Mathematical Statistics I
Assignment 7 - Questions

Due by June 21, 2024

1. Let (X, Y) has the joint density
$$f(x, y; \theta) = \exp\{-(\theta x + y/\theta)\}, \quad x > 0, \quad y > 0.$$
(a) For an iid sample of size n, show that the Fisher information is $I(\theta) = 2n/\theta^2$.
(b) For the estimators
$$T = \sqrt{\sum Y_i \Big/ \sum X_i} \text{ and } \quad U = \sqrt{\sum X_i \sum Y_i},$$
show that
(i) the information in T alone is $[n/(2n+1)]I(\theta)$;
(ii) the information in (T, U) is $I(\theta)$;
(iii) (T, U) is jointly sufficient but not complete.

2. Let $X_1, \ldots, X_n$ be i.i.d. with the p.d.f. $f_\theta(x) = \theta f_1(x) + (1-\theta) f_2(x)$, where f_j's are two different known p.d.f.'s and $\theta \in (0, 1)$ is unknown.
(a) Provide a necessary and sufficient condition for the likelihood equation to have a unique solution and show that if there is a solution, it is the MLE of θ.
(b) Derive the MLE of θ when the likelihood equation has no solution.

3. Suppose that T is a UMVUE of an unknown parameter θ. Show that T^k is a UMVUE of $E\left(T^k\right)$, where k is any positive integer for which $E\left(T^{2k}\right) < \infty$.

4. Let W_n denote a random variable with mean μ and variance b/n^p, where $p > 0$, μ, and b are constants (not functions of n). Prove that W_n converges in probability to μ.

5. Let $X_1, \ldots, X_n$ be i.i.d. random variables having the p.d.f.
$$f_\theta(x) = \exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^4 - \xi(\theta)\right\},$$
where $\theta = (\mu, \sigma) \in \Theta = \mathcal{R} \times (0, \infty)$. Show that $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ is an exponential family, where P_θ is the joint distribution of $X_1, \ldots, X_n$, and that the statistic $T = \left(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2, \sum_{i=1}^n X_i^3, \sum_{i=1}^n X_i^4\right)$ is minimal sufficient for $\theta \in \Theta$.