Marginal Utility Shocks and the Precautionary Saving Puzzle

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Abstract

Keywords: ...

1 Saving at the Tail

To understand the reason our framework can give rise to a saving function that is increasing in wealth, after some threshold—in contrast to the Bewley (1977), Aiyagari (1994), and Huggett (1993) (BAH)'s framework where saving becomes negative after some threshold—we consider two stylized environments with uninsurable idiosyncratic shocks in this section: An income fluctuations problem, *a lá* BAH, where individuals have to self-insure against fluctuations in their income, and a modified framework where income shocks are replaced by longevity shocks—or, more accurately, shocks to continuation marginal utility.

Consider the problem of an individual in a standard income fluctuations framework, where income can take one of two values, $\left\{\underline{y}, \bar{y}\right\}$, where $\underline{y} < \bar{y}$. If we refer to these two as the "low" and "high states," and denoting the value and policy functions in each state by means of lower and upper bars, respectively, we can write the individual's problem in high state as

$$\begin{aligned} \bar{V}\left(b\right) = \max_{c,b'} \quad & \left\{u\left(c\right) + \rho\left[\pi \bar{V}\left(b'\right) + \left(1-\pi\right)\underline{V}\left(b'\right)\right]\right\} \\ s.t. \quad & b' = \left(1+r\right)b + \bar{y} - c, \\ & b' \geq \underline{b}. \end{aligned} \tag{BAH}$$

In this problem, π is the conditional probability of staying in the high state in the next period, conditioned on being in the high state in the current period. The problem in the low state is rather similar to (BAH), and we omit it here.

In our framework, we replace income shocks by longevity shocks. To this end, we denote an individual's underlying health by h, and assume it can take two values, , $\left\{\underline{h}, \bar{h}\right\}$, where $\underline{h} < \bar{h}$. With some abuse of terminology, we will refer to these two as the "low" and "high states," and keep denoting them via lower and upper bars as in problem (BAH).[1]

To make sure the proceeding arguments deal explicitly and entirely with these two different mechanisms, in this section, we assume that the low state is an absorbing state in this latter

1. This choice of notation serves a simplifying purpose, as it becomes apparent when writing the first order conditions. It should be clear which problem a function refers to from the context. We make the reference explicit if not.

framework: Once an individual enters the low state, he remains there forever. The probability of an individual transitioning to this state from the high state is denoted by $1-\pi$, as in problem (BAH). In addition, we assume that an individual does not face any chance of mortality, whereas this probability in the low state is determined by the *health production function*, $\chi(m)$, where m captures what the individual spends on his health.

As such, the individual's problem in the high state is quite similar to that in (BAH) in this alternative setting:

$$\begin{aligned} \bar{V}(b) = \max_{c,b'} \quad & \left\{u(c) + \rho\left[\pi\bar{V}(b') + (1-\pi)\underline{V}(b')\right]\right\} \\ s.t. \quad & b' = (1+r)b + y - c, \\ & b' \geq \underline{b}, \end{aligned} \tag{BL-H}$$

where y denotes the individuals' income, which is independent of their health state (so that we can focus on marginal utility shocks, as states earlier). In the low state, this problem becomes:

$$\begin{aligned} \underline{V}(b) = \max_{c,m,b'} \quad & \left\{u(c) + \rho\cdot\chi(m)\cdot\underline{V}(b')\right\} \\ s.t. \quad & b' = (1+r)b + y - c - m, \\ & b' \geq \underline{b}. \end{aligned} \tag{BL-L}$$

Note that, in this problem, we have implicitly normalized the value upon death to zero.

In all that follows, we are going to assume that the Bernoulli utility takes the CRRA functional form with a constant term representing the value of being alive:

$$u(c) = \nu + \frac{c^{1-\sigma}}{(1-\sigma)},$$

for some $\sigma > 0$ and $\nu > 0$ such that, for some minimal sustainable level of consumption, $\underline{c}$, we have $u(\underline{c}) > 0$.[2] Moreover, we assume the following form for the health production

2. Otherwise, (BL-L) turns into a trivial problem where individuals prefer death to being alive. The CRRA

function for the sake of tractability:

$$\chi(m) = 1 - \left(\frac{1}{\alpha \cdot m + \underline{h}}\right)^{\beta}, \tag{1}$$

for some $\alpha > 0$ and $\beta \leq 1$.

Consider the Euler equation for an *interior solution* to problems (BAH) and (BL-H):

$$\bar{V}_b(b) = \rho \cdot (1+r) \cdot \left[\pi \bar{V}_b(b') + (1-\pi) \underline{V}_b(b')\right], \tag{HEE}$$

where subscripts show first order derivatives, and we have used the envelope condition to replace for marginal utility of consumption:

$$\bar{V}_b(b) = (1+r) \cdot u_c(c). \tag{HEC}$$

Under the assumption that the value function is concave in wealth in both states, the Euler equation determines the direction of saving at any current level of wealth. To see this more clearly, let us momentarily assume that $\pi = 1$. When so, when $\rho(1+r) = 1$, we must have $b = b'$ and saving will be zero: $c = c' = rb + \bar{y}$ (or $c = c' = rb + y$, when income is constant). In other words, if time discount rate and the rate of return cancel out, individual would "smooth consumption" perfectly. (When $\rho(1+r) < 1$, individual front loads consumption: In this case, for (HEE) to hold, $\bar{V}_b(b)$ must be greater than $\bar{V}_b(b')$. Assuming that the value function is concave, we must have $b' < b$ for this to be the case. The inverse of this argument holds when $\rho(1+r) > 1$.)

In the presence of uninsurable risk, saving might not be negative at lower levels of wealth even when $\rho(1+r) < 1$, as long as the arrival of the shock (whether it is to income or underlying health) significantly increases the marginal effect of assets on future value. We will refer to this—*i.e.* $V_b(b')$—as the *marginal continuation value.* To see this, let us write (HEE) as

$$\pi \left[\frac{\bar{V}_b(b')}{\bar{V}_b(b)}\right] + (1-\pi)\left[\frac{\underline{V}_b(b')}{\bar{V}_b(b)}\right] = \frac{1}{\rho \cdot (1+r)}. \tag{2}$$

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utility form is indispensable to Huggett (1993)'s proof of negative saving above some level.

When value function is concave, the first term in brackets in this equation will be less than one when $b' > b$. Therefore, the only possibility for this equality to hold when $b' > b$ and $\rho(1+r) < 1$ is for $\underline{V}_b(b')$ to greater than $\bar{V}_b(b)$ by a “large enough” margin, and this is the logic behind the terminology “precautionary saving”: If the value of having one more unit of wealth is high enough *when an unfavourable shock is realized*, individual tends to save.

One can summarize this argument as follows: Under the concavity assumption for the value function, saving is positive at any current level of wealth b if, and only if,

$$\pi\left[\frac{\bar{V}_b(b)}{\bar{V}_b(b)}\right] + (1-\pi)\left[\frac{\underline{V}_b(b)}{\bar{V}_b(b)}\right] > \frac{1}{\rho\cdot(1+r)}.$$ [3]

This inequality can be rearranged as

$$(1-\pi)\left[\frac{\underline{V}_b(b) - \bar{V}_b(b)}{\bar{V}_b(b)}\right] > \frac{1}{\rho\cdot(1+r)} - 1. \tag{3}$$

The right-hand side of this equality is a positive constant when $\rho(1+r) < 1$. As such, the shape of optimal saving as a function of current wealth, b, is determined by the marginal value of wealth in the low state *relative to that in the high state* (and not by the difference between the two).

This is rather intuitive: Individual saves as a “precaution” against the possibility of low shocks, and as long as the marginal benefits of one additional unit of saving is high enough should the low shock arrives. Saving is a precaution against fluctuations in marginal continuation value when value function is concave. On the other hand, when the ratio of marginal continuation value in the low state to that in the high state tends to one, this precautionary saving motive vanishes: When there are not much fluctuations in the marginal continuation value in the first place, there are no incentives to insure against such fluctuations either.[4]

3. Similarly, saving will be zero or negative, respectively, when wealth is equal to b if, and only if, this inequality holds as strict equality or the left-hand side is strictly less than the constant term on the right-hand side.

4. This is not to say that the only incentive for saving is precautionary motive. The ratio of time discount rate to the risk-free rate is still a strong determinant of saving behaviour. Moreover, the precautionary saving motive is determined by the probability of the realization of unfavourable shocks.

One can use condition (HEC) to write (3) as

$$(1-\pi)\left[\frac{u_c\left(\underline{c}\left(b\right)\right)-u_c\left(\bar{c}\left(b\right)\right)}{u_c\left(\bar{c}\left(b\right)\right)}\right] > \frac{1}{\rho\cdot(1+r)}-1, \tag{4}$$

where $\underline{c}\left(\cdot\right)$ and $\bar{c}\left(\cdot\right)$ are the policy functions in low and high states, respectively. If we replace the utility function with its CRRA form, we can write this inequality as:

$$\left[\frac{\bar{c}\left(b\right)}{\underline{c}\left(b\right)}\right]^{\sigma} > \left[\frac{\rho^{-1}(1+r)^{-1}-\pi}{1-\pi}\right]. \tag{5}$$

Note that the first term of the numerator on the right-hand side of this inequality is greater than one. As such, the right-hand side is a constant, greater than one.

To see the intuition behind this inequality, we can use a linear expansion around $\bar{c}\left(b\right)$ to write the numerator in (4) in terms of consumption in two states:

$$(1-\pi)\left[\frac{u_{cc}\left(\bar{c}\left(b\right)\right)\left[\underline{c}\left(b\right)-\bar{c}\left(b\right)\right]}{u_c\left(\bar{c}\left(b\right)\right)}\right] > \frac{1}{\rho\cdot(1+r)}-1.$$

Using the CRRA utility form, the ratio on the left-hand side of this inequality can be written as:

$$1-\frac{\underline{c}\left(b\right)}{\bar{c}\left(b\right)} > \frac{1-\rho\cdot(1+r)}{\sigma\cdot(1-\pi)\cdot\rho\cdot(1+r)}.\text{[5]} \tag{6}$$

As such, the shape of saving function depends critically on σ, π, and $\rho\left(1+r\right)$, as well as the ratio $\underline{c}\left(b\right)/\bar{c}\left(b\right)$ given these parameters. As this inequality suggests, if $1-\pi$ or σ are too small, it is well possible for saving to remain negative at all levels of wealth. This is rather intuitive: When the possibility of a negative shock is "too small," or when individuals are "too risk neutral," precautionary saving motive would also be too small. In such cases, if the incentive to front-load consumption is strong enough—when $\rho\left(1+r\right)$ is far from unity—individual might not have any incentives to save against unlikely adverse shocks in the future.

5. Note that, since $u_c\left(\cdot\right)$ itself is a convex function, this inequality is the sufficient condition for the saving rate to be positive and not a necessary condition, at least unless consumption is not "too large."

On the other hand, when $1-\pi$ and σ are large enough, it is possible for the precautionary saving motive to dominate the motive to front-load consumption, leading to positive saving. When this is the case, we'd expect the ratio of consumption in two states to remain far from one. The same forces that determine how this ratio changes as wealth increases (in the optimal solution) also determine the shape of optimal saving as a function of wealth.

The aforementioned arguments are summarize in lemma 1.1, under the assumption that the solutions to the functional equations in (BL-H) and (BL-L) are concave. This assumption, however, cannot be proved analytically: The product of two concave functions on the right-hand side of functional equation (BL-L) is not necessarily concave. As a result, the usual inheritance arguments in dynamic programming do not carry over to this case. As such, throughout this section, we maintain the following assumption:

Assumption 1.1 *The solution to the functional equation* (BL-L), $\underline{V}(\cdot)$, *is concave.*

Note that, if $\underline{V}(\cdot)$ is concave, $\bar{V}(\cdot)$ will necessarily be concave. Moreover, we know from the envelope condition that assumption 1.1 is equivalent to the policy function, $\underline{c}(\cdot)$, being increasing. In our numerical experiments, we have not managed to come up with a set of parameter values that violate this latter condition.

Lemma 1.1 *Under assumption 1.1, optimal saving in* (BAH) *or* (BL-H) *is positive when wealth is b if, and only if, the ratio of optimal consumption in high state to that in low state satisfies* (5).

What about non-interior solutions to (BL-H)? Euler equation in high vs low states, binding of borrowing constraint

The preceding argument suggests that, to determine the shape of saving as a function of current wealth in either problem, one can focus on the ratio $\bar{c}(b)/\underline{c}(b)$ and how it changes with wealth. One can show that, in problem (BAH), this ratio converges to one as $b \to \infty$, and, as such, there exists some $\bar{b}$ above which saving inevitably becomes negative.[6]

6. This claim, stated by Huggett (1993) in Theorem 2, enables him to show that an individual's wealth does not exceed $\bar{b}$. As such, the state-space is bounded and a stationary distribution of wealth can emerge as a result. We postpone the proof of this claim for problem (BAH)—which is different from Huggett's proof—to appendix A.

In our environment with health shocks, on the other hand, the same mechanics but in the opposite direction indicate the exact inverse regarding the saving behaviour: that optimal saving may have to be positive above some threshold $\bar{b}$ (while it can be negative below $\bar{b}$). The reason that the ratio $\bar{c}(b)/\underline{c}(b)$ may be bounded away from one in problem (BL-H) is due to the role of health spending in (BL-L).

The intuition is as follows: If health spending is a luxury good, its share in total expenditures must increase with wealth compared to that of consumption. In other words, in the presence of this luxury good, the marginal value of saving remains high in the low state compared to that in the high state, even as individual becomes wealthier. As such, the precautionary saving motive does not vanish with wealth, as is the case in the income fluctuation problem in (BAH).

To formally derive the conditions under which inequality (5) holds, at least above some wealth threshold $\bar{b}$, we start by characterizing the optimal medical spending in the low-state, in (BL-L). When interior, the optimal health spending is given by the following first order condition:

$$\rho \cdot \chi_m(m) \cdot \underline{V}(b') = u_c(c).$$

Replacing the utility and health production functions by their functional forms, we can rearrange this equation to find optimal health spending as a function of optimal consumption in the low state. This is given by

$$\underline{m}(b) = \left[\frac{\rho \cdot \beta \cdot \underline{V}(\underline{b}'(b))}{\alpha^{\beta}}\right]^{\left(\frac{1}{1+\beta}\right)} \cdot [\underline{c}(b)]^{\left(\frac{\sigma}{1+\beta}\right)} - \left(\frac{h}{\alpha}\right), \tag{7}$$

when the right-hand side of (7) is positive, and zero otherwise. In this equation, $\underline{m}(\cdot)$, $\underline{c}(\cdot)$ and $\underline{b}'(\cdot)$ are the policy functions for medical spending, consumption and future wealth in the low state, respectively.

We are now in a position to prove the following theorem, the central result of this paper:

Theorem 1.2 *Under assumption 1.1, saving rate in problem* (BL-H) *cannot be negative above some threshold $\bar{b}$, when $\sigma > 1 + \beta$.*

Proof. We accept, without proof, that the value function in (BL-L) is non-decreasing in current wealth. As such, we can use (7) to conclude:

$$\begin{aligned}\underline{m}\left(b\right)=\left[\frac{\rho\cdot\beta\cdot\underline{V}\left(\underline{b}'\left(b\right)\right)}{\alpha^{\beta}}\right]^{\left(\frac{1}{1+\beta}\right)}\cdot\left[\underline{c}\left(b\right)\right]^{\left(\frac{\sigma}{1+\beta}\right)}-\left(\frac{h}{\alpha}\right)\\ \geq\left[\frac{\rho\cdot\beta\cdot\underline{V}\left(\underline{b}\right)}{\alpha^{\beta}}\right]^{\left(\frac{1}{1+\beta}\right)}\cdot\left[\underline{c}\left(b\right)\right]^{\left(\frac{\sigma}{1+\beta}\right)}-\left(\frac{h}{\alpha}\right)\\ =A\cdot\left[\underline{c}\left(b\right)\right]^{\left(\frac{\sigma}{1+\beta}\right)}-\left(\frac{h}{\alpha}\right),\end{aligned}\tag{8}$$

where

$$A\coloneqq\left[\frac{\rho\cdot\beta\cdot\underline{V}\left(\underline{b}\right)}{\alpha^{\beta}}\right]^{\left(\frac{1}{1+\beta}\right)}>0.$$

The claim that $A>0$ follows from the assumption that, even at $b=\underline{b}$, individual can maintain a sustenance level of consumption, delivering a utility that is strictly greater than zero—*i.e.* the utility upon death. When $\sigma>1$, the value of being alive, $\nu>0$, guarantees that this is the case. Since such a stream of consumption is sustainable as long as the individual lives, value function must remain strictly positive for all b.

Let us assume, for the sake of contradiction, that saving is non-positive above some arbitrary threshold $\tilde{b}$. Then, we must have

$$\bar{c}\left(b\right)\geq r\cdot b+y,\qquad\forall b\geq\tilde{b}.$$

If we let

$$\gamma\coloneqq\liminf_{b\to\infty}\frac{\bar{c}\left(b\right)}{\underline{c}\left(b\right)},$$

condition (5) implies that

$$\gamma\leq\left[\frac{\rho^{-1}(1+r)^{-1}-\pi}{1-\pi}\right]^{\frac{1}{\sigma}},\tag{9}$$

above some threshold of wealth. Therefore, we can assume without any loss of generality that, for all $b\geq\tilde{b}$,

$$\underline{c}\left(b\right)\geq\gamma\cdot\bar{c}\left(b\right)=\gamma\left(r\cdot b+y\right),\tag{10}$$

and—from (8),

$$\underline{m}(b) \geq A\left[\gamma \cdot \bar{c}(b)\right]^{\left(\frac{\sigma}{1+\beta}\right)} - \left(\frac{\underline{h}}{\alpha}\right) \geq A\left[\gamma\left(r \cdot b + y\right)\right]^{\left(\frac{\sigma}{1+\beta}\right)} - \left(\frac{\underline{h}}{\alpha}\right). \tag{11}$$

If we replace these in individual's budget constraint, we get:

$$\begin{aligned}
\underline{b}'(b) &= (1+r)\,b + y - \underline{c}(b) - \underline{m}(b) \\
&\leq (1+r)\,b + y + \left(\frac{\underline{h}}{\alpha}\right) - \gamma\left(r \cdot b + y\right) - A\left[\gamma\left(r \cdot b + y\right)\right]^{\left(\frac{\sigma}{1+\beta}\right)} \\
&= -A\left[\gamma\left(r \cdot b + y\right)\right]^{\left(\frac{\sigma}{1+\beta}\right)} + (1 + r - \gamma \cdot r)\,b + (1-\gamma)\,y + \left(\frac{\underline{h}}{\alpha}\right).
\end{aligned} \tag{12}$$

When $\sigma > 1 + \beta$, the polynomial on the right-hand side of this inequality is concave and, as such, decreases without bound with b. As a result, the borrowing constraint must bind for all $b \geq \hat{b}$, for some large enough $\hat{b}$. We let $\bar{b}$ denote the maximum of this threshold and $\tilde{b}$:

$$\bar{b} := \max\left\{\tilde{b}, \hat{b}\right\}.$$

The Kuhn-Tucker conditions for problem (BL-H) when the borrowing constraint binds imply that

$$u_c(c) = \rho \cdot \chi(m) \cdot \underline{V}_b(b') + \mu, \tag{13}$$

where $\mu \geq 0$ is the Lagrange multiplier on the borrowing constraint, which is strictly positive when this constraint binds. Thus, for all $b \geq \bar{b}$, we should have:

$$u_c(c) = \chi(m) \cdot \rho \cdot \underline{V}_b(\underline{b}) + \mu. \tag{14}$$

As noted earlier, the term $\rho \cdot \underline{V}_b(\underline{b})$ in this equality is a positive constant, independent of b. Moreover, from (10) and (11), we know that $\underline{c}(b)$ and $\underline{m}(b)$ both increase without bound with b. Therefore, $u_c(\underline{c}(b)) \searrow 0$ and $\chi(\underline{m}(b)) \nearrow 1$, as $b \to \infty$. Therefore, as long as μ is to remain positive, equation (14) is bound to get violated for some $b \geq \bar{b}$. This contradicts the optimality of $\underline{c}(b)$ and $\underline{m}(b)$, and completes our proof. □

In the proof of theorem 1.2, what causes a contradiction is not whether or not $\bar{c}(b)/\underline{c}(b)$

converges to a constant term *per se*. In fact, this ratio can theoretically converge to a constant even when saving remains positive for all b above some threshold $\bar{b}$, as long as this ratio is above the threshold prescribed by (5). What leads to the contradiction is the assumption that saving becomes negative for large enough b. As such, $\bar{c}(b)$ must increase with $rb + y$ to keep the saving negative. Moreover, negative saving implies that $\bar{c}(b)$ and $\underline{c}(b)$ cannot diverge. It is the combination of these two that lead to a contradiction. For saving to remain positive, then, we need the ratio $\bar{c}(b) / \underline{c}(b)$ to remain above

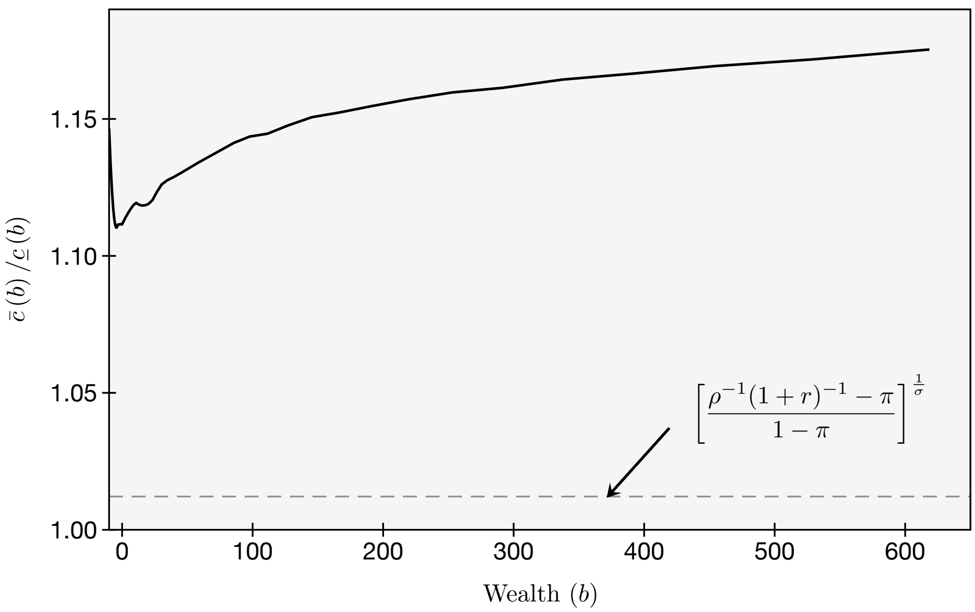
$$\left[\frac{\rho^{-1}(1+r)^{-1} - \pi}{1-\pi}\right]^{\frac{1}{\sigma}}.$$

As figure 1 demonstrates how, for our sample choice of parameter values in ..., consumption in the high state diverges from total income, leading to excessive saving. Panel (b) in this graph suggests that the ratio $\bar{c}(b) / \underline{c}(b)$ remains consistently above the right-hand side of (5). (In the numerical solutions to the problem in (BL-H) and (BL-L), this ratio seems to be exploding, though at a slow rate, as can be seen in figure 1 too.)

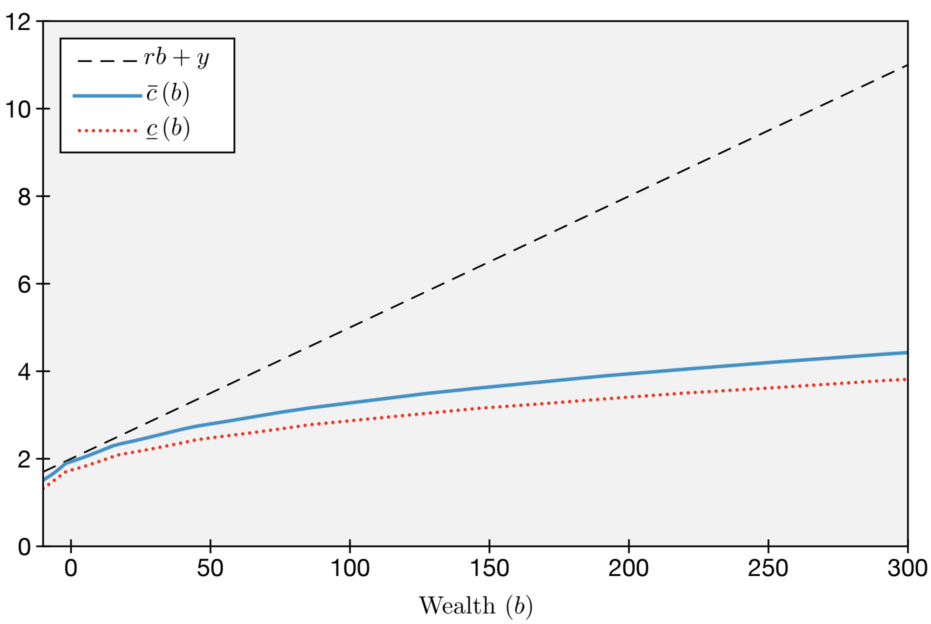
The condition $\sigma > 1 + \beta$ in theorem 1.2 has an intuitive and straightforward interpretation: As long as the *curvature* of the marginal utility (and not the marginal utility itself) is greater than the *curvature* of the marginal product of health spending (in the health production function), then health is a luxury good. It is important to note that it is not the result that health spending is a luxury that causes precautionary saving to increase without bound, *per se*; But It is the that such spending is necessitated by an adverse health shock.

It is also worth mentioning that $\sigma > 1+\beta$ is not a necessary condition for health spending to increase faster than consumption with wealth, but a sufficient one: Any assumption implied by equation (7) that causes health spending to increase faster than consumption with b suffices for the proof of theorem 1.2 to go through. This can be a high enough value of being alive (ν) or a high enough share of health spending (α) in the health production function, as confirmed by our numerical results.

Figure 1. Consumption in Low and High States as Function of Wealth



Panel (a). Consumption in high state to low state



Panel (b). Consumption in two states vs income

Note: For $\bar{c}(b)/\underline{c}(b)$, centred moving averages are drawn to smooth out high-frequency fluctuations at lower wealth levels due to numerical error.

2 High Utilization State and Realized Health Spending

Life expectancy in low state:

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Appendices

A Huggett (1993): Theorem 2

Lemma A.1 *For all b, $b_l'(b) \leq b$.*

Proof. The Euler equation for the low-type with wealth b is

$$u'(c_l(b)) = \frac{\rho}{(1+r)}\left[\pi_l u'(c_l(b_l'(b))) + (1-\pi_l)\, u'(c_h(b_l'(b)))\right]$$
$$< \pi_l u'(c_l(b_l'(b))) + (1-\pi_h)\, u'(c_h(b_l'(b)))\,. \quad (15)$$

Since $c_l(b) \leq c_h(b)$, this last inequality holds only if

$$u'(c_l(b_l'(b))) > u'(c_l(b))\,,$$

or $c_l(b_l'(b)) < c_l(b)$. Since $c_l(b)$ is non-decreasing in b, this is true only if $b_l'(b) < b$. □

Note that $b_l'(-\underline{b}) \leq -\underline{b}$ can only hold as a strict equality. As a result, $b_l'(b)$ must pass through the point $(-\underline{b}, -\underline{b})$ and lie under the 45-degree line for all values of b.

Lemma A.2 *When $b_h'(b) > b$, then $c_h(b) - c_l(b) < e_h - e_l$.*

Proof. By lemma A.1, we know that $rb + e_l - c_l(b) \leq 0$, or

$$c_l(b) - e_l \geq rb.$$

On the other hand, for $b_h'(b) > b$, we need to have $c_h(b) - e_h < rb$, or

$$-[c_h(b) - e_h] > -rb.$$

If we add these two inequalities, we get the intended result. □

A corollary of this is that

$$0 < b_h'(b) - b_l'(b) = e_h - e_l - [c_h(b) - c_l(b)] < e_h - e_l. \tag{16}$$

Therefore,

$$b_l'(b) \geq b - (e_h - e_l). \tag{17}$$

If this were the case, then

If we let $\kappa := (e_h - e_l)$, from lemma A.2, we can write $c_h(b) - \kappa < c_l(b) \leq c_h(b)$, where the second inequality follows from the monotonicity of optimal consumption. Therefore,

$$u'(c_l(b)) < u'(c_h(b) - \kappa) \approx u'(c_h(b)) - \kappa u''(c_h(b)). \tag{18}$$

Let us plug this into the Euler equation for the high-types:

$$\begin{aligned} u'(c_h(b)) = \frac{\rho}{(1+r)} &[\pi_h u'(c_h(b_h'(b))) + (1-\pi_h) u'(c_l(b_h'(b)))] \\ &< \pi_h u'(c_h(b_h'(b))) + (1-\pi_h) u'(c_l(b_h'(b))) \\ &< \pi_h u'(c_h(b_h'(b))) + (1-\pi_h)[u'(c_h(b_h'(b))) - \kappa u''(c_h(b_h'(b)))]. \end{aligned} \tag{19}$$

The last inequality can be rearranged and written as

$$u'\left(c_h\left(b\right)\right)-u'\left(c_h\left(b_h'\left(b\right)\right)\right)<-\left(1-\pi_h\right)\kappa u''\left(c_h\left(b_h'\left(b\right)\right)\right). \tag{20}$$

If we replace for the first and second derivatives of the utility function, assuming a CRRA felicity, we have

$$\frac{1}{\left[c_h\left(b\right)\right]^{\sigma}}-\frac{1}{\left[c_h\left(b_h'\left(b\right)\right)\right]^{\sigma}}<\frac{\left(1-\pi_h\right)\kappa}{\left[c_h\left(b_h'\left(b\right)\right)\right]^{\sigma+1}}. \tag{21}$$

The next theorem uses this inequality to show that the individual's saving rate must eventually become negative, regardless of her level of current income.

Theorem A.3 *There must exist some* $\underline{b}<\bar{b}<\infty$ *for which, for all* $b\geq\bar{b}$*, we must have* $rb+e_h-c_h\left(b\right)\leq 0$.

Proof. Suppose, for the sake of contradiction, that this were not the case, and, for all $\underline{b}<b$, optimal policy satisfies $rb+e_h-c_h\left(b\right)>0$. Then, by the monotonicity of the optimal consumption in income, and by lemma A.1, we have

$$rb+e_l<c_l\left(b\right)\leq c_h\left(b\right)<rb+e_h. \tag{22}$$

In other words, as b □