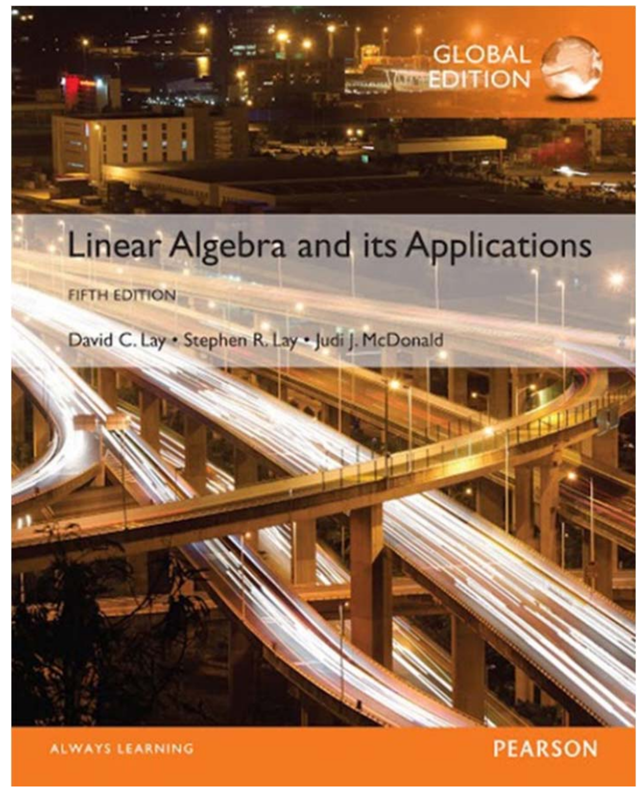
7 Symmetric Matrices and Quadratic Forms

7.4

THE SINGULAR VALUE DECOMPOSITION



A matrix P is orthogonal if $P^T = P^{-1}$

An n*n matrix A is said to be orthogonally diagonalizable if there are orthogonal matrix P and a diagonal matrix D such that

$A = PDP^T = PDP^{-1}$

An n*n matrix A is orthogonally diagonalizable iff A is a symmetric matrix.

THE SINGULAR VALUES OF AN $m \times n$ MATRIX

- Let A be an $m \times n$ matrix.
  - A^TA is symmetric and can be orthogonally diagonalized.
- Let $\{v_1, \ldots, v_n\}$ is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A^TA and $\lambda_1 \geq \cdots \geq \lambda_n$ be the associated eigenvalues of A^TA.

$$\begin{aligned} \|A\mathbf{v}_i\|^2 &= (A\mathbf{v}_i)^T A\mathbf{v}_i = \mathbf{v}_i^T A^T A\mathbf{v}_i \\ &= \mathbf{v}_i^T(\lambda_i \mathbf{v}_i) && \text{Since } \mathbf{v}_i \text{ is an eigenvector of } A^TA \\ &= \lambda_i && \text{Since } \mathbf{v}_i \text{ is a unit vector} \end{aligned}$$

- Hence, eigenvalues of A^TA are nonnegative.

THE SINGULAR VALUES OF AN $m \times n$ MATRIX

- **Singular values of A:**
  - Square roots of the eigenvalues of A^TA

- **Theorem 9** Suppose $\{v_1, \dots, v_n\}$ is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A^TA, arranged so that the corresponding eigenvalues of A^TA satisfy $\lambda_1 \geq \cdots \geq \lambda_n$, and suppose A has r nonzero singular values. Then $\{Av_1, \dots, Av_r\}$ is an orthogonal basis for Col A, and rank $A = r$.

Since $\|Av_i\|^2 = \lambda_i$

THE SINGULAR VALUES OF AN $m \times n$ MATRIX

- **Proof** Because v_i and $\lambda_j v_j$ are orthogonal for $i \neq j$,
  $$(Av_i)^T(Av_j) = v_i^T \underline{A^T A v_j} = v_i^T\left(\underline{\lambda_j v_j}\right) = 0$$

- Thus $\{Av_1, \ldots, Av_n\}$ is an orthogonal set.

- Since the lengths of the vectors $Av_1, \ldots, Av_n$ are the singular values of A, and since there are r nonzero singular values, $\underline{Av_i \neq 0 \text{ if and only if } 1 \leq i \leq r}$.

Recall: The column space of an m*n matrix A is the span (set of all possible linear combinations) of its column vectors. The dimension of the column space is called the rank of the matrix and is at most min(m,n).

- So $Av_1, \ldots, Av_r$ are linearly independent vectors, and they are in Col A.

- Finally, for any y in Col A—say, y = Ax—we can write $x = c_1v_1 + \cdots + c_nv_n$, and

$$y = Ax$$
$$= \underline{c_1Av_1 + \cdots + c_rAv_r} + c_{r+1}Av_{r+1} + \cdots + c_nAv_n$$

THE SINGULAR VALUES OF AN $m \times n$ MATRIX

$$= c_1 A v_1 + \cdots + c_r A v_r + 0 + \cdots + 0$$

- Thus **y** is in Span$\{Av_1, \ldots, Av_r\}$, which shows that

$$\{Av_1, \ldots, Av_r\}$$

is an (orthogonal) basis for Col A.

- Hence:

$$\text{rank } A = \dim \text{Col } A = r.$$

THE SINGULAR VALUE DECOMPOSITION

- **Theorem 10: The Singular Value Decomposition** Let A be an $m \times n$ matrix with rank r. Then there exists an $m \times n$ matrix Σ as:

$$\Sigma = \begin{array}{ll} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} & \leftarrow m - r \text{ rows} \\ \quad\quad\; \uparrow n - r \text{ columns} & \end{array} \qquad (3)$$

for which the diagonal entries in D are the first r singular values of A, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, and there exist an $m \times m$ orthogonal matrix U and an $n \times n$ orthogonal matrix V such that

$$\boxed{A = U\Sigma V^T}$$

THE SINGULAR VALUE DECOMPOSITION

- Any factorization $A = U\Sigma V^T$, with U and V orthogonal, Σ as in (3), and positive diagonal entries in D, is called a **singular value decomposition** (or **SVD**) of A.

- The columns of U in such a decomposition are called **left singular vectors** of A, and the columns of V are called **right singular vectors** of A.

- **Proof** Let λ_i and v_i be as in Theorem 9, so that $\{Av_1, \ldots, Av_r\}$ is an orthogonal basis for Col A.

THE SINGULAR VALUE DECOMPOSITION

- Normalize each Av_i to obtain an orthonormal basis $\{u_1, \dots, u_r\}$, where

$$u_i = \frac{1}{\|Av_i\|} Av_i = \frac{1}{\sigma_i} Av_i$$

$$\|Av_i\|^2 = \lambda_i \quad \downarrow \quad \|Av_i\| = \sigma_i$$

- And

$$\boxed{Av_i = \sigma_i u_i} \qquad (1 \leq i \leq r) \qquad (4)$$

- Now extend $\{u_1, \dots, u_r\}$ to an orthonormal basis $\{u_1, \dots, u_m\}$ of $\mathbb{R}^m$, and let

$$U = [u_1 \ u_2 \ \dots \ u_m] \quad \text{and} \quad V = [v_1 \ v_2 \ \dots \ v_n]$$

- By construction, U and V are orthogonal matrices.

The matrix U and matrix V both are orthogonal matrices.

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- Also, from (4),

  $AV = [Av_1 \ \ldots \ Av_r \ \ 0 \ \ldots \ 0] = [\sigma_1 u_1 \ldots \sigma_r u_r \ \ 0 \ldots 0]$

- Let D be the diagonal matrix with diagonal entries $\sigma_1, \ldots, \sigma_r$, and let Σ be as in (3) above. Then

$$U\Sigma = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_m] \left[\begin{array}{cccc|c} \sigma_1 & & & 0 & \\ & \sigma_2 & & & 0 \\ & & \ddots & & \\ 0 & & & \sigma_r & \\ \hline & & 0 & & 0 \end{array}\right]$$

$$= [\sigma_1\mathbf{u}_1 \quad \cdots \quad \sigma_r\mathbf{u}_r \quad \mathbf{0} \quad \cdots \quad \mathbf{0}]$$
$$= AV$$

- Since V is an orthogonal matrix, $U\Sigma V^T = AVV^T = A$.

$V^T = V^{-1}$

THE SINGULAR VALUE DECOMPOSITION

- **Example 3** Use the results of Examples 1 and 2 to construct a singular value decomposition of $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$.

- **Solution** A construction can be divided into three steps.
- ***Step 1*. *Find an orthogonal diagonalization of A^TA***. That is, find the eigenvalues of A^TA and a corresponding orthonormal set of eigenvectors. If A had only two columns, the calculations could be done by hand. Larger matrices usually require a matrix program. However, for the matrix A here, the eigendata for A^TA are provided in Example 2.

THE SINGULAR VALUE DECOMPOSITION

We find eigenvalues and their corresponding (orthonormal) eigenvectors of A^TA.

$$A^TA = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

The eigenvalues of A^TA are $\lambda_1 = 360$, $\lambda_2 = 90$, and $\lambda_3 = 0$. Corresponding unit eigenvectors are, respectively,

$$\mathbf{v}_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

THE SINGULAR VALUE DECOMPOSITION

- ***Step 2*. *Set up V and Σ*.** Arrange the eigenvalues of A^TA in decreasing order. In Example 1, the eigenvalues are already listed in decreasing order: 360, 90, and 0. The corresponding unit eigenvectors, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, are the right singular vectors of A. Using Example 1, construct

$$V = [v_1 \;\; v_2 \;\; v_3] = \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix}$$

- The square roots of the eigenvalues are the singular values:

$$\sigma_1 = 6\sqrt{10}, \quad \sigma_2 = 3\sqrt{10}, \quad \sigma_3 = 0$$

THE SINGULAR VALUE DECOMPOSITION

- The nonzero singular values are the diagonal entries of D. The matrix Σ is the same size as A, with D in its upper left corner and with 0's elsewhere.

$$D = \begin{bmatrix} 6\sqrt{10} & 0 \\ 0 & 3\sqrt{10} \end{bmatrix}, \qquad \Sigma = [D \;\; 0] = \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}$$

- ***Step 3. Construct U***. When A has rank r, the first r columns of U are the normalized vectors obtained from $A\mathrm{v}_1, \ldots, A\mathrm{v}_r$. In this example, A has two nonzero singular values, so rank A = 2. Recall from equation (2) and the paragraph before Example 2 that $\|Av_1\| = \sigma_1$ and $\|Av_2\| = \sigma_2$.

THE SINGULAR VALUE DECOMPOSITION

- Thus

$$u_1 = \frac{1}{\sigma_1}Av_1 = \frac{1}{6\sqrt{10}}\begin{bmatrix}18\\6\end{bmatrix} = \begin{bmatrix}3/\sqrt{10}\\1/\sqrt{10}\end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2}Av_2 = \frac{1}{3\sqrt{10}}\begin{bmatrix}3\\-9\end{bmatrix} = \begin{bmatrix}1/\sqrt{10}\\-3/\sqrt{10}\end{bmatrix}$$

- Note that $\{\mathrm{u}_1, \mathrm{u}_2\}$ is already a basis for $\mathbb{R}^2$. Thus no additional vectors are needed for U, and $U = [\mathrm{u}_1 \;\; \mathrm{u}_2]$. The singular value decomposition of A is

$$A = \begin{bmatrix}3/\sqrt{10} & 1/\sqrt{10}\\1/\sqrt{10} & -3/\sqrt{10}\end{bmatrix}\begin{bmatrix}6/\sqrt{10} & 0 & 0\\0 & 3/\sqrt{10} & 0\end{bmatrix}\begin{bmatrix}1/3 & 2/3 & 2/3\\-2/3 & -1/3 & 2/3\\2/3 & -2/3 & 1/3\end{bmatrix}$$

THE SINGULAR VALUE DECOMPOSITION

- **Theorem: The Invertible Matrix Theorem (concluded)**
- Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

u. $(\text{Col } A)^{\perp} = \{0\}$.

v. $(\text{Nul} A)^{\perp} = \mathbb{R}^n$

w. $\text{Row } A = \mathbb{R}^n$

x. A has n nonzero singular values.