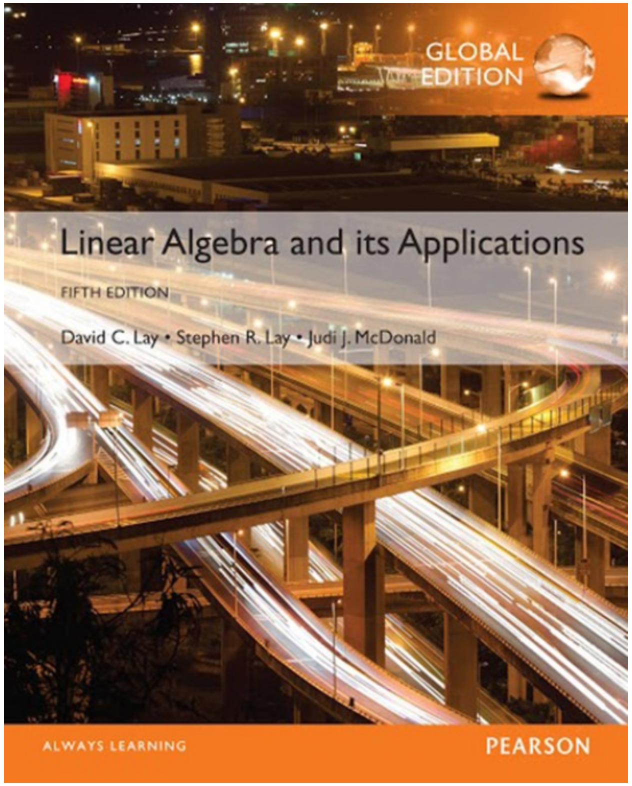
4 Determinants

4.7

CHANGE OF BASIS



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CHANGE OF BASIS

- **Example 1** Consider two bases $\beta = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ for a vector space V, such that

$$b_1 = 4c_1 + c_2 \quad \text{and} \quad b_2 = -6c_1 + c_2 \tag{1}$$

- Suppose

$$x = 3b_1 + b_2 \tag{2}$$

- That is, suppose $[x]_\beta = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $[x]_C$.

CHANGE OF BASIS

- **Solution** Apply the coordinate mapping determined by C to $\mathbf{x}$ in (2). Since the coordinate mapping is a linear transformation,

$$\begin{aligned}[x]_C &= [3\mathrm{b}_1 + \mathrm{b}_2]_C \\ &= [3\mathrm{b}_1]_C + [\mathrm{b}_2]_C\end{aligned}$$

- We can write the vector equation as a matrix equation, using the vectors in the linear combination as the columns of a matrix:

$$[x]_C = \left[[\mathrm{b}_1]_C [\mathrm{b}_2]_C \right] \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad (3)$$

CHANGE OF BASIS

- This formula gives $[x]_C$, once we know the columns of the matrix. From (1),

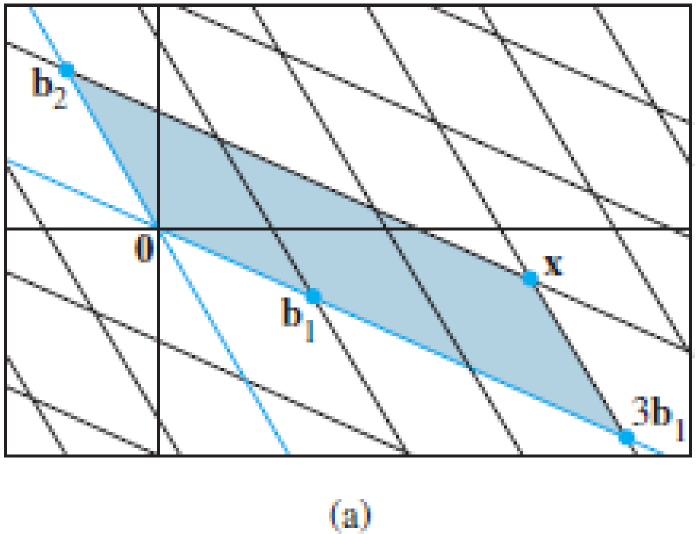
$$[\mathrm{b}_1]_C = \begin{bmatrix} \overset{4}{\cancel{3}} \\ 1 \end{bmatrix} \text{and } [\mathrm{b}_2]_C = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$$

- Thus, (3) provides the solution:

$$[x]_C = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

- The *C*-coordinates of **x** match those of the **x** in Fig. 1, as seen on the next slide.

CHANGE OF BASIS



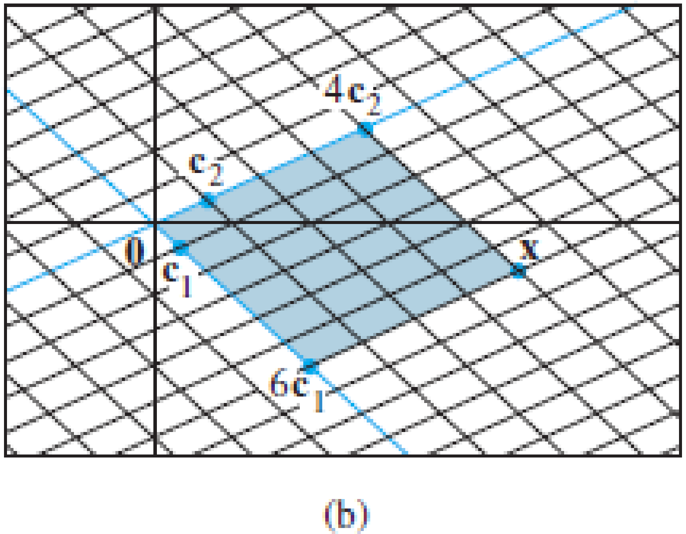


FIGURE 1 Two coordinate systems for the same vector space.

CHANGE OF BASIS

- **Theorem 15**: Let $\beta = \{b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, c_2\}$ for a vector space V. Then there is a unique n × n matrix $\underset{C \leftarrow \beta}{P}$ such that

$$[x]_C = \underset{C \leftarrow \beta}{P} [x]_\beta \tag{4}$$

- The columns of $\underset{C \leftarrow \beta}{P}$ are the C-coordinate vectors of the vectors in the basis β. That is,

$$\underset{C \leftarrow \beta}{P} = [\; [b_1]_C [b_2]_C \quad \ldots \quad [b_n]_C] \tag{5}$$

CHANGE OF BASIS

- The matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ in Theorem 15 is called the **change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.** Multiplication by $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ converts $\mathcal{B}$-coordinates into $\mathcal{C}$-coordinates.
- Figure 2 below illustrates the change-of-coordinates equation (4).

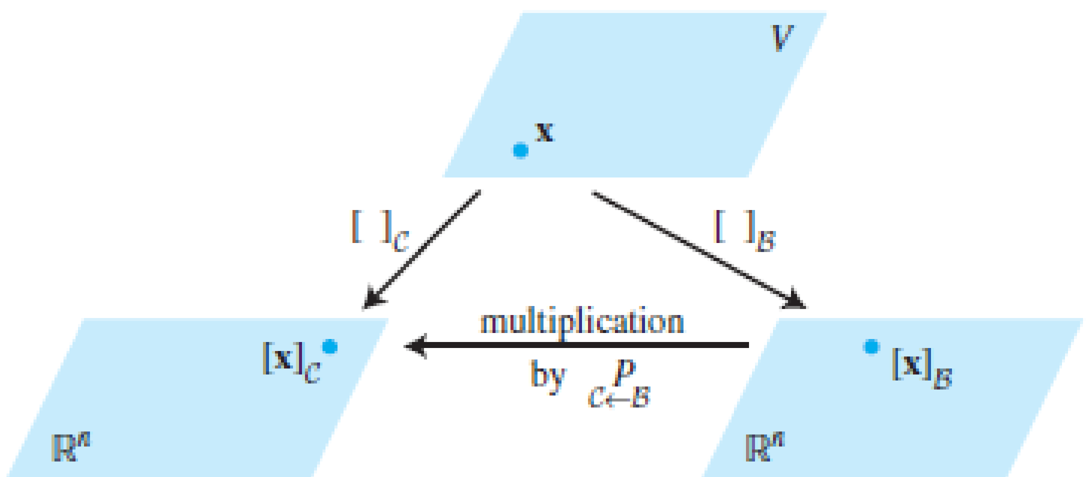


FIGURE 2 Two coordinate systems for V.

CHANGE OF BASIS

- The columns of $\underset{C\leftarrow\beta}{P}$ are linearly independent because they are the coordinate vectors of the linearly independent set β.
- Since $\underset{C\leftarrow\beta}{P}$ is square, it must be invertible, by the Invertible Matrix Theorem. Left-multiplying both sides of equation (4) by $(\underset{C\leftarrow\beta}{P})^{-1}$ yields

$$(\underset{C\leftarrow\beta}{P})^{-1}[x]_C = [x]_\beta$$

- Thus $(\underset{C\leftarrow\beta}{P})^{-1}$ is the matrix that converts C-coordinates into β–coordinates. That is,

$$(\underset{C\leftarrow\beta}{P})^{-1} = \underset{\beta\leftarrow C}{P} \tag{6}$$

CHANGE OF BASIS IN $\mathbb{R}^n$

- If $\beta = \{b_1, \ldots, b_n\}$ and $\mathcal{E}$ is the standard basis $\{e_1, \ldots, e_n\}$ in $\mathbb{R}^n$, then $[b_1]_{\mathcal{E}} = b_1$, and likewise for the other vectors in β. In this case, $\underset{\mathcal{E} \leftarrow \beta}{P}$ is thesame as the change-of-coordinates matrix P_β introduced in Section 4.4, namely,

$$P_\beta = [\, b_1 \quad b_2 \ldots b_n]$$

- To change coordinates between two nonstandard bases in $\mathbb{R}^n$,we need Theorem 15. The theorem shows that to solve the change-of-basis problem, we need the coordinate vectors of the old basis relative to the new basis.

CHANGE OF BASIS IN $\mathbb{R}^n$

- **Example 2** Let $\mathrm{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $\mathrm{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $\mathrm{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\mathrm{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ and consider the bases for $\mathbb{R}^n$ given by $\beta = \{\mathrm{b}_1, \mathrm{b}_2\}$ and $C = \{\mathrm{c}_1, \mathrm{c}_2\}$. Find the change-of-coordinates matrix from β to C.

- **Solution** The matrix $\beta \overset{P}{\leftarrow} C$ involves the C-coordinate vectors of b_1 and b_2. Let $[\mathrm{b}_1]_C = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $[\mathrm{b}_2]_C = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Then, by definition,

$$[c_1 \quad c_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathrm{b}_1 \quad \text{and} \quad [c_1 \quad c_2]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathrm{b}_2$$

- To solve both systems simultaneously, augment the coefficient matrix with b_1 and b_2, and row reduce:

$$[c_1 \quad c_2 \,\vdots\, b_1 \quad b_2] = \left[\begin{array}{cc:cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc:cc} \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I} & \underbrace{\begin{matrix} 6 & 4 \\ -5 & -3 \end{matrix}}_{C \xleftarrow{P} \beta} \end{array}\right] \qquad (7)$$

- Thus

$$[\mathrm{b}_1]_C = \begin{bmatrix} 6 \\ -5 \end{bmatrix} \text{and} \quad [\mathrm{b}_2]_C = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

- The desired change-of-coordinates matrix is therefore

$$\underset{c\leftarrow\beta}{P} = [[b_1]_c \quad [b_2]_c] = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$$