Deep Optimal Stopping

Vanessa Pizante

University of Toronto

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Optimal Stopping Problem

- $X = (X_n)_{n=0}^N$ is an Markov process in $\mathbb{R}^d$ on $(\Omega, \mathcal{F}, (\mathcal{F})_{n=0}^N, \mathbb{P})$.
- $\mathcal{T}$ is the set of all X-stopping times.
  - Recall: τ is an X-stopping time if $\{\tau = n\} \in \mathcal{F}_n, \ \forall n \in \{0, 1, ..., N\}$
- $g : \{0, 1, ..., N\} \times \mathbb{R}^d \to \mathbb{R}$ is a measurable, integrable function.

Objective

Find $V = \sup_{\tau \in \mathcal{T}} \mathbb{E} g(\tau, X_\tau)$.

Introduction

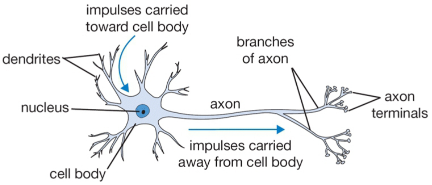
Primary Ressource

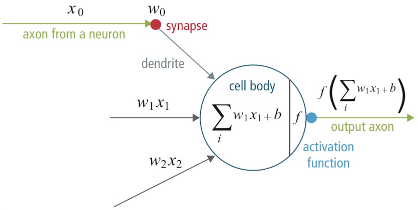
Becker, S., Cheridito, P., and Jentzen, A.: *Deep Optimal Stopping*. (2018).

- Optimal stopping problems with finitely many stopping times can be solved exactly.
  - Optimal V given by Snell Envelope.
- Difficult to approximate numerically in higher dimensions.
- [Becker et al., 2018] proposes decomposing τ into a sequence of $N+1$ 0-1 stopping decisions.
- Each decision is learned via a deep feedforward neural network.

What is a Neural Network?

- A neural network is a computational system modelled loosely after the human brain: a neuron receives input from other neurons or from an external source which it uses to generate output.
- Each input has a weight quantifying its relative significance.
- You can think of a neuron's output as its relative firing rate.

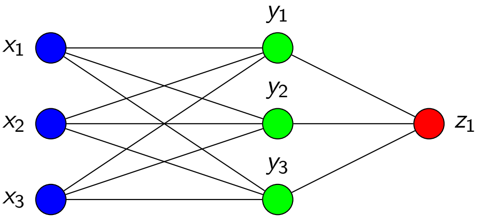




Source: [Hijazi et al., 2015]

Feedforward Neural Network

- **Input Layer**: $x = (x_1, x_2, x_3)$.
- **Hidden Layer**: Takes x, and outputs $y = (y_1, y_2, y_3)$ where $y_i = f_y(w_i x + b_i)$ and $w_i = (w_{i1}, w_{i2}, w_{i3})$ for $i = 1, 2, 3$.
- **Output Layer**: Take y and outputs $z_1 = f_z(w_z y + b_z)$ where $w_z = (w_{z1}, w_{z2}, w_{z3})$



- f_y and f_z are called **activation functions**.

Training a Neural Network

- The **training data** can consist of input and target pairs $(x^m, z_1^m)_{m=1}^M$ or just inputs $(x^m)_{m=1}^M$.

- **Training** the network refers to setting the **weight** w_i and **bias** b_i terms using the training data.
    - $\theta = \{w_1, w_2, w_3, w_z, b_1, b_2, b_3, b_z\}$ is called the model **parameters**.

- Training is done by minimizing a **loss function**, C, w.r.t θ.
    - e.g. mean squared-error loss function

$$C(\theta) = \frac{1}{2M} \sum_{m=1}^{M} \|z_1^\theta(x^m) - z^m\|^2$$

Training a Neural Network

- The optimal parameters are found by minimizing the loss function via a gradient descent algorithm.
  - e.g. Vanilla gradient descent with learning rate η update step:

$$\theta_{t+1} \leftarrow \theta^t - \eta \cdot \nabla_\theta C(\theta^t)$$

- $\nabla_\theta C(\theta^t)$ is computed using **backpropagation**.
  - Method for efficiently moving backwards through the neural network graph to compute each $\frac{\partial C}{\partial \theta_i}$ via the chain rule.
  - Special case of reverse-mode autodifferentiation.

- After training, we run the algorithm using a new dataset, this is the **testing data**.

- Note: Minimizing a loss function via gradient descent is equivalent to maximizing a reward function via gradient ascent.

Reinforcement Learning (Optional)

Problem set-up:

- Agent receives reward $r_t = r(s_t, a_t)$ depending on its actions a_t and the state s_t.
- Environment is a Markov Decison Process with *unknown* transition probabilites $p(s_{t+1}|s_t, a_t)$.

Reinforce:

- Agents aim to learn a policy $\pi_\theta(s_t, a_t)$ so they interact with their environment to maximize some cummulative reward function.
- Samples different rollouts $\rho = (s_1, a_1, ..., s_T, a_T)$ to obtain a policy where rollouts corresponding to a high reward are more likely to be chosen.

- *Model-free* approach to policy iteration.

Q-Learning (Optional)

- Q-function (a.k.a. action-value function): Expected rewards (discounted by factor γ) if you take action a then follow your policy:

$$Q^\pi(s,a) = \mathbb{E}\left[\sum_{i=0}^{\infty} \gamma_i r_{t+i} \middle| s_t = s, a_t = a\right]$$

- Q-learning is an iterative algorithm based on the recursive formula for Q derived via Bellman's equation:

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r(s_t, a_t) + \gamma \max_a Q(s_{t+1}, a) - Q(s_t, a_t))$$

- Note $r(s_t, a_t) + \gamma \max_a Q(s_t, a) - Q(s_t, a_t)$ is the Bellman error $\implies$ Q-Learning is equivalent to minimizing the Bellman error.

Optimal Stopping and Q-Learning (Optional)

Yu, H. and Bertsekas, D.P.: *Q-learning Algorithms for Optimal Stopping Based on Least Squares.* (2007).

- Given current state x_t we have two options:
  - Stop and incur *cost* $f(x_t)$.
  - Continue and incur *cost* $h(x_t, x_{t+1})$.
- Let $Q^*(x_t) = \min_a Q(x_t, a)$ where a is either to stop or continue.
- Optimal policy is to stop as soon as (X_t) enters the set: $\mathcal{D} = \{x_t | f(x_t) \leq Q^*(x_t)\}$

Optimal Stopping and Q-Learning (Optional)

- Consider approximations for Q that use a linear projection of transformed x_t [Yu and Bertsekas, 2007]:

$$Q^*(x_t) \equiv \phi(X_t)'\beta_t$$

- The corresponding projected Bellman error is then minimized w.r.t. β_t via a least-squares approximation based on the Q-learning algorithm as follows:

$$\beta_{t+1} = \beta_t + \alpha(\hat{\beta}_{t+1} - \beta_t)$$

where

$$\hat{\beta}_t = \arg\min_{\beta} \sum_{k=0}^{t} \left(\phi(x_k)'\beta - h(x_k, x_{k+1}) - \gamma \min\{f(x_{k+1}), \phi(x_k)'\beta_t\}\right)^2$$

Least-Squares Monte Carlo (LSMC)

Longstaff, F. and Schwartz E.: *Valuing American Options by Simulation: A Simple Least-Squares Approach*, (2001).

- Simulate M stock price paths $(x_n^m)_{n=0}^N$.
- Set $V_N^m = g(N, x_N^m), \forall m$ and then for $n = N-1, ..., 0$:
  - Approximate continuation value q_n via $\hat{q}_n(x) = \sum_{k=0}^K \beta_k x^k$ where β minimizes the least-squares error

$$\sum_{m=1}^{M} (e^{-rt_n} V_{n+1}^m - q_n(x_n^m))^2$$

  - Set $V_n^m = \begin{cases} g(n, x_n^m) & \text{if } q_n(x_n^m) < g(n, x_n^m) \\ e^{-rn\frac{T}{N}} V_{n+1}^m & \text{otherwise} \end{cases}$

f

Least-Squares Neural Network Approach

Kohler, Krzyzak and Todorovic: *Pricing of High-Dimensional American Options by Neural Networks*, (2006).

- Neural network applied to Longstaff-Schwartz algorithm to price higher dimensional American options.
- Replaces least-squares regression with neural network of the form:

$$q^{\theta_n} = a_2^{\theta_n} \circ \sigma \circ a_1^{\theta_n}$$

  where

  - $a_1 : \mathbb{R}^d \to \mathbb{R}^K$, $a_2 : \mathbb{R}^K \to \mathbb{R}$: $a_i(x) = W_i x + b_i$ and $\sum_{k=0}^{K} |w_{2,k}| \leq C_n$.
  - $\sigma : \mathbb{R}^k \to [0,1]^K$: $\sigma_k(y_k) = 1/(1 + e^{-y_k})$.

- Algorithm backwards recursively creates training data $(x_n^m, e^{-rt_n} \cdot V_{n+1}^m)$ and has a mean squared loss function.

Extension: More Complex Neural Networks

Hu, R.: *Deep Learning for Ranking Response Surfaces with Applications to Optimal Stopping Problems*, (2019).

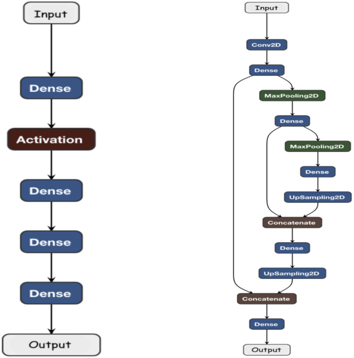


Figure 1: Left: Feed-forward NN, similar to NN applied in [Becker et al., 2018], Right: UNET, applied in [Hu, 2019].

Source: [Hu, 2019]

Ranking Response Surfaces

- Optimal stopping problem can be reframed as an image classification problem via surface ranking.
- Surface ranking problem: Extract index of minimal surface for each input x and treat it as a class label.
  - For optimal stopping labels are just stop or continue (at each time step).
- Can now **use convolution NNs** (e.g. UNET), which can be more computationally efficient [Hu, 2019].
- *Disadvantage*: Convergence theory has yet to have been completely developed for fully convolutional NN.

Expressing Stopping Times as a Series of 0-1 Decisions

Goal

Show optimal decision to stop $(X_n)_{n=0}^N$ can be made according to $\{f_n(X_n)\}_{n=0}^N$ where $f_n : \mathbb{R}^d \to \{0, 1\}$.

- Write time n stopping time, τ_n, as a function of $\{f_k(X_k)\}_{k=n}^N$.
- Let $\mathcal{T}_n$ be the set of all X-stopping times τ s.t. $n \leq \tau \leq N$.
- Clearly $\mathcal{T}_N = \{N\}$, so let $\tau_N = N \cdot f_N(X_N)$ where $f_N \equiv 1$.
- Now, for $n = N - 1, ..., 0$, we can define:

$$\tau_n = \sum_{k=n}^{N} k f_k(X_k) \prod_{j=n}^{k-1} (1 - f_j(X_j)) \in \mathcal{T}_n \tag{1}$$

Theorem 1

For $n \in \{0, ..., N-1\}$, let $\tau_{n+1} \in \mathcal{T}_{n+1}$ be of the form:

$$\tau_{n+1} = \sum_{k=n+1}^{N} k f_k(X_k) \prod_{j=n+1}^{k-1} (1 - f_j(X_j)) \tag{2}$$

Then there exists a measurable function $f_n : \mathbb{R}^d \to \{0, 1\}$ such that $\tau_n \in \mathcal{T}_n$ satisfies

$$\mathbb{E} g(\tau_n, X_{\tau_N}) \geq V_n - (V_{n+1} - \mathbb{E} g(\tau_{n+1}, X_{\tau_{n+1}})) \tag{3}$$

where V_n and V_{n+1} satisfy

$$V_n = \sup_{\tau \in \mathcal{T}_n} \mathbb{E} g(\tau, X_\tau) \tag{4}$$

Sketch of Theorem 1 Proof

- Fix stopping time $\tau \in \mathcal{T}_n$ and let $\epsilon = V_{n+1} - \mathbb{E}g(\tau_{n+1}, X_{\tau_{n+1}})$.
- Show $g(\tau_{n+1}, X_{\tau_{n+1}})$ is meas. using (2).
  $\implies \exists h_n$ meas. and Markovian s.t. $h_n(X_n) = \mathbb{E}[g(\tau_{n+1}, X_{\tau_{n+1}})|X_n]$.
- Define $D = \{g(n, X_n) \geq h_n(X_n)\}$, $E = \{\tau = n\} \in \mathcal{F}_n$ and:
  - $\tau_n = nI_D + \tau_{n+1} I_{D^c} \ \in \mathcal{T}_n$
  - $\tilde{\tau} = \tau_{n+1} I_E + \tau I_{E^c} \ \in \mathcal{T}_{n+1}$
- So $\mathbb{E}g(\tau, X_{\tau_{n+1}}) \geq \mathbb{E}g(\tilde{\tau}, X_{\tilde{\tau}}) - \epsilon \implies \mathbb{E}[g(\tau, X_{\tau_{n+1}}) I_{E^c}] \geq \mathbb{E}[g(\tau, X_\tau) I_{E^c}] - \epsilon$.
- *Exercise:* Show $\mathbb{E}g(\tau_n, X_{\tau_n}) \geq \mathbb{E}g(\tau, X_\tau) - \epsilon$.
- Since τ is arbitrary, we have $\mathbb{E}g(\tau_n, X_{\tau_n}) \geq V_n - \epsilon$, satisfying (3).
- Proof is completed by showing τ_n here satisfies (1).

What have we proven?

- Theorem 1 shows that τ_n from (1) can adequately be used to compute V_n.
  - Meaning τ_n is the time n optimal stopping time.
  - Follows from (3) due to the backwards recursive way τ_n is computed.

- Optimal stopping time corresponding to $V = \sup_{\tau \in \mathcal{T}} \mathbb{E} g(\tau, X_\tau)$ is:

$$\tau = \sum_{n=1}^{N} n f_n(X_n) \prod_{k=0}^{n-1} (1 - f_k(X_k))$$

- ...But how do we find the sequence $\{f_n\}_{n=0}^{N}$?

Introducing a Neural Network

- Using $f_N \equiv 1$ we construct a sequence of neural networks, $f^{\theta_n} : \mathbb{R}^d \to \{0, 1\}$, to approximate f_n for $n = N - 1, ..., 0$.
- We can then approximate τ_{n+1} via

$$\sum_{k=n}^{N} k \cdot f^{\theta_k}(X_k) \prod_{j=n}^{k-1} (1 - f^{\theta_j}(X_j)) \tag{5}$$

Problem:

- f^{θ_n} produces binary output $\implies$ it is not continuous w.r.t. θ.
- Need continuous output to train θ_n via a gradient-based optimization algorithm.

Neural Network with Continuous Output

Introduce $F^\theta : \mathbb{R}^d \to (0,1)$, a two-layer, feed-forward neural network of the form

$$F^\theta = \psi \circ a_3^\theta \circ \phi_{q_2} \circ a_2^\theta \circ \phi_{q_1} \circ a_1^\theta \tag{6}$$

where

- q_1 and q_2 are the number of nodes in the hidden layers.
- $a_1^\theta : \mathbb{R}^d \to \mathbb{R}^{q_1}$, $a_2^\theta : \mathbb{R}^{q_1} \to \mathbb{R}^{q_2}$, $a_3^\theta : \mathbb{R}^{q_2} \to \mathbb{R}$ satisfy

$$a_i^\theta(x) = W_i x + b_i$$

- $\phi_{q_i} : \mathbb{R}^{q_i} \to \mathbb{R}^{q_i}$ are ReLU activation function, $\phi_{q_i}(x_1, ..., x_{q_i}) = (x_1^+, ..., x_{q_i}^+)$.
- $\psi : \mathbb{R} \to \mathbb{R}$ is the logistic sigmoid function, $\psi(x) = 1/(1 + e^{-x})$.

Return to Binary Decisions

- Parameters are $\theta = \{(A_i, b_i)_{i=1}^3\} \in \mathbb{R}^q$, where $q = q_1(d + q_2 + 1) + 2q_2 + 1$.
- Can now use F^θ to find optimal θ_n via gradient descent for $n = N - 1, ..., 0$.
- After, we can compute $f^{\theta_n} : \mathbb{R}^d \to \{0, 1\}$ using:

$$f^{\theta_n} = I_{[0,\infty)} \circ a_3^{\theta_n} \circ \phi_{q_2} \circ a_2^{\theta_n} \circ \phi_{q_1} \circ a_1^{\theta_n} \tag{7}$$

- Note [Becker et al., 2018] does not provide formal proof that using F^θ to optimize θ will provide optimal parameters for f^θ.
- However, it does make sense intuitively when we consider $F^{\theta_n}(X_n)$ to be the probability that stopping is the optimal decision at time n (given X_n).

Selecting a Reward Function

Main Idea

Want to define a reward function that yields a stopping decision at time n that will maximize our expected future payoff.

If at time n we:

- Stop $\implies$ $f_n(X_n) = 1$ and we will receive payoff $g(n, X_n)$
- Continue and after proceed optimally $\implies$ $f_n(X_n) = 0$ and we will eventually receive payoff of $g(\tau_{n+1}, X_{\tau_{n+1}})$.

So we want our reward function at time n to approximate

$$\sup_{f \in \mathcal{D}} \mathbb{E}[g(n, X_n) f(X_n) + g(\tau_{n+1}, X_{\tau_{n+1}})(1 - f(X_n))] \quad (8)$$

where $\mathcal{D}$ is the set of all $f : \mathbb{R}^d \to \{0, 1\}$ measurable.

- ...But can we even replace f in (8) with neural network f^θ?

Theorem 2

Let $n \in \{0, ..., N-1\}$ and fix a stopping time $\theta_{n+1} \in \mathcal{T}_{n+1}$, then $\forall \epsilon > 0$, there exists $q_1, q_2 \in \mathbb{N}^+$ such that

$$\begin{aligned} &\sup_{\theta \in \mathbb{R}^q} \mathbb{E}[g(n, X_n) f^\theta(X_n) + g(\tau_{n+1}, X_{\tau_{n+1}})(1 - f^\theta(X_n))] \\ \geq &\sup_{f \in \mathcal{D}} \mathbb{E}[g(n, X_n) f(X_n) + g(\tau_{n+1}, X_{\tau_{n+1}})(1 - f(X_n))] - \epsilon \end{aligned} \tag{9}$$

where $\mathcal{D}$ is the set of all measurable functions $f : \mathbb{R}^d \to \{0, 1\}$.

Corrallary 1

For any $\epsilon > 0$, there exists $q_1, q_2 \in \mathbb{N}^+$ and neural network functions of the form (7) such that $f^{\theta_N} \equiv 1$ and the stopping time

$$\hat{\tau} = \sum_{n=1}^{N} n f^{\theta_n}(X_n) \prod_{k=0}^{n-1} (1 - f^{\theta_k}(X_k)) \tag{10}$$

satisfies $\mathbb{E} g(\hat{\tau}, X_{\hat{\tau}}) \geq \sup_{\tau \in \mathcal{T}} \mathbb{E} g(\tau, X_\tau) - \epsilon$.

- This means we can approximate the sequence of optimal stopping decisions $\{f_n\}_{n=0}^N$ with the sequence of optimized neural networks $\{f^{\theta_n}\}_{n=0}^N$.

Sketch of Theorem 2 Proof (Optional)

- By integrability of g, $\exists \tilde{f} : \mathbb{R}^d \to \{0,1\}$ meas. s.t.

$$\begin{aligned} &\mathbb{E}[g(n, X_n)\tilde{f}^\theta(X_n) + g(\tau_{n+1}, X_{\tau_{n+1}})(1 - \tilde{f}^\theta(X_n))] \\ \geq\ &\sup_{f \in \mathcal{D}} \mathbb{E}[g(n, X_n)f(X_n) + g(\tau_{n+1}, X_{\tau_{n+1}})(1 - f(X_n))] - \epsilon/4 \end{aligned} \tag{11}$$

- Let $\tilde{f} = I_A$, $A = \{x \in \mathbb{R}^d | \tilde{f} = 1\}$ and note

$$B \mapsto \mathbb{E}[|g(n, X_n)| I_B(X_n)] \qquad \text{and} \qquad B \mapsto \mathbb{E}[|g(\tau_{n+1}, X_{\tau_{n+1}})| I_B(X_n)]$$

  are finite Borel measures on $\mathbb{R}^d$.

- So $\exists K \subseteq A$ compact s.t.

$$\begin{aligned} &\mathbb{E}[g(n, X_n)I_K(X_n) + g(\tau_{n+1}, X_{\tau_{n+1}})(1 - I_K(X_n))] \\ \geq\ &\mathbb{E}[g(n, X_n)\tilde{f}^\theta(X_n) + g(\tau_{n+1}, X_{\tau_{n+1}})(1 - \tilde{f}^\theta(X_n))] - \epsilon/4 \end{aligned} \tag{12}$$

Sketch of Theorem 2 Proof Cont. (Optional)

- Let $\rho_K(x) = \inf_{y \in K} \|x - y\|_2$ and note $k_j(x) = \max\{1 - j \cdot \rho_K(x),\ -1\}$, $j \in \mathbb{N}$ converges pointwise to $I_K - I_{K^c}$. So by DCT:

$$\begin{aligned} &\mathbb{E}[g(n, X_n) I_{k_j(X_n) \geq 0} + g(\tau_{n+1}, X_{\tau_{n+1}})(1 - I_{k_j(X_n) \geq 0})] \\ &\geq \mathbb{E}[g(n, X_n) I_K(X_n) + g(\tau_{n+1}, X_{\tau_{n+1}})(1 - I_K(X_n))] - \epsilon/4 \end{aligned} \tag{13}$$

- By **[Leshno et al., 1993]** since k_j can be approx. uniformly on compact sets by $h(x) = \sum_{i=1}^{r} (v_i^T x + c_i)^+ - \sum_{i=1}^{s} (w_i^T x + d_i)^+$ and thus:

$$\begin{aligned} &\mathbb{E}[g(n, X_n) I_{h(X_n) \geq 0} + g(\tau_{n+1}, X_{\tau_{n+1}})(1 - I_{h(X_n) \geq 0})] \\ &\geq \mathbb{E}[g(n, X_n) I_{k_j(X_n) \geq 0} + g(\tau_{n+1}, X_{\tau_{n+1}})(1 - I_{k_j(X_n) \geq 0})] - \epsilon/4 \end{aligned} \tag{14}$$

- By considering $I_{[0,\infty)} \circ h$ as an NN of the form f^θ, eq.(14) becomes:

$$\begin{aligned} &\mathbb{E}[g(n, X_n) f^\theta(X_n) + g(\tau_{n+1}, X_{\tau_{n+1}})(1 - f^\theta(X_n))] \\ &\geq \mathbb{E}[g(n, X_n) I_{k_j(X_n) \geq 0} + g(\tau_{n+1}, X_{\tau_{n+1}})(1 - I_{k_j(X_n) \geq 0})] - \epsilon/4 \end{aligned} \tag{15}$$

- Combining (11), (12), (13) and (15) we obtain (9), as required.

Computing Estimates via the Neural Network

- Simulate M independent paths of Markov process, $(x_n^m)_{n=0}^N$.
- θ_N is chosen so $f^{\theta_N} \equiv 1$ and for $n = N-1, ..., 0$:
  - Use $\theta_{n+1}, ..., \theta_N$ to compute $\bar{\tau}_{n+1}^m$ along each of the m paths via

$$\bar{\tau}_{n+1}^m = \sum_{k=n+1}^{N} k f^{\theta_k}(x_n^m) \prod_{j=n+1}^{k-1} (1 - f^{\theta_j}(x_j^m))$$

  - At time n along path m, if we stop w.p. $F^{\theta_n}(x_n^m)$ and then adhere to $\{f^{\theta_k}(x_k^m)\}_{k=n+1}^N$, the realized reward is

$$r_n^m(\theta_n) = g(n, x_n^m) F^{\theta}(x_n^m) + g(\bar{\tau}_{n+1}^m, x_{\bar{\tau}_{n+1}^m}^m)$$

  - For sufficiently large M

$$\frac{1}{M} \sum_{m=1}^{M} r_n^m(\theta_n) \tag{16}$$

approximates $\mathbb{E}[g(n, X_n) F^{\theta_n}(X_n) + g(\tau_{n+1}, X_{\tau_{n+1}})(1 - F^{\theta_n}(X_n))]$.

Computing Estimates via the Neural Network

- Note (16) acts as the reward function we want to maximize w.r.t. θ_n via a gradient ascent algorithm.
- Next translate $F^{\theta_n}(x_n^m)$ into 0-1 stopping decisions, $f^{\theta_n}(x_n^m)$.

- Now generate our testing sample paths $(y_n^m)_{n=0}^N$ for $m = 1, ..., M$.
- For $n = N-1, ..., 0$, using the $\{\theta_n\}$ found during training, compute

$$\tilde{\tau}_{n+1}^m = \sum_{k=n+1}^{N} k f^{\theta_k}(y_k^m) \prod_{j=n+1}^{k-1} (1 - f^{\theta_j}(y_j^m))$$

along each of the m sample paths.

Computing Estimates via the Neural Network

- Overall optimal stopping time (estimate of $\hat{\tau}$ from (10)):

$$\tilde{\tau}^m = \sum_{n=1}^{N} k f^{\theta_n}(y_n^m) \prod_{k=1}^{n-1} (1 - f^{\theta_k}(y_k^m))$$

- Corresponding Monte Carlo estimate of $V = g(\hat{\tau}, X_{\hat{\tau}})$:

$$\hat{V} = \frac{1}{M} \sum_{m=1}^{M} g(\tilde{\tau}_m, y_{\tilde{\tau}_m}^m) \tag{17}$$

  - By CLT a $1 - \alpha$, $\alpha \in (0, 1)$ confidence interval for V is:

$$\left[\hat{V} - z_{\alpha/2} \frac{\hat{\sigma}}{\sqrt{M}}, \hat{V} + z_{\alpha/2} \frac{\hat{\sigma}}{\sqrt{M}}\right]$$

  where $z_{\alpha/2}$ is the $1 - \alpha/2$ quantile of $\mathcal{N}(0, 1)$ and

$$\hat{\sigma} = \frac{1}{M-1} \sum_{m=1}^{M} \left(g(\tilde{\tau}_m, y_{\tilde{\tau}_m}^m) - \hat{V} \right)^2$$

Application: Bermudan Max Call Option

Example

Bermudan max-call option expiring at time T with strike price K written on d assets, $X^1, ..., X^d$, with $N+1$ equidistant exercise times $t_n = nT/N,\ n = 0, 1, ..., N$.

- Payoff function time t is: $\left(\max_{i \in \{1,\dots,d\}} X_t^i - K\right)^+$.
- So its price at time t is $\sup_\tau \mathbb{E}\left[e^{-r\tau}\left(\max_{i \in \{1,\dots,d\}} X_\tau^i - K\right)^+\right]$.
- Frame this as an optimal stopping problem $\sup_{\tau \in \mathcal{T}} \mathbb{E} g(\tau, X_\tau)$ where

$$g(n, x) = e^{-rt_n}\left(\max_{i \in \{1,\dots,d\}} x^i - K\right)^+ \tag{18}$$

Application: Bermudan Max Call Option

- Assume a Black-Scholes market model with d uncorrelated assets.
- For $i = 1, ..., d$ set:

$$x_0^i = 90, \quad K = 100, \quad \sigma_i = 0.2, \quad \delta_i = 0.1, \quad r = 0.05, \quad T = 3, \quad N = 9$$

- Asset price paths can be simulated via

$$x_{n,i}^m = x_{0,i} \cdot \exp\left\{ \sum_{k=0}^{n} \left((r - \delta_i - \sigma_i^2/2)\Delta t + \sigma_i \sqrt{\Delta t} \cdot Z_{k,i}^m \right) \right\} \tag{19}$$

where $\Delta t = T/N$ and $Z_{k,i}^m \sim \mathcal{N}(0,1)$.

Building Neural Network using PyTorch

Constructing a neural network of the form F^{θ} from (6) is simple using `Pytorch`!

```
import torch.nn as nn

class NeuralNet(torch.nn.Module):
    def __init__(self, d, q1, q2):
        super(NeuralNet, self).__init__()
        self.a1 = nn.Linear(d, q1)
        self.relu = nn.ReLU()
        self.a2 = nn.Linear(q1, q2)
        self.a3 = nn.Linear(q2, 1)
        self.sigmoid=nn.Sigmoid()

    def forward(self, x):
        out = self.a1(x)
        out = self.relu(out)
        out = self.a2(out)
        out = self.relu(out)
        out = self.a3(out)
        out = self.sigmoid(out)

        return out
```

Training the Network in Pytorch

```
def loss(y_pred,s, x, n, tau):
    r_n=torch.zeros((s.M))
    for m in range(0,s.M):
        r_n[m]=-s.g(n,m,x)*y_pred[m] - s.g(tau[m],m,x)*(1-y_pred[m])
    return(r_n.mean())

def NN(n,x,s, tau_n_plus_1):
    epochs=50
    model=NeuralNet(s.d,s.d+40,s.d+40)
    optimizer = torch.optim.Adam(model.parameters(), lr = 0.0001)

    for epoch in range(epochs):
        F = model.forward(X[n])
        optimizer.zero_grad()
        criterion = loss(F,S,X,n,tau_n_plus_1)
        criterion.backward()
        optimizer.step()

    return F,model
```

- `s.g` is g, as defined in equation (18).

Adaptive Moment Estimation (Adam) Optimization

Input: α, $r(\theta)$, θ^0

Set $\beta_1 = 0.9; \beta_2 = 0.99; \epsilon = 10^{-8}$

$m_0 = v_0 = 0;\ t = 0$

while $r(\theta_t)$ *not minimized* **do**

- $t = t + 1$
- $G_t = \nabla_\theta r(\theta_{t-1})$
- $m_t = \beta_1 \cdot m_{t-1} + (1 - \beta_1) \cdot G_t$
- $v_t = \beta_2 \cdot v_{t-1} + \beta_2 \cdot (G_t)^2$
- $\hat{m}_t = m_t / (1 - (\beta_1)^t)$
- $\hat{v}_t = v_t / (1 - (\beta_2)^t)$
- $\theta_t = \theta_{t-1} - \alpha \cdot \hat{m}_t / (\sqrt{\hat{v}_t} + \epsilon)$

Output: θ^t

- Adaptive: Performs smaller updates (lower learning rate) for frequently occurring features.
- Moment Estimation: Stores exponentially decaying average of gradient's mean (m_t) and uncentered variance (v_t).

Results

d	Actual	Estimate	Standard Error	95 % Confidence Interval
2	8.04	7.50	0.23	(7.05, 7.95)
5	16.64	16.80	0.32	(16.18,17.43)
10	26.20	23.83	0.29	(23.25, 24.40)

Table 1: Tabulated estimates for V. Results computed using $M = 5000$ sample paths.

Note: Actual estimate for $d = 2$ is from [Hu, 2019] and for $d = 5, 10$ are from [Becker et al., 2018].

Optimally Stopping a Fractional Brownian Motion (Optional)

- Fractional Brownian motion with Hurst Parameter H is a Gaussian process with mean zero and covariance structure

$$\mathbb{E}[W_t^H W_s^H] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) \tag{20}$$

- Standard Brownian Motion is a fractional Brownian motion with $H = 1/2$.

Objective

We want to evaluate $\sup_{0\leq\tau\leq 1} \mathbb{E} W_\tau^H$.

Challenge of Optimally Stopping FBM (Optional)

Optional Stopping Theorem (OST)

Let $(X_t)_{t\geq 0}$ be a martingale and τ be a stopping time with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, then if either of the following hold:

- τ is bounded
- $X_{\tau\wedge t}$ is bounded

it follows that $\mathbb{E}X\tau = \mathbb{E}X_0$.

- So by OST $\mathbb{E}W_\tau^{1/2} = 0$ for any stopping time τ s.t. $W_{\tau\wedge t}^{1/2}$ is bounded above by a constant.
- However if $H \neq 1/2$, W^H is not a martingale, so the OST does not apply.

Optimally Stopping a Fractional Brownian Motion (Optional)

- Discretize the time interval $[0, 1]$ into time steps, $t_n = \frac{n}{100}$.
- Create a 100-dimensional Markov process, $(X_n)_{n=0}^{100}$ to describe $(W_{t_n}^H)_{n=0}^{100}$:

$$\begin{aligned} X_0 &= (0, 0, ..., 0) \\ X_1 &= (W_{t_1}^H, 0, ..., 0) \\ X_2 &= (W_{t_1}^H, W_{t_2}^H, ..., 0) \\ &\vdots \\ X_{100} &= (W_{t_{100}}^H, W_{t_{99}}^H, ..., W_{t_1}^H) \end{aligned}$$

- Letting $g : \mathbb{R}^{100} \to \mathbb{R}$ be $g(x_1, ..., x_{100}) = x_1$, we now aim to compute

$$\sup_{\tau \in \mathcal{T}} \mathbb{E} g(X_\tau) \tag{21}$$

where τ is the set of all X-stopping times.

Optimally Stopping a Fractional Brownian Motion (Optional)

- [Becker et al., 2018] simulated $(X_n)_{n=0}^{100}$ by defining $Y_n^H = W_{t_n}^H - W_{t_{n-1}}^H$ for each $n = 1, ..., N$, which forms a stationary Gaussian process with autocovariance

$$\mathbb{E}[Y_n^H Y_{n+k}^H] = \frac{|k+1|^{2H} - |k|^{2H} + |k-1|^{2H}}{2(100^{2H})}$$

- Sample paths of (X_n) are then computed via $W_{t_n}^H = \sum_{k=1}^{n} Y_k^H$.

- Then train neural networks of the form (7) with $d = 100$, $q_1 = 110$ and $q_2 = 55$ and approximate (21) via Monte Carlo.

Optimally Stopping a Fractional Brownian Motion (Optional)

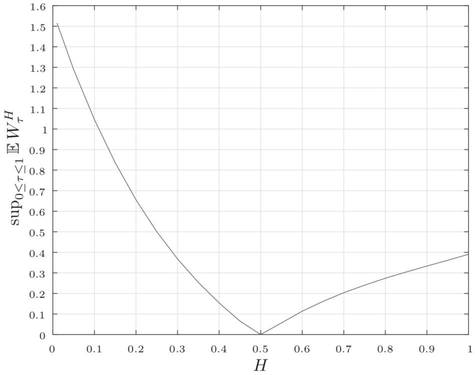


Figure 2: Results of [Becker et al., 2018]. Expected value of optimally stopped FBM w.r.t. its Hurst parameter H.

Source: [Becker et al., 2018]

Concluding Remarks

- Neural networks (and machine learning algorithms in general) can aid in solving the optimal stopping problem in higher dimensions.
- New area of research:
  - Neural Network employed by [Becker et al., 2018] has one of the simplest architectures.
  - More complex architectures lack fully developed convergence theory [Hu, 2019].

Questions for further research:

- Can we strategically select the number of nodes per hidden layer and/or the number of epochs for this algorithm?
- What other neural network architectures might be worth trying?
- What other problems in mathematical finance can be approximated in higher dimensions by applying neural networks?

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