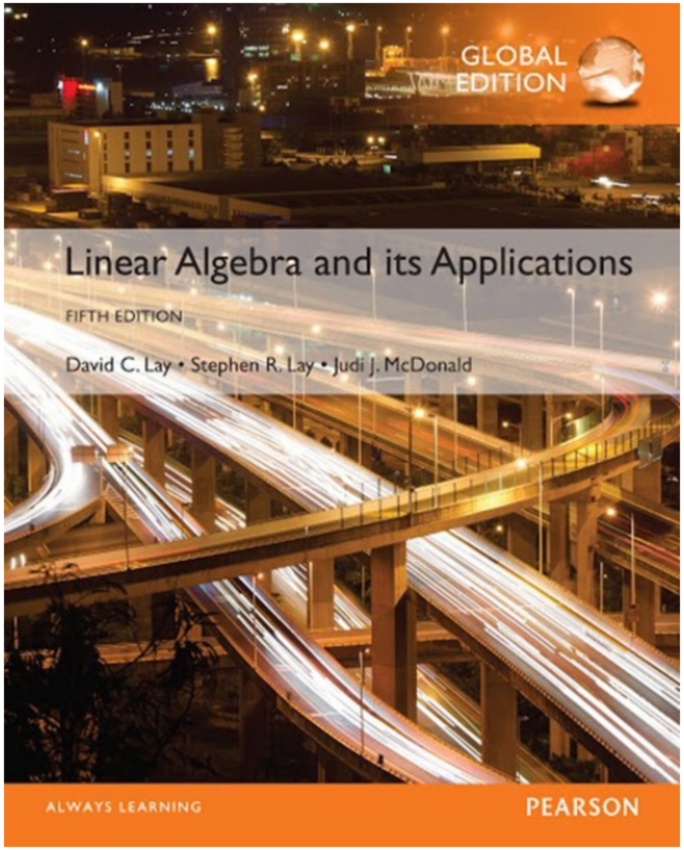
1 Linear Equations in Linear Algebra

1.4

THE MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$



MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- **Definition:** If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of** A **and** $\mathbf{x}$, denoted by $A\mathbf{x}$, **is the linear combination of the columns of** A **using the corresponding entries in x as weights**; that is,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n.$$

- Note that $A\mathbf{x}$ is defined only if the number of columns of A equals the number of entries in $\mathbf{x}$.

MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- **Example 2:** For $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbb{R}^m$, write the linear combination $3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3$ as a matrix times a vector.
- **Solution:** Place $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ into the columns of a matrix A and place the weights $3, -5$, and 7 into a vector $\mathbf{x}$.
- That is,

$$3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3 = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix} = A\mathbf{x}.$$

MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- Now, write the system of linear equations as a vector equation involving a linear combination of vectors.
- For example, the following system

$$x_1 + 2x_2 - x_3 = 4 \qquad (1)$$
$$-5x_2 + 3x_3 = 1$$

is equivalent to

$$x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}. \qquad (2)$$

MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- As in the example, the linear combination on the left side is a matrix times a vector, so that (2) becomes

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}. \qquad (3)$$

- Equation (3) has the form $A\mathbf{x} = \mathbf{b}$. Such an equation is called a **matrix equation**, to distinguish it from a vector equation such as shown in (2).

MATRIX EQUATION $A\mathrm{x} = \mathrm{b}$

A system of linear equations may be viewed in three different but equivalent ways: matrix equation, vector equation or a system of linear equations. Either is solved in the same way - by row reducing the augmented matrix.

THEOREM 3

If A is an $m \times n$ matrix, with columns $\mathrm{a}_1, \ldots, \mathrm{a}_n$, and if b is in $\mathbb{R}^n$, then the matrix equation

$$\mathrm{A}\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = b$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[a_1 \quad a_2 \;\ldots\; a_n \quad \mathrm{b}]$$

EXISTENCE OF SOLUTIONS

- ~~The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.~~ Correct, but may be misleading here

$\boxed{\text{THEOREM 4}}$

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x}=\mathbf{b}$ has a solution.
b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
c. The columns of A span $\mathbb{R}^m$.
d. A has a pivot position in every row.

Note: A system is inconsistent if [A b] has a pivot in the last column. However, a system is consistent if the matrix A has a pivot in every row. Because then it cannot have a row such as [0 0 0 b], where b is nonzero.

COMPUTATION OF *A*x

- **Example 4:** Compute $A\mathbf{x}$, where $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.
- **Solution:** From the definition,

$$\begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$$

COMPUTATION OF $A\mathbf{x}$

$$= \begin{bmatrix} 2x_1 \\ -x_1 \\ 6x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 5x_2 \\ -2x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -3x_3 \\ 8x_3 \end{bmatrix} \quad (1)$$

$$= \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix}.$$

- The first entry in the product $A\mathbf{x}$ is a sum of products (sometimes called *a dot product*), using the first row of A and the entries in $\mathbf{x}$.

COMPUTATION OF $A\mathbf{x}$

- That is, $\begin{bmatrix} 2 & 3 & 4 \\ & & \\ & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ \\ \end{bmatrix}.$

- Similarly, the second entry in $A\mathbf{x}$ can be calculated by multiplying the entries in the second row of A by the corresponding entries in $\mathbf{x}$ and then summing the resulting products.

$$\begin{bmatrix} & & \\ -1 & 5 & -3 \\ & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ -x_1 + 5x_2 - 3x_3 \\ \\ \end{bmatrix}$$

ROW-VECTOR RULE FOR COMPUTING $A\mathbf{x}$

- Likewise, the third entry in $A\mathbf{x}$ can be calculated from the third row of A and the entries in $\mathbf{x}$.
- If the product $A\mathbf{x}$ is defined, then the ith entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vector $\mathbf{x}$.
- The matrix with 1's on the diagonal and 0's elsewhere is called an **identity matrix** and is denoted by I.
- For example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an identity matrix.

PROPERTIES OF THE MATRIX-VECTOR PRODUCT $A\mathbf{x}$

- **Theorem 5:** If A is an $m \times n$ matrix, u and v are vectors in $\mathbb{R}^n$, and c is a scalar, then

  $$A(\mathrm{u} + \mathrm{v}) = A\mathrm{u} + A\mathrm{v};$$

  a. $A(c\mathrm{u}) = c(A\mathrm{u})$.

- **Proof:** For simplicity, take $n = 3$, $A = \begin{bmatrix} \mathrm{a}_1 & \mathrm{a}_2 & \mathrm{a}_3 \end{bmatrix}$, and **u**, **v** in $\mathbb{R}^3$.

- For $i = 1, 2, 3$, let u_i and v_i be the ith entries in **u** and **v**, respectively.

PROPERTIES OF THE MATRIX-VECTOR PRODUCT $A\mathbf{x}$

THEOREM 5

If A is an $m \times n$ matrix, u and v are vectors in $\mathbb{R}^n$, and c is a scalar, then

a. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;

b. $A(c\mathbf{u}) = c(A\mathbf{u})$.

- **Proof:** For simplicity, take $n = 3$, $A = [\mathrm{a}_1 \;\; \mathrm{a}_2 \;\; \mathrm{a}_3]$, and **u**, vin $\mathbb{R}^3$.
- For $i = 1, 2, 3$, let u_i and v_i be the ith entries in **u** and **v**, respectively.

PROPERTIES OF THE MATRIX-VECTOR PRODUCT $A\mathbf{x}$

- To prove statement (a), compute $A(\mathbf{u}+\mathbf{v})$ as a linear combination of the columns of A using the entries in $\mathbf{u}+\mathbf{v}$ as weights.

$$A(\mathbf{u}+\mathbf{v}) = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}$$

Entries in $\mathbf{u}+\mathbf{v}$

$$= (u_1+v_1)\mathbf{a}_1 + (u_2+v_2)\mathbf{a}_2 + (u_3+v_3)\mathbf{a}_3$$

Columns of A

$$= (u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3) + (v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + v_3\mathbf{a}_3)$$

$$= A\mathbf{u} + A\mathbf{v}$$

PROPERTIES OF THE MATRIX-VECTOR PRODUCT *A***x**

- To prove statement (b), compute $A(c\mathbf{u})$ as a linear combination of the columns of A using the entries in $c\mathbf{u}$ as weights.

$$A(c\mathbf{u}) = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} = (cu_1)\mathbf{a}_1 + (cu_2)\mathbf{a}_2 + (cu_3)\mathbf{a}_3$$

$$= c(u_1\mathbf{a}_1) + c(u_2\mathbf{a}_2) + c(u_3\mathbf{a}_3)$$

$$= c(u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3)$$

$$= c(A\mathbf{u})$$