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Pattern Recognition HW3

Python 3.5; Libraries: Numpy, Matplotlib; Documentation: HTML, Bootstrap 4, pylatex

Problem 1

a.

$$p_{x_i|\theta} = \left[\frac{1}{\theta}\right]^{x_i} \left[\frac{1}{\theta}\right]$$

$$\text{LHF} = \prod_{i=1}^{n} \left[\frac{1}{\theta}\right]^{x_i} \left[\frac{1}{\theta}\right]$$

$$\text{RR} = \text{argmin} = \left[\prod_{i=1}^{n} \left[\frac{1}{\theta}\right]^{x_i} \left[\frac{1}{\theta}\right]\right]$$

$$\text{RR} = \sum_{i=1}^{n} \left[\left[\frac{1}{\theta}\right]^{x_i} \left[\frac{1}{\theta}\right]\right]$$

$$\text{RR} = \sum_{i=1}^{n} \left[\left[\frac{1}{\theta}\right]^{x_i} + \text{argmin}\left[\frac{1}{\theta}\right]\right]$$

$$\text{RR} = \sum_{i=1}^{n} \text{argmin}\left[\frac{1}{\theta}\right] + \sum_{i=1}^{n} \left[\frac{1}{\theta}\right]^{x_i}$$

$$\text{RR} = \sum_{i=1}^{n} \text{argmin}\left[\frac{1}{\theta}\right] + \sum_{i=1}^{n} \left[\frac{1}{\theta}\right]^{\sum_{i=1}^{n} x_i}$$

$$\text{RR} = n \sum_{i=1}^{n} \text{argmin}\left[\frac{1}{\theta}\right] + \left[\frac{1}{\theta}\right]^{\sum_{i=1}^{n} x_i}$$

$$\frac{\partial \text{RR}}{\partial \theta} = -n\left[\frac{1}{\theta^2}\right] + \left[\frac{1}{\theta}\right]^{\sum_{i=1}^{n} x_i} \cdot -\sum_{i=1}^{n} x_i \cdot \frac{1}{\theta^2}$$

$$\frac{\partial \text{RR}}{\partial \theta} = -\frac{n}{\theta^2} + \frac{\sum_{i=1}^{n} x_i}{\theta^{\sum_{i=1}^{n} x_i + 2}}$$

$$\theta^{\sum_{i=1}^{n} x_i + 2} \cdot -\frac{n}{\theta^2} = -\sum_{i=1}^{n} x_i$$

$$\theta = \sqrt[\sum_{i=1}^{n} x_i]{\frac{n}{\sum_{i=1}^{n} x_i}}$$

b.

$$p_{x_i|\theta} = \frac{1}{\theta} e^{-\frac{x_i}{\theta}}$$

$$\text{LHF} = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{x_i}{\theta}}$$

$$\text{RR} = \text{argmin} = \left[\prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{x_i}{\theta}}\right]$$

$$\text{RR} = \sum_{i=1}^{n} \text{argmin}\left[\frac{1}{\theta} e^{-\frac{x_i}{\theta}}\right]$$

$$\text{RR} = \sum_{i=1}^{n} \text{argmin}\left[\frac{1}{\theta} e^{-\frac{x_i}{\theta}} + \text{argmin}\left[\frac{1}{\theta}\right]\right]$$

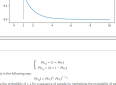
$$\text{RR} = n \cdot \text{argmin}\left[\frac{1}{\theta}\right] + \sum_{i=1}^{n} x_i$$

$$\frac{\partial \text{RR}}{\partial \theta} = -\frac{n}{\theta^2} + \sum_{i=1}^{n} x_i \cdot \frac{1}{\theta^2}$$

$$\theta^2 \cdot -\frac{n}{\theta^2} = -\sum_{i=1}^{n} x_i$$

$$\theta = \frac{\sum_{i=1}^{n} x_i}{n}$$

c. I ordinate samples where drawn from the distribution with $\theta = 1$ and the plot shows the result of the estimator (refer to Problem 3.1.a.)



As can be seen the estimation is quite close to 1.

Problem 3.3

a. We know that p_x is a beta variable it has beta α, β parameters.

$$\left(\frac{\alpha p_x + \beta - p_x}{\alpha + \beta - 2 p_x} \right)$$

So we can combine three probability like the Bernoulli density in the following way:

$$p_x \cdot p_x^{\alpha} \cdot (1-p_x)^{\beta} \cdot (1-p_x)^{-1}$$

And since each sample is drawn independently, we can derive the probability of p_x by multiplying the probability of each of them (exactly like equation 4 Bernoulli trial that leads to Binomial distribution)

$$\prod_{i=1}^{n} (p_x)^{\alpha} (1-p_x)^{-1}$$

b.

$$\text{LHF}(p) = \text{argmax}_{p_x} \{ \text{PR}(\theta)$$

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$$\text{LHF}(p) = \left[\prod_{i=1}^{n} (p_x)^{\alpha} (1-p_x)^{-1} \right]$$

$$\text{LHF}(p) = \sum_{i=1}^{n} \{ \alpha \log(p_x) - \log(1-p_x) \} + \text{PR}(\theta)$$

$$\text{LHF}(p) = \text{argmax}_{p_x} \{ \sum_{i=1}^{n} \alpha \log(p_x) - \log(1-p_x) \} + \text{PR}(\theta)$$

$$\frac{\partial \text{LHF}(p)}{\partial \text{Data}(p)} = \frac{\partial}{\partial p_x} \{ \sum_{i=1}^{n} \alpha \cdot \frac{1}{p_x} - \frac{1}{1-p_x} \} \cdot \sum_{i=1}^{n} (1-p_x) + 0$$

$$\sum_{i=1}^{n} \alpha \cdot \frac{1}{p_x} - \frac{1}{1-p_x}$$

$$\sum_{i=1}^{n} \alpha \cdot \frac{1}{p_x} - \frac{1}{1-p_x} \cdot \sum_{i=1}^{n} (1-p_x) = 0$$

$$\alpha = \frac{p_x}{1-p_x} \cdot \sum_{i=1}^{n} (1-p_x) = \sum_{i=1}^{n} p_x$$

$$\alpha = \text{PR}(p) \cdot \sum_{i=1}^{n} p_x$$

$$\text{PR}(p) = \frac{\alpha}{\sum_{i=1}^{n} p_x}$$

Problem 3.4

From least 5 of n samples, then the likelihood of θ is:

$$\text{LPRP}(\theta) = \left[\prod_{i=1}^{5} (\theta^{-1} \cdot e^{-\theta^{-1}} \right]$$

$$\text{SPRP}(\theta) = \left[\prod_{i=1}^{n} (\theta^{-1} \cdot e^{-\theta^{-1}} \right]$$

$$\text{SPRP}(\theta) = \sum_{i=1}^{n} \{ \alpha \cdot (-\text{ln}(\theta^{-1}) - \theta^{-1}) + \text{ln}(-\theta^{-1}) \}$$

$$\{ \alpha = 5, \theta \cdot \frac{\text{SPRP}(\theta)}{\partial \theta} = \sum_{i=1}^{n} \{ \alpha \cdot \frac{1}{\theta^{-1}} \cdot \frac{\theta^{-1}}{\theta^2} - \frac{1}{\theta^2} \} + \frac{1}{\theta^{-1}} \cdot \frac{1}{\theta^2} \}$$

$$\sum_{i=1}^{n} \{ \frac{\alpha}{\theta} - \frac{1}{\theta^2} \} + \frac{1}{\theta^2} = 0