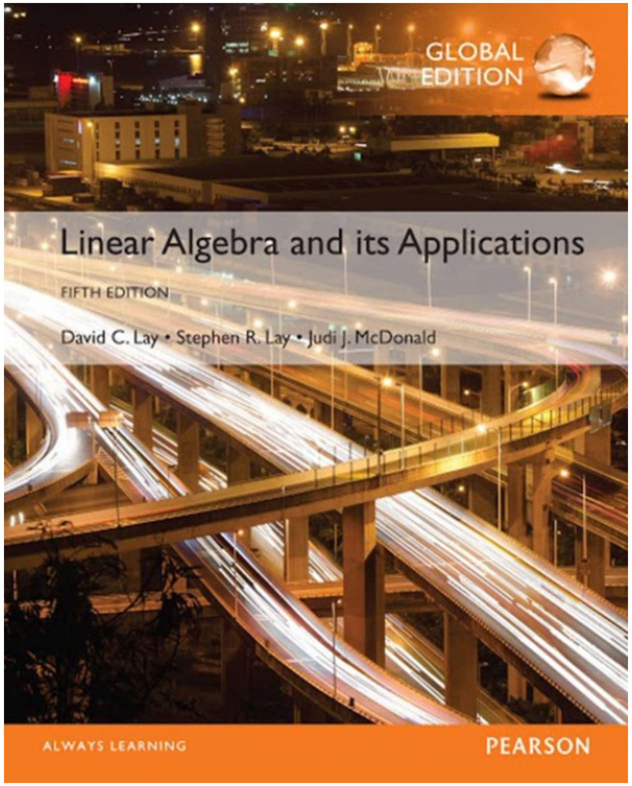
6 Orthogonality and Least Squares

6.7

THE GRAM-SCHMIDT PROCESS



INNER PRODUCT SPACES

- **Definition**An **inner product** on a vector space V is a function that, to each pair of vectors **u** and **v** in V, associates a real number $\langle u, v \rangle$ and satisfies the following axioms,for all **u**, **v**, **w** in V and all scalars c:

1. $\langle \mathrm{u}, \mathrm{v} \rangle = \langle \mathrm{v}, \mathrm{u} \rangle$
2. $\langle \mathrm{u} + \mathrm{v}, \mathrm{w} \rangle = \langle \mathrm{u}, \mathrm{w} \rangle + \langle \mathrm{v}, \mathrm{w} \rangle$
3. $\langle c\mathrm{u}, \mathrm{v} \rangle = c\langle \mathrm{u}, \mathrm{v} \rangle$
4. $\langle \mathrm{u}, \mathrm{u} \rangle \geq 0$ and $\langle \mathrm{u}, \mathrm{u} \rangle = 0$ if and only u = 0

- A vector space with an inner product is called an **inner product space.**

INNER PRODUCT SPACES

- **Example 1**Fix any two positive numbers—say, 4 and 5—and for vectors u = (u_1, u_2) and v = (v_1, v_2) in $\mathbb{R}^2$, set

$$\langle \mathrm{u}, \mathrm{v} \rangle = 4u_1v_1 + 5u_2v_2 \tag{1}$$

- Show that equation (1) defines an inner product.

- **Solution** Certain Axiom 1 is satisfied, because $\langle \mathrm{u}, \mathrm{v} \rangle = 4u_1v_1 + 5u_2v_2 = 4u_1v_1 + 5u_2v_2 = \langle \mathrm{v}, \mathrm{u} \rangle$.

INNER PRODUCT SPACES

- If $w = (w_1, w_2)$, then

$$\langle \mathrm{u} + \mathrm{v}, \mathrm{w} \rangle = 4(u_1 v_1) w_1 + 5(u_2 v_2) w_2$$
$$= 4u_1 w_1 + 5u_2 w_2 + 4v_1 w_1 + 5v_2 w_2$$
$$= \langle \mathrm{u}, \mathrm{w} \rangle + \langle \mathrm{v}, \mathrm{w} \rangle$$

- This verifies Axiom 2. For Axiom 3, compute

$$\langle c\mathrm{u}, \mathrm{v} \rangle = 4(cu_1) v_1 + 5(cu_2) v_2 = c(4u_1 v_1 + 5u_2 v_2) = c\langle \mathrm{u}, \mathrm{v} \rangle$$

INNER PRODUCT SPACES

- For Axiom 4, note that $\langle \mathrm{u}, \mathrm{u} \rangle = 4u_1^2 + 5u_2^2 \geq 0$, and $4u_1^2 + 5u_2^2 = 0$ only if $\mathrm{u}_1 = \mathrm{u}_2 = 0$, that is, if $\mathrm{u} = 0$.

- Also, $\langle 0, 0 \rangle = 0$. So (1) defines an inner product on $\mathbb{R}^2$.

LENGTHS, DISTANCES, AND ORTHOGONALITY

- Let V be an inner product space, with the inner product denoted by $\langle \mathrm{u}, \mathrm{v} \rangle$. Just as in $\mathbb{R}^n$, we define the length, or norm, of a vector v to be the scalar

$$\|v\| = \sqrt{\langle v, v \rangle}$$

- Equivalently, $\|v\|^2 = \langle \mathrm{v}, \mathrm{v} \rangle$.

- A **unit vector** is one whose length is 1. The **distance between u and v** is $\|\boldsymbol{u} - \boldsymbol{v}\|$. Vectors **u** and **v** are **orthogonal** if $\langle \mathrm{u}, \mathrm{v} \rangle = 0$.

TWO INEQUALITIES

- Given a vector **v** in an inner product space *V* and given a finite-dimensional subspace *W*, we may apply the Pythagorean Theorem to the orthogonal decomposition of **v** with respect to *W* and obtain

$$\|v\|^2 = \|proj_W v\|^2 + \|v - proj_W v\|^2$$

- See Fig 2 on the next slide. In particular, this shows that the norm of the projection of **v** onto *W* does not exceed the norm of **v** itself. This simple observation leads to the following important inequality.

TWO INEQUALITIES

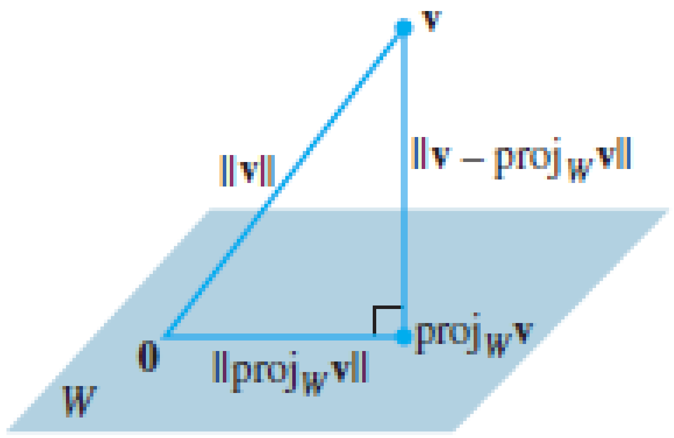


FIGURE 2
The hypotenuse is the longest side.

- **Theorem 16 The Cauchy-Schwarz Inequality:** For all u, v in V,

$$|(u, v)| \leq \|u\|\|v\| \qquad (4)$$

TWO INEQUALITIES

- **Proof** If u = 0, then both sides of (4) are zero, and hence the inequality is true in this case.

- If u ≠ 0, let *W* be the subspace spanned by u.

- Recall that $\|cu\| = |c|\ \|u\|$ for any scalar c. Thus

$$\|proj_W v\| = \left\| \frac{\langle \mathrm{v},\mathrm{u}\rangle}{\langle \mathrm{u},\mathrm{u}\rangle} u \right\| = \frac{|\langle \mathrm{v},\mathrm{u}\rangle|}{|\langle \mathrm{u},\mathrm{u}\rangle|} \|u\| = \frac{|\langle \mathrm{v},\mathrm{u}\rangle|}{\|u\|^2} \|u\| = \frac{|\langle \mathrm{u},\mathrm{v}\rangle|}{\|u\|}$$

- Since $\|proj_W v\| \le \|v\|$, we have $\frac{|\langle \mathrm{u},\mathrm{v}\rangle|}{\|u\|} \le \|v\|$, which gives (4).

TWO INEQUALITIES

- **Theorem 17 The Triangle Inequality:** For all u, v in V,

$$\|u + v\| \leq \|u\| + \|v\|$$

- **Proof** $\|u + v\|^2 = \langle \mathrm{u} + \mathrm{v}, \mathrm{u} + v \rangle = \langle \mathrm{u}, \mathrm{v} \rangle + 2\langle \mathrm{u}, \mathrm{v} \rangle + \langle \mathrm{v}, \mathrm{v} \rangle$

$$\leq \|u\|^2 + 2|\langle \mathrm{u}, \mathrm{v} \rangle| + \|v\|^2$$

$$\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$$

$$= (\|u\| + \|v\|)^2$$

- The triangle inequality follows immediately by taking square roots of both sides.