ECO 555 report: Gale-Shapley Stable Marriage Problem Revisited - Strategic issues and Applications

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Abstract

We focus in this report on the paper [3] of Teo, Sethuraman and Tan which tackles the problem of cheating in the Gale-Shapley stable marriage model, and then applies these theoretical results to the school allocation system in Singapore. We summed up the main ideas it depicts and coded the optimal cheating algorithm, which is available *here*. Finally we tried to challenge the french allocation system Parcoursup by confronting it to the ideas presented in [3].

1 Optimal cheating in the Gale-Shapley model

In the Gale-Shapley solution to the stable marriage problem (see *here* for an implementation), men keep asking the women according to their preference list. If a man asks a woman that is not already matched, he matches with her. If she is not single, the woman decides according to her own preference if she keeps her current partner or she changes it to the new one. This process keeps running till no one is single. One can easily prove that this algorithm works properly [1], as it gives a particular stable matching among the variety of matchings that exists. In fact, the Gale-Shapley solution is the men-optimal matching, i.e each man is paired with the best woman he can have in a stable matching. However, this matching is also the most unfavorable matching for the women [1]. We focus on the optimal cheating strategy for women, which are only allowed to lie in their preference list.

1.1 Optimal Partner

The first step of computing the optimal strategy is finding the optimal partner of a given woman w, with her true preference list. If she keeps rejecting men, this optimal partner will be among the men who asked her. The algorithm is thus fairly simple: we run Gale-Shapley algorithm with w rejecting everyone. We end the algorithm when there is only one single man. The optimal partner is then the most preferred man who asked w according to her preference. In [3] a rigorous proof of this algorithm is given. It computes indeed the optimal partner given a preference list. You can find an implementation of the optimal partner search *here*.

1.2 Optimal cheating strategy

Given the optimal partner algorithm, one can declare that there is a fairly optimal strategy for w to have the best partner possible. Indeed, w can run the Optimal Partner algorithm before the man-optimal Gale-Shapley algorithm where she will keep rejecting men till the optimal man asks her. She then can accept him and reject all the others that may ask her after. Although this strategy allows w to have her optimal partner, it doesn't allow her to get the *best* partner. We can convince ourselves with this example given in [3]:

Men	Men's preferences	Women	Women's preferences
1	2 3 4 5 1	1	1 2 3 5 4
2	3 4 5 1 2	2	2 1 4 5 3
3	5 1 4 2 3	3	3 2 5 1 4
4	3 1 2 4 5	4	4 5 1 2 3
5	1 5 2 3 4	5	5 1 2 3 4

In fact, let's focus on the woman 1. If we apply the optimal partner algorithm on her, we would see that only men 4 and 5 ask her, and therefore her optimal partner is 5. Yet, let's say she lies and say that her preference list is:

1	3 4 1 2 5

Then the Gale-Shapley men-optimal algorithm matches woman 1 with man 3, who is more desirable than man 5 according to woman 1's true preference. Therefore, the optimal partner algorithm doesn't give the *best* partner as women are allowed to lie in their preferences. Instead, [3] presents a concrete strategy to compute the fake preference list in order to get the *best* partner. The main idea is that if during the optimal partner search, a man asks w, then there is a fake list that makes w and that man match during the Gale-Shapley algorithm. The algorithm of [3] uses this principle to navigate through all the men that ask w - they are called *potential partners* -, create new list and computes the new optimal partners and so on... In the end, we have access to a subset of the set of men, that gathers all the men that ask w with a specific fake list.w can then choose the best of those men, change here list to the fake preference list where this best man is at top, and manage to be with him in the matching. In the example above, man 3 is the *best* man for woman 1 with the list:

1	3 4 1 2 5

An implementation of the optimal cheating strategy is *here*.

2 Strategic issues in the Gale Shapley Problem

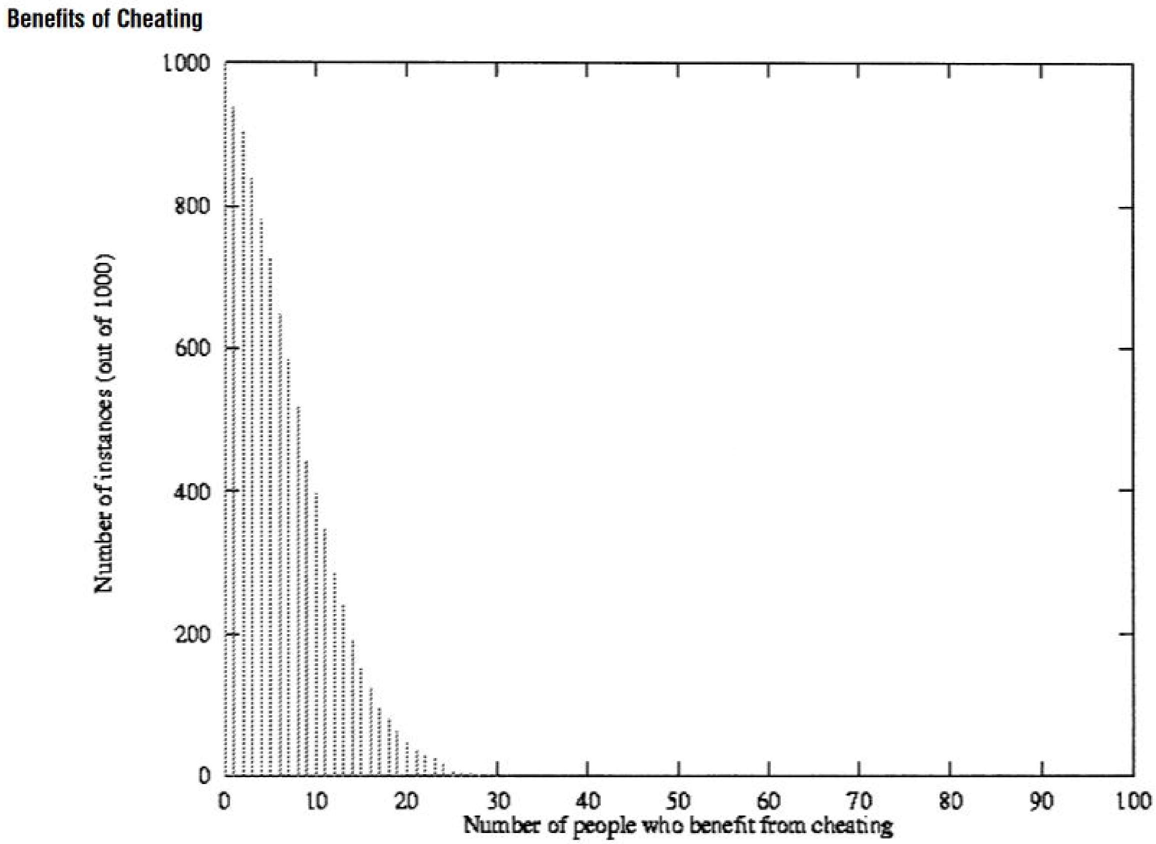
In this section we will discuss different examples in order to show that the strategic behavior of the women vary under the models with and without rejection.

2.1 The Best Man (attained by cheating) may not be women-optimal

If rejection is accepted in the two-sided matching model, women can always work their way in the men-optimal model in order to get the women-optimal solution. It suffices for each woman to reject men who are inferior to her true women-optimal partner. In [3] the choice of forbidding rejection was made. The influence of the women is therefore reduced as they can only lie on their preferences. For example, in the case where each woman is ranked first by exactly one man, in this case each man, in the men-optimal solution, is matched to the woman he ranks first. The algorithm will terminate with the men-optimal matching regardless of how the women rank the men in their lists.

2.2 Is cheating a strategy that pays?

We know that under men-optimal mechanism the men has no incentive to change their true preference lists. In the opposite side, we know that in the stable marriage game under rejection, it's more beneficial for women to not reveal their true preferences. However, these studies are not realistic in practice as the model assumes that the women have complete knowledge of all other participants as well as their preferences lists. For the Gale-Shapley model considered, the average percentage of women who benefit from cheating (by asking information about the preferences of the remaining participants in the game) is low. In order to look at the impact of strategic behavior, the author of [3] ran heuristics on 1,000 random instances for n=100 (number of men/women). The cumulative plot 2.2 show that in more than 96 % of the instances at most 20 women out of 100 benefited from cheating. The average women who benefited from cheating is 9.515 %. The chances that a typical woman can benefit from acquiring perfect information is slim in the Gale-Shapley model.



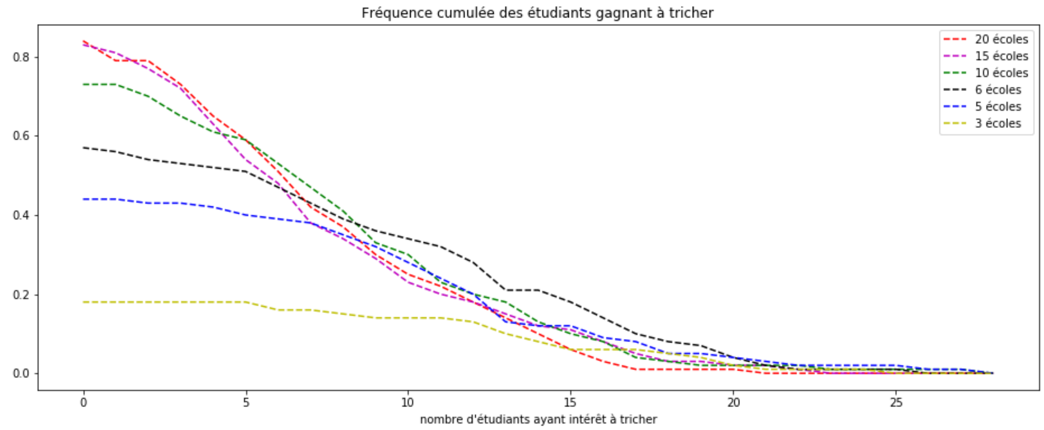
3 The school allocation problem

3.1 The example of the article

We repeated the experiment (*here*) of the impact of cheating in the school allocation model in Singapore. First of all, let us briefly recall the conditions of the experiment.

The framework of the experiment is as follows : n students are involved, with k schools, and each school has m_k places available. The game is carried out so that each student can have one place in the end, so there are as many places as there are requests. At first, we took exactly the values of [3], i.e. we chose $n = 60$ students, a variable number of schools, included in the set $\{3, 5, 6, 10, 15, 20\}$, (we added 6 and 15 to the initial set of the article). Moreover, places are the same for each school, so $60/k$ places in each school.

Preferences were created randomly in the many-to-one model, meaning that students "randomly ranked" the k schools, as well as schools "randomly ranked" the n students. From there, fictitious schools were created with a single place, to represent the m_k places of each school, and thus move to the one-to-one model, which is simpler to process. Each multiplied school will have the same preferences and the student preferences are modified so that an artificial order is created between the same places in a school, but the order remains the same between schools.
We will then look at the benefits that students (represented as women in the algorithm of stable marriages) could have to cheat according to the algorithm presented previously.



It can be seen from the previous graph that at most 80 % of the students benefit from cheating in the case of 20 schools (unrealistic due to the population distribution of the schools that is to say only 3 students in each school). If we take the case of 6 schools, we observe that only 60 % of the students benefit from cheating. And this is an upper bound, because it is based on the order recreated to get into the one-to-one model.

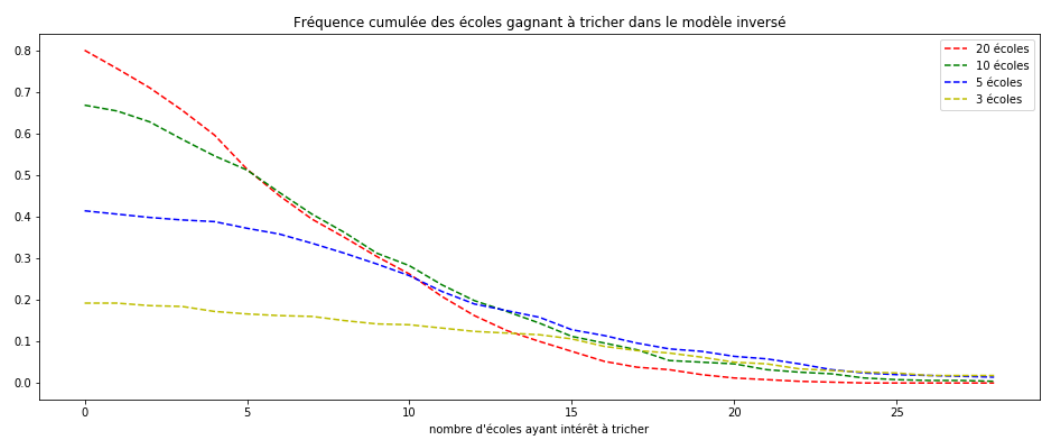
For each k value of schools, we get the average number of women/students who benefit from cheating to get a better school. This table should be understood as such: for $k = 5$, there is an average of 9.21% of students who benefit from cheating.

number of schools k	3	5	6	10	15	20
Students inclined to cheat	4.35%	9.21%	11.51%	11.95%	11.7%	11.68%

Thus, there is an increase in the number of students who would benefit from cheating to get a better choice based on the number of schools. However, there seems to be a ceiling at 12.00%, beyond which the number of schools is no longer taken into account.

3.2 Extension of the example

Now we wanted to see if it could be more profitable to reverse the roles, keeping the same algorithm: the schools become women and the students choose the schools that refuse them successively with the position of the men.

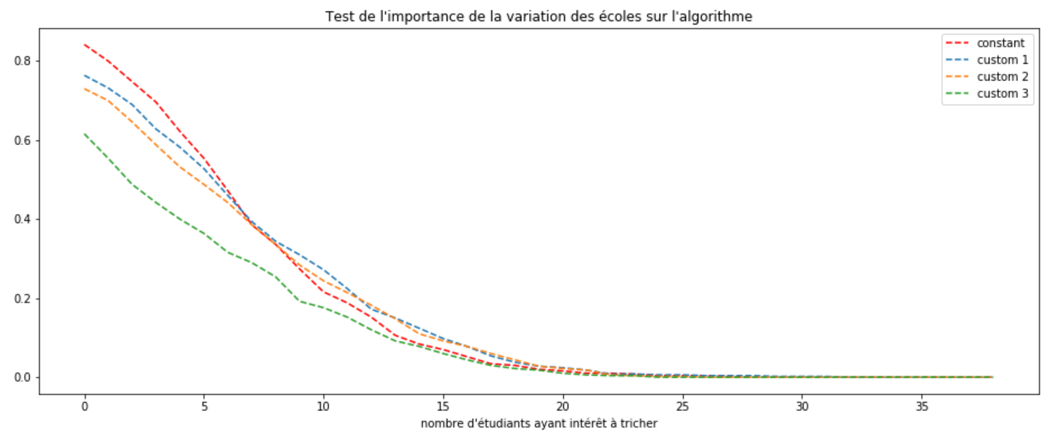


Exchanging roles does not seem to be viable because despite a slightly different situation, because of the existence of identical preferences in some schools, nothing changes the proportions. The schools would have the same gains in cheating as the students in position of women. Therefore, the payoffs for cheating do not depend on the gambler.

Finally, there is the question of the distribution of places. It seems unlikely that all schools would have the same number of places. In the graph below, we have compared the cheating winnings for different distributions of places with 20 schools, but in such a way that there are always 60 places in all, i.e. the number of students :

- custom1 := {1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 6, 6, 6, 6, 6} (5 schools with 1 place, 5 schools with 2 places, 5 schools with 3 places and 5 schools with 6 places)
- custom2 :={1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 3, 3, 3, 7, 7, 7, 7, 7}
- custom3 :={1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 5, 5, 7, 14, 14}

We wanted to gradually increase the concentration of places in certain schools and thus increase the variability in the distribution of places.



Increasing the concentration of seats in certain schools seems to cause the earnings profile to move towards the curves for a smaller number of schools. This means that *custom3* are close to the profile for 6 schools. It can therefore be concluded that increasing variability in the distribution of student places leads to a situation with fewer schools present.

In summary, the payoffs for students to cheat are increasing with the number of schools in the model but on average the number of students winning at cheating never exceeds 12% and it's relatively constant from 6 schools on. Moreover, reversing the roles in the algorithm by putting the students as male and vice versa does not change anything because the schools would gain as much from cheating as female as the initial situation. Finally, increasing the variability in the distribution of places causes the gains to tend towards situations with fewer schools and thus, according to our models, reduces the gains to cheating. The initial distribution of preferences could be improved among students by incorporating preference biases for certain popular schools. Indeed, it does not seem realistic to assume that students rank their schools randomly, nor does it seem realistic to assume that schools rank students randomly.

4 The same problem in France : Parcoursup

As explained in the article of *sciences et avenir* [2], the algorithm used by the site Parcoursup is the Gale-Shapey stable marriage one. Nonetheless, it is a bit more complex than that. Indeed, what was reproached to the previous system of assignment, APB, was its black box character, which nobody seemed to understand anything about. This is why the researchers tried to limit this aspect by increasing the students' interactions with the algorithm, ie by asking them in each round of assignment their successive preferences. The problem was that students could only act once through the ranking and then they tried to make strategies that sometimes backfired.

This is not the only novelty of Parcoursup, now the algorithm is also required to respect new criteria such as the maximum percentage of non-residents, the minimum percentage of scholarship holders or the management of boarding places (based on social and academic criteria) in preparatory classes.

To link these considerations to our previous results, it is possible to see this as a solution to limit the gains of cheating, since as the students are being questioned each time, the notion of strategy disappears. In the same way to manage social or distancing criteria, it is sufficient to give a list of preferences accordingly for non-selective degrees (selective degrees create their lists themselves). This list would then be generated by the software so as to highlight close students and scholarship holders in the preference list with coefficients to be specified. The difficulty here lies much more in the management of preference lists of non-selective degrees than in the management of the algorithm as such.

References

[1] B. Doerr. *Design and Analysis of Algorithms, INF 421*. Ecole Polytechnique - Département d'Informatique, 2019.

[2] H. Gimbert and C. Mathieu. L'algorithme de parcoursup décrypté par les deux chercheurs qui l'ont conçu. *Sciences et Avenir*, 06 2018.

[3] C.-P. Teo, J. Sethuraman, and W.-P. Tan. Gale-shapley stable marriage problem revisited: Strategic issues and applications. *Management Science*, 47(9):1252–1267, 2001.