Mathematical Statistics I
Assignment 1 - Solutions

1. By definition we write,

$$\mathrm{E}(|X-\mu|) = \int_{-\infty}^{\infty} |x-\mu| \left(\sqrt{2\pi\sigma^2}\right)^{-1} e^{-(x-\mu)^2/(2\sigma^2)} dx. \tag{1}$$

The substitution $t = x - \mu$ turn (1) into

$$\left(\sqrt{2\pi\sigma^2}\right)^{-1} \int_{-\infty}^{\infty} |t| e^{-t^2/(2\sigma^2)} dx,$$

since $f(t) = |t| \exp(-t^2)$ is an even function, we have,

$$\left(\sqrt{2\pi\sigma^2}\right)^{-1} \int_{-\infty}^{\infty} |t| e^{-t^2/(2\sigma^2)} dx = \left(\sqrt{2\pi\sigma^2}\right)^{-1} \cdot 2 \int_{0}^{\infty} t e^{-t^2/(2\sigma^2)} dx. \tag{2}$$

Note that,

$$\int_0^{\infty} t e^{-t^2/(2\sigma^2)} dx = -\sigma^2 e^{-t^2/(2\sigma^2)} \Big|_0^{\infty} = \sigma^2,$$

and then the result follows immediately from (2).

2. Let $M_Z(t)$ be the moment generating function of the random variable Z, we can write,

$$M_{X_1}(t) M_Y(t) = M_{X_1+Y}(t) = M_{X_2}(t), \tag{3}$$

where the first equality comes from the independence of X_1 and Y. Hence (3) implies,

$$M_Y(t) = \frac{M_{X_2}(t)}{M_{X_1}(t)}.$$

Note that if X be a chi-square random variable with r degrees of freedom, then the moment generating function of X is, $M_X(t) = (1-2t)^{-r/2}$ for $t < 1/2$. Hence for $t < 1/2$,

$$M_Y(t) = (1-2t)^{-(r_2-r_1)/2},$$

which is the moment generating function of a chi-square random variable with $r_2 - r_1$ degrees of freedom.

3. Note that $F_{X_i}(x) = 1 - e^{(x-a)/\theta}$ for $i = 1, 2, \ldots, n$. Hence we can write,

$$F_{X_{(1)}}(x) = \mathrm{P}(X_{(1)} \le x) = 1 - \mathrm{P}(X_{(1)} > x) = 1 - \left(1 - F_{X_1}(x)\right)^n,$$

note that in the last equality we use the independence assumption. We have,

$$F_{X_{(1)}}(x) = 1 - e^{n(x-a)/\theta},$$

therefore $X_{(1)} \sim E(a, \theta/n)$. It can be easily seen that $X_{(1)} - a \sim E(\theta/n)$, therefore $2n(X_{(1)} - a)/\theta \sim E(1/2)$ which is the same as χ^2_2. On the other hand, for $i = 1, 2, \ldots, n$, $(X_i - a)/\theta \sim E(1)$, hence $2\sum_{i=1}^n (X_i - a)/\theta \sim \Gamma(n, 1/2)$ which is the same as χ^2_{2n}. Since $X_{(1)}$ is a complete sufficient statistic for parameter θ, and $Y = 2\sum_{i=1}^n \left(X_i - X_{(1)}\right)/\theta$ is an ancillary statistic (its distribution does not depend on θ), they are independent of each other according to Basu's theorem. (Don't forget that this independence was given to us as a presumption). Hence we can apply the theorem that we proved in previous question to write,

$$Y = \left[2\sum_{i=1}^n (X_i - a)/\theta - 2n(X_{(1)} - a)/\theta\right] \sim \chi^2_{2n-2}.$$

4. Let Z be a random variable with distribution G, and let $X = \sigma Z$ for $\sigma > 0$. Then the distributions of X form a scale family. Now consider the distributions of $\log X = \log \sigma + \log Z$.

$$\mathrm{P}(\log X \leq y) = \mathrm{P}(\log \sigma + \log Z \leq y) = \mathrm{P}(\log Z \leq y - \log \sigma) = F(y - \mu),$$

where $\mu = \log \sigma$ and F is the distribution of $\log Z$. Thus the distributions of $\log X$ are functions of $y - \mu$ and therefore form a location family.

5. (i) The probability density function of X is

$$\frac{1}{\Gamma(\alpha)\gamma^\alpha} x^{\alpha-1} e^{-x/\gamma} I_{(0,\infty)}(x).$$

Therefore the probability density function for $Y = \sigma \log X$ is

$$\frac{1}{\Gamma(\alpha)\sigma} e^{\alpha(y-\sigma\log\gamma)/\sigma} \exp\left\{-e^{(y-\sigma\log\gamma)/\sigma}\right\} = \frac{1}{\sigma} g\left(\frac{y-\eta}{\sigma}\right),$$

where $\eta = \sigma \log \gamma$ and $g(t) = \Gamma(\alpha)^{-1} \exp\{\alpha t - e^t\}$. Hence it belongs to a location-scale family with location parameter η and scale parameter σ.
(ii) When σ is known, we rewrite the density of Y as

$$\frac{1}{\sigma\Gamma(\alpha)} \exp\left\{\alpha y/\sigma - \frac{e^{y/\sigma}}{\gamma} - \alpha \log \gamma\right\}.$$

Therefore, the distribution of Y is from an exponential family.