = 1) \\ &= \mathrm{Var}(X_i) \\ &= \frac{1}{(\lambda_i + \mu_i)^2} \end{aligned} \tag{6}$$

where the preceding uses the fact that the time until transition is independent of the next state visited. Also,

$$\begin{aligned} \mathrm{Var}(T_i \mid I_i = 0) &= \mathrm{Var}(X_i + \text{ time to get back to } i + \text{ time to then reach } i+1) \\ &= \mathrm{Var}(X_i) + \mathrm{Var}(T_{i-1}) + \mathrm{Var}(T_i) \end{aligned} \tag{7}$$

where the foregoing uses the fact that the three random variables are independent. We can rewrite equations (6) and (7) as

$$\mathrm{Var}(T_i \mid I_i) = \mathrm{Var}(X_i) + (1 - I_i)\left[\mathrm{Var}(T_{i-1}) + \mathrm{Var}(T_i)\right]$$

So

$$E\left[\mathrm{Var}(T_i \mid I_i)\right] = \frac{1}{(\mu_i + \lambda_i)^2} + \frac{\mu_i}{\mu_i + \lambda_i}\left[\mathrm{Var}(T_{i-1}) + \mathrm{Var}(T_i)\right] \tag{8}$$

Hence, using the conditional variance formula, which states that $\mathrm{Var}(T_i)$ is the sum of equations (8) and (5), we obtain

$$\begin{aligned} \mathrm{Var}(T_i) = & \frac{1}{(\mu_i + \lambda_i)^2} + \frac{\mu_i}{\mu_i + \lambda_i}\left[\mathrm{Var}(T_{i-1}) + \mathrm{Var}(T_i)\right] \\ & + \frac{\mu_i \lambda_i}{(\mu_i + \lambda_i)^2}\left(E[T_{i-1}] + E[T_i]\right)^2 \end{aligned}$$

or, equivalently,

$$\mathrm{Var}(T_i) = \frac{1}{\lambda_i(\lambda_i + \mu_i)} + \frac{\mu_i}{\lambda_i}\mathrm{Var}(T_{i-1}) + \frac{\mu_i}{\mu_i + \lambda_i}\left(E[T_{i-1}] + E[T_i]\right)^2$$

Starting with $\mathrm{Var}(T_0) = 1/\lambda_0^2$ and using the former recursion to obtain the expectations, we can recursively compute $\mathrm{Var}(T_i)$.

3. Let $X(t), 0 \le t < \infty$, be a Markov pure jump process. If x is a non-absorbing state, then F_x has an exponential density f_x. Let q_x denote the parameter of this density. Then $q_x = 1/E_x(\tau_1) > 0$. We know that,

$$q_{xy} = \begin{cases} -q_x, & y = x \\ q_x Q_{xy}, & y \neq x \end{cases} \tag{9}$$

when the process leaves x it goes to y with probability Q_{xy} such that $Q_{xx} = 0$ and $\sum_y Q_{xy} = 1$.

   (a) We know that $\tau_1 | X_0 = 1$ have exponential density with parameter q_1, hence by (9) $q_1 = -q_{11} = 3$ and $\tau_1 | X_0 = 1 \sim E(3)$.

   (b) By (6) the embedded chain has transition matrix,

$$Q = \begin{bmatrix} 0 & Q_{01} & Q_{02} \\ Q_{10} & 0 & Q_{12} \\ Q_{20} & Q_{21} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -q_{01}/q_{00} & -q_{02}/q_{00} \\ -q_{10}/q_{11} & 0 & -q_{12}/q_{11} \\ -q_{20}/q_{22} & -q_{21}/q_{22} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 2/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 \end{bmatrix}$$

   (c) Let T_i represent the amount of time that the process stays in the ith state, hence $T_0 \sim E(2)$, $T_1 \sim E(3)$ and we can write $P_0(\tau_2 > 1 \mid X(\tau_1) = 1) = P(T_0 + T_1 > 1)$. For $z > 0$ we have,

$$\begin{aligned} P(T_0 + T_1 > z) &= \int_0^\infty 2e^{-2x} P(T_1 > z - x)dx \\ &= \int_0^z 2e^{-2x} P(T_1 > z - x)dx + \int_z^\infty 2e^{-2x}dx \\ &= 2e^{-3z}\int_0^z e^x dx + e^{-2z} = 2e^{-3z}(e^z - 1) + e^{-2z} \end{aligned}$$

   In particular, $P_0(\tau_2 > 1 \mid X(\tau_1) = 1) = P(T_0 + T_1 > 1) = 3e^{-2} - 2e^{-3}$.

4. Let T_i denote the time between the $(i-1)$th and the ith job completion. Then the T_i's are independent, with $T_i, i = 1, \ldots, n-1$ being exponential with rate $\mu_1 + \mu_2$. With probability $\mu_1/(\mu_1+\mu_2)$, T_n is exponential with rate μ_2, and with probability $\mu_2/(\mu_1+\mu_2)$ it is exponential with rate μ_1. Therefore,

$$\begin{aligned}
E[T] &= \sum_{i=1}^{n-1} E\left[T_i\right] + E\left[T_n\right] \\
&= (n-1)\frac{1}{\mu_1+\mu_2} + \frac{\mu_1}{\mu_1+\mu_2}\frac{1}{\mu_2} + \frac{\mu_2}{\mu_1+\mu_2}\frac{1}{\mu_1} \\
\operatorname{Var}(T) &= \sum_{i=1}^{n-1} \operatorname{Var}\left(T_i\right) + \operatorname{Var}\left(T_n\right) \\
&= (n-1)\frac{1}{\left(\mu_1+\mu_2\right)^2} + \operatorname{Var}\left(T_n\right),
\end{aligned}$$

for $\operatorname{Var}(T_n)$ we have,

$$\begin{aligned}
\operatorname{Var}\left(T_n\right) =& E\left[T_n^2\right] - \left(E\left[T_n\right]\right)^2 \\
=& \frac{\mu_1}{\mu_1+\mu_2}\frac{2}{\mu_2^2} + \frac{\mu_2}{\mu_1+\mu_2}\frac{2}{\mu_1^2} - \left(\frac{\mu_1}{\mu_1+\mu_2}\frac{1}{\mu_2} + \frac{\mu_2}{\mu_1+\mu_2}\frac{1}{\mu_1}\right)^2.
\end{aligned}$$

5. Let S_i denote the service time at server $i, i = 1, 2$ and let X denote the time until the next arrival. Then, with p denoting the proportion of customers that are served by both servers, we have

$$\begin{aligned}
p &= P\left(X > S_1 + S_2\right) \\
&= P\left(X > S_1\right) P\left(X > S_1 + S_2 \mid X > S_1\right) \\
&= P\left(X > S_1\right) P\left(X > S_2\right) \qquad (10) \\
&= \frac{\mu_1}{\mu_1+\lambda}\frac{\mu_2}{\mu_2+\lambda}
\end{aligned}$$

where equality (10) follows from memoryless property of exponential distributions.

6. Let X_i, $i = 1, 2$ denote the time it takes for team i to score a goal, then X_i is exponentially distributed with parameter λ_i. This is the gambler's ruin probability that, starting with k, the gambler's fortune reaches $2k$ before 0, when the probability of winning each bet is

$$p = P(X_1 < X_2) = \lambda_1/\left(\lambda_1+\lambda_2\right),$$

hence the desired probability is

$$P_k\left(T_{2k} < T_0\right) = \frac{\sum_{y=0}^{k-1}\gamma^y}{\sum_{y=0}^{2k-1}\gamma^y}$$

where $\gamma_y = \left(\frac{1-p}{p}\right)^y = (\lambda_2/\lambda_1)^y$. Hence we have,

$$P_k\left(T_{2k} < T_0\right) = \frac{1-\left(\lambda_2/\lambda_1\right)^k}{1-\left(\lambda_2/\lambda_1\right)^{2k}}.$$