Mathematical Statistics I
Assignment 2 - Questions

Due by March 13, 2024

1. Let $X_1, \ldots, X_n$ be a random sample from the beta distribution $B(a, b)$,
   (i) Show that $(\Pi X_i, \Pi(1 - X_i))$ is a sufficient statistic for (a, b).
   (ii) Determine the sufficient statistic when $a = b$.

2. Let $X_1, \ldots, X_n$ be a random sample from a population with density $f_\theta(x) = a(\theta)h(x)I_{(\theta_1,\theta_2)}(x)$, where $h(x) \geq 0, \theta = (\theta_1, \theta_2)$ with $-\infty < \theta_1 \leq \theta_2 < \infty$, and $a(\theta) = \left[\int_{\theta_1}^{\theta_2} h(x)dx\right]^{-1}$ is assumed to exist. Find a two-dimensional sufficient statistic for θ and apply your result to the $U(\theta_1, \theta_2)$ family of distributions.

3. Let $X_1, \ldots, X_n$ be a random sample from rectangular distribution $R(\theta_1, \theta_2), \theta_1 < \theta_2$.
   (i) Apply the Factorization Theorem to prove that $X_{(1)}$ and $X_{(n)}$ are sufficient statistics.
   (ii) Derive the conditional density of $\mathbf{X} = (X_1, \ldots, X_n)$ given $\left(X_{(1)}, X_{(n)}\right)$.

4. Suppose that $X_1, \ldots, X_n$ be a random sample from a location-scale family with distribution function $F((x - a)/b)$. A statistic $T(X)$ is said to be ancillary if its distribution does not depend on parameters of the model
   (i) If b is known, show that the differences $(X_1 - X_i)/b, i = 2, \ldots, n$, are ancillary.
   (ii) If a is known, show that the ratios $(X_1 - a)/(X_i - a), i = 2, \ldots, n$, are ancillary.
   (iii) If neither a or b are known, show that the quantities $(X_1 - X_i)/(X_2 - X_i), i = 3, \ldots, n$, are ancillary.

5. (i) Let $X_1, \ldots, X_n$ be a random sample from the uniform distribution $U(0, \theta)$, $0 < \theta < \infty$, and let $T = \max(X_1, \ldots, X_n)$. Show that T is sufficient, once by using the definition of sufficiency and once by using the factorization criterion. Compare your answer with the results obtained from question 2.
   (ii) Let $X_1, \ldots, X_n$ be a sample from the exponential distribution $E(a, b)$ with density $(1/b)e^{-(x-a)/b}$ when $x \geq a$ $(-\infty < a < \infty, 0 < b)$. Use the factorization criterion to prove that $\left(X_{(1)}, \sum_{i=1}^{n} X_i\right)$ is sufficient for a, b.