Master's Thesis

Pontryagin Ring of Loop Spaces

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Abstract

Loop spaces are of fundamental importance in Algebraic Topology. They lie at the very foundation of the field, having motivated Marston Morse to develop his theory of geodesics on spheres, as well as Raoul Bott on his path toward his periodicity theorem. In geometry, they appear almost everywhere. Their homology has a ring structure known as the Pontryagin ring. This ring is computed using three main methods: the Cobar construction, the Adams-Hilton model (AH), and the Milnor-Moore theorem (MM).

Resumé

Les espaces de lacets occupent une place fondamentale en topologie algébrique. Ils jouent un rôle majeur dans ce domaine, ayant motivé Marston Morse à développer sa théorie des géodésiques sur les sphères, ainsi que Raoul Bott dans sa démarche vers son théorème de périodicité. En géométrie, ils apparaissent presque partout. Leur homologie possède une structure d'anneau connue sous le nom d'anneau de Pontryagin. Cet anneau est calculé à l'aide de trois méthodes principales : la construction de Cobar, le modèle d'Adams-Hilton (AH), et le théorème de Milnor-Moore (MM).

1 Introduction

The primary aim of this thesis is to present the principal methods for computing homology of loop spaces while respecting the chronological order in which they were developed.

In Section [2], we recall the algebraic notions on which our work rests, and in Section [3], we review certain topological constructions that serve as our framework. Throughout these sections, we try to include as many examples as possible to clarify the concepts.

Section 4 introduces a fundamental object in algebraic topology: the H-space. We study the Pontryagin algebra of its homology, which naturally leads us to the notion of the loop space and its rich structure. At the end of this section, we state Milnor-Moore's structure theorem on graded, connected, cocommutative Hopf algebras over a field of characteristic zero, from which one deduces the homology of the loop space with rational coefficients.

Before addressing explicit methods for computing the homology of these Pontryagin algebras, we introduce in Section 5 the notions of fibration and cofibration, which will be useful later. In Section [6], we present a variant of the standard loop space: the space of Moore loops, which is a deformation retract of the standard loop space ΩX and more suitable for certain constructions.

Sections [7], [8], and [9] present three important methods, in historical order. In Section [7], we describe the James construction, through which we study the homology of the loop space of a suspension of a CW-complex. In Section [8], we introduce the Adams–Hilton method and perform some computations. In Section [9], we turn to the Cobar construction, highlighting how it overcomes some limitations of Adams–Hilton.

2 Algebraic Background

2.1 Graded algebra

Let R be a commutative ring with unit. We will work with graded modules over R. Here, graded will mean graded over the non-negative integers, so such an object M has a decomposition

$$M = \bigoplus_{n \geq 0} M_n$$

where each M_n is an R-module. If $x \in M_n$, then we write $|x| = n$ to denote its degree. We will work with modules of finite type, which means that each M_n will be a finitely generated R-module. A homomorphism $f : M \to N$ between graded modules M, N consists of a family of R-module homomorphisms

$$f_n : M_n \longrightarrow N_n.$$

Definition 2.1. An **R-algebra** with unit is a module M (over R) with a bilinear multiplication map

$$\mu : M \otimes M \longrightarrow M,$$

and a unit, given by a linear unit map

$$\eta : R \longrightarrow M,$$

such that the following diagrams commute

$$\begin{tikzcd}
M \otimes M \otimes M \arrow[r, "\mu\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mu"'] & M \otimes M \arrow[d, "\mu"] \\
M \otimes M \arrow[r, "\mu"] & M
\end{tikzcd}$$

and

$$\begin{tikzcd}
R \otimes M \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[dr, "\cong"'] & M \otimes M \arrow[d, "\mu"] & M \otimes R \arrow[l, "\mathrm{id}\otimes\eta"'] \arrow[dl, "\cong"] \\
& M &
\end{tikzcd}$$

where $\cong$ is an isomorphism

Definition 2.2. A linear map $f : M \longrightarrow N$ between two algebras is a morphism if the following diagram commutes:

$$\begin{tikzcd}
M \otimes M \arrow[r, "f\otimes f"] \arrow[d, "\mu_M"'] & N \otimes N \arrow[d, "\mu_N"] \\
M \arrow[r, "f"] & N
\end{tikzcd}$$

Definition 2.3. A **graded R-algebra** M is a graded module, equipped with bilinear products

$$M_p \otimes_R M_q \longrightarrow M_{p+q}$$

and an element $1 \in M_0$ making M_0 and $\bigoplus_{p\geq 0} M_p$ into associative R-algebras with unit.

The R-algebra M is graded-commutative if we have

$$a \cdot b = (-1)^{|a||b|}\, b \cdot a$$

Definition 2.4. A morphism between two graded algebras is a linear map

$$f : M \longrightarrow N$$

such that $f(M_i) \subseteq N_i$ for all $i \geq 0$, and satisfies $f(a \cdot b) = f(a) \cdot f(b)$ for all $a, b \in M$.

Definition 2.5. A linear map

$$f : M \longrightarrow N$$

between graded algebras is said to be of degree n if

$$f(M_i) \subseteq N_{i+n} \quad \text{for all } i.$$

Definition 2.6. A **differential graded algebra (DG-algebra)** is a graded R-algebra M, equipped with a map

$$d : M_* \longrightarrow M_*$$

of degree (-1), satisfying $d^2 = 0$ and the graded Leibniz rule:

$$d(a \cdot b) = d(a) \cdot b + (-1)^{|a|}\, a \cdot d(b) \tag{1}$$

Definition 2.7. A morphism of differential graded algebras is a morphism of graded algebras $f : M_* \longrightarrow N_*$ that commutes with the differential.

Example 2.8. (Cohomology Ring). The cup product defines an associative (and distributive) graded commutative product operation in cohomology as follows

$$\smile : H^p(X;R) \otimes H^q(X;R) \longrightarrow H^{p+q}(X;R)$$

turning the cohomology of a space X into a graded ring $H^*(X)$ called the cohomology ring.

Example 2.9. [**Tensor algebra**] If M, N are graded modules, then their graded tensor product is the graded module $M \otimes N$ with

$$(M \otimes N)_n = \bigoplus_{i+j=n} M_i \otimes_R N_j.$$

Let $f : M \longrightarrow M'$ and $g : N \longrightarrow N'$ be linear maps of degree k and l, respectively. Then the tensor product map

$$f \otimes g : M \otimes N \longrightarrow M' \otimes N'$$

is the linear map of degree $k + l$ defined by

$$(f \otimes g)(a \otimes b) = (-1)^{k.l} f(a) \otimes g(b).$$

The graded module structure over a ring R of the tensor product $M \otimes N$ is also a graded algebra with the product given by:

$$(a \otimes a') \cdot (b \otimes b') = (-1)^{|a'||b|}(ab \otimes a'b').$$

Let M be a graded module. The tensor algebra on M is the graded algebra $T(M)$ defined as follows

$$T(M) = \bigoplus_{n \geq 0} M^{\otimes n}.$$

2.2 Graded Coalgebra

Definition 2.10. A **coalgebra** over a ring R is a triple (C, Δ, ε), where C is a R-module,

$$\Delta : C \longrightarrow C \otimes C$$

is the comultiplication map, and

$$\varepsilon : C \longrightarrow R$$

is the counit map, both are morphisms of R-module, satisfying the following commutative diagrams:

- Coassociativity

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathrm{id}\otimes\Delta"] \\ C \otimes C \arrow[r, "\Delta\otimes\mathrm{id}"] & C \otimes C \otimes C \end{tikzcd}$$

- Counit condition

$$\begin{tikzcd} R\otimes C & C\otimes C \arrow[l, "\varepsilon\otimes\mathrm{id}"'] \arrow[r, "\mathrm{id}\otimes\varepsilon"] & C\otimes R \\ & C \arrow[ul, "\cong"] \arrow[u, "\Delta"] \arrow[ur, "\cong"'] & \end{tikzcd}$$

Definition 2.11. Let (C, Δ, ε) and $(C', \Delta', \varepsilon')$ be coalgebras. A linear map $f : C \to C'$ between two coalgebras is a morphism if the following diagram commutes :

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta"'] & C' \arrow[d, "\Delta'"] \\ C\otimes C \arrow[r, "f\otimes f"] & C'\otimes C' \end{tikzcd}$$

Example 2.12. Let $T(V)$ be the tensor algebra on a graded vector space V over a field k. We define the comultiplication

$$\Delta\colon T(V) \longrightarrow T(V)\otimes T(V)$$

and the counit

$$\varepsilon\colon T(V) \longrightarrow k$$

as follows

$$\begin{aligned} \Delta(v_1\otimes v_2\otimes\cdots\otimes v_n) &= \sum_{i=0}^{n}(v_1\otimes\cdots\otimes v_i)\otimes(v_{i+1}\otimes\cdots\otimes v_n), \\ \varepsilon(1) &= 1, \\ \varepsilon(v_1\otimes v_2\otimes\cdots\otimes v_n) &= 0 \quad \text{if } n\geq 2. \end{aligned}$$

$(T(V), \Delta, \varepsilon)$ is a graded coalgebra.

Example 2.13. The homology groups $H_*(X)$ of any topological space, with coefficients in a field, so that $H_*(X)\otimes H_*(X) \xrightarrow{\simeq} H_*(X\times X)$, carries a coalgebra structure, where the comultiplication

$$\Delta : H_*(X) \longrightarrow H_*(X)\otimes H_*(X)$$

is induced by the diagonal map $\Delta_X : X \longrightarrow X\times X$.

Example 2.14. If G is a group, then the group algebra kG is a **coalgebra**, with comultiplication and counit defined by:

$$\Delta(g) = g\otimes g, \quad \varepsilon\left(\sum \lambda_g g\right) = \sum \lambda_g.$$

Example 2.15. A Lie algebra over a field k is a vector space g over k equipped with a bilinear map

$$[\,,\,] : g\times g \longrightarrow g$$

called the Lie bracket, satisfying the following properties for all $x, y, z \in g$:

1. Antisymmetry: $[x, y] = -[y, x]$
2. Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

Definition 2.16. The enveloping algebra of g, with g a Lie algebra, is the unital associative algebra quotient $\mathcal{U}(g) := T(g)/I$, where I is the ideal of $T(g)$ generated by the elements $x \otimes y - y \otimes x - [x, y]$ with $x, y \in g$.

For a Lie algebra g over a field k, the universal enveloping algebra $\mathcal{U}(g)$ is a co-algebra with the co-multiplication

$$\Delta(x) = x \otimes 1 + 1 \otimes x \quad \text{for} \quad x \in g.$$

2.3 Hopf algebra

Over a field k, a Hopf algebra is a graded vector space having the structure of a bialgebra where the algebra and coalgebra structures must be compatible in some precise way spelled out below:

- The comultiplication Δ and counit ϵ must be algebra morphisms.
- The multiplication μ and unit η must be coalgebra morphisms.

Example 2.17. : Let G be a Lie group and k a field. The **cohomology algebra** $H^*(G, k)$ of the Lie group G with coefficients in k has a natural Hopf algebra structure. This structure is induced by the group multiplication and the cup product. The multiplication in $H^*(G, k)$ is given by the **cup product**, which is the standard product in cohomology. The comultiplication is given by the map:

$$\Delta : H^*(G, k) \to H^*(G \times G, k) \cong H^*(G, k) \otimes H^*(G, k)$$

This map is induced by the group multiplication map $G \times G \to G$, reflecting the diagonal map $G \to G \times G$ and the group multiplication in G.

Using this comultiplication, the cohomology algebra $H^*(G, k)$ is equipped with a Hopf algebra structure.

3 H-spaces and Pontryagin Algebras

3.1 H-spaces

For X and Y pointed spaces,the set (X, Y) of pointed maps from X to Y is endowed with compact open topology.

Definition 3.1. A topological space X is an H-space, 'H' standing for 'Hopf', if it is a pointed space equipped with a continuous map $\mu : X \times X \longrightarrow X$, called the H-multiplication, such that if

$$e : X \longrightarrow X$$

is the constant map whose value is the base point $e(X) = x_0$, then it is an identity up to homotopy, or an H-identity; that is, the composites

$$\mu \circ (e, \mathrm{id}_X) \sim \mathrm{id}_X \quad \text{and} \quad \mu \circ (\mathrm{id}_X, e) \sim \mathrm{id}_X,$$

where $\sim$ denotes homotopy.

This can be represented by the following commutative diagram

$$\begin{tikzcd} X \vee X \arrow[rr, hook] \arrow[dr, "\nabla"'] & & X \times X \arrow[dl, "\mu"] \\ & X & \end{tikzcd}$$

Example 3.2. A topological group is a topological space such that:
• The multiplication map

$$m : X \times X \longrightarrow X, \quad (x, y) \mapsto x \cdot y$$

is continuous.
• The inverse map : $i : X \longrightarrow X, \quad x \mapsto x^{-1}$, is continuous.

Topological groups are classical examples of H-spaces.

Definition 3.3. We say that X is homotopy associative or H-associative if the composites

$$\mu \circ (\mu \times \mathrm{id}), \quad \mu \circ (\mathrm{id} \times \mu) : X \times X \times X \longrightarrow X$$

are homotopic. That is, if the following diagram commutes up to homotopy

$$\begin{tikzcd} X \times X \times X \arrow[r, "\mu\times\mathrm{id}"] \arrow[d, "\mu\times\mathrm{id}"'] & X \times X \arrow[d, "\mu"] \\ X \times X \arrow[r, "\mu"] & X \end{tikzcd}$$

Definition 3.4. Let (X, μ) and (Y, μ') be H-spaces.
A map $f : X \longrightarrow Y$ is said to be an H-map if the following diagram commutes up to homotopy:

$$\begin{tikzcd} X \times X \arrow[r, "f\times f"] \arrow[d, "\mu"'] & Y \times Y \arrow[d, "\mu'"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

3.2 Pontryagin Product

A key advantage of H-spaces is that the multiplication

$$\mu : X \times X \longrightarrow X,$$

induces a ring structure on their homology $H_*(X, k)$ as follows.

Given two topological spaces X and Y, there is a chain-level map on the singular chain complexes

$$\times \colon C_p(X; R) \otimes C_q(Y; R) \to C_{p+q}(X \times Y; R)$$

defined by sending the tensor product of singular simplices $\sigma \otimes \tau$ with

$$\sigma \colon \Delta^p \to X \quad , \quad \tau \colon \Delta^q \to Y$$

to the singular map $\sigma \times \tau \colon \Delta^p \times \Delta^q \to X \times Y$ given by:

$$(u, v) \mapsto (\sigma(u), \tau(v))$$

However, since $\Delta^p \times \Delta^q$ is not a standard simplex, we apply the Eilenberg–Zilber map to obtain a chain map $C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$ which induces a bilinear map in homology, called the cross product $\times \colon H_p(X; k) \otimes H_q(Y; k) \to H_{p+q}(X \times Y; k)$. sending a pair of classes of degrees p and q to a class of degree $p+q$. By composing, we obtain the pairing

$$H_p(X) \otimes H_q(X) \xrightarrow{\times} H_{p+q}(X \times X) \xrightarrow{\mu_*} H_{p+q}(X). \tag{2}$$

where $\times$ is the cross product in homology. This is the so-called the **Pontryagin product**.

Note that when working with coefficients in a field, the Künneth theorem ensures that the cross product is an isomorphism.

Lemma 3.5. *The Pontryagin product is associative*

Proof. Let m denote the **Pontryagin product**,to prove the associativity of m, we consider first this diagram

$$\begin{tikzcd}
H_*(X) \otimes H_*(X) \otimes H_*(X) \arrow[r, "1\otimes m"] \arrow[d, "1\otimes\times"'] & H_*(X) \otimes H_*(X) \arrow[r, "m"] \arrow[d, "id"'] & H_*(X) \arrow[d, "id"] \\
H_*(X) \otimes H_*(X \times X) \arrow[r, "1\times\mu_*"] \arrow[d, "\times"] & H_*(X) \otimes H_*(X) \arrow[r, "m"] \arrow[d, "\times"] & H_*(X) \arrow[d, "id"] \\
H_*(X \times X \times X) \arrow[r, "(1\times\mu)_*"] & H_*(X \times X) \arrow[r, "\mu_*"] & H_*(X)
\end{tikzcd}$$

Since homotopic maps induce the same maps in homology, we get

$$\mu_* \circ (1 \times \mu)^* = (\mu \circ (1 \times \mu))_* = (\mu \circ (\mu \times 1))_* = \mu_* \circ (\mu \times 1)^*.$$

So we consider a second diagram

$$\begin{tikzcd}
H_*(X) \otimes H_*(X) \otimes H_*(X) \arrow[r, "m\otimes 1"] \arrow[d, "\times\otimes 1"'] & H_*(X) \otimes H_*(X) \arrow[r, "m"] \arrow[d, "="'] & H_*(X) \arrow[d, "="] \\
H_*(X \times X) \otimes H_*(X) \arrow[r, "\mu_*\times 1"] \arrow[d, "\times"] & H_*(X) \otimes H_*(X) \arrow[r, "m"] \arrow[d, "\times"] & H_*(X) \arrow[d, "="] \\
H_*(X \times X \times X) \arrow[r, "(\mu\times 1)_*"] & H_*(X \times X) \arrow[r, "\mu_*"] & H_*(X)
\end{tikzcd}$$

adding this diagram

$$\begin{tikzcd}
H_*(X \times X \times X) \arrow[r, "(\mu\times 1)^*"] \arrow[d, "(1\times\mu)^*"'] & H_*(X \times X) \arrow[d, "\mu_*"] \\
H_*(X \times X) \arrow[r, "\mu_*"] & H_*(X)
\end{tikzcd}$$

$\square$

The Pontryagin product endows the homology groups (with coefficients in a field) of an H-space X, that is path-connected and homotopy associative, with the structure of a **Pontryagin algebra**. The diagonal map on the other hand induces a comultiplication in homology

$$\Delta_* : H_*(X) \longrightarrow H_*(X) \otimes H_*(X)$$

This makes $H_*(X)$ into a graded coassociative counital coalgebra. The algebra and coalgebra structures are compatible, satisfying the bialgebra axioms so $H_*(X)$ is a graded connected Hopf algebra.

In particular, the homology of a connected finite dimensional Lie group is a graded Hopf algebra.

3.3 The Based Loop Space

A fundamental example of an H-space is the loop space of a pointed topological space.

Definition 3.6. Let (X, x_0) be a pointed connected space with base point x_0. The *based loop space* ΩX is the set of all continuous maps

$$\alpha : I = [0,1] \longrightarrow X$$

such that $\alpha(0) = \alpha(1) = x_0$. These maps are called *based loops* in X based at x_0, and the space ΩX is equipped with the compact-open topology. The space ΩX is itself a pointed space with base point the constant loop at x_0, i.e., the loop $\alpha(t) = x_0$ for all $t \in [0,1]$.

If X is a pointed space with base point x_0, then its loop space ΩX has the structure of an H-space, as follows. Let

$$\mu : \Omega X \times \Omega X \longrightarrow \Omega X$$

be such that for loops $\alpha, \beta \in \Omega X$, the H-multiplication is defined by

$$\mu(\alpha,\beta)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \beta(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

The map μ is a H-multiplication called concatenation.
If $e : \Omega X \longrightarrow \Omega X$ is the constant map whose value is the constant loop $\alpha_0 : I \longrightarrow X$ defined by

$$\alpha_0(t) = x_0,$$

we see that this is an H-unit. That is, for every loop β, we have:

$$\mu(\beta, \alpha_0) \simeq \beta \quad \text{and} \quad \mu(\alpha_0, \beta) \simeq \beta.$$

The first homotopy is given by

$$F : \Omega X \times I \longrightarrow \Omega X,$$

where

$$F(\beta, s)(t) = \begin{cases} \beta(2t), & \text{if } 0 \leq t \leq \frac{1+s}{2}, \\ \beta(2t-1), & \text{if } \frac{1+s}{2} \leq t \leq 1. \end{cases}$$

Proposition 3.7. *The H-multiplication μ is H-associative. That is, the two compositions*

$$\mu \circ (\mu \times id), \quad \mu \circ (id \times \mu) : \Omega X \times \Omega X \times \Omega X \longrightarrow \Omega X$$

are homotopic.

Proof. The homtopy

$$G : \Omega X \times \Omega X \times \Omega X \times I \longrightarrow \Omega X$$

between $\mu \circ (\mu \times \mathrm{id})$ and $\mu \circ (\mathrm{id} \times \mu)$ is given by:

$$G(\alpha, \beta, \gamma, s)(t) = \begin{cases} \alpha\left(\frac{4t}{1+s}\right), & \text{if } 0 \leq t \leq \frac{1+s}{4}, \\ \beta\left(4t - 1 - s\right), & \text{if } \frac{1+s}{4} \leq t \leq \frac{2+s}{4}, \\ \gamma\left(4t - 2 - s\right), & \text{if } \frac{2+s}{4} \leq t \leq 1. \end{cases}$$

□

3.4 Homotopy Type

Since this thesis is about understanding the homology of ΩX, it would be a very useful property if sometimes ΩX can break into simpler spaces, or loops on simpler spaces.

For example, it is evident to see that there is a homeomorphism

$$\Omega(X \times Y) = \Omega X \times \Omega Y$$

and so the loop space of a product of based spaces is obtained from the product of loop spaces. The identification above is one of H-spaces as well.

There are many theorems in the literature that give splittings of ΩX for various spaces, or after "localizations", a theme we will not approach.

A very useful, yet elementary splitting, is obtained for loops on the complex projective space.

Lemma 3.8. *There is a homotopy equivalence*

$$\Omega \mathbb{C}P^n \simeq \Omega S^{2n+1} \times S^1$$

This is not an equivalence of H-spaces only if $n = 3$.

Proof. Consider the Hopf fibration

$$S^1 \longrightarrow S^{2n+1} \to \mathbb{C}P^n$$

and loop it once to get a new fibration

$$\Omega S^{2n+1} \xrightarrow{i} \Omega \mathbb{C}P^n \longrightarrow S^1$$

This fibration has a section $s : S^1 \to \Omega \mathbb{C}P^n$ which is the adjoint of the inclusion $S^2 = \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^n$. Since $\Omega \mathbb{C}P^n$ is an H-space, there is a well defined map from the product

$$S^1 \times \Omega S^{2n+1} \longrightarrow \Omega \mathbb{C}P^n \tag{3}$$

which sends (x, y) to the loop product $s(x) * i(y)$. This map is a weak homotopy equivalence by the long exact sequence in homotopy. Since all spaces have the homotopy type of CW-complexes, this map is a homotopy equivalence. The second statement is a deep result of Stasheff. □

4 Topological and Homotopical Background

4.1 CW-complex

A CW complex is a space X constructed in the following way:
First, start with a discrete set X_0, the 0-cells of X.
Next, inductively form the n-skeleton X_n from X_{n-1} by attaching n-cells e_α^n via maps

$$\varphi_\alpha : S^{n-1} \to X_{n-1}.$$

This means that X_n is the quotient space of $X_{n-1} \sqcup \bigsqcup_\alpha D_\alpha^n$ under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D_\alpha^n$. The cell e_α^n is the homeomorphic image of $D_\alpha^n \setminus S_\alpha^{n-1}$ under the quotient map.

The space $X = \bigcup_n X_n$ is given the weak topology and is called a CW complex.

Each cell e_α^n in a CW complex X has a characteristic map

$$\Phi_\alpha : D_\alpha^n \to X$$

which extends the attaching map

$$\varphi_\alpha : S^{n-1} \to X_{n-1}$$

and is a homeomorphism from the interior of D_α^n onto e_α^n.

Namely, we can take Φ_α to be the composition

$$D_\alpha^n \hookrightarrow X_{n-1} \sqcup \bigsqcup_\alpha D_\alpha^n \to X_n \hookrightarrow X.$$

This composition ensures that Φ_α is continuous, and its restriction to the interior of D_α^n is a homeomorphism onto e_α^n.

An alternative way to describe the topology on X is to say that a set $A \subset X$ is open (or closed) if and only if $\Phi_\alpha^{-1}(A)$ is open (or closed) in D_α^n for each characteristic map Φ_α.

One great use of CW-structures is in deriving homotopy equivalences from homology isomorphisms. As is understood, homology groups are relatively the easiest invariants to compute, but they are also a relatively weak invariant. The following remarkable theorem of J.H.C. Whitehead is perhaps one of the most cited results in the field. It works for simply connected CW-complexes having isomorphic homology, but it is imperative that this isomorphism be induced by a map.

Theorem 4.1 (J.H.C. Whitehead)**.** *Let $f : X \to Y$ be a continuous map between simply connected CW complexes. Assume that*

$$f_* : H_n(X;\mathbb{Z}) \to H_n(Y;\mathbb{Z})$$

is an isomorphism for each integer $n \geq 0$. Then f is a homotopy equivalence.

4.2 Fibrations

A map $p\colon E \longrightarrow B$ is said to satisfy the **homotopy lifting property** with respect to a space X if for any homotopy $g_t\colon X \longrightarrow B$ (with $t \in I = [0,1]$) and any lift $\tilde{g}_0\colon X \longrightarrow E$ of the initial map g_0, that is,

$$p \circ \tilde{g}_0 = g_0,$$

there exists a homotopy $\tilde{g}_t\colon X \longrightarrow E$ such that

$$p \circ \tilde{g}_t = g_t \quad \text{for all } t \in [0,1].$$

This property is illustrated by the following commutative diagram:

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\tilde{g}_0"] \arrow[d, hook] & E \arrow[d, "p"] \\
X \times I \arrow[r, "g_t"'] \arrow[ur, dashed, "\tilde{g}_t"] & B
\end{tikzcd}$$

Definition 4.2. A map $p\colon E \longrightarrow B$ is called a **fibration** if it satisfies the **homotopy lifting property** (HLP) with respect to all topological spaces X.

Over a path-connected B, if $p : E \to B$ is a fibration, then the preimages of points of B all have the same homotopy type, and this is one of the reasons why fibrations are very useful. If $x_0 \in B$ is a prefered basepoint, we call $F := p^{-1}(x_0)$ the fiber of p. As we mentioned, for path-connected B, $F \simeq \pi^{-1}(x)$ for all $x \in B$. It becomes therefore convenient to write a "fibration sequence" as follows

$$F \longrightarrow E \xrightarrow{p} B$$

where F is understood to be the fiber.

One main application of fibrations is the existence of a long exact sequence (LES) in homotopy groups (as sets for π_0 which takes the form

$$\cdots \pi_{i+1}B \xrightarrow{\partial} \pi_1(F) \longrightarrow \pi_i(E) \longrightarrow \pi_i(B) \xrightarrow{\partial} \pi_{i-1}(F) \cdots$$

4.2.1 Path fibration

This is a key construction in algebraic topology, due presumably to J.P. Serre. Let (X, x_0) be a pointed topological space. Let PX denote the space of continuous maps $\alpha\colon [0,1] \longrightarrow X$ such that $\alpha(0) = x_0$, endowed with the compact-open topology.

We define a continuous map

$$p\colon PX \longrightarrow X, \quad \alpha \longmapsto \alpha(1),$$

and claim that p is a fibration.

To justify this, consider the commutative diagram

$$\begin{tikzcd}
I^n \times \{0\} \arrow[r, "f_0"] \arrow[d, hook] & PX \arrow[d, "p"] \\
I^n \times I \arrow[r, "f"'] \arrow[ur, dashed] & X
\end{tikzcd}$$

We construct a continuous map $F'\colon I^n \times I \times I \longrightarrow X$ defined by

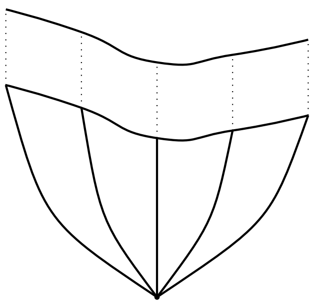
$$F'(x,t,s) = \begin{cases} f_0(x)(s(t+1)), & \text{if } 0 \le s \le \frac{1}{t+1}, \\ f(x, s(t+1)-1), & \text{if } \frac{1}{t+1} \le s \le 1. \end{cases}$$

This function is continuous and defines a continuous map $\tilde{f}\colon I^n \times I \longrightarrow PX$, which sends (x,t) to the path $\tilde{f}(x,t)$ defined by

$$\tilde{f}(x,t)(s) = F'(x,t,s).$$

This path satisfies:

- $\tilde{f}(x,t)(0) = f_0(x)(0) = x_0$, so $\tilde{f}(x,t) \in PX$,
- $\tilde{f}(x,0)(s) = f_0(x)(s)$, hence $\tilde{f} \circ j = f_0$,
- $p(\tilde{f}(x,t)) = F'(x,t,1) = f(x,t)$, thus $p \circ \tilde{f} = f$.



Therefore, $p\colon PX \longrightarrow X$ satisfies the homotopy lifting property and is a fibration with $\Omega(X,x_0)$ as the fibre.

Remark. There is a group isomorphism

$$\pi_n(\Omega X) \cong \pi_{n+1}(X).$$

This is a direct consequence of the long exact sequence of homotopy groups for the path-loop fibration, together with the fact that PX is contractible.

4.3 Cofibrations

A map $i\colon A \to B$ is called a **cofibration** if it satisfies the homotopy extension property (HEP) property:

Given a map $f\colon B \longrightarrow X$ and a homotopy $g_t\colon A \longrightarrow X$ such that

$$f_0 \circ i = g_0,$$

there exists a homotopy $G\colon B \times I \longrightarrow X$ such that

$$G(a,t) = g_t(a) \quad \text{for all } a \in A,\ t \in I, \quad \text{and} \quad G(b,0) = g_0(b) \text{ for all } b \in B.$$

This can be represented by the following commutative diagram

$$\begin{tikzcd} A \arrow[r, "g_t"] \arrow[d, "i"'] & X^I \arrow[d] \\ B \arrow[r, "f_0"] \arrow[ur, dashed, "G"] & X. \end{tikzcd}$$

Remark. In the definition of a fibration as a map satisfying the homotopy lifting property (HLP), if we reverse the direction of all arrows, we obtain the dual notion of a **cofibration**.

4.4 Whitehead and Samelson Products

Given elements $f \in \pi_m(X), g \in \pi_n(X)$, the **Whitehead bracket**

$$[f, g] \in \pi_{m+n-1}(X)$$

is defined as follows:

The product $S^m \times S^n$ can be obtained by attaching a $(m+n)$-cell to the wedge sum $S^m \vee S^n$ via an attaching map $S^{m+n-1} \xrightarrow{\phi} S^m \vee S^n$.

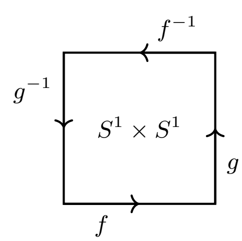
Represent f and g by maps of the same name $f\colon S^m \to X$ and $g\colon S^n \to X$, then compose their wedge with the attaching map, as

$$S^{m+n-1} \xrightarrow{\phi} S^m \vee S^n \xrightarrow{f \vee g} X.$$

The homotopy class of the resulting map does not depend on the choices of representatives, and thus one obtains a well-defined element

$$[f, g] \in \pi_{m+n-1}(X).$$

Example 4.3. Ine the case, $m = n = 1$, $[f, g]$ is the commutator $fgf^{-1}g^{-1}$.



The following computation will be handy in applications.

Lemma 4.4. *Let $X = S^2$, the 2-sphere. Consider the element $\iota \in \pi_2(S^2)$ represented by the identity map $\mathrm{id}_{S^2}\colon S^2 \to S^2$, and let $\eta \in \pi_3(S^2)$ denote the class of the Hopf fibration. Then the Whitehead product of ι with itself is given by:*

$$[\iota, \iota] = 2\eta \in \pi_3(S^2) = \mathbb{Z}.$$

Proof. (Sketch). The argument below is pictorial. It is known by Hopf that $\pi_3(S^2) = \mathbb{Z}$, with generator η. Given a map $f : S^3 \to S^2$, to find its class $[f] = d \in \pi_3(S^2)$, one can proceed as follows: approximate up to homotopy f by a smooth function, then consider the preimages of two regular values. These preimages form two copies of S^1 in S^3, they have a linking number d. This is the same as the class of the map f. It turns out that the Whitehead product produces a linking number of 2. $\square$

Properties: It is well known that **the Whitehead product**

$$[\ ,\ ] : \pi_m X \otimes \pi_n X \to \pi_{m+n-1} X$$

satisfies the following properties:

1. It is bilinear:

$$[f, g + h] = [f, g] + [f, h], \quad [f + g, h] = [f, h] + [g, h].$$

2. Graded commutativity [15]:

$$[f, g] = (-1)^{mn}[g, f], \quad f \in \pi_m X, \quad g \in \pi_n X, \quad m, n \geq 2.$$

3. Graded Jacobi Identity [14] :

$$(-1)^{ml}[[f, g], h] + (-1)^{mn}[[g, h], f] + (-1)^{ln}[[h, f], g] = 0,$$

where
$$f \in \pi_m X, \quad g \in \pi_n X, \quad h \in \pi_l X, \quad m, n, l \geq 2.$$

Definition 4.5. Let $f : X \to Z$ and $g : Y \to Z$ where Z is an H-group. The restriction to $X \vee Y$ of the map $X \times Y \to Z$ given by the commutator

$$(x, y) \mapsto f(x)g(y)f(x)^{-1}g(y)^{-1}$$

is null-homotopic, so there is an induced map $\langle f, g\rangle : X \wedge Y \to Z$. The sequence

$$[X \vee Y, Z] \to [X \times Y, Z] \to [X \wedge Y, Z]$$

splits, so the homotopy class of $\langle f, g\rangle$ is uniquely determined by those of f and g and is called the **Samelson product** of f and g.

The next result by Hans Samelson is pivotal in applications as it relates the Whitehead product to the commutator map in homology, via the Hurewicz homomorphism. More precisely, and for any space X, there is a *Hurewicz homomorphism*:

$$\tau : \pi_n(X) \to H_n(X; \mathbb{Z}) \tag{4}$$

sending the homotopy class of $f : S^n \to X$ to the image of the top dimensional generator $\tau([f]) = f_*([S^n]) \in H_n(G, \mathbb{Z})$.

Theorem 4.6 (Samelson 1953 [13])**.** *If $f \in \pi_{m+1}(X)$, $g \in \pi_{n+1}(Y)$, with $m, n \geq 1$, then*

$$\tau[f, g] = (-1)^m \left(\tau f \star \tau g - (-1)^{mn} \tau g \star \tau f\right),$$

where τ denotes the image under the Hurewicz map.

More on the Hurewicz homomorphism in the next section.

5 The Rational Homology of Loop Spaces

Since simply connected rationalized spaces are completely encoded in a commutative differential graded algebra (i.e. a CDGA, this is the core of Rational Homotopy Theory), one expects to find a very explicit model, or even a formula for the rational homology of the loop space of a simply connected space, and indeed this is the case as we here explain.

We start recalling that an element x in a Hopf algebra A (or more generally in a coalgebra A) is said to be **primitive** if

$$\varepsilon(x) = 0 \quad \text{and} \quad \Delta(x) = x \otimes 1 + 1 \otimes x.$$

When A is a Hopf algebra, the set of primitive elements $P(A)$ forms a graded Lie algebra under the commutator map

$$[x, y] = x \star y - (-1)^{|x||y|} y \star x$$

where $\star$ is the product in A.

We next revisit the Hurewicz homomorphism τ (4) and state some of its key properties below.

Lemma 5.1. *Let k be a field, and $\tau : \pi_*(X) \to H_*(X, k)$ the Hurewicz homomorphism.*
(a) The image of the Hurewicz homomorphism lies in the submodule of primitives.
(b) When $X = G$ is an H-group, the Hurewicz map induces a morphism of graded Lie algebras

$$\tau(\langle f, g\rangle) = [\tau(f), \tau(g)]$$

The product on the right being the Pontryagin product.

Proof. Let $f : S^n \to X$ be a map, then $f_*([S^n]) \in H_n(X)$ is called a *spherical class*. We wish to show that spherical classes are primitive. Write $\Delta : X \to X \times X$, $x \mapsto (x, x)$ the diagonal map, and Δ_* its induced map in homology. The following diagram commutes

$$\begin{tikzcd}
H_*(S^n) \arrow[r, "\Delta_*"] \arrow[d, "\tau"'] & H_*(S^n) \otimes H_*(S^n) \arrow[d, "\tau\times\tau"] \\
H_*(X) \arrow[r, "\Delta_*"] & H_*(X) \otimes H_*(X)
\end{tikzcd}$$

Since the orientation class of the sphere $[S^n] \in H_n(S^n)$, which is the only non-trivial generator, is necessarily primitive $\Delta_([S^n]) = [S^n] \otimes 1 + 1 \otimes [S^n]$, part (a) follows immediately by tracing through the diagram. $\square$

The following is the main structural result of this section.

Theorem 5.2 (Milnor–Moore Theorem [9])**.** *Over a field of characteristic zero, every connected Hopf algebra A with commutative and associative diagonal is primitively generated and is isomorphic to the universal enveloping algebra on its Lie algebra $P(A)$ of primitives, that is, the natural Hopf algebra map*

$$U(P(A)) \longrightarrow A$$

is an isomorphism.

This is a remarkable result establishing that primitives are the only generators. This structure result can be very effectively applied to H-spaces and loop spaces, in characteristic zero again. Let $\mathbb{Q}$ be the field of rational numbers, then the graded vector space $\pi_*(G) \otimes \mathbb{Q}$ becomes a Lie algebra over $\mathbb{Q}$, and the induced morphism $\lambda : \pi_k(G) \otimes \mathbb{Q} \to H_k(G, \mathbb{Q})$ is a morphism of Lie algebras which must map to the primitives by Lemma 5.1. The following is a consequence of the existence of Sullivan minimal models or Quillen models in rational homotopy theory.

Proposition 5.3. *Let G be a connected group-like space (that is, a connected homotopy associative H-space with a homotopy inverse). Then the rational Hurewicz map is an isomorphism*

$$\pi_*(G) \otimes \mathbb{Q} \xrightarrow{\cong} P(H_*(G; \mathbb{Q}))$$

of Lie algebras.

As a consequence, when G is a loop space ΩX, the Milnor-Moore theorem asserts that the rational Pontryagin algebra $H_*(\Omega X, \mathbb{Q})$ is isomorphic to envelopping algebra of the Lie algebra $\pi_*(\Omega X) \otimes \mathbb{Q}$. This is quite a complete result since if one has knowledge of the homotopy Lie algebra, then one can completely describe the Pontryagin ring rationally. This is done in [16] and we explain their method next.

Let g be a Lie algebra over a field k, and let X be a totally ordered basis of g. A canonical monomial over X is a finite sequence $(x_1, x_2, \ldots, x_n)$ of elements of X which is non-decreasing in the order $\leq$, that is,

$$x_1 \leq x_2 \leq \cdots \leq x_n.$$

Let h denote the canonical k-linear map from a Lie algebra g into its universal enveloping algebra $\mathcal{U}(g)$. Extend h to all canonical monomials as follows: if $(x_1, x_2, \ldots, x_n)$ is a canonical monomial, then define

$$h(x_1, x_2, \ldots, x_n) := h(x_1) \cdot h(x_2) \cdots h(x_n).$$

The **Poincaré–Birkhoff–Witt theorem** states that h is injective on the set of canonical monomials, and the image of this set

$$\{h(x_1, \ldots, x_n) \mid x_1 \leq \cdots \leq x_n\}$$

forms a basis of $\mathcal{U}(g)$ as a k-vector space.

Going back to the Milnor–Moore theorem, we know that if X is simply-connected, then its loop space homology of is isomorphic to the universal enveloping algebra of its homotopy Lie algebra $H_*(\Omega X; \mathbb{Q}) \cong U g_X$. We can deduce the following calculation.

Theorem 5.4. *Let $r_i = \dim \pi_i(\Omega X) \otimes \mathbb{Q} = \dim \pi_{i+1}(X) \otimes \mathbb{Q}$. Then*

$$P_{\Omega X} = \frac{\displaystyle\prod_{i=0}^{\infty} (1 + z^{2i+1})^{r_{2i+1}}}{\displaystyle\prod_{i=1}^{\infty} (1 - z^{2i})^{r_{2i}}}.$$

where $P_{\Omega X}$ is the Poincaré series for ΩX as the formal power series.

As pointed out in [16], this formula provides algorithms for computing the integers $\dim H_i(\Omega X; \mathbb{Q})$, $1 \leq i \leq N$, from the integers $\dim \pi_{i+1}(X) \otimes \mathbb{Q}$, and for computing the integers $\dim \pi_{i+1}(X) \otimes \mathbb{Q}$, $1 \leq i \leq N$, from the integers $\dim H_i(\Omega X; \mathbb{Q})$, $1 \leq i \leq N$. This is why we mentioned that this computation is quite optimal.

6 Loop Spaces of Suspensions

The next very well-understood computation of the Pontryagin ring of a space X is when X is a suspension of a based connected space Y. In that case, the computation is due to Ioan James, and it is very explicit. It consists in constructing "a combinatorial model" for $\Omega X = \Omega\Sigma Y$ with explicit homology. James' construction introduces a novel and potent set of ideas for the first time in algebraic topology with great generalizations some of which are disussed in [4].

6.1 Moore Loops

As a starting point, we give a construction of a space of loops on X that is strictly associative. This is called the space of **Moore loops**.

Given a topological space X, define

$$\Omega^M(X, x_0) = \{(\alpha, v) \in C([0,\infty), X) \times [0,\infty) \mid \alpha(0) = x_0 \text{ and } \alpha(t) = x_0 \text{ for all } t \geq v\},$$

topologized as a subspace of $C([0,\infty), X) \times [0,\infty)$.

We define a product

$$\Omega^M X \times \Omega^M X \longrightarrow \Omega^M X$$

by

$$(\alpha_1, v_1).(\alpha_2, v_2) = (\alpha_1 \star \alpha_2, v_1 + v_2),$$

where the concatenated path $\alpha_1 \star \alpha_2 \colon [0,\infty) \longrightarrow X$ is given by

$$(\alpha_1 \star \alpha_2)(t) = \begin{cases} \alpha_1(t) & \text{if } 0 \leq t \leq v_1, \\ \alpha_2(t - v_1) & \text{if } v_1 \leq t \leq v_1 + v_2, \\ x_0 & \text{if } t \geq v_1 + v_2. \end{cases}$$

The constant loop of length zero, $(x_0, 0)$, serves as a strict unit for this product.

Moreover, the operation is strictly associative.

The main point in introducing $\Omega^M(X)$ is that it an equivalent space to ΩX but now, the associativity has been strictified.

Proposition 6.1. *$\Omega(X, x_0)$ is a deformation retract of $\Omega^M(X, x_0)$.*

Proof. First consider the subspace $\tilde{\Omega}(X, x_0) \subset \Omega^M(X, x_0)$, consisting of all (α, v) with $v \leq 1$. A deformation retraction H from $\Omega^M(X, x_0)$ to $\tilde{\Omega}(X, x_0)$ is given by the following formulas

$$H(s, (\alpha, v)) = \begin{cases} (\alpha, v + s), & \text{if } v + s \leq 1, \\ (\alpha, 1), & \text{if } v \leq 1 \text{ and } v + s \geq 1, \\ (\alpha, v), & \text{if } v \geq 1. \end{cases}$$

Now we define a deformation retraction G from $\tilde{\Omega}(X, X_0)$ to $\Omega(X, x_0)$ by the formula

$$G(s, (\alpha, v)) = (\alpha_s, (1 - s)v + s),$$

where the reparametrized map α_s is defined by:

$$\alpha_s(t) = \alpha\left(\frac{v}{(1-s)v+s}t\right)$$

This defines a continuous deformation retraction, and thus $\Omega(X, X_0)$ is a deformation retract of $\Omega^M(X, X_0)$. $\square$

Remark. The inclusion of length one loops $\varphi : \Omega X \xrightarrow{\simeq} \Omega^M X$ is an H-map which means that the following diagram commutes up to homotopy:

$$\begin{tikzcd} \Omega X \times \Omega X \arrow[r, "\varphi\times\varphi"] \arrow[d, "\text{concatenation}"'] & \Omega^M X \times \Omega^M X \arrow[d, "\text{concatenation}"] \\ \Omega X \arrow[r, "\varphi"] & \Omega^M X \end{tikzcd}$$

6.2 The James construction

A **topological monoid** is, by definition, a monoid object in the category of topological spaces. More precisely, it consists of:

- A topological space M.
- A continuous multiplication map

$$\mu : M \times M \longrightarrow M$$

  such that

$$\mu(\mu(x, y), z) = \mu(x, \mu(y, z)) \quad \text{for all } x, y, z \in M \quad \text{(associativity).}$$

- An identity element $e \in M$ such that

$$\mu(e, x) = \mu(x, e) = x \quad \text{for all } x \in M.$$

Example 6.2. Topological groups, which have been mentioned in example[3.2], are examples of topological monoids.

Definition 6.3. Let (X, x_0) be a pointed topological space. The **free topological monoid** on X, denoted $M(X)$, is constructed as the disjoint union

$$M(X) = \bigsqcup_{n=0}^{\infty} X^n$$

where

- X^n is the n-fold Cartesian product of X (with $X^0 = x_0$)
- Each X^n is equipped with the product topology.
- The total space $M(X)$ is given the disjoint union topology.

The monoid operation is the concatenation of tuples; that is, for tuples

$$(x_1, \dots, x_m) \in X^m \quad \text{and} \quad (y_1, \dots, y_n) \in X^n$$

their product is defined as

$$(x_1, \dots, x_m) \cdot (y_1, \dots, y_n) = (x_1, \dots, x_m, y_1, \dots, y_n).$$

Let (X, x_0) be a pointed topological space. The **James construction** $J(X)$ is defined as the free topological monoid generated by X, where the basepoint $x_0 \in X$ acts as the unit. Formally, $J(X)$ is the disjoint union

$$J(X) = \coprod_{n \geq 0} X^n / \sim$$

where the equivalence relation $\sim$ is generated by identifying any tuple with the basepoint in any coordinate

$$(x_1, \dots, x_i, x_0, x_{i+2}, \dots, x_n) \sim (x_1, \dots, x_i, x_{i+2}, \dots, x_n)$$

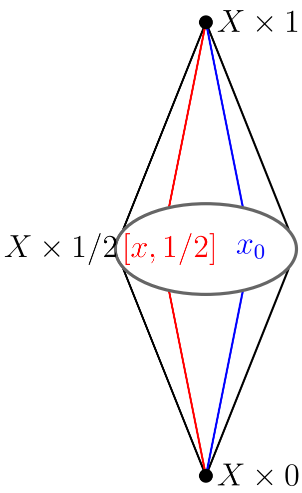
for all $n \geq 2$.
Explicitly two points $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are equivalent if and only if, after deleting all entries x_0, they are exactly the same list of elements in exactly the same order. For example, if $x_0, x, y \in X$, then

$$(x_0, y, x, x_0, y, x_0, x_0) \sim (y, x_0, x_0, x, , x_0, x_0, y) \quad \text{in } X^7.$$

The important thing to observe is that there is a natural map

$$J(X) \longrightarrow \Omega^M \Sigma X,$$

where ΣX denotes the reduced suspension of X.



$$\Sigma X := (X \times [0,1]) / (X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times [0,1])$$

There is a canonical inclusion $\lambda\colon X \hookrightarrow \Omega^M \Sigma X$ sending $x \in X$ to the loop $J(x)(t)$, defined as the path in ΣX which sends $t \in [0,1]$ to the image of $(x,t) \in X \times I \subset \Sigma X$. Because $\Omega^M \Sigma X$ is a topological monoid, this map extends uniquely to a map from $J(X)$ to $\Omega^M \Sigma X$, which we call the *James map* .

$$\tilde{\lambda}\colon J(X) \hookrightarrow \Omega^M \Sigma X \tag{5}$$

The following is the main result of this section.

Theorem 6.4. *(*James*). $\tilde{\lambda}$ is a homotopy equivalence.*

To prove this result, we follow the outline of [4, 1]. We will show that both spaces have the same homology (Propositions 6.6 and 6.5 next), and that the map induces an isomorphism in homology. We can then invoke the Whitehead theorem (Theorem 4.1).

We work first over field coefficients, so that $H_*(X \times Y) \cong H_*(X) \otimes H_*(Y)$ by the Kunneth theorem.

Since JX is a monoid, $H_*(JX)$ is a Pontryagin algebra. There is a map $H_*(X) \to H_*(JX)$, induced by the canonical inclusion $X \to JX$, and since $H_*(JX)$ is an algebra, this map extends multiplicatively to

$$T(H_*(X)) \longrightarrow H_*(JX), \mathbb{Z}).$$

Proposition 6.5. $\varphi : T(\tilde{H}_*(X)) \longrightarrow H_*(J(X)$ *is an isomorphism.*

Proof. $\varphi : T\widetilde{H}_*(X) \longrightarrow H_*(J(X))$ is a homomorphism whose restriction to the n-fold tensor product $\widetilde{H}_*(X)^{\otimes n}$ is the composition:

$$\widetilde{H}_*(X)^{\otimes n} \hookrightarrow H_*(X)^{\otimes n} \xrightarrow{\times} H_*(X^n) \longrightarrow H_*(J_n(X)) \longrightarrow H_*(J(X))$$

To show that φ is an isomorphism, consider the following commutative diagram of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & T_{n-1}\widetilde{H}_*(X) \arrow[r] \arrow[d, "\varphi"] & T_n\widetilde{H}_*(X) \arrow[r] \arrow[d, "\varphi"] & \widetilde{H}_*(X)^{\otimes n} \arrow[r] \arrow[d, "\times", "\cong"'] & 0 \\
0 \arrow[r] & H_*(J_{n-1}(X)) \arrow[r] & H_*(J_n(X)) \arrow[r] & \widetilde{H}_*(X^{\wedge n}) \arrow[r] & 0
\end{tikzcd}$$

We have $J_n(X)/J_{n-1}(X) = X^{\wedge n}$ and using induction and the five lemma, we get $T_n\tilde{H}_*X = H_*J_nX$. Letting $n \to \infty$, this implies that

$$\varphi : TH_*(X) \to H_*(J(X))$$

is an isomorphism since, in any fixed degree, the truncated tensor algebra

$$T_nH_*(X) = \bigoplus_{k=0}^{n}(H_*(X))^{\otimes k}$$

stabilizes; that is, $T_nH_*(X)$ becomes independent of n when n is sufficiently large. we have isomorphisms on homology:

$$H_*(J_n(X)) \cong T_nH_*(X),$$

and since homology commutes with colimits,

$$H_*(J(X)) \cong \varinjlim_n H_*(J_n(X)) \cong \varinjlim_n T_nH_*(X) = TH_*(X).$$

is an isomorphism as claimed. □

Proposition 6.6. $H_*(\Omega^M\Sigma X) \cong T(\tilde{H}_*(X))$

Proof. To compute the homology groups of the space $\Omega\Sigma X$, we consider the path fibration:

$$\Omega\Sigma X \longrightarrow P\Sigma X \longrightarrow \Sigma X$$

The suspension of X splits into a postive cone $C_+X = X \times [\frac{1}{2}, 1]$ and a negative cone $C_-X = X \times [0, \frac{1}{2}]$ which are contractible and intersect in $X \subset \Sigma X$.
Consider the inverse images $p^{-1}(C_+X)$ and $p^{-1}(C_-X)$ where p is the evaluation map at 1. We write

$$P_\pm \Sigma X = p^{-1}(C_\pm X).$$

$P_+\Sigma X$ consists of paths in ΣX starting at the basepoint and ending in C_+, and similarly for $P_-\Sigma X$.

To understand the homology $H_*(\Omega\Sigma X)$ we use the Mayer–Vietoris sequence in homology for the decomposition of $P\Sigma X$ as $P_+\Sigma X \cup P_-\Sigma X$and the fact that $P\Sigma X$ is contractible. we obtain an isomorphism

$$\Phi : H_*(P_+\Sigma X \cap P_-\Sigma X; k) \longrightarrow H_*(P_+\Sigma X; k) \otimes H_*(P_-\Sigma X; k)$$

The fibration $P_+\Sigma X \to \Sigma X_+$ is fiber-homotopically trivial, and similarly for the fibration $P_-\Sigma X \to \Sigma X_-$. So we obtain an isomorphism

$$H_*(\Omega\Sigma X; k) \otimes \tilde{H}_*(X; k) \xrightarrow{1\otimes\lambda^*} H_*(\Omega\Sigma X; k) \otimes \tilde{H}_*(\Omega\Sigma X; k) \longrightarrow \tilde{H}_*(\Omega\Sigma X; k)$$

To finish the calculation of $H_*(\Omega\Sigma X, k)$ as the tensor algebra $T\tilde{H}_*(X, k)$, we apply the following lemma □

Lemma 6.7. *Let A be a graded algebra over a field k with $A_0 = k$, and let V be a graded vector space over k with $V_0 = 0$. Suppose we have a linear map*

$$i : V \longrightarrow A$$

preserving grading, such that the multiplication map

$$\mu : A \otimes V \longrightarrow A, \quad \mu(a \otimes v) = ai(v),$$

is an isomorphism. Then the canonical algebra homomorphism

$$i : TV \longrightarrow A$$

extending the previous i is an isomorphism.

Proof. Since μ is an isomorphism, each element $a \in A_n$ with $n > 0$ can be written uniquely in the form $\mu\left(\sum_j a_j \otimes v_j\right) = \sum_j a_j i(v_j)$ for $v_j \in V$ and $a_j \in A_{n(j)}$, with $n(j) < n$ since $V_0 = 0$.

By induction on n, we have $a_j = i(\alpha_j)$ for a unique $\alpha_j \in (TV)_{n(j)}$. Thus,

$$a = i\left(\sum_j \alpha_j \otimes v_j\right),$$

so i is surjective. Since these representations are unique, i is also injective.

The induction starts with the hypothesis that $A_0 = k$, the scalars in TV. □

We are now ready to deduce the Theorem of James.

Proof. (Theorem 6.4). The map λ defined in (5) induces an isomorphism in homology over any field of coefficients. By the Universal Coefficient Theorem for homology, λ_* is also an isomorphism on integral homology. Combining results 6.6, 6.5, and 4.1, we obtain James' theorem. □

A direct consequence of James' theorem is the Freudhental suspension theorem.

Corollary 6.8. *Let X be n-connected, $n \geq 1$. Then the map*

$$\lambda : X \longrightarrow \Omega(\Sigma X), x \longmapsto [t, x]$$

induces a map on homotopy groups

$$\pi_k(X) \longrightarrow \pi_k(\Omega(\Sigma X)),$$

that is

- *an isomorphism if $k \leq 2n$,*
- *an epimorphism if $k = 2n + 1$.*

Proof. The argument combines James' theorem with the Hurewicz theorem. The homology $\tilde{H}_*(X)$ starts in degree $n + 1$. This means that the products of classes in $H_*(\Omega\Sigma X)$ start at the lowest degree $2(n + 1) = 2n + 2$. This means in turn that the map λ induces a homology isomorphism up to degree $2n + 1$. By the relative Hurewicz theorem, this implies an isomorphism in homotopy groups up to degree $2n$ and an epimorphism in degre $2n + 1$. □

As is well-known, the bounds $2n$ and $2n + 1$ are sharp.

Some illustrative computations with James' theorem are given below.

Example 6.9. (Spheres and Morse original computation). This is the computation of $H_*(\Omega S^n)$ for $n \geq 2$, which was originally given by M. Marston, using his newly developed theory (see [8]). When $n = 1$, ΩS^1 has a discrete set of contractible components indexed by $\mathbb{Z}$.

Let $Y = S^n$. Its reduced homology is $\widetilde{H}_*(S^n) = \mathbb{Z}$ in degree n. We denote the generator by a with $|a| = n$. The tensor algebra has a unital element and then all generators a^k for $k \geq 1$. By James' theorem,

$$\tilde{H}_*(\Omega S^n) = \mathbb{Z}\{a, a^2, a^3, \ldots\}$$

with the degree $|a^k| = kn$. There is a single torsion free generator in every degree that is a multiple of n. In this case, the Poincaré series is given by $P_t(\Omega S^n) = \frac{1}{1-t^n}$

Example 6.10. : Calculation of $H_*(\Omega(S^{n+1} \vee S^{m+1})), n, m \geq 1$.

Let $Y = S^n \vee S^m$. Then

$$\widetilde{H}_*(S^n \vee S^m) = \mathbb{Z}a \oplus \mathbb{Z}b$$

where a is the generator of degree n and b is a generator of degree m. By James' theorem, the homology of the loop space is

$$T(\mathbb{Z}a \oplus \mathbb{Z}b) = \bigoplus_{i,j,r,s \geq 0} \mathbb{Z}\{a^i b^j, b^r a^s\}$$

where the degrees are $|a^i b^j| = i + j$, $|b^r a^s| = r + s$. The Poincaré series has the form

$$P_t(\Omega(S^{n+1} \vee S^{m+1})) = \frac{1}{1 - t^n} \cdot \frac{1}{1 - t^m} \cdot \prod_{i,j \geq 1} \frac{1}{1 - t^{in+jm}}$$

7 Adams-Hilton Model

7.1 Introduction

After understanding the James construction, which models the homology of loop spaces like $\Omega\Sigma X$ as a tensor algebra on the reduced homology of X, it is natural to look for a more general algebraic model of loop spaces, especially when X is not a suspension.

The Adams-Hilton model provides such a construction: it produces a differential graded algebra that models ΩX for X a CW-complex space with one 0-cell and no 1-cells. Such a complex is called 1-reduced. Given a 1-reduced CW-complex X, the Adams-Hilton model is a free differential graded algebra built as follows

- Start with a CW decomposition of X.
- For each n cell, introduce an algebraic generator of degree $n-1$.
- Build the free tensor algebra on these generators.
- Define a differential determined by the attaching maps of the CW-complex.

This model is then a graded tensor algebra TV (see Example 2.9), with a differential d governed by the attaching map of the CW structure.

Before we explain the details, we need a lemma. Let X be obtained from a space Y by attaching a single cell of dimension $n+1$. We write $Y = X \cup_f e^{n+1}$, where $f : S^n \longrightarrow X$ is the attaching map. Let $g : S^{n-1} \to \Omega X$ be the adjoint map, and let $i : X \hookrightarrow Y$ be the inclusion.

Lemma 7.1. *The class $\beta = (\Omega i)_* \circ g_*[S^{n-1}]$ is trivial in $H_{n-1}(Y)$.*

Proof. Consider the following diagram

$$\begin{tikzcd}
\Omega S^n \arrow[r, "\Omega f"] & \Omega X \arrow[r, "\Omega i"] & \Omega Y \\
S^{n-1} \arrow[u, "ad"] \arrow[ur, "g"] & &
\end{tikzcd}$$

Since $i \circ f : S^n \to X \hookrightarrow Y$ is null-homotopic, S^n being the boundary of a cell in Y, it follows that $\Omega(i \circ f) = \Omega i \circ \Omega f$ is null-homotopic, and therefore so is $\Omega i \circ g$. The claim follows. $\square$

We now give the construction of Adams and Hilton which consists in producing a quasi-isomorphism between a free tensor algebra model and the complex of singular chains

$$\theta_X : (TV, d) \longrightarrow C_*(\Omega X)$$

such that θ_X restricts to quasi-isomorphisms

$$(TV^{\leq n}, d) \longrightarrow C_*(\Omega X^{n+1})$$

This model is constructed inductively, skeleton by skeleton.

Let X be a CW complex with a single vertex and no 1-cells

$$\begin{aligned} X &= * \cup e_1^2 \cup e_2^2 \cup \cdots \cup e_r^2 \cup e_1^3 \cup \cdots, \\ &= * \cup_{\alpha \in A} e_\alpha^{n_\alpha+1} \end{aligned}$$

Build V to be the free module with a degree-homogeneous basis $\{v_\alpha : \alpha \in A\}$ such that

$$\deg(v_\alpha) = n_\alpha.$$

There is one generator for every cell.

Suppose we have already constructed a quasi-isomorphism $\theta_{X^{(n)}}$ on the n-skeleton, and we attach a cell e of dimension $n+1$ to $X^{(n)}$ along a continuous map $f : S^n \longrightarrow X^{(n)}$ to make a new space $Y = X \cup_f D^{n+1}$. We denote by $\beta \in H_{n-1}(\Omega X^{(n)}; k)$ the image under $adf : S^{n-1} \to \Omega X^{(n)}$ of the generator of $H_{n-1}(S^{n-1}; k)$. Since $H_{n-1}(\theta_{X^{(n)}})$ is an isomorphism, we can choose a cycle $a \in (T(v_\alpha), d)$, $\deg v_\alpha \leq n-1$, that maps to a representative cycle of β. But then, by Lemma 7.1, the class β maps to zero in $H_{n-1}(\Omega X^{(n+1)})$, which means that in the model, it has to "die out".

Extend the tensor algebra $(T(v_\alpha) \amalg T(v), d)$, $\deg v_\alpha \leq n-1$, by a class v of degree $|v| = n$ and such that $dv = a$. Adams and Hilton manage to extend the $\theta_{X^{(n)}}$ to a map θ_Y so as to obtain a commutative diagram

$$\begin{array}{ccc} (T(v_\alpha), d) & \xrightarrow{\theta_{X^{(n)}}} & C_*(\Omega X^{(n)}; k) \\ \downarrow & & \downarrow \\ (T(v_\alpha) \amalg T(v), d) & \xrightarrow{\theta_Y} & C_*(\Omega Y; k) \end{array}$$

By construction of v, this map induces a quasi-isomorphism. This construction can be iterated over all cells, by increasing dimension. It is not functorial as it depends on various choices of cycles and lifts.

We can record this entire construction below.

Theorem 7.2. *(Adams-Hilton 1955) Let X be a CW-complex such that X has exactly one 0-cell and no 1-cells and such that every attaching map is based with respect to the unique 0-cell of X. There exists a morphism of differential graded algebras inducing a quasi-isomorphism*

$$\theta_X : (TV, d) \longrightarrow C_*(X)$$

such that θ_X restricts to quasi-isomorphisms $(TV_{\leq n}, d) \to C_*(X^{(n+1)})$, where $X^{(n+1)}$ denotes the $n+1$-skeleton of X, TV denotes the free (tensor) algebra on a free, graded $\mathbb{Z}$-module V whose generators are in to one correspondence with the cells of X, ΩX is the space of Moore loops on X and C_* denotes the cubical chains.

This model has several properties summarized in [17]. We will write $AH(X) = (TV, d)$ any such Adams-Hilton model of X. This choice is unique only up to isomorphisms.

We will illustrate this model on some examples next.

7.2 Some computations

Example 7.3. Let S^n be the n-sphere with $n > 1$, and let k be a field. We can write $S^n = e_0 \cup e^n$, therefore the Adams-Hilton model

$$AH(S^n) = (T(v), d) \ , \ |v| = n-1, d = 0$$

which gives that

$$H_*(\Omega S^n; k) \cong T(v)$$

where $T(v)$ is the free associative algebra generated by v.

Example 7.4. More generally, if X is a wedge of spheres, that is

$$X = \bigvee_{i \in I} S^{n_i+1},$$

then the homology of the based loop space is given by

$$H_*(\Omega X; k) \cong T(v_i, i \in I), \quad |v_i| = n_i,$$

where $T(v_i, i \in I)$. This a special case of a result that is easy to prove that

$$H_*(\Omega(X \vee Y), k) \cong T(V \amalg W, d_V \amalg d_W) \tag{6}$$

where $AH(X) = (TV, d_V)$ and $AH(Y) = (TW, d_W)$. Indeed, the cells of X and the cells of Y appear in the bouquet separately and the Adams-Hilton differential is defined on each cell separately.

Example 7.5. We give the Adams-Hilton model for the product of two spheres $S^p \times S^q$ where $p, q > 1$.

The cellular decomposition of this product space is given by

$$S^p \times S^q = e^0 \cup e^p \cup e^q \cup e^{p+q}$$

and the $p+q$ cell attaches via the Whitehead product. We write the cellular generators as $a, b, c,$ where $|a| = p-1, |b| = q-1, |c| = p+q-1$. $AH(S^p \times S^q)$ is given by $TV = (T(a, b, c)$ with a differential d that we now determine. Clearly $da = 0, db = 0$. To determine dc, let $f : S^{p+q-1} \to S^p \vee S^q$ be the attachment map of the top-dimensional cell on the lower skeleton. By adjunction, we get the following diagram

$$\begin{tikzcd}
\Omega S^{p+q-1} \arrow[r, "\Omega f"] & \Omega(S^p \vee S^q) \\
S^{p+q-2} \arrow[u] \arrow[ur, "g"] &
\end{tikzcd}$$

and a map in homology:

$$g_* : H_*(S^{p+q-2}) \to H_*(\Omega(S^p \vee S^q)) = T(a, b)$$

where the last equality comes from (6). So $dc \in T(a, b)$ has the form $dc = kab + k'ba$. However, $[\theta(dc)]$ maps to the adjoint of the Whitehead product of $[id_p, id_q]$ in $H_{p+q-2}(\Omega(S^p \vee S^q)$, which by Samelson's theorem (Theorem **??**) is the graded commutator

$$ab + (-1)^{p+q} ba$$

This gives the entire model

$$AH(S^p \times S^q) = (T(a, b, c), da = db = 0, dc = ab + (-1)^{p+q} ba)$$

Not surprisingly, this gives exactly the homology of

$$T(a) \times T(b) = \Omega(S^p) \times \Omega(S^q) \cong \Omega(S^p \times S^q)$$

Example 7.6. The complex projective plane $\mathbb{CP}^2$ has a standard cellular decomposition consisting of cells in dimensions 0, 2, and 4:

$$\mathbb{CP}^2 = e^0 \cup e^2 \cup e^4$$

The Adams-Hilton model has two generators a, b $|a| = 1$, $|b| = 3$, coming from the cells in positive degree. The 2-cell e^2 attaches to the 0-cell e^0 trivially. The nontrivial differential reflects the attaching map of the 4-cell to the 2-skeleton.

The 4-cell e^4 attaches to the 2-skeleton $S^2 = \mathbb{CP}^1$ via the Hopf map $\eta : S^3 \to S^2$. By adjunction, we get the following diagram

$$\begin{tikzcd} \Omega S^3 \arrow[r] & \Omega(\mathbb{CP}^1) \\ S^2 \arrow[u] \arrow[ur, "g"] & \end{tikzcd}$$

This induces a map in homology:

$$g_* : H_*(S^2) \to H_*(\Omega(\mathbb{CP}^1))$$

By Adams-Hilton we know that $d(b) = c$ where c is a cycle in $T(a, d)$ such that

$$[\theta_{S^2}(c)] = g_*[S^2]$$

Since g is the adjoint of the Hopf map $h\colon S^3 \to S^2$, and using the Whitehead product As shown in lemma 4.4

$$[\iota, \iota] = 2\eta \in \pi_3(S^2)$$

Via adjunction we obtain

$$ad[\iota, \iota] = 2 \cdot ad(\eta) = 2g$$

After passing to homology, we get

$$[ad(\iota), ad(\iota)] = 2g_*$$

Where $[ad(\iota), (ad\iota)]$ is **the Samelson product** on ΩX. As stated in theorem 4.6, Passing through the Hurewicz map to homology we obtain

$$h(ad[-, -]) : H_1\Omega S^2 \times H_1\Omega S^2 \longrightarrow H_2\Omega S^2$$

$$(a, a) \longrightarrow aa + aa = 2a^2$$

We conclude that $d(b) = a^2$
This model gives the loop space homology:

$$H_*(\Omega\mathbb{CP}^2) \cong H_*(T(a, b), d(a) = 0, d(b) = a^2, |a| = 1, |b| = 3) \tag{7}$$

The next Lemma confirms the splitting in Lemma 3.8.

Lemma 7.7. *There is an isomorphism of algebras*

$$\begin{aligned} H_*(\Omega\mathbb{CP}^2) &\cong T(a, c)/(a^2, ac - ca), \quad c = ab + ba, \quad |c| = 4 \\ &\cong \Lambda([a]) \otimes \mathbb{Z}([c]) \end{aligned}$$

Proof. We compute the homology of the differential tensor algebra given in (7). Since $db = a^2$, both b and a^2 vanish in homology, and there are no degree 2 and 3 homology in $H_*(\Omega\mathbb{C}P^2)$. In degree 4, $c = ab + ba$ is a cycle, and $d(b^2) = ac - ca$ by the Leibniz rule 1. In fact, $ac^n - c^n a$ is a boundary for all $n \geq 1$. By taking homology, we account for all classes as given in the Lemma. □

Example 7.8. The complex projective space $\mathbb{CP}^n$.

$$\mathbb{CP}^n = \{\text{complex lines through 0 in } \mathbb{C}^{n+1}\}$$

is obtained from the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ by quotienting out the free S^1–action $\lambda\cdot(z_0, \ldots, z_n) = (\lambda z_0, \ldots, \lambda z_n)$. Thus $\mathbb{CP}^n = S^{2n+1}/S^1$ and is a $2n$–dimensional real manifold.

There are natural inclusions

$$\mathbb{CP}^0 \subset \mathbb{CP}^1 \subset \mathbb{CP}^2 \subset \cdots \subset \mathbb{CP}^n \subset \cdots$$

Inductively, $\mathbb{CP}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching a single $2n$–cell, giving a CW–structure with one cell in each even dimension $0, 2, \ldots, 2n$.

Proposition 7.9. *The Adams–Hilton model of $\mathbb{CP}^n$ is given by the differential graded algebra*

$$(T(a_1, a_2, \ldots, a_n), d)$$

where:

- $|a_i| = 2i - 1$ *for each* $i = 1, \ldots, n$,
- *the differential is defined by*

$$d(a_1) = 0, \quad d(a_i) = \sum_{k=1}^{i-1} a_k \otimes a_{i-k}. \tag{8}$$

Proof. In the previous example, we analyzed the case $n = 2$, that is, $\mathbb{CP}^2$, which satisfies the conditions described in (8). For the general case ,we proceed by induction as in [17]. Assume that we have already constructed Adams–Hilton models for $\mathbb{CP}^1, \mathbb{CP}^2, \ldots, \mathbb{CP}^{n-1}$, and that each model $AH(\mathbb{CP}^j)$ extends the previous one $AH(\mathbb{CP}^i)$ for $i < j < n$. In particular, we suppose that

$$AH(\mathbb{CP}^{n-1}) = (T(a_1, \ldots, a_{n-1}), \tilde{d})$$

is given in the form described earlier.

Consider the element

$$v := \sum_{i=1}^{n-1} a_i \otimes a_{n-i}, \qquad \text{degree } |v| = 2n - 2.$$

We aim to show that we can choose an Adams–Hilton model

$$AH(\mathbb{CP}^n) = (T(a_1, \ldots, a_n), d)$$

such that $d(a_n) = v$ and $d(a_i) = \tilde{d}(a_i), \quad i = 1, ...n - 1$

A straightforward computation shows that $d(v) = 0$. Next, we use the homotopy equivalence $\Omega\mathbb{CP}^n \simeq \Omega S^{2n+1} \times S^1$ from Lemma 3.8. From this, it follows that the homology group $H_{2n-2}(\Omega\mathbb{CP}^n)$ vanishes. Hence, v must be a boundary. This means there exist an integer $k \in \mathbb{Z}$ and an element $w \in T(a_1, \ldots, a_{n-1})$ such that

$$v = kd(a_n) + d(w).$$

Because the differential on w increases tensor length by one, and w is at least of length two, then no term of the form $a_i \otimes a_{n-i}$ can occur in $d(w)$. Thus, all such terms in v must come from $kd(a_n)$, forcing $k = \pm 1$. We then conclude:

$$d(a_n) = \pm v + d(w), \quad w \in T(a_1, \ldots, a_{n-1}).$$

By adjusting the orientation of the cell corresponding to a_n, we can assume that $d(a_n) = v + d(w)$.Since we have $[v + d(w)] = [v]$ we choose an Adams–Hilton model for $\mathbb{CP}^n$ with $d(a_n) = v$, that extends the model of $\mathbb{CP}^{n-1}$. □

8 Cobar construction

For a **graded R-module M** The **r-suspension** s^r is defined by

$$(s^r M)_n = M_{n-r}.$$

A (co)algebra A is called **connected** if $A_n = 0$ for $n \leq -1$ and $A_0 \cong R$.

8.1 Introduction

To compute the boundary in the chain complex of $AH(X)$ for general X is a major problem in homotopy theory. For certain special types of complexes this has been done,in the special case that X is given as a simplicial complex with no edges and only one vertex, Adams built an explicit model with an explicit boundary map.

8.2 The main theorem

The cobar constrution Ω is a functor from differential graded coalgebras to differential graded algebras.

$$\Omega : \mathbf{DGC}_1 \to \mathbf{DGA}_0 \quad (C, d_C, \nabla_C) \mapsto \Omega C = (T(s^{-1}C), d_\Omega)$$

from the category of 1-connected DG-coalgebras to the category of connected DG-algebras. Here, $T(s^{-1}C)$ is the free tensor algebra on the module $s^{-1}C$ and d_Ω is the unique derivation such that

$$d_\Omega^2 = 0$$

$$d_\Omega(s^{-1}c) = -s^{-1}d_C(c) + (s^{-1} \otimes s^{-1})\Delta_C(c)$$

for all $c \in C$.

Theorem 8.1. *Let X be simply connected. Then there is an isomorphism of graded algebra $H_*(\Omega C_* X) = H_*(\Omega X, k)$, where C_* is the reduced singular chain complex of X.*

8.3 Homology of Spheres

Before computing the homology of the loop space of a sphere using the Cobar construction, we should first introduce the notion of formality.

Definition 8.2. A space X is said to be **formal** if there is a differential graded commutative algebra (A, d) which maps to both cochains and cohomology of X by quasi-isomorphisms

$$(H^*(X), 0) \longleftarrow (A, d) \longrightarrow (C^*(X), d)$$

Proposition 8.3. *The sphere S^n is a **formal** space.*

Proof. see example (2.80) in [12]

□

Let $C = H_0(S^n) = \mathbb{Z}\{a\}$ be the reduced homology of the n-sphere.
The coproduct structure is given by:

$$\Delta(a) = a \otimes 1 + 1 \otimes a$$

$$\Delta(a) = 0 \quad \text{(in reduced homology)}$$

Consider the tensor algebra:

$$T(s^{-1}(\tilde{H}_*(S^n)), a)$$

Generated by words $|a|a| \cdots |a|$ with the boundary operator:

$$d(|a| \cdots |a|) = 0$$

The homology of this complex is:

$$H_*(T(s^{-1}\tilde{H}_*(S^n), a)) \cong T(a)$$

where $|a|$ has degree $n - 1$.

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