Mathematical Statistics I
Recitation Session 5

Definition. Suppose that random variables $X_1, \ldots, X_n$ have a joint density or frequency function $f(x_1, x_2, \ldots, x_n; \theta)$. Given observed values $X_i = x_i$, where $i = 1, \ldots, n$, the likelihood of θ as a function of $x_1, x_2, \ldots, x_n$ is defined as

$$L(\theta; x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n; \theta)$$

Note that we consider the joint density as a function of θ rather than as a function of the x_i.

Definition. For each sample point $\mathbf{x}$, let $\hat{\theta}(\mathbf{x})$ be a parameter value at which $L(\theta; \mathbf{x})$ attains its maximum as a function of θ, with $\mathbf{x}$ held fixed. A maximum likelihood estimator (MLE) of the parameter θ based on a sample $\mathbf{X}$ is $\hat{\theta}(\mathbf{X})$.

Theorem. *(Invariance property of MLEs) If $\hat{\theta}$ is the MLE of θ, then for any function $\tau(\theta)$, the MLE of $\tau(\theta)$ is $\tau(\hat{\theta})$.*

Exercise. Let $X_1, \ldots, X_n$ be iid with one of two pdfs. If $\theta = 0$, then

$$f(x; \theta) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

while if $\theta = 1$, then

$$f(x; \theta) = \begin{cases} 1/(2\sqrt{x}) & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the MLE of θ.

Exercise. Let $X_1, \ldots, X_n$ be iid with pdf

$$f(x; \theta) = \frac{1}{\theta}, \quad 0 \leq x \leq \theta, \quad \theta > 0.$$

Estimate θ using both the method of moments and maximum likelihood. Calculate the means and variances of the two estimators. Which one should be preferred and why?

Exercise. The independent random variables $X_1, \ldots, X_n$ have the common distribution

$$F(x; \alpha, \beta) = \begin{cases} 0 & \text{if } x < 0 \\ (x/\beta)^\alpha & \text{if } 0 \leq x \leq \beta \\ 1 & \text{if } x > \beta, \end{cases}$$

where the parameters α and β are positive.
(a) Find a two-dimensional sufficient statistic for (α, β).
(b) Find the MLEs of α and β.

Exercise. Let $X = (X_1, \ldots, X_n)$ be a random sample of random variables with probability density f_θ. Find an MLE (maximum likelihood estimator) of θ in each of the following cases.
(i) $f_\theta(x) = \theta^{-1} I_{\{1,\ldots,\theta\}}(x), \theta$ is an integer between 1 and θ_0.
(ii) $f_\theta(x) = e^{-(x-\theta)} I_{(\theta,\infty)}(x), \theta > 0$.
(iii) $f_\theta(x) = \theta(1-x)^{\theta-1} I_{(0,1)}(x), \theta > 1$.
(iv) $f_\theta(x) = \frac{\theta}{1-\theta} x^{(2\theta-1)/(1-\theta)} I_{(0,1)}(x), \theta \in \left(\frac{1}{2}, 1\right)$.
(v) $f_\theta(x) = \theta^x (1-\theta)^{1-x} I_{\{0,1\}}(x), \theta \in \left[\frac{1}{2}, \frac{3}{4}\right]$.