Stochastic Processes
Assignment 4 - Questions

Due on April 22, 2023

1. A particle moves according to a Markov chain on $\{1, 2, \ldots, c+d\}$, where c and d are positive integers. Starting from any one of the first c states, the particle jumps in one transition to a state chosen uniformly from the last d states; starting from any of the last d states, the particle jumps in one transition to a state chosen uniformly from the first c states.
(a) Show that the chain is irreducible.
(b) Find the stationary distribution.

2. Consider a Markov chain on $\{0, 1, 2, 3, 4\}$ having transition matrix

$$\begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 3/4 \\ 0 & 0 & 0 & 1/4 & 3/4 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(a) Show that the chain is irreducible.
(b) Find the stationary distribution.

3. Suppose we have two boxes and $2d$ balls, of which d are black and d are red. Initially, d of the balls are placed in box 1 , and the remainder of the balls are placed in box 2. At each trial a ball is chosen at random from each of the boxes, and the two balls are put back in the opposite boxes. Let X_0 denote the number of black balls initially in box 1 and, for $n \geq 1$, let X_n denote the number of black balls in box 1 after the nth trial.
(a) Find the transition function of the Markov chain X_n, $n \geq 0$.
(b) Find the stationary distribution of the chain. *Hint:* Use the formula

$$\binom{d}{0}^2 + \cdots + \binom{d}{d}^2 = \binom{2d}{d}.$$

4. Let $X_n, n \geq 0$, be the Ehrenfest chain on $\{0, 1, \ldots, d\}$.
(a) Find the stationary distribution.
(b) Find the mean and variance of this distribution.
(c) Suppose that initially all of the balls are in the second box. Find the expected amount of time until the system returns to that state.

5. Suppose that every man in a certain society has exactly three children, which independently have probability one-half of being a boy and one-half of being a girl. Suppose also that the number of males in the nth generation forms a branching chain.
(a) Find the probability that the male line of a given man eventually becomes extinct.
(b) If a given man has two boys and one girl, what is the probability that his male line will continue forever?

6. Consider a branching chain where each particle gives rise to ξ particles in the next generation, where ξ is a random variable having density $f(x) = p(1-p)^x, x \geq 0$, where $0 < p < 1$. Show that $\rho = 1$ if $p \geq 1/2$ and that $\rho = p/(1-p)$ if $p < 1/2$ where ρ is the probability that the descendants of a given particle eventually become extinct i.e. $\rho = \rho_{10} = P_1(T_0 < \infty)$.