Monitoring of Structural Health and Geohazards

CEGM2008

TU Delft

Workshop 3.1 Dynamic analysis: Full- and reduced-order solutions

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Tasks

- Briefly familiarize yourself with the FE model of the beam ->

  model.py

- Try to answer the following question:
  - If this beam is excited at the tip in the frequency range 0-500 Hz, how many modes do you expect to contribute to its response?

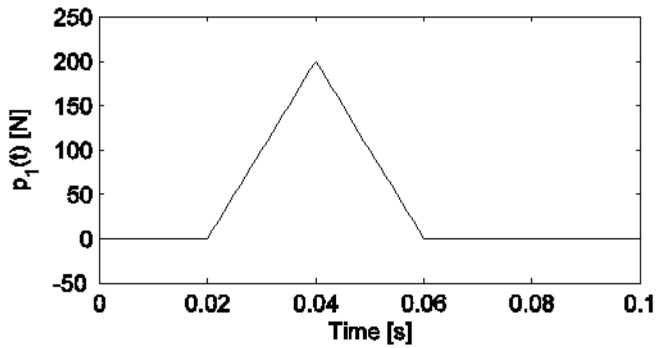
- Calculate the dynamic response of the cantilever to two transient forces, in the frequency domain:

  full-_and_reduced-order_solutions.ipynb

  a. using the frequency response function matrix,

  b. using modal superposition.

$$p_1(t) = 2p_2(t)$$



Theory needed for the full-order solution: 3.1a

1. General solution in the frequency domain
2. Construction of the damping matrix
3. Forward and inverse Fast Fourier Transforms (FFTs) [MUDE 2.4.2]
4. Appropriate choice of sampling parameters
5. Conjugation property

1. General solution in the frequency domain

- SDOF equation of motion:

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = p(t)$$

- MDOF equation of motion:

$$\mathbf{M}\ddot{\mathbf{U}}(t) + \mathbf{C}\dot{\mathbf{U}}(t) + \mathbf{K}\mathbf{U}(t) = \mathbf{P}(t)$$

- Transformation to frequency domain (Fourier integral):

$$\int_{-\infty}^{\infty} (-\mathbf{M}\omega^2 + i\omega\mathbf{C} + \mathbf{K})\tilde{\mathbf{U}}(\omega)e^{i\omega t}d\omega = \int_{-\infty}^{\infty} \tilde{\mathbf{P}}(\omega)e^{i\omega t}d\omega$$

$$\Rightarrow (-\mathbf{M}\omega^2 + i\omega\mathbf{C} + \mathbf{K})\tilde{\mathbf{U}}(\omega) = \tilde{\mathbf{P}}(\omega)$$

$$\tilde{\mathbf{U}}(\omega) = \tilde{\mathbf{H}}(\omega)\tilde{\mathbf{P}}(\omega)$$

- Frequency response function:

$$\tilde{\mathbf{H}}(\omega) = (-\mathbf{M}\omega^2 + i\omega\mathbf{C} + \mathbf{K})^{-1}$$

2. Construction of the damping matrix

Derivation assuming mass-normalized mode shapes -> on board

$$\mathbf{C} = \mathbf{M\Phi C^{*}\Phi^{T}M}$$

$$\mathbf{C}^{*} = \begin{bmatrix} 2\omega_1\xi_1 & & \\ & \ddots & \\ & & 2\omega_n\xi_n \end{bmatrix}$$

3. Forward & inverse FFTs

- Continuous Fourier transform pair:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{x}(\omega)e^{i\omega t}d\omega$$

$$\tilde{x}(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt$$

- Discrete Fourier transform pair: → *N-periodic functions!*

$$x_n = \frac{1}{N}\sum_{k=1}^{N} \tilde{x}_k e^{i2\pi(k-1)(n-1)/N}/\Delta t, \qquad n = 1, \dots, N$$

$$\tilde{x}_k = \sum_{n=1}^{N} x_n e^{-i2\pi(k-1)(n-1)/N}\Delta t, \qquad k = 1, \dots, N$$

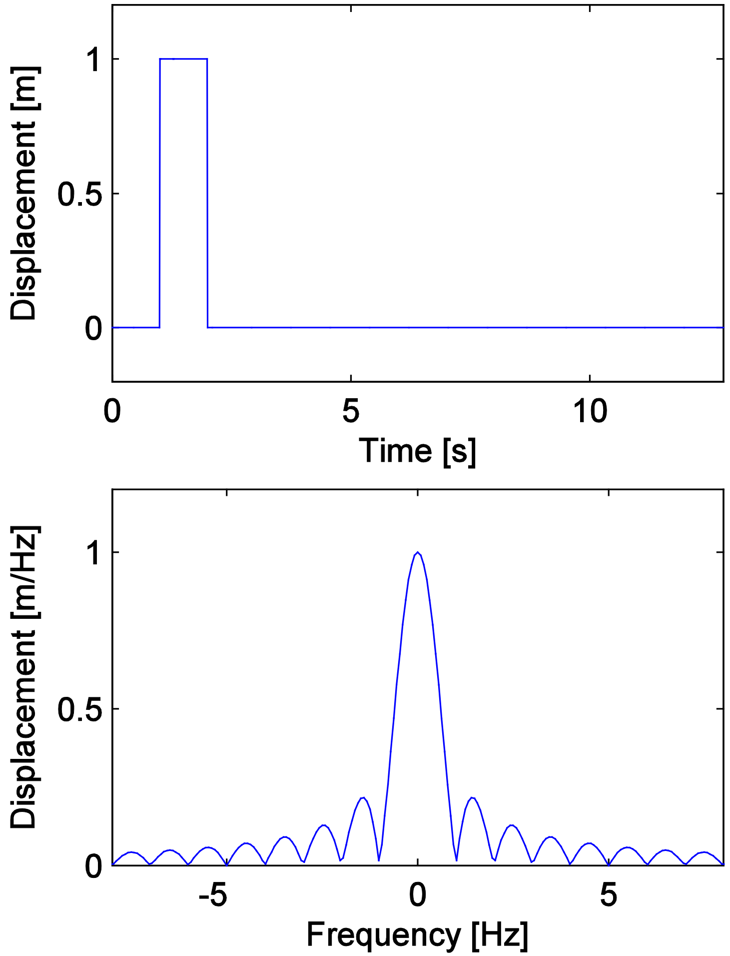
3. Forward & inverse FFTs

$$u(t) = \begin{cases} 0, & |t - t_0| > t_d/2 \\ 1, & |t - t_0| < t_d/2 \end{cases}$$

$$t_0 = 1{,}5 \text{ s} \quad \text{and} \quad t_d = 1 \text{ s}$$

$$\tilde{u}(\omega) = \int_{-\infty}^{\infty} u(t) e^{-i\omega t} dt$$

$$= t_d \text{sinc}\left(\frac{\omega t_d}{2}\right) e^{-i\omega t_0}$$



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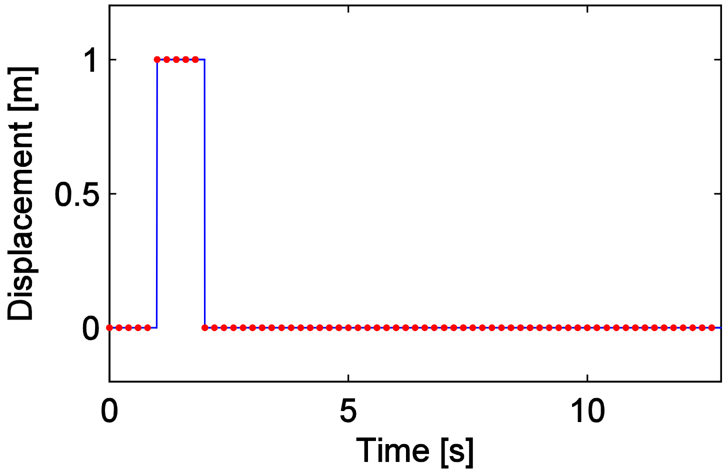
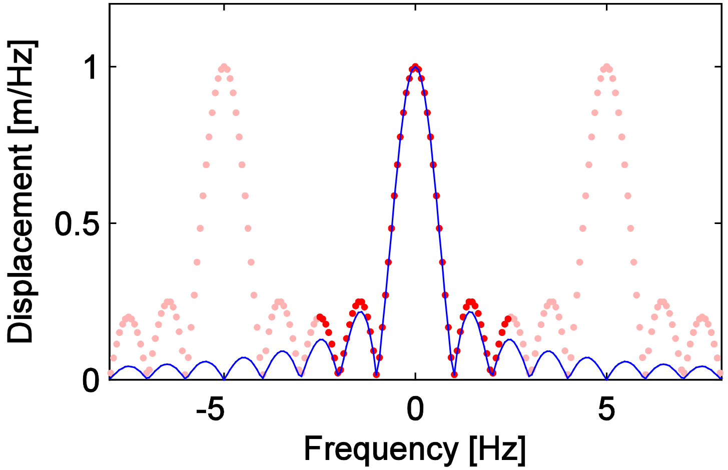
3. Forward & inverse FFTs

$$u_n = u((n-1)\Delta t), \qquad n = 1, \dots, N$$



$$\tilde{u}_k = \sum_{n=1}^{N} u_n e^{-i2\pi(k-1)(n-1)/N} \Delta t$$

$$k = 1, \dots, N$$



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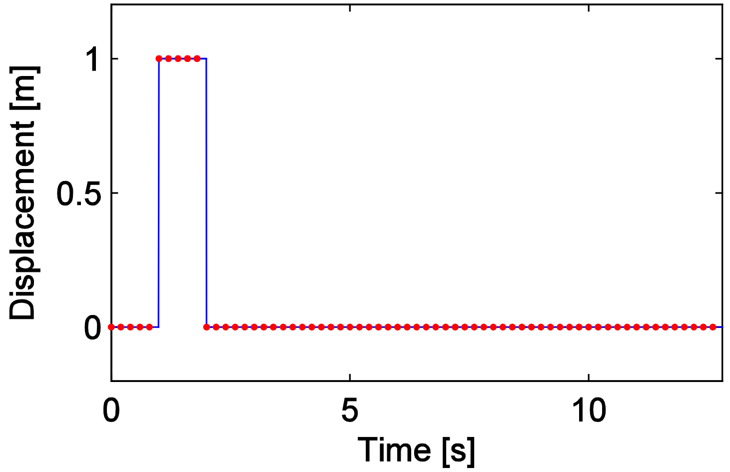
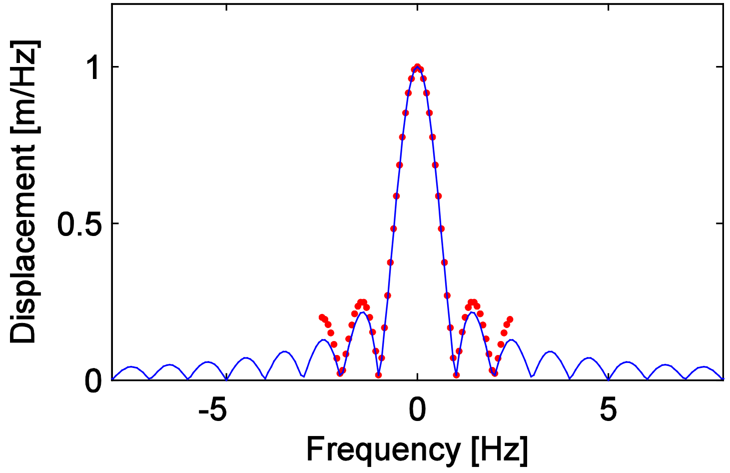
See also MUDE FFT notebook

3. Forward FFTs in Python

```
N = 64
dt = 0.2
T = N*dt
t = np.arange(N)*dt                 # Time axis
df = 1/T                            # Frequency bin
f = np.concatenate((np.arange(0, N//2), np.arange(-N//2, 0)))*df
                                    # Frequency axis

t0= 1.5
td= 1
u = np.zeros(N)
u[(t >= t0 - td / 2) & (t < t0 + td / 2)] = 1
```

```
U = np.fft.fft(u)

plt.figure()
plt.plot(t, u)

plt.figure()
plt.plot(f, np.abs(U) * dt)
```

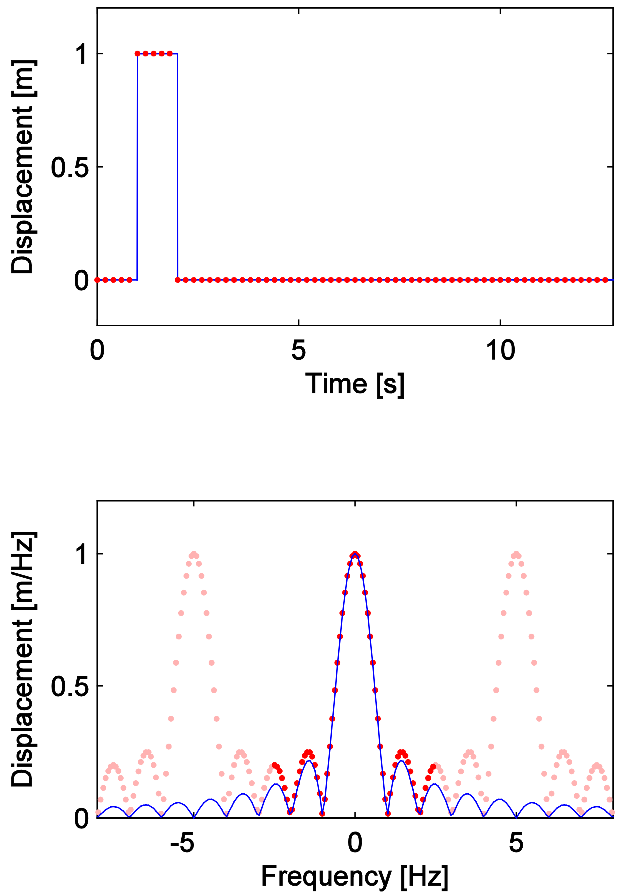




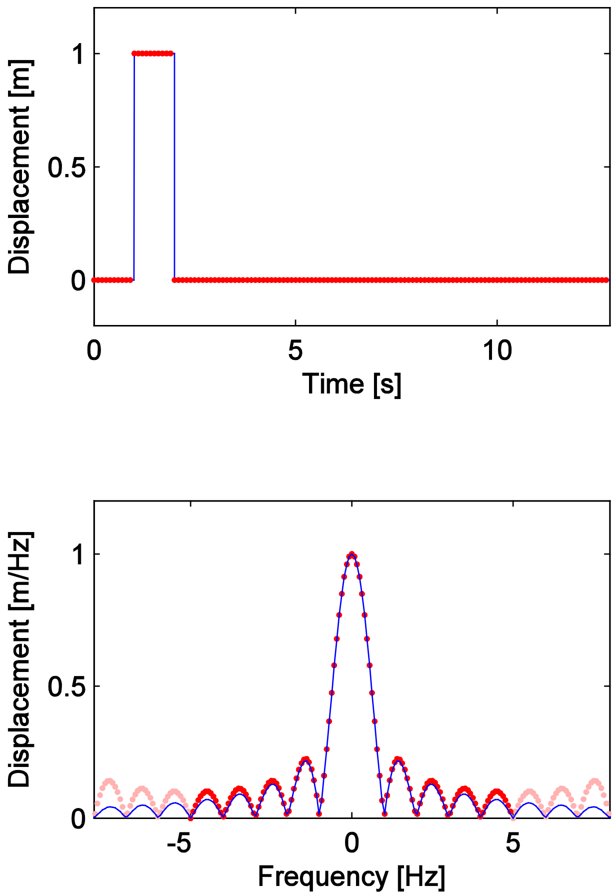
4. Choice of sampling parameters

- Influence of the time step

Δt = 0.2 s   T = 12.8 s



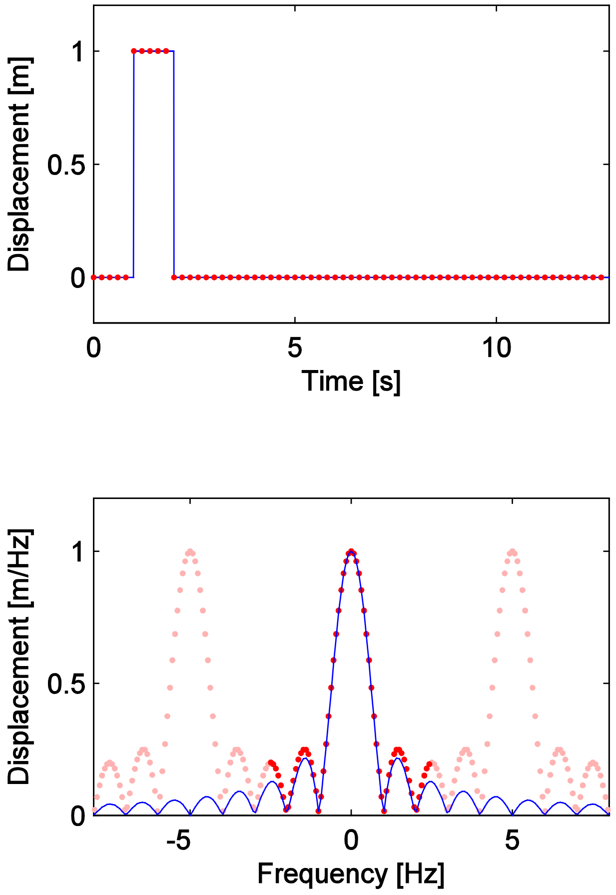
Δt = 0.1 s   T = 12.8 s



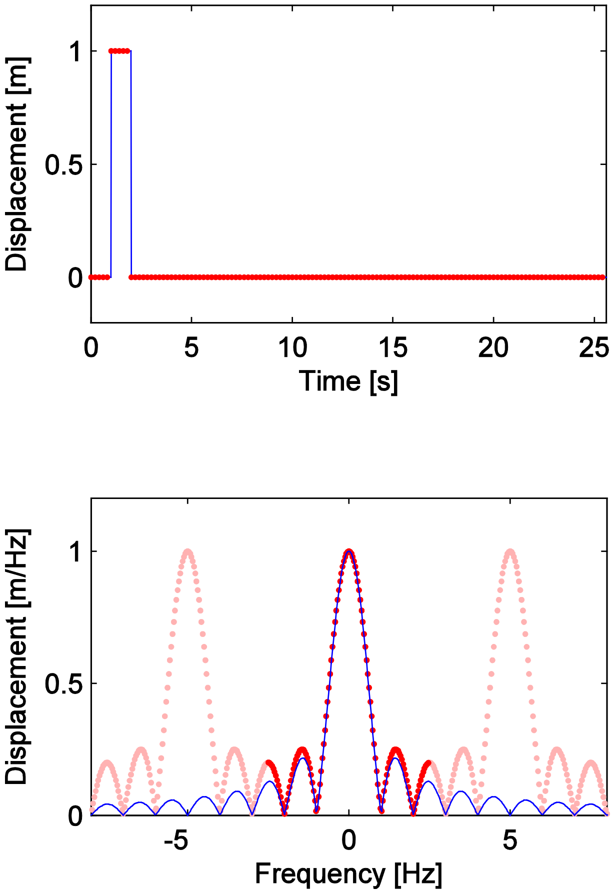
4. Choice of sampling parameters

- Influence of the period

Δt = 0.2 s   T = 12.8 s



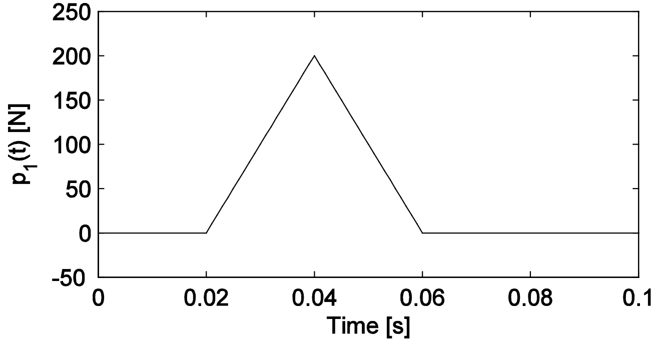
Δt = 0.2 s   T = 25.6 s



4. Choice of sampling parameters

- **Δt?**
  - Sampled signals must closely represent continuous signals
  - Maximum frequency content included $\left(\mathrm{F_{Nyq}} = \mathrm{F}/2\right)$
- **Period T?**
  - Excitation AND response
  - Logarithmic decrement

$$ln\left(\frac{u_i}{u_{i+n}}\right) \approx 2\pi n\xi$$



- **N?**
  - 2^n for FFT
- **Check with zero-padding?**

5. Conjugation property

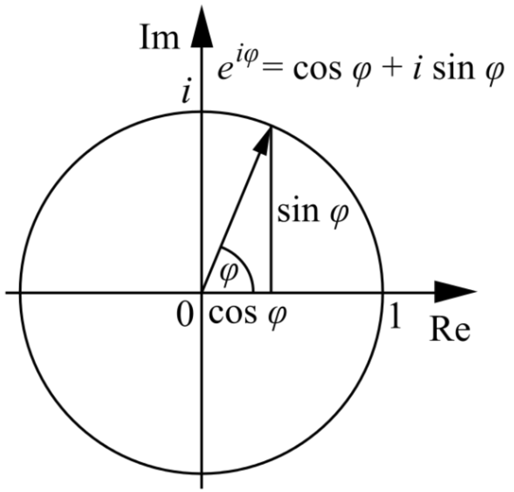
$$\tilde{x}(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt$$

$$= \int_{-\infty}^{\infty} x(t)[\cos(\omega t) - i\sin(\omega t)]\, dt$$

$$\mathrm{Re}\big(\tilde{x}(\omega)\big) = \int_{-\infty}^{\infty} x(t)\cos(\omega t)\, dt$$

$$\mathrm{Im}\big(\tilde{x}(\omega)\big) = -\int_{-\infty}^{\infty} x(t)\sin(\omega t)\, dt$$

$$\tilde{x}(-\omega) = \overline{\tilde{x}(\omega)}$$



Theory needed for the reduced-order solution: 3.1b

Model reduction / modal decomposition

- Second-order equations of motion:

$$\mathbf{M\ddot{u}}(\mathrm{t}) + \mathbf{C\dot{u}}(\mathrm{t}) + \mathbf{Ku}(\mathrm{t}) = \mathbf{P}(\mathrm{t}) = \mathbf{S}_{\mathrm{p}}\mathbf{p}(\mathrm{t})$$

- Transformation to modal coordinates:

$$\mathbf{u}(\mathrm{t}) = \mathbf{\Phi z}(\mathrm{t}) \;\Rightarrow \mathbf{\Phi}^{\mathrm{T}}\mathbf{M\Phi\ddot{z}}(\mathrm{t}) + \mathbf{\Phi}^{\mathrm{T}}\mathbf{C\Phi\dot{z}}(\mathrm{t}) + \mathbf{\Phi}^{\mathrm{T}}\mathbf{K\Phi z}(\mathrm{t}) = \mathbf{\Phi}^{\mathrm{T}}\mathbf{P}(\mathrm{t})$$

$$\mathbf{M}^*\mathbf{\ddot{z}}(\mathrm{t}) + \mathbf{C}^*\mathbf{\dot{z}}(\mathrm{t}) + \mathbf{K}^*\mathbf{z}(\mathrm{t}) = \mathbf{P}^*(\mathrm{t})$$

$$\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}\mathbf{\ddot{z}}(\mathrm{t}) + \begin{bmatrix} 2\omega_1\xi_1 & & \\ & \ddots & \\ & & 2\omega_n\xi_n \end{bmatrix}\mathbf{\dot{z}}(\mathrm{t}) + \begin{bmatrix} \omega_1^2 & & \\ & \ddots & \\ & & \omega_n^2 \end{bmatrix}\mathbf{z}(\mathrm{t}) = \mathbf{P}^*(\mathrm{t})$$

$$\ddot{z}_j(t) + 2\omega_j\xi_j\dot{z}_j(t) + \omega_j^2 z_j(t) = p_j^*(t)$$

⇒ all SDOF solution techniques applicable to MDOF systems

Needed for the reduced-order solution: 3.1b

Model reduction / modal decomposition

- In frequency domain:

$$\tilde{z}_j(\omega) = \tilde{h}_j(\omega)\tilde{p}_j^*(\omega)$$

$$\tilde{h}_j(\omega) = \frac{1}{-\omega^2 + 2i\omega\omega_j\xi_j + \omega_j^2}$$

Additional questions

- What is the smallest allowable model order that would allow an accurate representation of the response of the beam to the defined loading?
- When working with reduced-order models, what are the factors influencing the allowable model order?