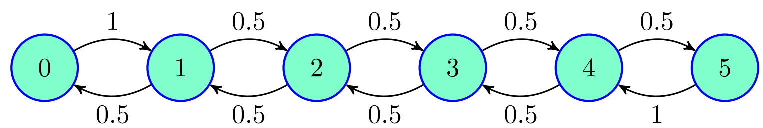
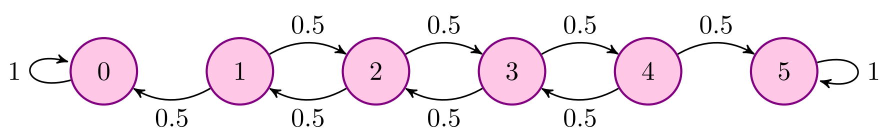
Stochastic Processes
Assignment 2 - Solutions

1. a) The probability that two pennies match or don't match is $\frac{1}{2}$. Let x be the number of pennies that A has. Then with probability $\frac{1}{2}$ he will next have x + 1 pennies or with probability $\frac{1}{2}$ he will next have x - 1 pennies. The exception is when x = 0, in which case, he gets, for free, a penny from B and he next has 1 penny. Also, if x = 5 he gives a penny to B and he next has 4 pennies. Thus:



b) To do this, modify the chain and make it stop once one of the players loses his pennies. After all, we are NOT interested in the behaviour of the chain after this time. The modification is an absorbing chain:



Let $\rho_{10}(i)$ be the probability that player A with i dollars, reaches state 0 before 1.

$$\rho_{10}(i)= \mathbb{P}_i(\text{hit 0 before 1})$$

We want to compute the absorbing probability $\rho_{10}(2)$ (for brevity write $\rho_{10}(i)= \rho(i)$) .Using first-step analysis:

$$\rho(0) = 1$$

$$\rho(1) = \frac{1}{2}\rho(0) + \frac{1}{2}\rho(2)$$

$$\rho(2) = \frac{1}{2}\rho(1) + \frac{1}{2}\rho(3)$$

$$\rho(3) = \frac{1}{2}\rho(2) + \frac{1}{2}\rho(4)$$

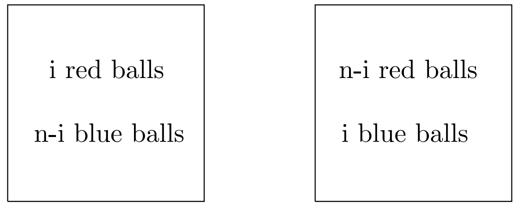
$$\rho(5) = 0$$

Solve and find:

$$\begin{aligned}\rho &= (\rho(0), \rho(1), \rho(2), \rho(3), \rho(4), \rho(5)) \\ &= (1, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, 0)\end{aligned}$$

So $\rho(2)= \frac{3}{5}$.

2. Consider:

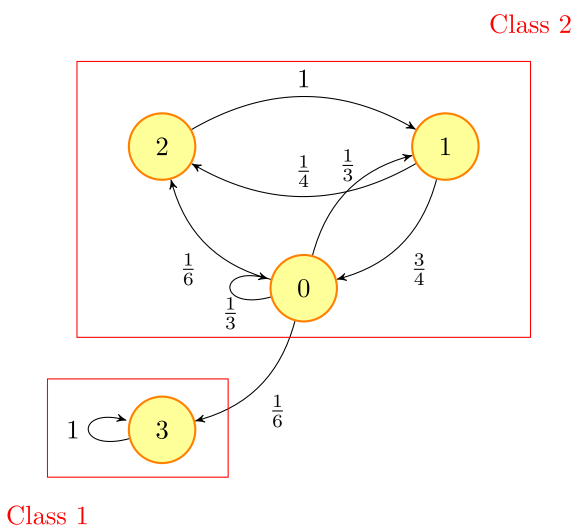


first box second box

$$\rho_{ij} = \mathbb{P}(X_{m+1} = j | X_m = i) = \begin{cases} (\frac{n-i}{n})^2 & j = i+1 \\ 2(\frac{i}{n})(\frac{n-i}{n}) & j = i \\ (\frac{i}{n})^2 & j = i-1 \\ 0 & o.w \end{cases} \qquad \rho_{0j} = \begin{cases} 1 & j = i+1 \\ 0 & o.w \end{cases} \qquad \rho_{nj} = \begin{cases} 1 & j = n-1 \\ 0 & o.w \end{cases}$$

3. **a)**

The accessibility diagram:



So there are two distinct communication classes: [3], [0]=[1]=[2]={0,1,2} . Clearly, State 3 is an absorbing state, hence it is recurrent .For another class, we need to compute one of the ρ_{00}, ρ_{11}, ρ_{22} .

$$\begin{aligned}
\rho_{00} &= \sum_{n=1}^{\infty} \rho_{00}^n \\
&= \frac{1}{3} + (\frac{1}{3} \cdot \frac{3}{4}) + (\frac{1}{6} \cdot 1 \cdot \frac{3}{4}) + (\frac{1}{3} \cdot \frac{1}{4} \cdot 1 \cdot \frac{3}{4}) + (\frac{1}{6} \cdot 1 \cdot \frac{1}{4} \cdot 1 \cdot \frac{3}{4}) + (\frac{1}{3} \cdot \frac{1}{4} \cdot 1 \cdot \frac{1}{4} \cdot 1 \cdot \frac{3}{4}) + (\frac{1}{6} \cdot 1 \cdot \frac{1}{4} \cdot 1 \cdot \frac{1}{4} \cdot 1 \cdot \frac{3}{4}) + \cdots \\
&= \frac{1}{3} + \frac{1}{3} \cdot \frac{3}{4} \sum_{k=0}^{\infty} (\frac{1}{4})^k + \frac{1}{6} \cdot \frac{3}{4} \sum_{k=0}^{\infty} (\frac{1}{4})^k \\
&= \frac{1}{3} + (\frac{1}{4} + \frac{1}{8}) \cdot \frac{4}{3} \\
&= \frac{5}{6}
\end{aligned}$$

$\rho_{00} = \frac{5}{6} < 1$,Hence [0] is a transient state. So {1,2} are transient too (according to the communication classes property).

b) We want to find the expected time until MC reaches state 3.

$$\mathbb{E}[T_3] = \mathbb{E}[T_3|X_0 = 0] \cdot \mathbb{P}(X_0 = 0) + \mathbb{E}[T_3|X_0 = 1] \cdot \mathbb{P}(X_0 = 1) + \mathbb{E}[T_3|X_0 = 2] \cdot \mathbb{P}(X_0 = 2) + \mathbb{E}[T_3|X_0 = 3] \cdot \mathbb{P}(X_0 = 3)$$

$$= \mu \cdot \pi_0(0) + \nu \cdot \pi_0(1) + \gamma \cdot \pi_0(2) + 0$$

To compute μ , ν , γ :

$$\begin{aligned}
\mu &= \sum_{i=0}^{3} \mathbb{E}_0[T_3|X_1 = i] \cdot \rho_{0i} \\
&= \mathbb{E}_0[T_3|X_1 = 0] \cdot \rho_{00} + \mathbb{E}_0[T_3|X_1 = 1] \cdot \rho_{01} + \mathbb{E}_0[T_3|X_1 = 2] \cdot \rho_{02} + \mathbb{E}_0[T_3|X_1 = 3] \cdot \rho_{03} \\
&= (1+\mu) \cdot \frac{1}{3} + (1+\nu) \cdot \frac{1}{3} + (1+\gamma) \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} \\
&= \frac{2\mu + 2\nu + \gamma + 6}{6} \qquad (1)
\end{aligned}$$

$$\begin{aligned}
\nu &= \sum_{i=0}^{3} \mathbb{E}_1[T_3|X_1 = i] \cdot \rho_{1i} \\
&= \mathbb{E}_1[T_3|X_1 = 0] \cdot \rho_{10} + \mathbb{E}_1[T_3|X_1 = 1] \cdot \rho_{11} + \mathbb{E}_1[T_3|X_1 = 2] \cdot \rho_{12} + \mathbb{E}_1[T_3|X_1 = 3] \cdot \rho_{13} \\
&= (1+\mu) \cdot \frac{3}{4} + 0 + (1+\gamma) \cdot \frac{1}{4} + 0 \\
&= \frac{3\mu + \gamma + 4}{4} \qquad (2)
\end{aligned}$$

$$\begin{aligned}
\gamma &= \sum_{i=0}^{3} \mathbb{E}_2[T_3|X_1 = i] \cdot \rho_{2i} \\
&= \mathbb{E}_2[T_3|X_1 = 0] \cdot \rho_{20} + \mathbb{E}_2[T_3|X_1 = 1] \cdot \rho_{21} + \mathbb{E}_2[T_3|X_1 = 2] \cdot \rho_{22} + \mathbb{E}_2[T_3|X_1 = 3] \cdot \rho_{23} \\
&= 0 + (1+\nu) \cdot 1 + 0 + 0 \\
&= 1 + \nu \qquad (3)
\end{aligned}$$

By solving (1) , (2) , (3) find:

$$\mu = 12,$$

$$\nu = \frac{41}{3},$$

$$\gamma = \frac{44}{3}$$

Finally:

$$\begin{aligned}
\mathbb{E}[T_3] &= \mu \cdot \pi_0(0) + \nu \cdot \pi_0(1) + \gamma \cdot \pi_0(2) \\
&= 12\pi_0(0) + \frac{41}{3}\pi_0(1) + \frac{44}{3}\pi_0(2)
\end{aligned}$$

c)

$$\mathbb{P}_1(X_2 = 3) = \sum_{i=0}^{3} \rho_{1i} \cdot \rho_{i3}$$

$$= \rho_{10} \cdot \rho_{03} + \rho_{11} \cdot \rho_{13} + \rho_{12} \cdot \rho_{23} + \rho_{13} \cdot \rho_{33}$$

$$= \frac{3}{4} \cdot \frac{1}{6} + 0 + 0 + 0$$

$$= \frac{1}{8}$$

4. Consider the Markov chain X_n with S={0,1} (State 0: Sunny , State 1:Rainy). So the transition matrix is :

$$\text{P} = \begin{pmatrix} \frac{9}{10} & \frac{1}{10} \\ \frac{75}{100} & \frac{25}{100} \end{pmatrix}$$

We want to find $\mathbb{P}(X_1 = 0|X_0 = 0, X_2 = 0)$. Know that:

$$\mathbb{P}(X_1 = 0|X_0 = 0, X_2 = 0) = \frac{\mathbb{P}(X_0 = 0, X_1 = 0, X_2 = 0)}{\mathbb{P}(X_0 = 0, X_2 = 0)}$$

First compute:

$$\mathbb{P}(X_1 = 0, X_0 = 0, X_2 = 0) = \mathbb{P}(X_0 = 0, X_1 = 0, X_2 = 0)$$

$$= \mathbb{P}(X_2 = 0|X_1 = 0, X_0 = 0)\mathbb{P}(X_1 = 0|X_0 = 0)\mathbb{P}(X_0 = 0)$$

$$= \mathbb{P}(X_2 = 0|X_1 = 0)\mathbb{P}(X_1 = 0|X_0 = 0)\mathbb{P}(X_0 = 0) \qquad [Markov Property]$$

$$= \rho_{00}\rho_{00}\pi_0(0) = \frac{9}{10} \cdot \frac{9}{10}\pi_0(0)$$

And ;

$$\mathbb{P}(X_0{=}0\ ,\ X_2{=}0) = \mathbb{P}(X_0{=}0\ ,\ X_1{=}0\ ,\ X_2{=}0) + \mathbb{P}(X_0{=}0\ ,\ X_1{=}1\ ,\ X_2{=}0)$$

Similarly:

$$\mathbb{P}(X_0 = 0, X_1 = 1, X_2 = 0) = \mathbb{P}(X_0 = 0, X_1 = 0, X_2 = 0) + \mathbb{P}(X_0 = 0, X_1 = 1, X_2 = 0)$$

$$= \mathbb{P}(X_2 = 0|X_1 = 1)\mathbb{P}(X_1 = 1|X_0 = 0)\mathbb{P}(X_0 = 0)$$

$$= \rho_{10}\rho_{01}\pi_0(0) = \frac{75}{100} \cdot \frac{1}{10}\pi_0(0)$$

So:

$$\mathbb{P}(X_0{=}0\ ,\ X_2{=}0) = 0.81\ \pi_0(0) + 0.075\ \pi_0(0) = 0.885\pi_0(0)$$

Finally:

$$\mathbb{P}(X_1 = 0|X_1 = 1, X_2 = 0) = \frac{\mathbb{P}(X_0 = 0, X_1 = 0, X_2 = 0)}{\mathbb{P}(X_0 = 0, X_2 = 0)}$$

$$= \frac{0.81\pi_0(0)}{0.885\pi_0(0)} \simeq 0.92$$

5.

$$P = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

If A={1,2} and X_0=0 :

a) We want to find the distribution of T_A, suppose T_A=1 :

$$\mathbb{P}(T_A = 1) = \mathbb{P}(X_1 \in A | X_0 = 0)$$

$$= \mathbb{P}(X_1 = 1 | X_0 = 0) + \mathbb{P}(X_1 = 2 | X_0 = 0)$$

$$= 0 + \frac{1}{2} = \frac{1}{2}$$

Now suppose T_A=2 :

$$\mathbb{P}(T_A = 2) = \mathbb{P}(X_2 \in A, X_1 = 0 | X_0 = 0)$$

$$= \mathbb{P}(X_2 \in A | X_1 = 0, X_0 = 0) \cdot \mathbb{P}(X_1 = 0 | X_0 = 0)$$

$$= \mathbb{P}(X_2 \in A | X_1 = 0) \cdot \mathbb{P}(X_1 = 0 | X_0 = 0) \qquad [Markov Property]$$

$$= \mathbb{P}(X_2 = 1 | X_1 = 0) \cdot \mathbb{P}(X_1 = 0 | X_0 = 0) + \mathbb{P}(X_2 = 2 | X_1 = 0) \cdot \mathbb{P}(X_1 = 0 | X_0 = 0)$$

$$= 0 + \frac{1}{2} \cdot \frac{1}{2} = (\frac{1}{2})^2$$

Similarly , for T_A=k:

$$\mathbb{P}(T_A = k) = \mathbb{P}(X_k \in A, X_{k-1} = 0, X_{k-2} = 0, \cdots, X_1 = 0 | X_0 = 0)$$

$$= \mathbb{P}(X_k \in A | X_{k-1} = 0) \cdot \mathbb{P}(X_{k-1} = 0 | X_{k-2} = 0) \cdots \mathbb{P}(X_1 = 0 | X_0 = 0)$$

$$= [\mathbb{P}(X_k = 1 | X_{k-1} = 0) + \mathbb{P}(X_k = 2 | X_{k-1} = 0)] \cdot (\rho_{00})^{k-1}$$

$$= [0 + \rho_{02}] \cdot (\rho_{00})^{k-1}$$

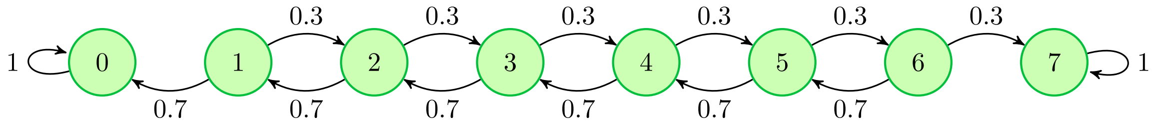
$$= \frac{1}{2} \cdot (\frac{1}{2})^{k-1} = (\frac{1}{2})^k$$

So T_A has a geometric distribution with parameter p=$\frac{1}{2}$.

b) In part(a) we show that T_A has a geometric distribution with p=$\frac{1}{2}$, hence $\mathbb{E}[T_A] = 2$.

6.

a) The Markov chain (X_n,n = 0,1,...) representing the evolution of Charlie's money has diagram :



Let $\rho(i)$ be the probability that the chain reaches state 7 before reaching state 0, starting from state i, In other words, if S_j is the first n $\geq$ 0 such that X_n = j,

$$\varphi(\mathbf{i}) = \mathbb{P}_i(S_7 < S_0) = \mathbb{P}(S_7 < S_0 | X_0 = i)$$

Using first-step analysis, we have:

$$\varphi(i) = 0.3\varphi(i+1) + 0.7\varphi(i-1), \qquad i = 1,2,3,4,5,6$$

$$\varphi(0) = 0$$

$$\varphi(7) = 1$$

We want to find $\varphi(2)$:

$$\varphi(1) = 0.3\varphi(2) + 0.7\varphi(0) = 0.3\varphi(2)$$

$$\varphi(2) = 0.3\varphi(3) + 0.7\varphi(1) = 0.3\varphi(3) + 0.7 \cdot 0.300\varphi(2) \Rightarrow \varphi(2) = \frac{0.3}{0.79}\varphi(3) \simeq 0.381\varphi(3)$$

$$\varphi(3) = 0.3\varphi(4) + 0.7\varphi(2) = 0.3\varphi(4) + 0.7 \cdot 0.381\varphi(3) \Rightarrow \varphi(3) = \frac{0.3}{0.733}\varphi(4) \simeq 0.41\varphi(4)$$

$$\varphi(4) = 0.3\varphi(5) + 0.7\varphi(3) = 0.3\varphi(5) + 0.7 \cdot 0.410\varphi(4) \Rightarrow \varphi(4) = \frac{0.3}{0.713}\varphi(5) \simeq 0.421\varphi(5)$$

$$\varphi(5) = 0.3\varphi(6) + 0.7\varphi(4) = 0.3\varphi(6) + 0.7 \cdot 0.421\varphi(5) \Rightarrow \varphi(5) = \frac{0.3}{0.705}\varphi(6) \simeq 0.425\varphi(6)$$

$$\varphi(6) = 0.3\varphi(7) + 0.7\varphi(5) = 0.3 \cdot 1 + 0.7 \cdot 0.425\varphi(6) \Rightarrow \varphi(6) = \frac{0.3}{0.703} \simeq 0.43$$

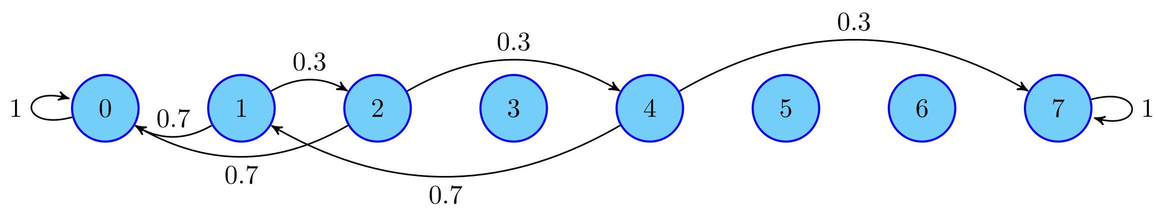
Finally:

$$\varphi = (\varphi(0), \varphi(1), \varphi(2), \varphi(3), \varphi(4), \varphi(5), \varphi(6), \varphi(7))$$

$$= (0, 0.0036, 0.0122, 0.032, 0.077, 0.183, 0.43, 1)$$

So $\varphi(2) = 0.0122$

b) Now the chain is:



The equations are:

$$\varphi(0) = 0$$

$$\varphi(7) = 1$$

$$\varphi(1) = 0.3\varphi(2) + 0.7\varphi(0) = 0.3\varphi(2)$$

$$\varphi(2) = 0.3\varphi(4) + 0.7\varphi(0) = 0.3\varphi(4)$$

$$\varphi(4) = 0.3\varphi(7) + 0.7\varphi(1) = 0.3 + 0.7\varphi(1)$$

We solve and find:

$$\varphi(1)= 0.0288\ , \quad \varphi(2)= 0.096\ , \quad \varphi(4)=0.32$$

So $\varphi(2) = 0.096$

c) By comparing the $\varphi(2)$ we find that the **second strategy** gives Charlie a better chance to be free .