Stochastic Processes
Assignment 1 - Questions

Due on February 28, 2023

1. Let U_1, U_2, and U_3 be independent random variables uniform on $[0,1]$. Find the probability that the roots of the quadratic $U_1x^2 + U_2x + U_3$ are real.

2. Let
$$f(x,y) = c\left(x^2 - y^2\right)e^{-x}, \quad 0 \le x < \infty, \quad -x \le y < x$$
a. Find c.
b. Find the marginal densities (i.e. $f_X(x)$ and $f_Y(y)$.)
c. Find the conditional densities (i.e. $f_{X|Y=y}(x)$ and $f_{Y|X=x}(y)$.)

3. Let T be an exponential random variable, and conditional on T, let U be uniform on $[0,T]$. Find the unconditional mean and variance of U.

4. If X is a non-negative integer-valued random variable, the probability-generating function of X is defined to be
$$G(s) = \sum_{k=0}^{\infty} s^k \; \mathrm{P}(X = k).$$
a. Show that
$$\mathrm{P}(X = k) = \frac{1}{k!}\frac{d^k}{ds^k}G(s)\bigg|_{s=0}$$
b. Show that
$$\frac{dG}{ds}\bigg|_{s=1} = E(X)$$
$$\frac{d^2G}{ds^2}\bigg|_{s=1} = E[X(X-1)]$$
c. Express the probability-generating function in terms of moment-generating function.
d. Find the probability-generating function of the Poisson distribution.

5. At a party n men throw their hats into the center of a room. The hats are mixed up and each man randomly selects one.
a. Find the expected number of men who select their own hats.
b. Show that the variance of the number of men who select their own hats is 1.
c. Find the conditional expected number of matches given that the first person did not have a match.

6. Suppose the random variable (X, Y, Z) has the following moment-generating function,
$$M_{(X,Y,Z)}(t,u,v) = \frac{e^{t+t^2+2u^2}}{1-v}, \qquad t,u \in \mathbb{R}, \quad v < 1$$
a. Show that X, Y and Z are independent.
b. Find $\mathrm{E}\left[e^{2X}(Y^2 + Z)\right]$