Stochastic Processes
Assignment 3 - Questions

Due on April 8, 2023

1. Let $X_n, n \geq 0$, be a Markov chain whose state space $\mathscr{S}$ is a subset of $\{0, 1, 2, \ldots\}$ and whose transition function P is such that

$$\sum_y yP(x, y) = Ax + B, \quad x \in \mathscr{S}$$

for some constants A and B.
(a) Show that $E(X_{n+1}) = AE(X_n) + B$.
(b) Show that if $A \neq 1$, then

$$E(X_n) = \frac{B}{1-A} + A^n \left(E(X_0) - \frac{B}{1-A}\right)$$

(c) Let $X_n, n \geq 0$, be the Ehrenfest chain on $\{0, 1, \ldots, d\}$. Show that the assumption of the Exercise holds and use (c) to compute $E_x(X_n)$

2. Consider a Markov chain on the nonnegative integers such that, starting from x, the chain goes to state $x + 1$ with probability p, $0 < p < 1$, and goes to state 0 with probability $1 - p$.
(a) Show that this chain is irreducible.
(b) Find $P_0(T_0 = n), n \geq 1$.
(c) Show that the chain is recurrent.

3. Consider a Markov chain having state space $\{0, 1, \ldots, 6\}$ and transition matrix

$$\begin{array}{c c}
\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\
\begin{pmatrix}
1/2 & 0 & 1/8 & 1/4 & 1/8 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 \\
0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/2 & 1/2
\end{pmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\end{array}$$

(a) Determine which states are transient and which states are recurrent.
(b) Find $\rho_{0y}, y = 0, \ldots, 6$

4. Consider a birth and death chain on the nonnegative integers or on the finite set $\mathscr{S} = \{0, \ldots, d\}$. In the former case we set $d = \infty$. The transition function is of the form

$$P(x, y) = \begin{cases} q_x, & y = x - 1, \\ r_x, & y = x, \\ p_x, & y = x + 1, \end{cases}$$

where $p_x + q_x + r_x = 1$ for $x \in \mathscr{S}, q_0 = 0$, and $p_d = 0$ if $d < \infty$. We assume additionally that p_x and q_x are positive for $0 < x < d$. For a and b in $\mathscr{S}$ such that $a < b$, set

$$u(x) = P_x\left(T_a < T_b\right), \quad a < x < b$$

and set $u(a) = 1$ and $u(b) = 0$. Show that

$$P_x\left(T_a < T_b\right) = \frac{\sum_{y=x}^{b-1} \gamma_y}{\sum_{y=a}^{b-1} \gamma_y}, \quad a < x < b$$

where $\gamma_0 = 1$ and

$$\gamma_y = \prod_{k=1}^{y} \frac{q_k}{p_k}, \quad 0 < y < d.$$

5. A gambler playing roulette makes a series of one dollar bets. He has respective probabilities 9/19 and 10/19 of winning and losing each bet. The gambler decides to quit playing as soon as he either is one dollar ahead or has lost his initial capital of \$1000.
(a) Find the probability that when he quits playing he will have lost \$1000. *Hint*: Use Exercise 4
(b) Find his expected loss.

6. Let X_n, $n \geq 0$ be a branching process where the off-spring distribution is Poisson with parameter $\lambda = 2$. Compute the following quantities:
(a) $P_1(X_1 = 0)$,
(b) $E_1(X_1)$
(c) $E_1(X_1 X_2)$.