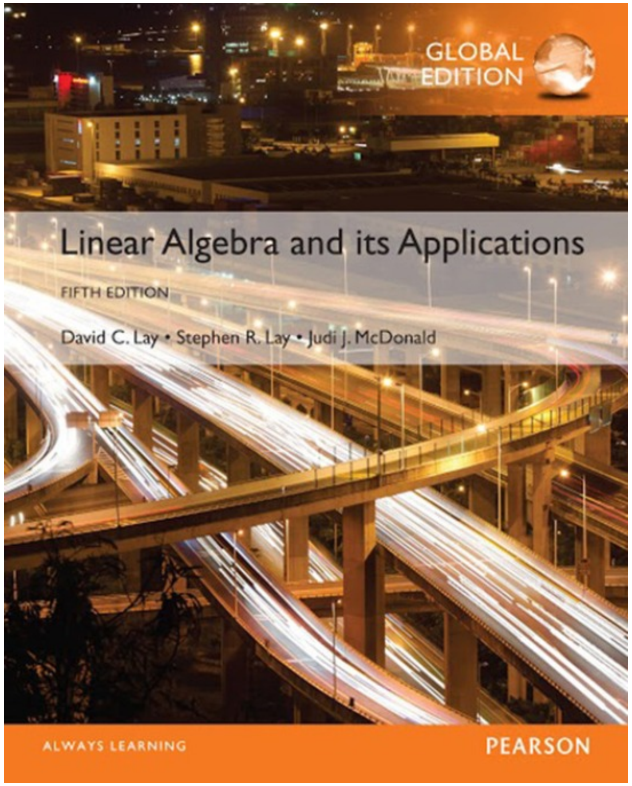
3 Determinants

3.2

PROPERTIES OF DETERMINANTS



PROPERTIES OF DETERMINANTS

- **Theorem 3**: Let A be a square matrix
  a) If a multiple of one row of A is added to another row to produce a matrix B, then det B = det A.
  b) If two rows of A are interchanged to produce B, then det B = - det A.
  c) If one row of A is multiplied by k to produce B, then det $B = k \cdot$ det A

PROPERTIES OF DETERMINANTS

- We can express the theorem as follows:

If A is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then

$$\det EA = (\det E)(\det A)$$

where

$$\det E = \begin{cases} 1 & \textit{if } E \textit{ is a row replacement} \\ -1 & \textit{if } E \textit{ is an interchange} \\ r & \textit{if } E \textit{ is a scale by } r \end{cases}$$

- We refer to *EA* as *B*.

PROPERTIES OF DETERMINANTS

Let A_ij denote the submatrix formed by deleting the i-th row and jth columns of A.

- The proof is done by induction on the size of *A*.
- For the case of 2*2, the correctness is obvious.
- Assume that for $n=k-1$, the theorem hold. We prove its correctness for $n=k+1$
- A row operation might affect 1 or 2 rows. So for $n>2$, there is at least one *unaffected* row (e.g., row *i*) in *A*.
- We perform co-factor expansion around row *i*.
- Sub-matrices A_{ij} and B_{ij} are $k*k$. Therefore, the induction assumption implies that: $\det B_{ij} = \alpha \cdot \det A_{ij}$
- We have:

$$\begin{aligned}
\det EA &= a_{i1}(-1)^{i+1} \det B_{i1} + \cdots + a_{in}(-1)^{i+n} \det B_{in} \\
&= \alpha a_{i1}(-1)^{i+1} \det A_{i1} + \cdots + \alpha a_{in}(-1)^{i+n} \det A_{in} \\
&= \alpha \cdot \det A
\end{aligned}$$

PROPERTIES OF DETERMINANTS

- **Example 1** Compute det A, where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

- **Solution** The strategy is to reduce A to echelon form and then to use the fact that the determinant of a triangular matrix is the product of the diagonal entries. The first two row replacements in column 1 do not change the determinant:

$$detA = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix}$$

PROPERTIES OF DETERMINANTS

- An interchange of rows 2 and 3 reverses the sign of the determinant, so

$$detA = -\begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -(1)(3)(-5) = 15$$

PROPERTIES OF DETERMINANTS

- **Theorem 4**: A square matrix A is invertible if and only if det $A \neq 0$.

- **Example 3** Compute det A, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$

- **Solution** Add 2 times row 1 to row 3 to obtain

$$detA = det\begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{bmatrix} = 0$$

because the second and third rows of the second matrix are equal.

COLUMN OPERATIONS

- **Theorem 5:** If A is a $n \times n$ matrix, then det A^T = det A.
- **Proof**: The theorem is obvious for $n = 1$. Suppose the theorem is true for $k \times k$ determinants and let $n = k + 1$.
- Then the cofactor of a_{1j} in A equals the cofactor of a_{j1} in A^T, because the cofactors involve $k \times k$ determinants.
- Hence the cofactor expansion of det A along the first row equals the cofactor expansion of det A^T down the first column. That is, A and A^T have equal determinants.
- Thus the theorem is true for $n = 1$, and the truth of the theorem for one value of n implies its truth for the next value of n. By the principle of induction, the theorem is true for all $n \geq 1$.

DETERMINANTS AND MATRIX PRODUCTS

- **Theorem 6:** If A and B are $n \times n$ matrices, then det AB= (det A)(det B).

- **Example 5** Verify Theorem 6 for $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$.
- Solution

$$AB = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix}$$

- and

$$\det AB = 25 \cdot 13 - 20 \cdot 14 = 325 - 280 = 45$$

Since det A = 9 and det B = 5,

$$(\det A)(detB) = 9 \cdot 5 = 45 = \det AB$$

PROOF OF THEOREM 6

- If A is not invertible, neither is AB (this is an exercise). So:

$$0 = \det A = \det AB = 0$$

- If A is invertible, A is equivalent to I_n. Therefore:

$$A = E_p E_{p-1} \cdots E_1 \cdot I_n = E_p E_{p-1} \cdots E_1$$

- For brevity, we write $|A|$ for $\det A$.

- We have:

$$\begin{aligned} |AB| &= |E_p \cdots E_1 B| = |E_p||E_{p-1} \cdots E_1 B| = \cdots \\ &= |E_p| \cdots |E_1||B| = \cdots = |E_p \cdots E_1||B| \\ &= |A||B| \end{aligned}$$