Stochastic Processes
Assignment 5 - Solutions

1. **a)** We only need to consider the case when $0{=}t_0 < t_1 < t_2 \cdots < t_n < t_{n+1} = t$. in that case,

$$\mathbb{P}(T_1 \le t_1, \cdots, T_n \le t_n | X(t) = n) = \frac{\mathbb{P}(T_1 \le t_1, \cdots, T_n \le t_n, X(t) = n)}{\mathbb{P}(X(t) = n)}$$

$$= \frac{\sum\limits_{x_1+\cdots+x_{n+1}=n} \mathbb{P}(X(t_1) = x_1, X(t_2) - X(t_1) = x_2, \cdots, X(t) - X(t_n) = x_{n+1})}{\mathbb{P}(X(t) = n)}$$

$$= \frac{\sum\limits_{x_1+\cdots+x_{n+1}=n} \mathbb{P}(X(t_1) = x_1)\mathbb{P}(X(t_2) - X(t_1) = x_2)\mathbb{P}(X(t) - X(t_n) = x_{n+1})}{\mathbb{P}(X(t) = n)}$$

$$= \sum_{x_1+\cdots+x_{n+1}=n} \frac{\prod_{i=1}^{n+1} \frac{(\lambda(t_i - t_{i-1}))^{x_i}}{x_i!} e^{-\lambda(t_i - t_{i-1})}}{\frac{(\lambda t)^n}{n!} e^{-\lambda t}}$$

$$= \sum_{x_1+\cdots+x_{n+1}=n} \frac{\lambda^{\sum_{i=1}^{n+1} x_i} e^{-\lambda \sum_{i=1}^{n+1}(t_i - t_{i-t})}}{\lambda^n e^{-\lambda t}} \frac{\binom{n}{x_1 \cdots x_{n+1}} \prod_{i=1}^{n+1} (t - t_{i-1})^{x_i}}{t^n}$$

$$= \sum_{x_1+\cdots+x_{n+1}=n} \frac{\binom{n}{x_1 \cdots x_{n+1}} \prod_{i=1}^{n+1} (t - t_{i-1})^{x_i}}{t^n}$$

which is free of λ as we desired.

b) Conditioning on the value of T yields

$$\mathbb{P}(X(T) = n) = \int_0^\infty f_T(t)\mathbb{P}(X(T) = n \mid T = t)dt = \int_0^\infty f_T(t)\mathbb{P}(X(t) = n)dt.$$

According to Poisson process property, $X(t)$ has a Poisson distribution with parameter λt. Substituting $f_{X(t)}(n) = P(X(t) = n)$ and $f_T(t)$ yields

$$\int_0^\infty \nu e^{-\nu t} \frac{(\lambda t)^n e^{-\lambda t}}{n!} dt = \frac{\nu \lambda^n}{n!} \frac{\Gamma(n+1)}{(\nu+\lambda)^{n+1}} \int_0^\infty \frac{(\nu+\lambda)^{n+1}}{\Gamma(n+1)} t^n e^{-(\nu+\lambda)t} dt,$$

notice that the integrand of the right hand side integral is gamma density with parameters $n+1$ and $\nu+\lambda$, hence we have,

$$\mathbb{P}(X(T) = n) = \frac{\nu \lambda^n}{n!} \frac{\Gamma(n+1)}{(\nu+\lambda)^{n+1}} = \frac{\nu \lambda^n}{(\nu+\lambda)^{n+1}},$$

where the last equality comes from the fact that for every positive integer n, $\Gamma(n+1) = n!$.

2. **a)** Let t ≥ 0, s $\in \mathbb{R}$. Recall $X(t) \sim \text{Poisson}(t\lambda_X)$

$$m_{X(t)}(s) = \mathbb{E}[e^{sX(t)}] = \sum_{k=0}^{\infty} e^{sk}\mathbb{P}(X(t) = k) = \sum_{k=0}^{\infty} e^{sk}e^{-t\lambda_x}\frac{(t\lambda_x)^k}{k!}$$
$$= e^{-t\lambda_x}\sum_{k=0}^{\infty}\frac{(e^s t\lambda_x)^k}{k!} = e^{-t\lambda_x}(exp(e^s t\lambda_x)) = e^{t\lambda_x(e^s-1)}$$

b) Let $0 < s < t$. Pick $n \in \{0, 1, 2, \cdots\}$ and observe that $\mathbb{P}(X(s) = k|X(t) = n) = 0$ for all $k > n$. To get something nontrivial let $k \in \{0, 1, 2, \cdots, n\}$. We use the independent-increment property of a Poisson process:

$$\mathbb{P}(X(s) = k|X(t) = n) = \frac{\mathbb{P}(X(s) = k, X(t) = n)}{\mathbb{P}(X(t) = n)}$$
$$= \frac{\mathbb{P}(X(s) = k, X(t-s) = n-k)}{\mathbb{P}(X(t) = n)}$$
$$= \frac{\mathbb{P}(X(s) = k)\cdot\mathbb{P}(X(t-s) = n-k)}{\mathbb{P}(X(t) = n)}$$
$$= \frac{e^{-\lambda_x s}\frac{(\lambda_x s)^k}{k!}e^{-\lambda_x(t-s)}\frac{(\lambda_x(t-s))^{n-k}}{(n-k)!}}{e^{-\lambda_x t}\frac{(\lambda_x t)^n}{n!}}$$
$$= \binom{k}{n}(\frac{s}{t})^k(1-\frac{s}{t})^{n-k}$$

Thus, given $X(t) = n, X(s)$ is Binomial with parameters n and $p = \frac{s}{t}$. (no matter λ_x)
c) Total probability law (conditioning on T = t) for $n \in \{0, 1, 2, \cdots\}$:

$$\mathbb{P}(X(T) = n) = \int_0^{\infty}\mathbb{P}(X(T) = n|T = t)f_T(t)dt = \int_0^{\infty}\mathbb{P}(X(t) = n|T = t)f_T(t)dt$$
$$= \int_0^{\infty}\mathbb{P}(X(t) = n)f_T(t)dt = \int_0^{\infty}e^{-\lambda_x t}\frac{(\lambda_x t)^n}{n!}\nu e^{-\nu t}dt$$
$$= \nu\frac{(\lambda_x)^n}{n!}\int_0^{\infty}e^{-(\lambda_x+\nu)t}\cdot t^n dt$$
$$= \nu\frac{(\lambda_x)^n}{n!}\frac{n!}{(\lambda_x+\nu)^{n+1}} = \frac{\nu}{\lambda_x+\nu}(\frac{\lambda_x}{\lambda_x+\nu})^n$$
$$= \frac{\nu}{\lambda_x+\nu}(1-\frac{\nu}{\lambda_x+\nu})^n$$

Thus, $X(T)$ is geometric with parameter $p = \frac{\nu}{\lambda_x+\nu}$.
d) Let $T_Y := inf\{t \geq 0 : X(t) = 1\}$ and $T_Y := inf\{t \geq 0 : Y(t) = 1\}$. These are the holding times of state 0 for X(t) and Y(t) respectively. Consequently, both T_X and T_Y are exponential random variables with parameters λ_x and λ_y , respectively. As X and Y are independent, so are T_X and T_Y
Note $\mathbb{P}(T_Y > t) = \int_t^{\infty}\lambda_y e^{-\lambda_y s}ds = e^{-\lambda_y t}$ for $t \geq 0$ (tail of the exponential distribution). Again, with the total probability law: (conditioning on $T_X = t$)

$$\mathbb{P}(T_X > T_Y) = \int_0^{\infty}\mathbb{P}(T_X > T_Y|T_X = t)f_{T_X}(t)dt = \int_0^{\infty}\mathbb{P}(t > T_Y|T_X = t)f_{T_X}(t)dt$$
$$= \int_0^{\infty}\mathbb{P}(t > T_Y)f_{T_X}(t)dt = \int_0^{\infty}e^{-\lambda_y t}\lambda_x e^{-\lambda_x t}dt$$
$$= \lambda_x\int_0^{\infty}e^{-(\lambda_x+\lambda_y)t}dt = \frac{\lambda_x}{\lambda_x+\lambda_y}.$$

3. **a)** Accessibility diagram: $[0] = [1] = [4] = \{0, 1, 4\} \leftarrow [2] = [3] = [5] = \{2, 3, 5\}$ There are two communication classes, $[0] = \{0, 1, 4\}$ and $[2] = \{2, 3, 5\}$. As [0] is closed and finite Class, it is positive recurrent. [2] is transient. As the state space is finite there are no null recurrent states.

   b) Period of states {2,3,5} is 1.(because $\rho^1_{ii} > 0$ for $i \in \{2, 3, 5\}$) And we know If a MC has at least one state s with self-transition $\rho_{ss} > 0$ then the chain is aperiodic. but the chain isn't irreducible, because we have 2 communication classes. So the chain isn't regular.

   c) The chain has a unique positive recurrent class. Thus, there exists a corresponding unique stationary distribution π. π is concentrated on the positive recurrent class [0], $\pi(2) = \pi(3) = \pi(5) = 0$ for the transient states. To find $\pi(0)$, $\pi(1)$, $\pi(4)$, we solve $\pi = \pi$P subject to $\pi(0) + \pi(1) + \pi(4) = 1$:

$$\pi(0) = \rho_{00}\pi(0) + \rho_{10}\pi(1) + \rho_{40}\pi(4) = \frac{1}{2}\pi(4)$$

$$\pi(1) = \rho_{01}\pi(0) + \rho_{11}\pi(1) + \rho_{41}\pi(4) = \pi(0) + \frac{1}{2}\pi(4)$$

$$\pi(4) = \rho_{04}\pi(0) + \rho_{14}\pi(1) + \rho_{44}\pi(4) = \pi(1)$$

$$1 = \pi(0) + \pi(1) + \pi(4)$$

   Finally, $\pi = (\frac{1}{5}, \frac{2}{5}, 0, 0, \frac{2}{5}, 0)$

   d) For $n \to \infty$: $(N_n(0)/n, N_n(1)/n, N_n(2)/n, N_n(3)/n, N_n(4)/n; N_n(5)/n) \approx \pi = (\frac{1}{5}, \frac{2}{5}, 0, 0, \frac{2}{5}, 0)$

4. **a)** Let X_1 be the arrival time in **minutes** of the first taxi, then X_1 is an exponential random variable with parameter $\lambda_1 = \frac{12}{60} = \frac{1}{5}$, and X_2 be the arrival time in **minutes** of the first bus then X_2 is an exponential random variable with parameter $\lambda_2 = \frac{6}{60} = \frac{1}{10}$. Your waiting time W is equal to $min\{X_1, X_2\}$. We want to find the distribution of W:

$$\begin{aligned}
\mathbb{P}(W > x) &= \mathbb{P}(min\{X_1, X_2\} > x) \\
&= \mathbb{P}(X_1, X_2 > x) \\
&= \mathbb{P}(X_1 > x)\mathbb{P}(X_2 > x) \\
&= e^{-\lambda_1 x}e^{-\lambda_2 x} \\
&= e^{-(\lambda_1+\lambda_2)x}
\end{aligned}$$

   So $W \sim exp(\lambda_1 + \lambda_2)$

   $\lambda_1 + \lambda_2 = \frac{1}{5} + \frac{1}{10} = 0.3$

   b) By using part(a):

$$\mathbb{P}(W > 10) = \int_{10}^{\infty} 0.3e^{-0.3x}dx = e^{-3} \simeq 0.04978$$

5. **a)** $\mathbb{P}(N(3) = 5|N(7) = 2) = 0$ because $7 > 3$ and $N(7) < N(3)$.

   $\mathbb{E}[S_k] = \sum_{i=2}^{k} \mathbb{E}[S_i - S_{i-1}] + \mathbb{E}[S_1]$

   Note that : $S_i - S_{i-1} \sim exp(\lambda)$. So:

   $\mathbb{E}[S_k] = (k-1)\frac{1}{\lambda} + \frac{1}{\lambda} = \frac{k}{\lambda}$.

   b) We make use of the **fact** that given the number of events between $[0, T]$ for a Poisson process, $N(T) = k$, the arrival times $S_1, S_2, \cdots, S_k$ are distributed as order statistic of a $Unif(0, T)$ distribution.Suppose passenger i arrives at the station at time S_i, his waiting time is then $W_i = T - S_i$. Therefore,

$$W = \sum_{i=1}^{N(T)} (T - S_i)$$

   is a compound Poisson process. Thus

$$\begin{aligned}
\mathbb{E}[W] &= \mathbb{E}[T - S_1 + T - S_2 + \cdots + T - S_N(T)] \\
&= \mathbb{E}[N(T)]\mathbb{E}[T - S_1] \\
&= \lambda T\mathbb{E}[T - S_1]
\end{aligned}$$

Using the **fact** , we have

$$\mathbb{E}[T-S_1] = \int_0^T (T-s_1)\frac{1}{T}ds_1 = T - \frac{1}{T}\int_0^T s_1 ds_1 = T - \frac{T}{2} = \frac{T}{2}$$

Therefore, $\mathbb{E} = \lambda T \cdot \frac{T}{2} = \frac{\lambda T^2}{2}$.

6. Let $S = \{0,1,\ldots,d\}$ or $S = \{0,1,2,\ldots\}$. By a birth and death process on S we mean a Markov pure jump process on S having infinitesimal parameters q_{xy} such that

$$q_{xy} = 0, \quad |y-x| > 1.$$

Thus a birth and death process starting at x can in one jump go only to the states $x-1$ or $x+1$. The parameters $\lambda_x = q_{x,x+1}, x \in S$, and $\mu_x = q_{x,x-1}, x \in S$, are called respectively the birth rates and death rates of the process. It can be shown, an irreducible birth and death process on $\{0,1,2,\ldots\}$ is transient if and only if

$$\sum_{x=1}^{\infty} \frac{\mu_1 \cdots \mu_x}{\lambda_1 \cdots \lambda_x} < \infty \tag{1}$$

holds, positive recurrent if and only if

$$\sum_{x=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{x-1}}{\mu_1 \cdots \mu_x} < \infty \tag{2}$$

holds, and null recurrent if and only if (1) and (2) each fail to hold.

a) Note that,

$$\sum_{x=1}^{\infty} \frac{\mu_1 \cdots \mu_x}{\lambda_1 \cdots \lambda_x} = \sum_{x=1}^{\infty} \frac{\prod_{k=1}^{x} k}{\prod_{k=2}^{x+1} k} = \sum_{x=1}^{\infty} \frac{1}{x+1}$$

diverges and

$$\sum_{x=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{x-1}}{\mu_1 \cdots \mu_x} = \sum_{x=1}^{\infty} \frac{\prod_{k=1}^{x} k}{\prod_{k=1}^{x} k} = \sum_{x=1}^{\infty} 1$$

also diverges, hence the process is null recurrent.

b) We have,

$$\sum_{x=1}^{\infty} \frac{\mu_1 \cdots \mu_x}{\lambda_1 \cdots \lambda_x} = \sum_{x=1}^{\infty} \frac{\prod_{k=1}^{x} k}{\prod_{k=3}^{x+2} k} = \sum_{x=1}^{\infty} \frac{1}{(x+1)(x+2)} = \sum_{x=1}^{\infty} \left(\frac{1}{x+1} - \frac{1}{x+2}\right)$$

which is a telescoping series and converges to $1/2$. Hence the process is transient.