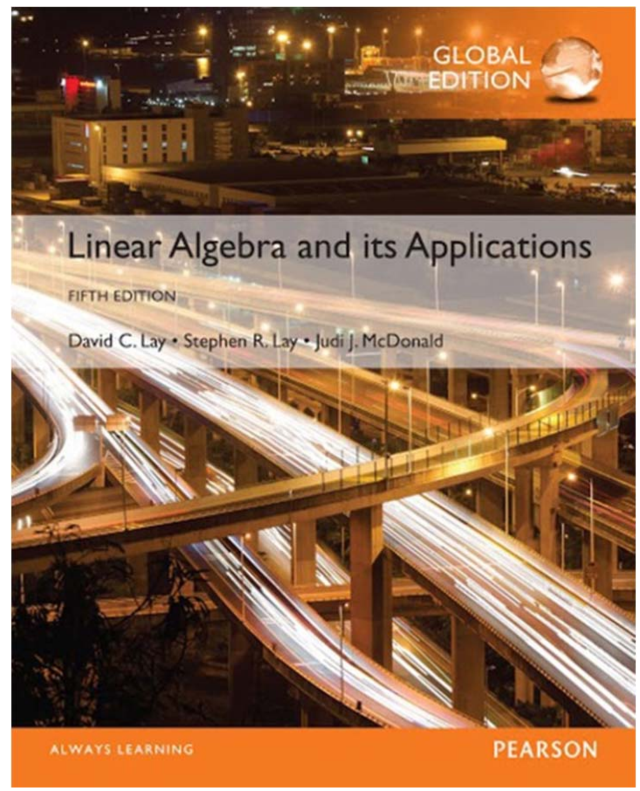
3 Determinants

3.3

CRAMER'S RULE, VOLUME, AND LINEAR TRANSFORMATIONS



CRAMER'S RULE

- **Theorem 7**: Let A be an invertible $n \times n$ matrix. For any b in $\mathbb{R}^n$, the unique solution x of $Ax=b$ has entries given by

$$x_i = \frac{\det A_i(b)}{\det A}, \qquad i = 1, 2, \ldots, n \qquad (1)$$

- **Proof** Denote the columns of A by $a_1, \ldots, a_n$ and the columns of the $n \times n$ identity matrix I by $e_1, \ldots, e_n$. If $Ax = b$, the definition of matrix multiplication shows that

$$\begin{aligned} A \cdot I_i(x) &= A[e_1 \ \ \ldots \ \ x \ \ \ldots \ \ e_n] = [Ae_1 \ \ \ldots \ \ Ax \ \ \ldots \ \ Ae_n] \\ &= [a_1 \ \ \ldots \ \ b \ \ \ldots \ \ a_n] = A_i(b) \end{aligned}$$

CRAMER'S RULE

- By the multiplicative property of determinants,

$$(detA)\big(detI_i(x)\big) = detA_i(b)$$

- The second determinant on the left is simply x_i. Hence $(detA) \cdot x_i = detA_i(b)$. This proves (1) because A is invertible and det $A \neq 0$.

- **Example 1** Use Cramer's rule to solve the system

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8 \end{aligned}$$

CRAMER'S RULE

- **Solution** View the system as $Ax = b$. Using the notation introduced above,

$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}, \quad A_1(b) = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}, \quad A_2(b) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}$$

- Since $\det A = 2$, the system has a unique solution. By Cramer's rule,

$$x_1 = \frac{detA_1(b)}{detA} = \frac{24 + 16}{2} = 20$$

$$x_2 = \frac{detA_2(b)}{detA} = \frac{24 + 30}{2} = 27$$

A FORMULA FOR A^{-1}

- **Theorem 8**: Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

- **Example 3** Find the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$.
- **Solution** The nine cofactors are

$$C_{11} = +\begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} = -2, \quad C_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3, \quad C_{13} = +\begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = 5$$

$$C_{21} = -\begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = 14, \quad C_{22} = +\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -7, \quad C_{23} = -\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = -7$$

$$C_{31} = +\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 4, \quad C_{32} = -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1, \quad C_{33} = +\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$$

A FORMULA FOR A^{-1}

- The adjugate matrix is the *transpose* of the matrix of cofactors. Thus

$$adj A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}$$

- We could compute det A directly, but the following computation provides a check on the calculations above and produces det A:

$$(adj A) \cdot A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix} = \cancel{\begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}} = 14I$$

$$\begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix}$$

A FORMULA FOR A^{-1}

- Since $(\text{adj } A)A = 14I$, Theorem 8 shows that $\det A = 14$ and

1

$$A^{-1} = \frac{1}{14}\begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} = \begin{bmatrix} -1/7 & 1 & 2/7 \\ 3/14 & -1/2 & 1/14 \\ 5/14 & -1/2 & -3/14 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \text{adj } A \longrightarrow (\det A) \cdot I = (\text{adj } A) \cdot A \quad 2$$

$$1, 2 \Longrightarrow \det A = 14$$

PROOF OF A FORMULA FOR A^{-1}

- Let $\mathbf{e}_j$ be the j^{th} column of identity matrix and $\mathbf{x}$ be the j^{th} column of A^{-1}. we have: $A\mathbf{x} = \mathbf{e}_j$
- j^{th} entry of $\mathbf{x}$ is the (i,j)-entry of A^{-1}. By Cramer's rule:

$$\{(i,j)\text{-entry of } A^{-1}\} = x_i = \frac{\det A_i(\mathbf{e}_j)}{\det A}$$

- A cofactor expansion down column i:

$$\det A_i(\mathbf{e}_j) = (-1)^{i+j} \det A_{ji} = C_{ji}$$

- So, $\{(i,j)\text{-entry of } A^{-1}\}$ is equal to C_{ji} divided by $\det A$.
- Therefore:

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

DETERMINANTS AS AREA OR VOLUME

- **Theorem 9**: If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is [det A]. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

- **Proof** The theorem is obviously true for any 2×2 diagonal matrix:

$$\left| det \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \right| = |ad| = \begin{Bmatrix} area\ of \\ rectangle \end{Bmatrix}$$

- See Fig. 1 on the next slide.

DETERMINANTS AS AREA OR VOLUME

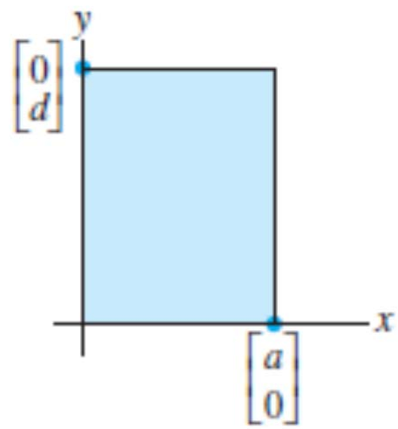
- 

FIGURE 1
Area $= |ad|$.

- It will suffice to show that any 2×2 matrix $A = [a_1 \ \ a_2]$ can be transformed into a diagonal matrix in a way that changes neither the area of the associated parallelogram nor $|\det A|$.

DETERMINANTS AS AREA OR VOLUME

- It suffices to prove the following simple geometric observation that applies to vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$:
  - Let a_1 and a_2 be nonzero vectors. Then for any scalar c, the area of the parallelogram determined by a_1 and a_2 equals the area of the parallelogram determined by a_1 and a_2+ca_1.
  - To prove this statement, we may assume that a_2 is not a multiple of a_1, for otherwise the two parallelograms would be degenerate and have zero area.
  - If L is the line through 0 and a_1, then $a_2 + L$ is the line through a_2 parallel to L, and $a_2 + \mathrm{c}a_1$ is on this line. See Fig. 2 on the next slide.

DETERMINANTS AS AREA OR VOLUME

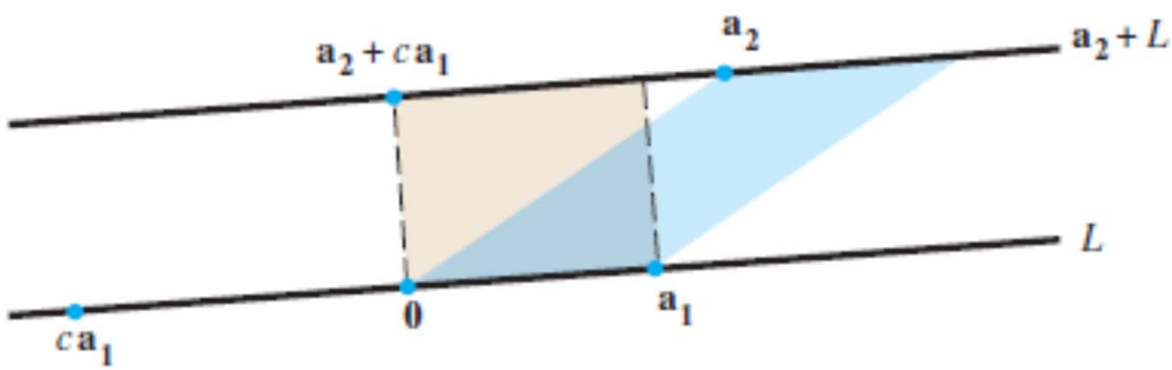


FIGURE 2 Two parallelograms of equal area.

- The points a_2 and $a_2 + ca_1$ have the same perpendicular distance to L. Hence the two parallelograms in Fig. 2 have the same area, since they share the base from 0 to a_1.

DETERMINANTS AS AREA OR VOLUME

- **Example 4** Calculate the area of the parallelogram determined by the points (-2, -2), (0, 3), (4, -1), and (6, 4). See Fig. 5(a) below:

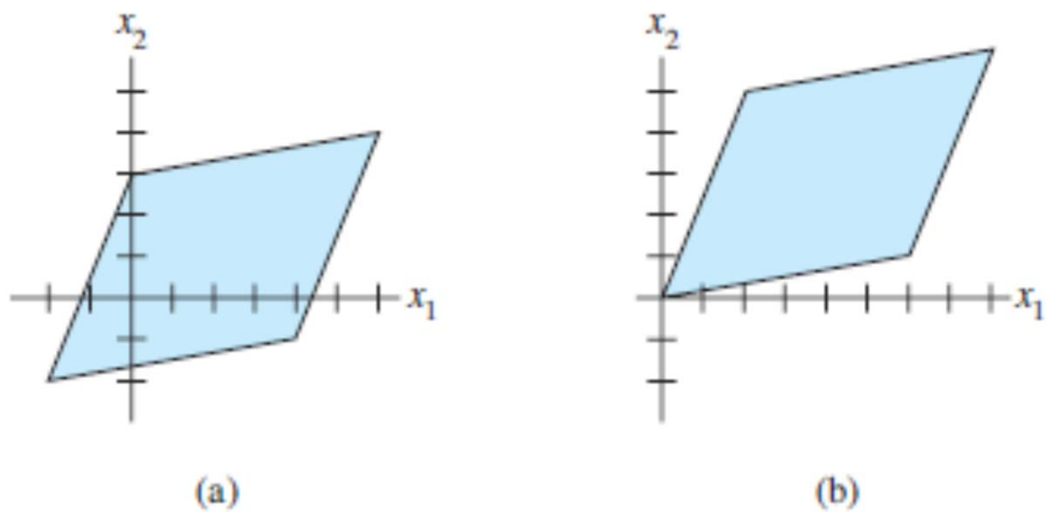


FIGURE 5 Translating a parallelogram does not change its area.

DETERMINANTS AS AREA OR VOLUME

- **Solution** First translate the parallelogram to one having the origin as a vertex. For example, subtract the vertex (-2, -2) from each of the four vertices.
- The new parallelogram has the same area, and its vertices are (0, 0), (2, 5), (6, 1), and (8, 6). See Fig. 5(b) on the previous slide.
- This parallelogram is determined by the columns of

$$A = \begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix}$$

- Since $|\det A| = |-28|$, the area of the parallelogram is 28.

LINEAR TRANSFORMATIONS

- **Theorem 10**: Let T: $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{area\ of\ T(S)\} = |detA| \cdot \{area\ of\ S\} \qquad (5)$$

- If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\{volume\ of\ T(S)\} = |detA| \cdot \{volume\ of\ S\} \qquad (6)$$

- **Proof** Consider the 2×2 case, with $A = [a_1\ \ a_2]$. A parallelogram at the origin in $\mathbb{R}^2$ determined by vectors b_1 and b_2 has the form

$$S = \{s_1 b_1 + s_2 b_2 \colon\ 0 \leq s_1 \leq 1,\ 0 \leq s_2 \leq 1\}$$

LINEAR TRANSFORMATIONS

- The image of S under T consists of points of the form

$$T(s_1b_1 + s_2b_2) = s_1T(b_1) + s_2T(b_2)$$
$$= s_1Ab_1 + s_2Ab_2$$

- where $0 \le s_1 \le 1,\ 0 \le s_2 \le 1$. It follows that $T(S)$ is the parallelogram determined by the columns of the matrix $[Ab_1 \ Ab_2]$. This matrix can be written as AB, where $B = [b_1 \ b_2]$.
- By Theorem 9 and the product theorem for determinants,

$$\{area\ of\ T(S)\} = |detAB| = |detA| \cdot |detB|$$
$$= |detA| \cdot \{area\ of\ S\} \qquad (7)$$

LINEAR TRANSFORMATIONS

- An arbitrary parallelogram has the form $\mathbf{p} + S$, where $\mathbf{p}$ is a vector and S is a parallelogram at the origin.
- It is easy to see that T transforms $\mathbf{p} + S$ into $T(p) + T(S)$. Since translation does not affect the area of a set,

$$\begin{aligned}
\{\text{area of T}(p+S)\} &= \{\text{area of T}(p) + T(S)\} \\
&= \{\text{area of T}(S)\} && \text{Translation} \\
&= |detA| \cdot \{area\ of\ S\} && \text{By equation (7)} \\
&= |detA| \cdot \{area\ of\ \boldsymbol{p} + S\} && \text{Translation}
\end{aligned}$$

- This shows that (5) holds for all parallelograms in $\mathbb{R}^2$. The proof of (6) for the 3×3 case is analogous.