Mathematical Statistics I
Assignment 5 - Questions

Due by May 22, 2024

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution with mean θ. Find the conditional expectation $E\left(X_1 + 2X_2 + 3X_3 \mid \sum_i X_i\right)$.

2. Suppose that n observations are taken from $N(\mu, 1)$ with an unknown μ. Instead of recording all the observations, one records only whether the observation is less than 0. Find an MLE of μ.

3. If X_1, X_2 is a random sample of size 2 from a distribution having pdf $f(x;\theta) = (1/\theta)e^{-x/\theta} I_{(0,\infty)}(x)$. Find the joint pdf of the sufficient statistic $Y_1 = X_1 + X_2$ for θ and $Y_2 = X_2$. Show that Y_2 is an unbiased estimator of θ with variance θ^2. Find $E\left(Y_2 \mid Y_1 = y_1\right) = \varphi\left(y_1\right)$ and the variance of $\varphi\left(Y_1\right)$.

4. Let $f(x, y) = \left(2/\theta^2\right) e^{-(x+y)/\theta}, 0 < x < y < \infty$, zero elsewhere, be the joint pdf of the random variables X and Y.
   (a) Show that the mean and the variance of Y are, respectively, $3\theta/2$ and $5\theta^2/4$.
   (b) Show that $E(Y \mid X = x) = x + \theta$. In accordance with the theory, the expected value of $X + \theta$ is that of Y, namely, $3\theta/2$, and the variance of $X + \theta$ is less than that of Y. Show that the variance of $X + \theta$ is in fact $\theta^2/4$.

5. Let $(X_1, \ldots, X_n)$ be a random sample from a population with discrete probability density $\left[x!\left(1 - e^{-\theta}\right)\right]^{-1} \theta^x e^{-\theta} I_{\{1,2,\ldots\}}(x)$, where $\theta > 0$ is unknown. Show that the likelihood equation has a unique root when the sample mean $\overline{X} > 1$. Show whether this root is an MLE of θ.