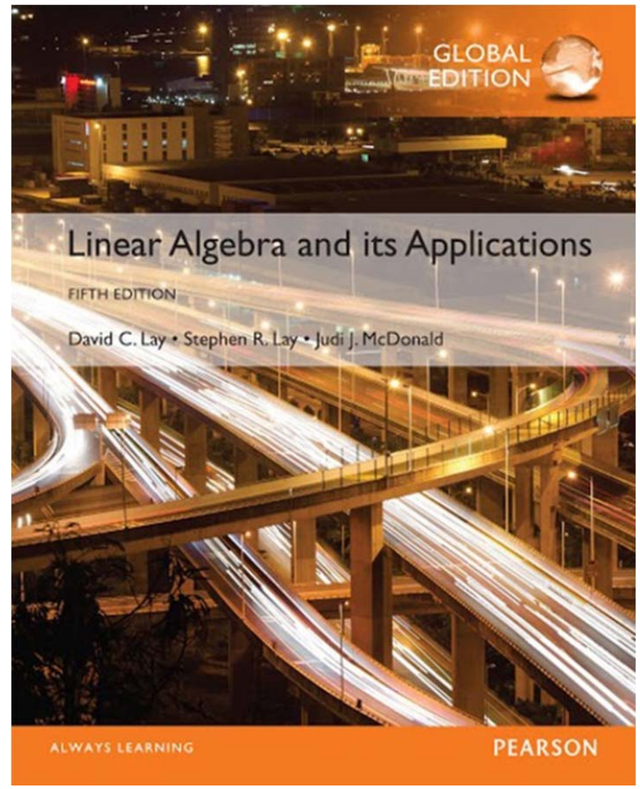
7 Symmetric Matrices and Quadratic Forms

7.3

CONSTRAINED OPTIMIZATION



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7.3 CONSTRAINED OPTIMIZATION

Engineers, economists, scientists, and mathematicians often need to find the maximum or minimum value of a quadratic form $Q(\mathbf{x})$ for $\mathbf{x}$ in some specified set. Typically, the problem can be arranged so that $\mathbf{x}$ varies over the set of unit vectors. This *constrained optimization problem* has an interesting and elegant solution.

The requirement that a vector $\mathbf{x}$ in $\mathbb{R}^n$ be a unit vector can be stated in several equivalent ways:

$$\|\mathbf{x}\| = 1, \qquad \|\mathbf{x}\|^2 = 1, \qquad \mathbf{x}^T\mathbf{x} = 1$$

and

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \tag{1}$$

The expanded version (1) of $\mathbf{x}^T\mathbf{x} = 1$ is commonly used in applications.

When a quadratic form Q has no cross-product terms, it is easy to find the maximum and minimum of $Q(\mathbf{x})$ for $\mathbf{x}^T\mathbf{x} = 1$.

EXAMPLE 1 Find the maximum and minimum values of $Q(\mathbf{x}) = 9x_1^2 + 4x_2^2 + 3x_3^2$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$.

SOLUTION Since x_2^2 and x_3^2 are nonnegative, note that

$$4x_2^2 \leq 9x_2^2 \qquad \text{and} \qquad 3x_3^2 \leq 9x_3^2$$

and hence

$$\begin{aligned} Q(\mathbf{x}) &= 9x_1^2 + 4x_2^2 + 3x_3^2 \\ &\leq 9x_1^2 + 9x_2^2 + 9x_3^2 \\ &= 9(x_1^2 + x_2^2 + x_3^2) \\ &= 9 \end{aligned}$$

whenever $x_1^2 + x_2^2 + x_3^2 = 1$. So the maximum value of $Q(\mathbf{x})$ cannot exceed 9 when $\mathbf{x}$ is a unit vector. Furthermore, $Q(\mathbf{x}) = 9$ when $\mathbf{x} = (1, 0, 0)$. Thus 9 is the maximum value of $Q(\mathbf{x})$ for $\mathbf{x}^T\mathbf{x} = 1$.

To find the minimum value of $Q(\mathbf{x})$, observe that

$$9x_1^2 \geq 3x_1^2, \qquad 4x_2^2 \geq 3x_2^2$$

and hence

$$Q(\mathbf{x}) \geq 3x_1^2 + 3x_2^2 + 3x_3^2 = 3(x_1^2 + x_2^2 + x_3^2) = 3$$

whenever $x_1^2 + x_2^2 + x_3^2 = 1$. Also, $Q(\mathbf{x}) = 3$ when $x_1 = 0$, $x_2 = 0$, and $x_3 = 1$. So 3 is the minimum value of $Q(\mathbf{x})$ when $\mathbf{x}^T\mathbf{x} = 1$. ∎

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- **Theorem 6** Let A be a symmetric matrix, and define m and M as in (2):

$$m = \min\{\mathbf{x}^{T}A\mathbf{x} : \|\mathbf{x}\|=1\} \quad \text{and} \quad M = \max\{\mathbf{x}^{T}A\mathbf{x} : \|\mathbf{x}\|=1\} \qquad (2)$$

- Then M is the greatest eigenvalue λ_1 of A and m is the least eigenvalue of A. The value of $x^{T}Ax$ is M when x is a unit eigenvector u_1 corresponding to M. The value of $x^{T}Ax$ is m when x is a unit eigenvector corresponding to m.

- **Proof** Orthogonally diagonalize A as PDP^{-1}. We know that (3)

$$x^{T}Ax = y^{T}Dy \quad \text{when} \quad x = Py$$

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- Also,

$$\|x\| = \|Py\| = \|y\| \text{ for all y}$$

- Because $P^T P = I$ and $\|Py\|^2 = (Py)^T(Py) = y^T P^T P y = y^T y = \|y\|^2$. In particular, $\|y\| = 1$ if and only if $\|x\| = 1$. Thus, $x^T Ax$ and $y^T Dy$ assume the same set of values as x and y range over the set of all unit vectors.

- To simplify notation, suppose that A is a 3×3 matrix with eigenvalues $a \geq b \geq c$. Arrange the columns of P so that $P = [\mathbf{u}_1 \;\; \mathbf{u}_2 \;\; \mathbf{u}_3]$ and

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$$D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

- Given any unit vector y in $\mathbb{R}^3$ with coordinates y_1, y_2, y_3, observe that

$$\begin{aligned} ay_1^2 &= ay_1^2 \\ by_2^2 &\le ay_2^2 \\ cy_3^2 &\le ay_3^2 \end{aligned}$$

- and obtain these inequalities:

$$\begin{aligned} y^T D y &= ay_1^2 + by_2^2 + cy_3^2 \\ &\le ay_1^2 + ay_2^2 + ay_3^2 \\ &= a(y_1^2 + y_2^2 + y_3^2) = a\|y\|^2 = a \end{aligned}$$

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- Thus $M \leq a$, by definition of M. However, $y^T D y = a$ when $y = e_1 = (1, 0, 0)$, so in fact $M = a$. By (3), the x that corresponds by $y = e_1$ is the eigenvector u_1 of A, because

$$x = Pe_1 = [u_1 \quad u_2 \quad u_3]\begin{bmatrix}1\\0\\0\end{bmatrix} = u_1$$

- Thus $M = a = e_1^T D e_1 = u_1^T A u_1$, which proves the statement about M. A similar argument shows that m is the least eigenvalue, c, and this value of $x^T A x$ is attained when $x = Pe_3 = u_3$.

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- **Example 3** Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$. Find the maximum value of the quadratic form $\mathrm{x}^T A \mathrm{x}$ subject to the constraint $\mathrm{x}^T\mathrm{x} = 1$, and find a unit vector at which this maximum value is attained.
- **Solution** By Theorem 6, the desired maximum value is the greatest eigenvalue of A. The characteristic equation turns out to be

$$0 = -\lambda^3 + 10\lambda^2 - 27\lambda + 18 = -(\lambda - 6)(\lambda - 3)(\lambda - 1)$$

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- The greatest eigenvalue is 6.

- The constrained maximum of $x^T A x$ is attained when x is a unit eigenvector for $\lambda = 6$. Solve $(A - 6I)x = 0$ and find an eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Set $u_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$.

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- **Theorem 7** Let A, λ_1, and u_1 be as in Theorem 6. Then the maximum value of x^TAx subject to the constraints

$$x^Tx = 1, x^Tu_1 = 0$$

- is the second greatest eigenvalue λ_2, and this maximum is attained when x is an eigenvector u_2 corresponding to λ_2.

- **Example 4** Find the maximum value of $9x_1^2 + 4x_2^2 + 3x_3^2$ subject to the constraints $x^Tx = 1$,and $x^Tu_1 = 0$, where $u_1 = (1, 0, 0)$. Note that u_1 is a unit eigenvector corresponding to the greatest eigenvalue $\lambda = 9$ of the matrix of the quadratic form.

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- **Solution** If the coordinates of x are x_1, x_2, x_3, then the constraint $x^T u_1 = 0$ means simply that $x_1 = 0$. For such a unit vector, $x_2^2 + x_3^2 = 1$, and

$$\begin{aligned} 9x_1^2 + 4x_2^2 + 3x_3^2 &= 4x_2^2 + 3x_3^2 \\ &\leq 4x_2^2 + 4x_3^2 \\ &= 4(x_2^2 + x_3^2) \\ &= 4 \end{aligned}$$

- Thus the constrained maximum of the quadratic form does not exceed 4. And this value is attained for x = (0, 1, 0) which is the eigenvector for the second greatest eigenvalue of the matrix of the quadratic form.

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- **Theorem 8** Let A be a symmetric $n \times n$ matrix with an orthogonal diagonalization $A = PDP^{-1}$, where the entries on the diagonal of D are arranged so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and where columns of P are corresponding unit eigenvectors $u_1, \dots, u_n$. Then for $k = 2, \dots, n$, the maximum value of $x^T A x$ subject to the constraints

$$x^T x = 1, x^T u_1 = 0, \qquad \dots, \qquad x^T u_{k-1} = 0$$

- is the eigenvalue λ_k, and this maximum is attained at $x = u_k$.