When to invest? Entry-exit market equilibria

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Abstract

In this report, we study a paper [3] written by Charles Bertucci, Louis Bertucci, Jean-Michel Lasry and Pierre-Louis Lions, dealing with market equilibria processes in bitcoin mining. First, we discuss the model used and highlight two refinements of the basic model. Then, we look at optimal stopping problems. These problems are related to the Hamilton-Jacobi-Bellman partial differential equations for which we propose numerical methods of resolution.

1 Introduction

[3] presents the problem of bitcoin mining, and more precisely the problem of entry or exit in this market.
The record of all Bitcoin transactions since its creation is called the blockchain. It is a decentralized record, i.e. there is no entity (banks, states...) that regulates the validity of the transactions recorded in it. In fact, this validity is based on the principle of general consensus, which itself results from the *proof-of-work*. It is a method that uses a cryptographic problem that can only be solved by brute force, and which obliges miners to verify transactions. The first miner to solve the problem receives a fixed remuneration in bitcoin, and a block of transactions is therefore added to the blockchain, as can be seen in 1.

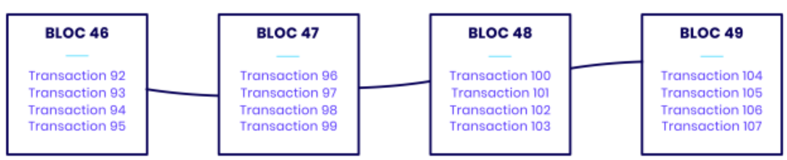

Figure 1: Example of blockchain

Miners are therefore in competition with each other. The mining market can therefore be modeled as a non-cooperative game with a very large number of players. Each player has the choice to invest in his computing power in order to gain an advantage over his competitors and be more likely to be the first to solve the problems. The strategy set is therefore $\mathbb{R}_+$ which is the total computing power that the player wishes to allocate to mining. Regarding winnings, the more computing power a player has, the higher the expectation of his income will be. However, high computing power also implies high energy costs, which may deter players from investing more. In fact, miners have an incentive to enter the market in order to receive bitcoins by mining. However, if the hashrate - the total amount of power dedicated to mining - is too high, the cost of mining exceeds the expected payout, and it is smarter to exit the market. This has the effect of reducing the hashrate and thus attracting other players to enter the market.
Since we are not interested in the strategy of each player, but more in the "global behavior" of all

players, the study of the problem under the prism of medium-field games [4] [6] seems relevant. Indeed, these tackle the study of large populations of agents in competition with medium-field interactions, i.e. each player is only interested in the average actions of the other players, and not in their individual strategies. In [6], the two authors proposed methods for producing approximations of Nash equilibria for symmetric stochastic games with a large number of players. It is in this context that our study is situated, as we are looking for Nash equilibria, i.e. situations where none of the players has an interest in changing strategies, to a game that we will model in 2.1.
We will also show that this modeling will lead us to study and then solve non-linear partial equations, called "master equation". [4] shows that this type of equations appears naturally through the chosen model and that they allow us to characterize the equilibria we are looking for. The last part of this report therefore focuses on the numerical methods for solving these equations. The finite difference methods studied in [1] and [5] are used to obtain approximations of the optimal stopping problems that we will describe.

2 Mathematical modeling of the problem

2.1 Formalization and study of the deterministic problem

Let us study the problem of mining in a deterministic way.
We note P_t the power allocated to the mining at the time t, for any $t \in \mathbb{R}_+$. Let us note δ the rate of innovation, i.e. the rate of evolution of the power produced at fixed cost over time. By noting K_t the adjusted power, we have:

$$\forall t \in \mathbb{R}_+, \;\; K_t = e^{-\delta t} P_t.$$

Here we consider machines that can only mine Bitcoin, and the remuneration is here normalized to 1. Let us therefore note U the remuneration corresponding to a computing power of K_t and λ the incentive coefficient to enter or leave the market. We have:

$$\forall t \in \mathbb{R}_+, \;\; \dot{K}_t = -\delta K_t + \lambda U_t.$$

It is clear that we must look for a form of U decreasing in K since the higher the hashrate, the lower the remuneration. Moreover, since U must encapsulate the information of the payoff between $t = 0$ and $t = +\infty$, we can imagine U as $U = \int_0^\infty \frac{1}{K_t} - c \, dt$. Nevertheless, we do not know a priori if K_t can be equal to 0 (everyone leaves the market) and the integral does not converge if K_t converges to a K_∞ value. We then assume that a hashrate unit gives a value $\frac{1}{K_t+\epsilon}$ with $\epsilon << c$. Moreover, we add the coefficients of innovation δ and risk aversion r. We therefore get:

$$U(K) = \int_0^{+\infty} e^{-(r+\delta)t} (\frac{1}{K_t + \epsilon} - c) dt. \tag{1}$$

K_t is therefore a solution to the following fixed point problem:

$$\begin{cases} dK_t = -\delta K_t dt + \lambda U(K_t) dt \\ K_0 = K \end{cases} \tag{2}$$

We notice that it is $U(K_t)$ that we have in the equation. This is the potential gain if the process started in K_t and not in K. In order to characterize the solutions of this problem, we will try to obtain an equivalent expression. Let us show that (1) and (2) imply:

$$\boxed{U'(K)(-\delta K + \lambda U(K)) + \frac{1}{K+\epsilon} - c + (r+\delta) U(K) = 0 \text{ for all } K \in [0, +\infty(} \tag{3}$$

Demonstration

Let's begin by deriving U to better understand $U(K_t)$. We have :

$$U(K_{dt}) - U(K_0) = \int_0^{+\infty} e^{-(r+\delta)t}(\frac{1}{\tilde{K}_t+\epsilon} - \frac{1}{K_t+\epsilon})dt,$$

where $\tilde{K}_t$ verifies:

$$\begin{cases} d\tilde{K}_t = -\delta\tilde{K}_t dt + \lambda U(L_t)dt \\ \tilde{K}_0 = K_{dt} \end{cases}$$

We have by change of variable:

$$U(K_{dt}) - U(K_0) = -\int_0^{dt} e^{-(r+\delta)t}\frac{e^{(r+\delta)dt}}{K_t+\epsilon}dt \;+\; \int_{dt}^{\infty} e^{-(r+\delta)t}\frac{e^{(r+\delta)dt}-1}{K_t+\epsilon}dt,$$

therefore by dividing by dt, we have:

$$U'(K)K'_0 = -\frac{1}{K+\epsilon} + c + (r+\delta)U(K),$$

and then:

$$U'(K)(-\delta K + \lambda U) + \frac{1}{K+\epsilon} - c + (r+\delta)U(K) = 0 \text{ pour tout } K \in [0,+\infty(.$$

In fact, there is equivalence between the existence and uniqueness of the fixed point problem and the existence and uniqueness of the solution of this non-linear differential equation. This equation is known as the 'master equation' in the literature of mean-field games. [6][4]. We can intuitively convince ourselves of the existence of a single solution to this equation. Indeed, the whole thing is based on the notion of competition. The more K_t increases, the less it will increase. There is a phenomenon of convergence that can be interpreted as a Nash equilibrium that ensures the existence and uniqueness of the master equation.
We can also show [3] that for any K_0, the K_t process will converge to a K_∞ value which depends explicitly on the model parameters. This is **the market equilibrium**. Thus, the dynamics of entry and exit market evolve until they converge towards a situation where the hashrate is constant. Under these conditions, miners have a zero gain because otherwise they would be tempted to enter or exit the market. [3] further studies the properties of this equilibrium as well as the speed of convergence of K_t according to the parameters of the model.

2.2 A first refinement: the probabilistic problem

Let's take the notations from the previous section. Let us fix $(\Omega, \mathcal{A}, \mathbb{P})$ a probabilistic space and look at the refinement where the unit $\frac{1}{K_t+\epsilon}$ of hashrate gives $g(P_t)$ in bitcoin with $g : \mathbb{R} \to \mathbb{R}$ a positive continuous function and P_t following the stochastic process:

$$\begin{cases} dP_t = \alpha(P_t)dt + \sqrt{2\nu}dW_t \\ P_0 = p \end{cases} \tag{4}$$

where (W_t) is a standard Brownian motion on $(\Omega, \mathcal{A}, \mathbb{P})$.
This time, U depends on K but also on p:

$$U(K,p) = \mathbb{E}\left[\int_0^{+\infty} e^{-(r+\delta)t}(\frac{g(P_t)}{K_t+\epsilon} - c)dt\right]. \tag{5}$$

Met's show that (4) and (5) imply:

$$\boxed{\partial_K U(-\delta K + \lambda U) + \alpha\partial_P U + \nu\partial_{PP}U + \frac{g(p)}{K+\epsilon} - c - (r+\delta)U = 0\ ,\ (K,p) \in \mathbb{R}_+ \times \mathbb{R}.} \tag{6}$$

Demonstration

According to 2.1, we have :

$$\partial_t U(K_t, p) = -\frac{g(p)}{K+\epsilon} + c + (r+\delta)U.$$

Itô's lemma gives in the present case:

$$dU(K,p) = \mathbb{E}\left(\partial_K U dK_t + \partial_P U dP_t + \frac{1}{2}\partial_{PP} U \sqrt{2\nu}^2 dt\right).$$

We therefore have:

$$dU(K,p) = \left(\partial_K U(-\delta K + \lambda U) + \partial_P U \alpha + \nu \partial_{PP} U\right) dt.$$

We conclude that:

$$\partial_K U(-\delta K+\lambda U)+\alpha\partial_P U+\nu\partial_{PP}U+\frac{g(p)}{K+\epsilon}-c-(r+\delta)U = 0 \text{ pour tout } (K,p) \in [0,+\infty(\times \mathbb{R}.$$

Again, there is equivalence between the initial problem and this partial differential equation. The article [3] shows that the problem is well posed, i.e. that there is a single solution to the problem. We can also expect that there is no equilibrium at $t = +\infty$. Indeed, although the incentives of miners push the hashrate to converge, the randomness of the gains can make the mining more or less profitable, leading to market imbalance.

2.3 A second refinement: the deterministic problem with two populations

Let's take the notations introduced in 2.1. We can focus on the refinement where two different populations are mining bitcoin at the same time. It is assumed that the rates of innovation are the same for these two populations. Nevertheless, different energy costs are chosen, c_1 and c_2, and different risk aversion coefficients, r_1 and r_2. It seems necessary to define two different hashrates, which we note K_t and L_t. Thus, the gain of each population is written as follows:

$$U(K,L) = \int_0^{+\infty} e^{-(r_1+\delta)t} \max\left(\frac{1}{K_t+L_t+\epsilon} - c; 0\right) dt, \tag{7}$$

$$V(K,L) = \int_0^{+\infty} e^{-(r_2+\delta)t} \max\left(\frac{1}{K_t+L_t+\epsilon} - c; 0\right) dt. \tag{8}$$

Indeed, we suppose here that minors have the right to turn off their machines, and thus to receive a zero payout. K_t and L_t also check the following processes:

$$\begin{cases} dK_t = -\delta K_t dt + \lambda_1 U(K_t, L_t) dt \\ dL_t = -\delta L_t dt + \lambda_2 V(K_t, L_t) dt \\ K_0 = K, L_0 = L \end{cases} \tag{9}$$

Let's show that (7), (8) and (9) imply:

$$\boxed{(-\delta K + \lambda_1 U)\partial_K U + (-\delta L + \lambda_2 V)\partial_L U - (r_1+\delta)U + \max(\frac{1}{K+L+\epsilon} - c_1; 0) = 0} \tag{10}$$

$$\boxed{(-\delta K + \lambda_1 U)\partial_K V + (-\delta L + \lambda_2 V)\partial_L V - (r_2+\delta)V + \max(\frac{1}{K+L+\epsilon} - c_2; 0) = 0} \tag{11}$$

Demonstration

Like in 2.1, we start by deriving $U(K, L)$:

$$U(K_{dt}, L_{dt}) - U(K_0, L_0) = -\int_0^{dt} e^{-(r_1+\delta)(t-dt)} \max\left(\frac{1}{K_t + L_t + \epsilon} - c_1; 0\right)$$
$$+ \int_{dt}^{\infty} e^{-(r_1+\delta)t}(e^{(r_1+\delta)dt} - 1) \max\left(\frac{1}{K_t + L_t + \epsilon} - c_1; 0\right).$$

By dividing by dt and writing:

$$U(K_{dt}, L_{dt}) - U(K_0, L_0) = U(K_{dt}, L_{dt}) - U(K_{dt}, L_0) + U(K_{dt}, L_0) - U(K_0, L_0),$$

we have:

$$\partial_K U K_0' + \partial_L U L_0' = (r_1 + \delta)U - \max\left(\frac{1}{K + L + \epsilon} - c_1; 0\right),$$

and therefore:

$$(-\delta K + \lambda_1 U)\partial_K U + (-\delta L + \lambda_2 V)\partial_L U - (r_1 + \delta)U + \max\left(\frac{1}{K + L + \epsilon} - c_1; 0\right) = 0,$$

and by symmetry:

$$(-\delta K + \lambda_1 U)\partial_K V + (-\delta L + \lambda_2 V)\partial_L V - (r_2 + \delta)V + \max\left(\frac{1}{K + L + \epsilon} - c_2; 0\right) = 0.$$

Once again, [3] shows that the problem is well posed. It is a 2-dimensional form of the master equation. It can be shown that in a stationary state, all machines are working. Indeed, if machines are switched off, then the miners concerned will stop buying machines and the hashrate will decrease to the stationary state. On the other hand, if we assume that $c_1 << c_2$, then only the miners in the first population will mine. If, on the contrary, the two costs are close, both populations will mine, including the one with the highest costs.

These three examples highlight the dynamics of market equilibrium in bitcoin mining. The study of these equilibria through the prism of medium field games proves to be relevant insofar as we find intuitive results *a priori* and that this approach allows us to complexify and enrich the model without significant computational cost.
Therefore, we could be more interested in the resolution of the partial differential equations that we have encountered.

3 Methods for numerical resolution of non-linear partial differential equations

3.1 Numerical method for linear partial differential equations

Before starting to solve equations of the type we have encountered, it is necessary to become familiar with the finite difference method presented in [1] and [2]. To do so, we consider the following introductory example:
Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probabilized space and X_t a stochastic process evolving according to:

$$\begin{cases} dX_t = \alpha dt + \sigma dW_t & \text{where } W_t \text{ is a Brownian motion and } (\alpha, \sigma) \in \mathbb{R}^2 \\ X_0 = x_0 \end{cases}$$

Here again, we are interested in a stopping problem through v defined by

$$\forall x \in [0, 1], v(x) = \mathbb{E}\left(\int_0^{\tau} e^{-\lambda t} f(X_t) d_t + e^{-\lambda \tau} \Psi(X_\tau) | X_0 = x\right),$$

with τ the first moment where the process leaves the set $[0,1]$,$\lambda \in \mathbb{R}$ et $f : \mathbb{R} \to \mathbb{R}$ et $\Psi : \mathbb{R} \to \mathbb{R}$ supposedly regular functions.
Then v is the solution of the following partial differential equation:

$$\begin{cases} \frac{\sigma^2}{2}\Delta v + \alpha \cdot \nabla v - \lambda v + f = 0 \text{ in } [0,1] \\ v = \Psi \text{ on } \partial[0,1] = \{0,1\} \end{cases} \tag{12}$$

It is a linear partial differential equation. The Lax-Milgram theorem assures the existence and uniqueness of the solutions provided that it is placed on the correct Hilbert space (here it is $\mathbb{H}_2$) and that f is fairly regular. In order to solve numerically this type of differential equations, the finite element method is used. We approximate v by the vector $V := \begin{pmatrix} v(0) \\ v(1/n) \\ \dots \\ v(1) \end{pmatrix}$ and f by: $F := \begin{pmatrix} f(0) \\ f(1/n) \\ \dots \\ f(1) \end{pmatrix}$. with $n \in \mathbb{N}^*$ representing the number of points in the method. We still have to write Δv and ∇v. Their writing depends on the conditions at the edges, which are here of Dirichlet type, i.e. they condition the value of v on the edge of the domain. To make things easier, it is assumed here that $\Psi = 0$ so as to have homogeneous Dirichlet conditions. The laplacian and the gradient are then written: $\begin{pmatrix} \frac{-2v(0)+v(1/n)}{(\frac{1}{n})^2} \\ \frac{-2v(1/n)+v(2/n)+v(0/n)}{(\frac{1}{n})^2} \\ \dots \\ \frac{-2v(1)+v(1-1/n)}{(\frac{1}{n})^2} \end{pmatrix}$ for the laplacian and $\begin{pmatrix} \frac{v(0)}{\frac{1}{n}} \\ \frac{v(1/n)-v(0)}{\frac{1}{n}} \\ \dots \\ \frac{v(1)-v(1-1/n)}{\frac{1}{n}} \end{pmatrix}$ for the gradient if α is positive. Indeed, if α is negative, we rather write the gradient as such: $\begin{pmatrix} \frac{v(1)-v(0)}{\frac{1}{n}} \\ \frac{v(2/n)-v(1/n)}{\frac{1}{n}} \\ \dots \\ \frac{-v(1)}{\frac{1}{n}} \end{pmatrix}$.

We therefore have as a new equation:

$$\frac{\sigma^2}{2} \begin{pmatrix} \frac{v(0)}{\frac{1}{n}} \\ \frac{v(1/n)-v(0)}{\frac{1}{n}} \\ \dots \\ \frac{v(1)-v(1-1/n)}{\frac{1}{n}} \end{pmatrix} + \alpha \cdot \begin{pmatrix} \frac{-2v(0)+v(1/n)}{(\frac{1}{n})^2} \\ \frac{-2v(1/n)+v(2/n)+v(0/n)}{(\frac{1}{n})^2} \\ \dots \\ \frac{-2v(1)+v(1-1/n)}{(\frac{1}{n})^2} \end{pmatrix} - \lambda \begin{pmatrix} v(0) \\ v(1/n) \\ \dots \\ v(1) \end{pmatrix} + \begin{pmatrix} f(0) \\ f(1/n) \\ \dots \\ f(1) \end{pmatrix} = 0.$$

The vector V is therefore the solution of a matrix equation of the type $AV = F$. We propose to see the problem as an equation of the type $H(V) = 0$ that we solve using Newton's algorithm, whose pseudo-code algorithm is recalled:

Algorithm 1 Newton

Require: f differentiable with $\forall x, f'(x) \neq 0, \epsilon > 0$
Ensure: x so as $f(x) \approx 0$
  $x \leftarrow 0$
  while $f(x) \geq \epsilon$ **do**
    $x \leftarrow x - \frac{f(x)}{f'(x)}$
  end while

For $f(x) = x, \alpha = 2, \sigma = 0.5, \lambda = 1$, we get the solution figure 2 which indicates that the expectation is maximal for $x = 0.25$ approximately. We see that the Dirichlet conditions are well verified.

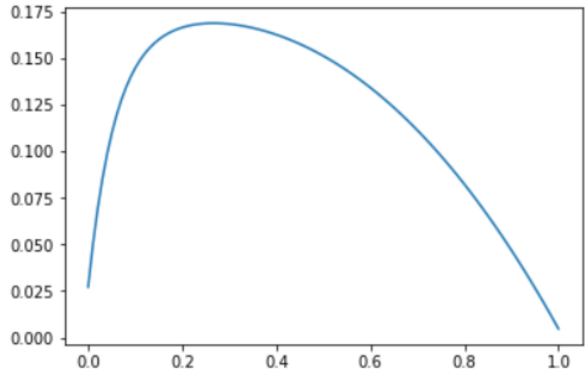

Figure 2: Solution for $f(x) = x$

If this time we keep the same values for the constants but change f for $f(x) = (x - \frac{1}{2})^2$, we get the graph figure 3 where we find the influence of the maximum in 0.5.

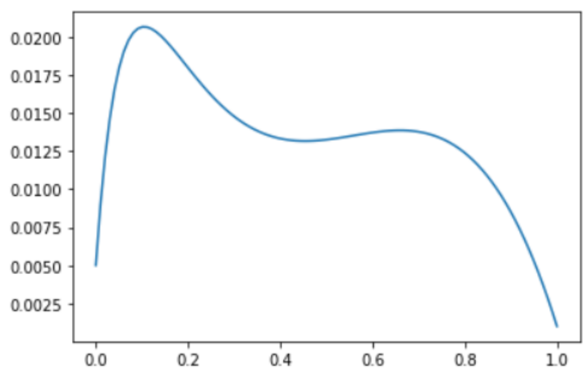

Figure 3: Solution four $f(x) = (x - \frac{1}{2})^2$

3.2 Numerical method to solveHJB in dimension 1

3.2.1 An optimal control problem

Let's start to look at non-linear partial differential equations. Let's study a slightly simpler problem than those encountered previously. We consider $(\Omega, \mathcal{A}, \mathbb{P})$ a probabilized space and the following process:

$$\begin{cases} dX_t = \alpha_t dt + \sqrt{2\nu} dW_t & \text{where } W_t \text{ is a Brownian motion} \\ X_0 = x_0 \end{cases}$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ and we focus on following the optimal control problem:

$$\inf_{\alpha_t} \mathbb{E} \left[\int_0^\infty e^{-rt} \left(f(X_s) + \frac{1}{2}\alpha_s^2 \right) d_s \right],$$

with $f : \mathbb{R} \to \mathbb{R}$ a known cost function assumed to be regular. We can imagine that an agent follows a trajectory described by the process X_t and pays a cost $e^{-rt} \left(f(X_s) + \frac{1}{2}\alpha_s^2 \right)$ at each instant. The objective is to reduce his cost, so we look for the best possible control α_t, that is to say the one minimizing the expected cost.

In order to solve this problem, we note

$$U(x) := \inf_{\alpha_t} \mathbb{E} \left[\int_0^\infty e^{-rt} \left(f(X_s) + \frac{1}{2}\alpha_s^2 \right) d_s | X_0 = x \right].$$

U is called the "value" function of the problem. It is somehow the value of the best answer knowing that at $t = 0$, the player is in x. Let's study U further: by applying Itô's lemma to U, U verifies the next problem:

$$rU + \frac{1}{2}U'^2 - \nu U'' = f.$$

This is a d'Hamilton-Jacobi equation in dimension 1.

3.2.2 A numerical solving method

Let's try to solve this equation in $[0.1]$, based on the finite element method seen 3.1. We still have to write the term U'^2. One would be tempted to write it as v' previously by squaring all the components. Nevertheless, in 3.1, the writing of the gradient depended on the sign of α. What to choose in this case to have convergence of Newton's method? In fact, according to [2] we write U'^2 as follows: $\begin{pmatrix} (\frac{U(1/n)-U(0)}{\frac{1}{n}})_+^2 \\ (\frac{U(1/n)-U(2/n)}{\frac{1}{n}})_+^2 + (\frac{U(1/n)-U(0)}{\frac{1}{n}})_+^2 \\ \dots \\ (\frac{U(1)-U(1-1/n)}{\frac{1}{n}})_+^2 \end{pmatrix}$ if we still place ourselves in the hypothesis of homogeneous Dirichlet conditions. The partial derivative equation then becomes:

$$\frac{1}{2}\begin{pmatrix} (\frac{U(1/n)-U(0)}{\frac{1}{n}})_+^2 \\ (\frac{U(1/n)-U(2/n)}{\frac{1}{n}})_+^2 + (\frac{U(1/n)-U(0)}{\frac{1}{n}})_+^2 \\ \dots \\ (\frac{U(1)-U(1-1/n)}{\frac{1}{n}})_+^2 \end{pmatrix} - \nu\begin{pmatrix} \frac{-2U(0)+U(1/n)}{(\frac{1}{n})^2} \\ \frac{-2U(1/n)+U(2/n)+U(0/n)}{(\frac{1}{n})^2} \\ \dots \\ \frac{-2U(1)+U(1-1/n)}{(\frac{1}{n})^2} \end{pmatrix} + r\begin{pmatrix} U(0) \\ U(1/n) \\ \dots \\ U(1) \end{pmatrix} + \begin{pmatrix} f(0) \\ f(1/n) \\ \dots \\ f(1) \end{pmatrix} = 0.$$

for $f(x) = (x - \frac{1}{2})^2, r = \frac{1}{2}, \nu = \frac{1}{2}$, we get the graph of Figure 4 :

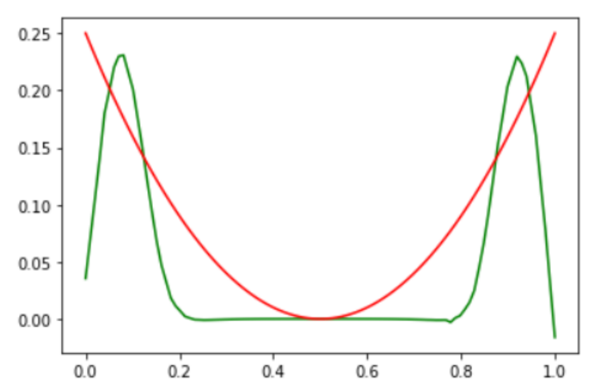

Figure 4: Solution for $f(x) = (x - \frac{1}{2})^2$

We find a symmetry of the solution with respect to the axis $x = \frac{1}{2}$ which results from the symmetry of f. We see that the Dirichlet conditions are well verified.

3.3 Numerical methods to solve HJB in dimension 2

3.3.1 An optimal stopping problem

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probabilized space. We consider the following process:

$$\begin{cases} dX_t = \alpha dt + \sigma dW_t & \text{where } W_t \text{ is a Brownian motion and} (\alpha, \sigma) \in \mathbb{R}_+ \\ X_0 = x_0 \end{cases}$$

and we focus on the optimal stopping problem that follows:

$$\inf_{\tau \leq T} \mathbb{E}\left(\int_0^\tau f(X_s)d_s + \Psi(X_\tau)\right),$$

with $f : \mathbb{R} \to \mathbb{R}$ and $\Psi : \mathbb{R} \to \mathbb{R}$ known cost functions and τ is a stopping suitable for the canonical filtration. We can imagine that an agent follows a trajectory described by the X_t process and pays an algebraic cost once it is on the market. The objective is to reduce its cost knowing that it has an exit cost Ψ. We are therefore looking for the best possible stopping time, that is to say, the one that minimizes the expected cost. In order to solve this problem, we use the principle of dynamic programming. Let's note

$$U(x,t) := \inf_{t \leq \tau \leq T} \mathbb{E}\left(\int_t^\tau f(X_s)d_s + \Psi(X_\tau)\, |X_t = x\right)$$

U is called the "value" function of the problem. It is sort of the value of the best answer knowing that at t, the player is in x. It is clear for example that $U(\cdot, T) = \Psi(\cdot)$ since if the agent is in x for $t = T$, then the only action he can do is to leave the market with a cost of $\Psi(x)$. The interest of studying U is that at any moment, if the agent knows U, he can decide to leave the game or not. For example, there is interest in staying in the game if $U < \Psi$, otherwise it would be better to quit the game.
Let's study more U. We notice that assuming that the optimal stopping time checks $\tau^* \geq t + dt$, we have:

$$U(x,t) = \mathbb{E}\left(\int_t^{t+dt} f(X_s)d_s + U(X_{t+dt}, t+dt)\right)$$

and the Itô lemma holds:

$$U(X_{t+dt}, t+dt) - U(t,x) = dt\left(\partial_t U + \frac{1}{2}\sigma^2 \Delta_x U + \alpha \partial_x U + \sigma \partial_x U dW_t\right)$$

Hence:

$$-\partial_t U - \frac{\sigma^2}{2}\Delta U - \alpha \partial_x U - f = 0$$

Now if $\tau^* = t$, then it is necessary to leave the market at t and we have $U = \Psi$. We conclude that U verifies the following problem:

$$\begin{cases} \max(-\partial_t U - \frac{\sigma^2}{2}\Delta U - \alpha \partial_x U - f, U - \Psi) = 0 \\ U(.,T) = \Psi \end{cases}$$

U is the solution of partial differential equation of Hamilton-Jacobi-Bellman (HJB) type. Such partial differential equations cannot be solved directly because of the $\max$. We opt for a constraint relaxation strategy, as described in [2]. The equation becomes:

$$-\partial_t U - \frac{\sigma^2}{2}\Delta U - \alpha \partial_x U - f + \frac{1}{\epsilon}(U - \Psi)_+ = 0$$

with ϵ that we take very small. Thus, if $U < \Psi$, then because of the $\max$, one must have: $-\partial_t U - \frac{\sigma^2}{2}\Delta U - \alpha \partial_x U - f = 0$ which is exactly what happens because in that case $(U - \Psi)_+ = 0$. If $U = \Psi$, then we know U and those maxima points are what we are looking for. Finally, if in the second formulation we have $U - \Psi > 0$, then the method does not converge for $\epsilon << 1$ and it's an impossible situation.
We implemented this method as before, writing the problem as $H(U)$ =0 and using Newton's method to solve this matrix equation. To do so, we encoded the square $[0,1]^2$ as a vector, and we performed all the calculations with vectors.
For $f(x) = (x - \frac{1}{2})^2 - 1, \sigma = 12, \alpha = 1, \epsilon = 10^{-5}$, we got the solution figure 5 :

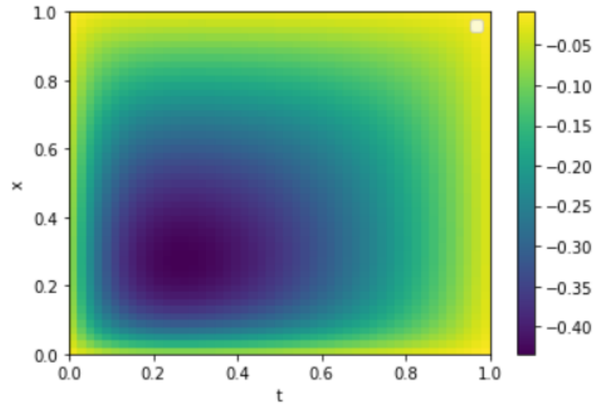

Figure 5: Solution for $f(x) = (x - \frac{1}{2})^2 - 1$

We can see that Dirichlet's conditions are well verified. By keeping the same values and changing f for $f(x) = 10(x - \frac{1}{2})^2 - 1$, we obtain the figure 6:

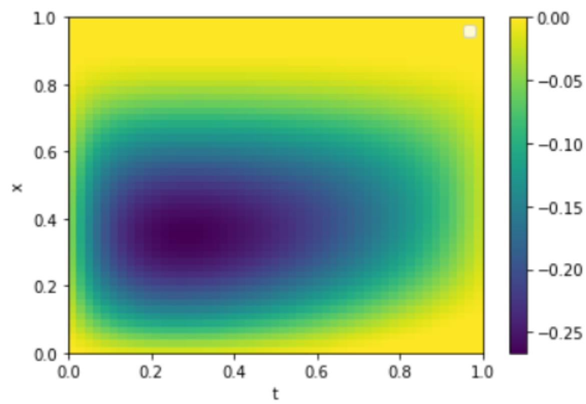

Figure 6: Solution for $f(x) = 10(x - \frac{1}{2})^2 - 1$

We can see that the values are significantly lower than before. Since f is much larger, it is smarter to leave the system earlier.

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