Ex 1

i) $\neg(p \Rightarrow q) = \neg(\neg p \vee q)$

p	q	$p \Rightarrow q$	$\neg(p \Rightarrow q)$	$\neg p$	$\neg p \vee q$	$\neg(\neg p \vee q)$
T	T	T	F	F	T	F
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	F	T	F	T	T	F

ii) $((p \wedge q) \vee r) = \{\text{De Morgan's law}\} = (\neg(\neg p \vee \neg q)) \vee r$

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$\neg(\neg p \vee \neg q)$	$(\neg(\neg p \vee \neg q)) \vee r$
T	T	T	T	T	F	F	F	T	T
T	T	F	T	T	F	F	F	T	T
T	F	T	F	T	F	T	T	F	T
T	F	F	F	F	F	T	T	F	F
F	T	T	F	T	T	F	T	F	T
F	T	F	F	F	T	F	T	F	F
F	F	T	F	T	T	T	T	F	T
F	F	F	F	F	T	T	T	F	F

iii) $\neg((p \wedge q) \Leftrightarrow r) = \neg(((p \wedge q) \rightarrow r) \wedge (r \rightarrow (p \wedge q))) =$

$= \neg((\neg(p \wedge q) \vee r) \wedge (\neg r \vee (p \wedge q))) =$

$= \neg(\neg(p \wedge q) \vee r) \vee \neg(\neg r \vee (p \wedge q)) =$

$= \neg(\neg p \vee \neg q \vee r) \vee \neg(\neg r \vee \neg(\neg p \vee \neg q))$

p	q	r	$p \wedge q$	$(p \wedge q) \Leftrightarrow r$	$\neg((p \wedge q) \Leftrightarrow r)$	$\neg p$	$\neg q$	$(\neg p \vee \neg q)$	$\neg(\neg p \vee \neg q)$
T	T	T	T	T	F	F	F	F	T
T	T	F	T	F	T	F	F	F	T
T	F	T	F	F	T	F	T	T	F
T	F	F	F	T	F	F	T	T	F
F	T	T	F	F	T	T	F	T	F
F	T	F	F	T	F	T	F	T	F
F	F	T	F	F	T	T	T	T	F
F	F	F	F	T	F	T	T	T	F

$\neg r$	$\neg r \vee \neg(\neg p \vee \neg q)$	$\neg(\neg r \vee \neg(\neg p \vee \neg q))$	$\neg p \vee \neg q \vee r$	$\neg(\neg p \vee \neg q \vee r)$
F	T	F	T	F
T	T	F	F	T
F	F	T	T	F
T	T	F	T	F
F	F	T	T	F
T	T	F	T	F
F	F	T	T	F
T	T	F	T	F

$\neg(\neg p \vee \neg q \vee r) \vee \neg(\neg r \vee \neg(\neg p \vee \neg q))$

F
T
T
F
T
F
T
F

Ex 2

i) Frodo has a ring

$\exists$ ring: has (Frodo, ring)

ii) Sauron does not have any rings

Not has (Sauron, ring) $\forall$ ring

iii) The One Ring rules all the other rings

rules (The One Ring, ring) $\forall$ ring

iv) The ring that Frodo has is the One Ring

$\exists$ ring: has (Frodo, ring) and ring = the One Ring

v) Whoever wears the ring, becomes invisible (i.e., no other ordinary human can see that person).

$\forall$ person: wears(person, ring), not sees (human, person)

$\forall$ human: ordinary (human)

vi) Bombadil Tom can see the ring-wearer, hence he is no ordinary human

$\forall$ person: wears (person, ring), sees (Tom, person)

$\Rightarrow$ not ordinary (Tom).

Ex 3

i) $R_\rho : A \leftrightarrow A$ is irreflexive if $\forall a \in A$: not $a \rho a$

ii) $R_\rho : A \leftrightarrow A$ is intransitive if $\exists a, b, c \in A : (a \rho b \wedge b \rho c \wedge \neg(a \rho c))$

iii) $R_\rho : A \leftrightarrow A$ is not a partial order if any of these conditions are not met:
- reflexive
- antisymmetric
- transitive

$R_\rho : A \leftrightarrow A$ is not a partial order if

$(\exists a \in A$: not $a \rho a)$ OR $(\exists a, b \in A : a \rho b \wedge b \rho a \wedge a \neq b)$ OR

$(\exists a, b, c \in A : (a \rho b \wedge b \rho c \wedge \neg(a \rho c)))$.

Ex 4

i) $f: \mathbb{Z} \to \mathbb{N}$, $\forall n \in \mathbb{Z}: f(n) = n^3 + 1$

$\forall a, b \in \mathbb{Z}$, $f(a) = f(b) \Rightarrow a^3 + 1 = b^3 + 1 \Rightarrow a^3 = b^3 \Rightarrow a = b$

$\Rightarrow$ this function is injective.

$f(n)$ is not surjective. For example, if $y = 7$, then

$f(n) = 7 \Rightarrow n^3 + 1 = 7 \Rightarrow n^3 = 6 \Rightarrow n = \sqrt[3]{6} \notin \mathbb{Z}$

ii) $g: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $\forall (n,k) \in \mathbb{N} \times \mathbb{N}: g(n,k) = 2^n 3^k 5^{n+k}$

This function is injective: $\forall (n_1, k_1), (n_2, k_2) \in \mathbb{N} \times \mathbb{N}$, $g(n_1, k_1) =$
$= g(n_2, k_2)$, $2^{n_1} 3^{k_1} 5^{n_1 + k_1} = 2^{n_2} 3^{k_2} 5^{n_2 + k_2} \Rightarrow$

{by uniqueness of prime factorisation} $\Rightarrow n_1 = n_2$, $k_1 = k_2$

$g(n,k)$ is not surjective. For example, if $y = 7$, then

$g(n,k) = 7 \Rightarrow 2^n 3^k 5^{n+k} = 7$ and it is impossible for $(n,k) \in \mathbb{N} \times \mathbb{N}$.

iii) $h: P(\mathbb{N}) \to P(\mathbb{N})$, $\forall A \in P(\mathbb{N}): h(A) = \mathbb{N} \setminus A$

This function is injective: $\forall A_1, A_2 \in P(\mathbb{N})$, $h(A_1) = h(A_2)$

$\Rightarrow \mathbb{N} \setminus A_1 = \mathbb{N} \setminus A_2 \Rightarrow A_1 = A_2$.

This function is also surjective: $\forall B \in P(\mathbb{N})$ $\exists A \in P(\mathbb{N})$:

$h(A) = \mathbb{N} \setminus A = B$.

iv) $k: \mathbb{N} \to \mathbb{Z}$, $\forall n \in \mathbb{N}: k(n) = (-1)^n$

This function is not injective. $\exists a, b \in \mathbb{N}: k(a) = k(b)$, $a \neq b$.

For example, $a = 5, b = 7$, $a \neq b$, but $k(a) = k(5) = (-1)^5 = -1 =$

$= k(7) = k(b)$.

This function is not surjective. for ~~$\forall$~~ y: $y \neq 1$ and $y \neq -1$

$\nexists x \in \mathbb{N}: k(x) = y$.

Ex 5.

Let's define possible events.

A_1: 0 white 4 black

A_2: 1 white 3 black

A_3: 2 white 2 black

A_4: 3 white 1 black

A: drawing of a white ball from the vessel.

$$P(A_1) = \frac{C_5^4}{C_8^4} = 1/14 \qquad P(A|A_1) = 0$$

$$P(A_2) = \frac{C_3^1 \cdot C_5^3}{C_8^4} = 3/7 \qquad P(A|A_2) = 1/4$$

$$P(A_3) = \frac{C_3^2 \cdot C_5^2}{C_8^4} = 3/7 \qquad P(A|A_3) = 1/2$$

$$P(A_4) = \frac{C_3^3 \cdot C_5^1}{C_8^4} = 1/14 \qquad P(A|A_4) = 3/4$$

By Bayes Theorem:

$$P(A_4|A) = \frac{1/14 \cdot 3/4}{3/7 \cdot 1/4 + 3/7 \cdot 1/2 + 1/14 \cdot 3/4} = \frac{1}{7} = 0.14$$

Ex 6

$P(\text{detected tub} \mid \text{ill}) = 1-\beta$

$P(\text{detected tub} \mid \text{healthy}) = a$

$c \sim \text{ill} \Rightarrow 1-c \sim \text{healthy}$

$P(\text{heathy} \mid \text{detected tub})$ – ?

Applying Bayes Theorem $P(A \mid B) = \dfrac{P(B \mid A) \cdot P(A)}{P(B)}$

$$P(\text{healthy} \mid \text{detected tub}) = \frac{P(\text{detected tub} \mid \text{healthy}) \cdot P(\text{healthy})}{P(\text{detected tub})} =$$

$$= \frac{P(\text{detected tub} \mid \text{healthy}) \cdot P(\text{healthy})}{P(\text{detected tub} \mid \text{healthy}) \cdot P(\text{healthy}) + P(\text{detected tub} \mid \text{ill}) \cdot P(\text{ill})} =$$

$$= \frac{a \cdot (1-c)}{a \cdot (1-c) + (1-\beta) \cdot c}$$

Ex 7.

$P(\text{two boys} \mid \text{at least one boy})$ – ?

$$P(\text{two boys} \mid \text{at least one boy}) = \frac{P(\text{two boys, at least one boy})}{P(\text{at least one boy})} =$$

G G

G B

B G

B B

$$= \frac{1/4}{3/4} = 1/3.$$

Ex 8

a) Beef: 186, 181, 176, 149, 184, 190, 158, 139, 175, 148, 152, 111

Poultry: 129, 132, 102, 106, 94, 102, 87, 99, 170, 113

	mean	std
beef	162.41	113.4
poultry	23.7	24.46

$$\text{mean} = \frac{\sum x_i}{n}$$

$$\text{std} = \sqrt{\frac{\sum (x-\bar{x})^2}{n-1}}$$

H_0: $\mu_B = \mu_p$ (There is no difference between the calorie content of beef and poultry)

H_1: $\mu_B > \mu_p$ (the calorie content of beef is higher than the calorie content of poultry)

$$t = \frac{\mu_B - \mu_p}{\sqrt{\frac{\sigma_B^2}{n_1} + \frac{\sigma_p^2}{n_2}}}$$

$$t = \frac{162.41 - 113.4}{\sqrt{\frac{23.7^2}{12} + \frac{24.46^2}{10}}} = \frac{49.01}{\sqrt{\frac{561.69}{12} + \frac{598.2916}{10}}} = \frac{49.01}{\sqrt{106.637}} =$$

$$= \frac{49.01}{10.3265} = 4.746.$$

$d = 20$

$P(t > 4.746) = 0.00024$

$0.00024 < 0.05 \Rightarrow$ we reject the null hypothesis

$\Rightarrow$ accept that the calorie content of beef is more that poultry.

b) 181, 191, 186, 129, 178, 194, 139, 122, 195, 158, 158, 104

$\mu = 161.25$

$\sigma = 31.26$

$H_0: \mu = \mu_{beef}$ (Mix of food has not an effect)

$H_1: \mu \neq \mu_{beef}$. (Mix of food has an effect)

Again, $t = \dfrac{\mu - \mu_{beef}}{\sqrt{\dfrac{\sigma}{n} + \dfrac{\sigma_{beef}}{n_{beef}}}} = \dfrac{162.41 - 161.25}{\sqrt{\dfrac{23.7^2}{12} + \dfrac{31.26^2}{12}}} = 0.1$

$p(t > 0.1) = 0.921$

$\Rightarrow$ we get high p-value $\Rightarrow$ we can accept null hypothesis

$\Rightarrow$ mix of food has not an effect in calorie of the beef hotdog.