Solution to 1. Convexity of the Exponent Region

Let

$$\mathcal{R}_n \;=\; \Big\{(\alpha,\beta) : \exists \text{ test } \phi_n \text{ s.t. } \pi_{1|0}(\phi_n) \le \alpha,\; \pi_{0|1}(\phi_n) \le \beta\Big\},$$

and define the exponent-region

$$\mathcal{E}_n \;=\; \Big\{(E_0,E_1) : \exists\, \phi_n \text{ s.t. } \pi_{1|0}(\phi_n) \le e^{-nE_0},\; \pi_{0|1}(\phi_n) \le e^{-nE_1}\Big\}.$$

We claim $\mathcal{E}_n$ is convex. Indeed, pick any two achievable pairs $(E_0^{(1)},E_1^{(1)})$ and $(E_0^{(2)},E_1^{(2)})$, realized by tests $\phi_n^{(1)}$ and $\phi_n^{(2)}$, respectively. For any $\lambda \in [0,1]$, form the randomized test

$$\phi_n = \begin{cases} \phi_n^{(1)}, & \text{with probability } \lambda, \\ \phi_n^{(2)}, & \text{with probability } 1-\lambda. \end{cases}$$

Then by the law of total probability,

$$\pi_{1|0}(\phi_n) = \;\lambda\,\pi_{1|0}(\phi_n^{(1)}) + (1-\lambda)\,\pi_{1|0}(\phi_n^{(2)}) \;\le\; \lambda\, e^{-nE_0^{(1)}} + (1-\lambda)\, e^{-nE_0^{(2)}},$$

and similarly for $\pi_{0|1}(\phi_n)$. Now observe the elementary inequality

$$\lambda\, e^{-nE_0^{(1)}} + (1-\lambda)\, e^{-nE_0^{(2)}} \;\le\; e^{-n\left(\lambda E_0^{(1)} + (1-\lambda)E_0^{(2)}\right)},$$

since the exponential is convex on $\mathbb{R}$. Hence

$$\pi_{1|0}(\phi_n) \le e^{-n\left(\lambda E_0^{(1)} + (1-\lambda)E_0^{(2)}\right)}, \quad \pi_{0|1}(\phi_n) \le e^{-n\left(\lambda E_1^{(1)} + (1-\lambda)E_1^{(2)}\right)},$$

showing that $(\lambda E_0^{(1)} + (1-\lambda)E_0^{(2)},\ \lambda E_1^{(1)} + (1-\lambda)E_1^{(2)}) \in \mathcal{E}_n$. This proves convexity.

Solution to 2. Exponential Bounds via the LLR Rule

Under $H_0 : P^n$ versus $H_1 : Q^n$, the log-likelihood ratio (LLR) for a sample X^n is

$$L_n(X^n) \;=\; \sum_{i=1}^{n} \log \frac{q(X_i)}{p(X_i)}.$$

The Neyman–Pearson test with threshold $\tau = n\,t$ is

$$\phi_n(x^n) = \begin{cases} 1, & L_n(x^n) \ge n\,t, \\ 0, & L_n(x^n) < n\,t. \end{cases}$$

Define the moment-generating-function (MGF)

$$M_P(\lambda) =_P \left[e^{\lambda \log \frac{q(X)}{p(X)}} \right] = \exp\big(\psi_P(\lambda)\big),$$

and analogously $M_Q(\lambda) = \exp(\psi_Q(\lambda))$. Note that condition (I)

$$-D_{\mathrm{KL}}(Q \,\|\, P) \;\le\; t \;\le\; D_{\mathrm{KL}}(P \,\|\, Q)$$

ensures both tail-probabilities are exponentially small. We mark each bound:

(a) Bound on Type I error, $\pi^{(n)}_{1|0}$. Under H_0, Markov's inequality gives for any $\lambda > 0$,

$$\Pr_{P^n}\big\{L_n \geq n\,t\big\} = \Pr\left\{e^{\lambda L_n} \geq e^{\lambda n\,t}\right\} \;\leq\; e^{-\lambda n\,t}\;{}_{P^n}\big[e^{\lambda L_n}\big] \;=\; \exp\big[-\lambda n\,t + n\,\psi_P(\lambda)\big].$$

Optimizing over λ yields the Chernoff bound:

$$\pi^{(n)}_{1|0} \;\leq\; \exp\big[-n\,\psi^*_P(t)\big], \quad \psi^*_P(t) = \sup_{\lambda}\{\lambda\,t - \psi_P(\lambda)\}.$$

(b) Bound on Type II error, $\pi^{(n)}_{0|1}$. Under H_1, set $Y_i = X_i$ but treat them as i.i.d. $\sim Q$. Then

$$\Pr_{Q^n}\{L_n < n\,t\} = \Pr\left\{e^{-\lambda L_n} \geq e^{-\lambda n\,t}\right\} \;\leq\; e^{\lambda n\,t}\;{}_{Q^n}\left[e^{-\lambda L_n}\right] = \exp\big[\lambda n\,t + n\,\psi_Q(-\lambda)\big].$$

Writing $\mu = -\lambda$ and optimizing over μ gives

$$\pi^{(n)}_{0|1} \;\leq\; \exp\big[-n\,\psi^*_Q(t)\big], \quad \psi^*_Q(t) = \sup_{\mu}\{\mu\,t - \psi_Q(\mu)\}.$$

Thus both bounds are established under condition (I).

Solution to 3. Achievable Exponent Pair

By item 2, for each threshold parameter $t \in [-D(Q \parallel P), D(P \parallel Q)]$ the NP test with $\tau = n\,t$ achieves

$$\pi^{(n)}_{1|0} \leq e^{-n\,\psi^*_P(t)}, \qquad \pi^{(n)}_{0|1} \leq e^{-n\,\psi^*_Q(t)}.$$

Hence the error-exponent pair

$$\big(E_0(t), E_1(t)\big) \;=\; \big(\psi^*_P(t),\; \psi^*_Q(t)\big)$$

is attainable. Noting that

$$\psi^*_Q(t) = \sup_{\mu}\{\mu\,t - \psi_Q(\mu)\} = \sup_{\lambda}\{(-\lambda)\,t - \psi_Q(-\lambda)\} = \psi^*_P(t) + t,$$

one often writes the pair as

$$E_0(t) = \psi^*_P(t), \quad E_1(t) = \psi^*_P(t) + t.$$

Solution to 4. Parametric Boundary Curve

The map

$$t \;\longmapsto\; \big(E_0(t), E_1(t)\big) = \big(\psi^*_P(t), \psi^*_P(t) + t\big)$$

traces the boundary of the convex region $\mathcal{E}_n$ as t varies over $[-D(Q\|P), D(P\|Q)]$. Indeed:

1. As t increases, $E_0(t) = \psi^*_P(t)$ is nondecreasing and $E_1(t) = E_0(t) + t$ is nonincreasing. 2. For any other achievable pair (E_0, E_1), one shows by convexity that it lies below this curve. 3. Thus the parametric curve is exactly the *upper-right* boundary ("ROC boundary") of $\mathcal{E}_n$.

Solution to 5. Optimal Exponent of Weighted Sum of Errors

We wish to find the best exponent for the Bayesian risk

$$R_n \;=\; \min_{0\le\phi_n\le 1}\Big\{\;\pi_0\,\pi_{1|0}(\phi_n)\;+\;\pi_1\,\pi_{0|1}(\phi_n)\Big\},$$

where $\pi_0, \pi_1 > 0$ are prior weights. By the Neyman–Pearson characterization and convexity, the minimum is attained by an LLR-rule with some threshold t. Hence asymptotically

$$R_n \;\approx\; \min_t\Big\{\pi_0\,e^{-n\,E_0(t)} \;+\; \pi_1\,e^{-n\,E_1(t)}\Big\} \;=\; \min_t\Big\{\pi_0\,e^{-n\,\psi_P^*(t)} \;+\; \pi_1\,e^{-n\,[\psi_P^*(t)+t]}\Big\}.$$

Taking $-\frac{1}{n}\log(\cdot)$ and letting $n\to\infty$, the dominant exponent is

$$E^* = \;\max_t\;\min\big\{\psi_P^*(t),\;\psi_P^*(t)+t\big\}.$$

Observe the two functions of t,

$$f_1(t) = \psi_P^*(t), \quad f_2(t) = \psi_P^*(t) + t,$$

satisfy f_1 increasing and f_2 decreasing, and they intersect exactly at $t = 0$. Hence

$$\max_t \min\{f_1(t), f_2(t)\} \;=\; f_1(0) \;=\; \psi_P^*(0).$$

But by definition,

$$\psi_P^*(0) = \sup_\lambda\big\{0\cdot\lambda - \psi_P(\lambda)\big\} = -\inf_\lambda \psi_P(\lambda) \;=\; -\min_\lambda \log_P\Big[\exp\!\big(\lambda\log\frac{q}{p}\big)\Big],$$

which is precisely the *Chernoff information* between P and Q. Therefore the optimal exponent of $\pi_0\,\pi_{1|0} + \pi_1\,\pi_{0|1}$ is

$$\boxed{E^* \;=\; \psi_P^*(0).}$$

This completes the detailed derivation. ■