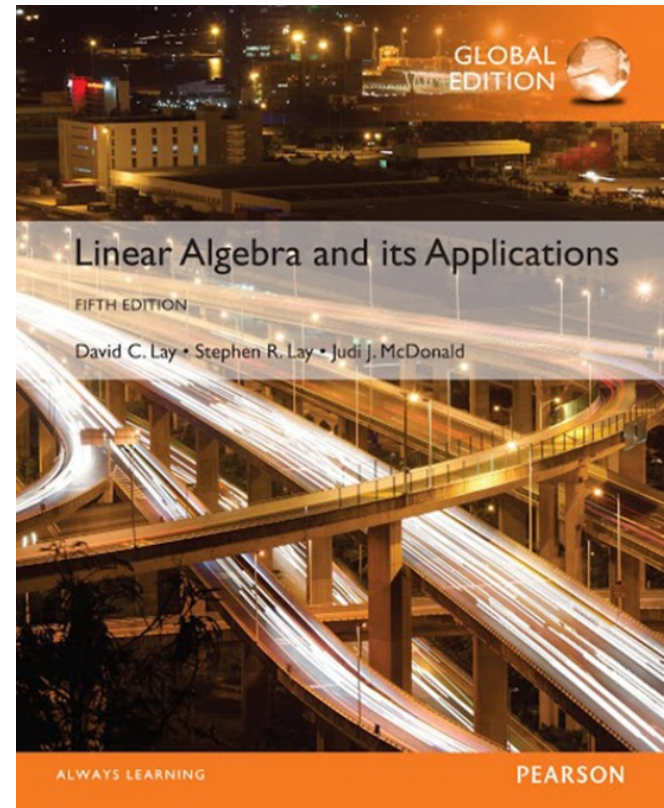


1

Linear Equations in Linear Algebra

1.7

LINEAR INDEPENDENCE



LINEAR INDEPENDENCE

- **Definition:** An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in \mathbb{R}^n is said to be **linearly independent** if the vector equation

$$x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + \dots + x_p \mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights c_1, \dots, c_p , not all zero, such that

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_p \mathbf{v}_p = \mathbf{0} \quad (2)$$

LINEAR INDEPENDENCE

- Equation (2) is called a **linear dependence relation** among $\mathbf{v}_1, \dots, \mathbf{v}_p$ when the weights are not all zero.
- An indexed set is linearly dependent if and only if it is not linearly independent.

- **Example 1:** Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

LINEAR INDEPENDENCE

- a. Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.
 - b. If possible, find a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2$, and \mathbf{v}_3 .
-
- **Solution:** We must determine if there is a nontrivial solution of the equation on the previous slide.

LINEAR INDEPENDENCE

- Row operations on the associated augmented matrix show that

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- x_1 and x_2 are basic variables, and x_3 is free.
- Each nonzero value of x_3 determines a nontrivial solution of (1).
- Hence, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent.

LINEAR INDEPENDENCE

- b. To find a linear dependence relation among \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 , completely row reduce the augmented matrix and write the new system:

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_1 - 2x_3 = 0 \\ x_2 + x_3 = 0 \\ 0 = 0 \end{array}$$

- Thus, $x_1 = 2x_3$, $x_2 = -x_3$, and x_3 is free.
- Choose any nonzero value for x_3 —say, $x_3 = 5$.
- Then $x_1 = 10$ and $x_2 = -5$.

LINEAR INDEPENDENCE

- Substitute these values into equation (1) and obtain the equation below.

$$10\mathbf{v}_1 - 5\mathbf{v}_2 + 5\mathbf{v}_3 = 0$$

- This is one (out of infinitely many) possible linear dependence relations among \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 .

LINEAR INDEPENDENCE OF MATRIX COLUMNS

- Suppose that we begin with a matrix $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ instead of a set of vectors.
- The matrix equation $A\mathbf{x} = \mathbf{0}$ can be written as
$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}.$$
- *Each linear dependence relation among the columns of A corresponds to a nontrivial solution of $A\mathbf{x} = \mathbf{0}$*
- The columns of matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution.

SETS OF ONE OR TWO VECTORS

- A set containing only one vector – say, \mathbf{v} – is linearly independent if and only if \mathbf{v} is not the zero vector.
- This is because the vector equation $x_1 \mathbf{v} = \mathbf{0}$ has only the trivial solution when $\mathbf{v} \neq \mathbf{0}$.
- The zero vector is linearly dependent because $x_1 \mathbf{0} = \mathbf{0}$ has many nontrivial solutions.

SETS OF ONE OR TWO VECTORS

- A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other.
- The set is linearly independent if and only if neither of the vectors is a multiple of the other.

SETS OF TWO OR MORE VECTORS

THEOREM 7

Characterization of Linearly Dependent Sets

An indexed set $S = \{v_1, \dots, v_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $v_1 \neq 0$, then some v_j (with $j > 1$) is a linear combination of the preceding vectors, v_1, \dots, v_{j-1} .

SETS OF TWO OR MORE VECTORS

- **Proof:** If some \mathbf{v}_j in S equals a linear combination of the other vectors, then \mathbf{v}_j can be subtracted from both sides of the equation, producing a linear dependence relation with a nonzero weight (-1) on \mathbf{v}_j .
- [For instance, if $\mathbf{v}_1 = c_2\mathbf{v}_2 + c_3\mathbf{v}_3$, then $0 = (-1)\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + 0\mathbf{v}_4 + \dots + 0\mathbf{v}_p$.]
- Thus S is linearly dependent.
- Conversely, suppose S is linearly dependent.
- If \mathbf{v}_1 is zero, then it is a (trivial) linear combination of the other vectors in S .

SETS OF TWO OR MORE VECTORS

- Otherwise, $v_1 \neq 0$, and there exist weights c_1, \dots, c_p , not all zero, such that

$$c_1 v_1 + c_2 v_2 + \dots + c_p v_p = 0.$$

- Let j be the largest subscript for which $c_j \neq 0$.
- If $j = 1$, then $c_1 v_1 = 0$, which is impossible because $v_1 \neq 0$.

SETS OF TWO OR MORE VECTORS

- So $j > 1$, and

$$c_1 \mathbf{v}_1 + \dots + c_j \mathbf{v}_j + 0\mathbf{v}_j + 0\mathbf{v}_{j+1} + \dots + 0\mathbf{v}_p = \mathbf{0}$$

$$c_j \mathbf{v}_j = -c_1 \mathbf{v}_1 - \dots - c_{j-1} \mathbf{v}_{j-1}$$

$$\mathbf{v}_j = \left(-\frac{c_1}{c_j} \right) \mathbf{v}_1 + \dots + \left(-\frac{c_{j-1}}{c_j} \right) \mathbf{v}_{j-1}.$$

SETS OF TWO OR MORE VECTORS

- Theorem 7 does *not* say that *every* vector in a linearly dependent set is a linear combination of the preceding vectors.
- A vector in a linearly dependent set may fail to be a linear combination of the other vectors. *example?*

- **Example 4:** Let $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$. Describe the

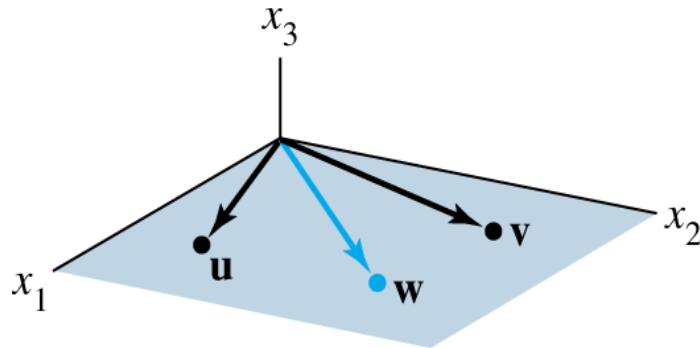
set spanned by \mathbf{u} and \mathbf{v} , and explain why a vector \mathbf{w} is in $\text{Span } \{\mathbf{u}, \mathbf{v}\}$ if and only if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

SETS OF TWO OR MORE VECTORS

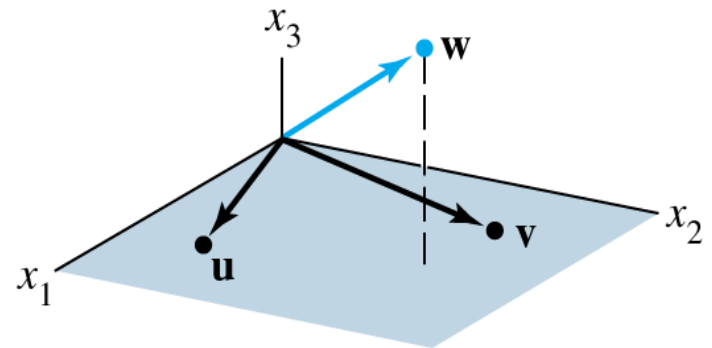
- **Solution:** The vectors \mathbf{u} and \mathbf{v} are linearly independent because neither vector is a multiple of the other, and so they span a plane in \mathbb{R}^3 .
- Span $\{\mathbf{u}, \mathbf{v}\}$ is the x_1x_2 -plane (with $x_3 = 0$).
- If \mathbf{w} is a linear combination of \mathbf{u} and \mathbf{v} , then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent, by Theorem 7.
- Conversely, suppose that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.
- By theorem 7, some vector in $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linear combination of the preceding vectors (since $\mathbf{u} \neq \mathbf{0}$).
- That vector must be \mathbf{w} , since \mathbf{v} is not a multiple of \mathbf{u} .

SETS OF TWO OR MORE VECTORS

- So \mathbf{w} is in $\text{Span}\{\mathbf{u}, \mathbf{v}\}$. Fig. 2 below



Linearly dependent,
 \mathbf{w} in $\text{Span}\{\mathbf{u}, \mathbf{v}\}$



Linearly independent,
 \mathbf{w} not in $\text{Span}\{\mathbf{u}, \mathbf{v}\}$

- Example 4 generalizes to any set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ in \mathbb{R}^3 with \mathbf{u} and \mathbf{v} linearly independent.
- The set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ will be linearly dependent if and only if \mathbf{w} is in the plane spanned by \mathbf{u} and \mathbf{v} .

SETS OF TWO OR MORE VECTORS

THEOREM 8

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in \mathbb{R}^n is linearly dependent if $p > n$.

- **Proof:** Let $A = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_p \end{bmatrix}$.
- Then A is $n \times p$, and the equation $A\mathbf{x} = \mathbf{0}$ corresponds to a system of n equations in p unknowns.
- If $p > n$, there are more variables than equations, so there must be a free variable.

SETS OF TWO OR MORE VECTORS

- Hence $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, and the columns of A are linearly dependent.
- See the figure below for a matrix version of this theorem.

$$\begin{matrix} & p \\ n & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \end{matrix}$$

If $p > n$, the columns are linearly dependent.

- Theorem 8 says nothing about the case in which the number of vectors in the set does *not* exceed the number of entries in each vector.

SETS OF TWO OR MORE VECTORS

THEOREM 9

If a set $S = \{v_1, \dots, v_p\}$ in \mathbb{R}^n contains the zero vector, then the set is linearly dependent.

- **Proof:** By renumbering the vectors, we may suppose $v_1 = 0$.
- Then the equation $1v_1 + 0v_2 + \dots + 0v_p = 0$ shows that S is linearly dependent.