1) Mathematical warm up

1.1) Matrices

1) $|A| = 1\times 2 - (-1)\times 1 = 3$ $\qquad A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ -1/3 & 1/3 \end{bmatrix}$

$|B| = 2\times(-2) - 1\times 1 = -5$ $\qquad B^{-1} = \frac{1}{-5}\begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2/5 & 1/5 \\ 1/5 & -2/5 \end{bmatrix}$

2) $AB = \begin{bmatrix} 2-1 & 1+2 \\ 2+2 & 1-4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix}$

$|AB| = 1\times(-3) - 3\times 4 = -15$ $\qquad$ (1) $(AB)^{-1} = \frac{1}{-15}\begin{bmatrix} -3 & -3 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 1/5 & 1/5 \\ 4/15 & -1/15 \end{bmatrix}$

(2) $B^{-1}A^{-1} = \begin{bmatrix} 2/5\times 2/3 + 1/5\times(-1/3) & 2/5\times 1/3 + 1/5\times \frac{1}{3} \\ 1/5\times 2/3 + (-2/5)\times(-1/3) & 1/5\times 1/3 + (-2/5)\times 1/3 \end{bmatrix} = \begin{bmatrix} 1/5 & 1/5 \\ 4/15 & -1/15 \end{bmatrix}$

(1) and (2) $\Longrightarrow (AB)^{-1} = B^{-1}A^{-1}$

3) $(AB)^t = \begin{bmatrix} 1 & 4 \\ 3 & -3 \end{bmatrix}$ (1) $\qquad A^t = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \qquad B^t = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

$B^t A^t = \begin{bmatrix} 2-1 & 2+2 \\ 1+2 & 1-4 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 3 & -3 \end{bmatrix}$ (2)

(1) and (2) $\Longrightarrow (AB)^t = B^t A^t$

4) $D-\lambda I = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ 2 & 0 & 3-\lambda \end{bmatrix}$

$\det(D-\lambda I) = +(1-\lambda)\Big((2-\lambda)(3-\lambda)-0\Big) - 0 + 0$

$= (1-\lambda)(2-\lambda)(3-\lambda)$

$\det(D-\lambda I) = 0$

$(1-\lambda)(2-\lambda)(3-\lambda) = 0 \Longrightarrow \begin{cases} \lambda = 1 \\ \lambda = 2 \\ \lambda = 3 \end{cases}$

5) $(D-\lambda I)V_\lambda = 0$ $\qquad \begin{bmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ 2 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

– In case $\lambda = 1$ : $\qquad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$\begin{cases} 0 = 0 \\ x + y = 0 \Rightarrow y = -x \\ 2x + 2z = 0 \Rightarrow z = -x \end{cases}$

$V_\lambda = \begin{bmatrix} x \\ -x \\ -x \end{bmatrix} \xrightarrow{\text{one special case}} V_\lambda = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

– In case $\lambda = 2$

$\begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$\begin{cases} -x = 0 \\ x = 0 \\ 2x + z = 0 \end{cases} \Longrightarrow \begin{cases} x = 0 \\ z = 0 \end{cases}$

$$V_\lambda = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} \xrightarrow{\text{one special case}} V_\lambda = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- in cas $\lambda = 3$:

$$\begin{bmatrix} -2 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{cases} -2x = 0 \\ x - y = 0 \\ 2x = 0 \end{cases} \Longrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

$$V_\lambda = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \xrightarrow{\text{one special case}} V_\lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

1.2) Calculus

1) $\nabla f_{(x,y)} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x - 4 \\ 2y + 6 \end{bmatrix}$

2) $\nabla f_{(x,y)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2x - 4 \\ 2y + 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{cases} 2x - 4 = 0 \Rightarrow x = 2 \\ 2y + 6 = 0 \Rightarrow y = -3 \end{cases}$$

the critical point for $f$: $(2, -3)$

$f(2,-3) = 4 + 9 - 8 + 18 + 13 = 0$

3) $f(x,y) = (x-2)^2 + (y+3)^2 \Rightarrow f(x,y) \geq 0$ (1)

$(2,-3)$ is the only critical point for $f(x,y)$. (2)

(1) and (2) $\Rightarrow$ $(2,-3)$ is a minimum and more precisely a Global minimum.

1.3) probabilities

1) – $p(\text{2nd year}) = \dfrac{300}{960}$

– $p(\text{girl and 1st year}) = \dfrac{176}{960}$

– $p(\text{3rd year}') = 1 - p(\text{3rd year}) = 1 - \dfrac{340}{960}$

2) A: girl B: 1st year

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{176/960}{320/960} = \frac{176}{320}$$

3) $$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)} = \frac{176/960}{576/960} = \frac{176}{576}$$

4) $$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B \cap A)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B)}$$

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$ Bayes' formula