# Lecture 14 : The Gamma Distribution and its Relatives

The gamma distribution is a continuous distribution depending on two parameters, $\alpha$ and $\beta$. It gives rise to three special cases

1. The exponential distribution ($\alpha = 1, \beta = \dfrac{1}{\lambda}$)
2. The $r$-Erlang distribution ($\alpha = r, \beta = \dfrac{1}{\lambda}$)
3. The chi-squared distribution ($\alpha = \dfrac{\nu}{2}, \beta = 2$)

## The Gamma Distribution

### Definition

*A continuous random variable $X$ is said to have gamma distribution with parameters $\alpha$ and $\beta$, both positive, if*

$$f(x) = \begin{cases} \dfrac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x > 0 \\ 0, & \textit{otherwise} \end{cases}$$

*What is $\Gamma(\alpha)$?*
*$\Gamma(\alpha)$ is the gamma function, one of the most important and common functions in advanced mathematics. If $\alpha$ is a positive integer $n$ then*

$$\Gamma(n) = (n-1)!$$

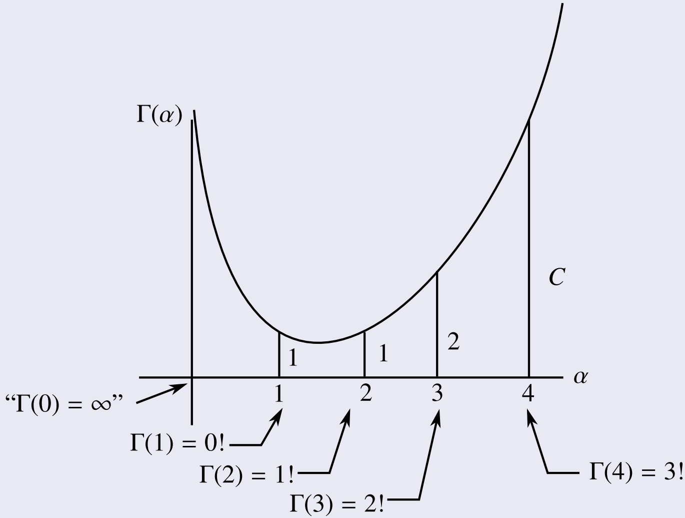
*(see page 17)*

## Definition (Cont.)

*So $\Gamma(\alpha)$ is an interpolation of the factorial function* <u>*to all real numbers.*</u>

**Z** $\lim_{\alpha \to 0} \Gamma(\alpha) = \infty$

### Graph of $\Gamma(\alpha)$



I will say more about the gamma function later. It isn't that important for Stat 400, here it is just a constant chosen so that

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

The key point of the gamma distribution is that it is of the form

$$\text{(constant) (power of } x\text{) } e^{-cx}, c > 0.$$

The $r$-Erlang distribution from Lecture 13 is almost the most general gamma distribution.

The only special feature here is that $\alpha$ is a whole number $r$.
Also $\beta = \dfrac{1}{\lambda}$ where $\lambda$ is the Poisson constant.

## Comparison Gamma distribution

$$\left(\frac{1}{\beta}\right)^{\alpha} \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$$

$r$-Erlang distribution $\alpha = r, \beta = \dfrac{1}{\lambda}$

$$\lambda^{r} \frac{1}{(r-1)!} x^{r-1} e^{-\lambda x}$$

## Proposition

*Suppose $X$ has gamma distribution with parameters $\alpha$ and $\beta$ then*

(i) $E(X) = \alpha\beta$

(ii) $V(X) = \alpha\beta^2$

*so for the $r$-Erlang distribution*

(i) $E(X) = \dfrac{r}{\lambda}$

(ii) $V(X) = \dfrac{r}{\lambda^2}$

### Proposition (Cont.)

*As in the case of the normal distribution we can compute general gamma probabilities by standardizing.*

### Definition

*A gamma distribution is said to be standard if $\beta = 1$. Hence the pdf of the standard gamma distribution is*

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

*The cdf of the standard*

## Definition (Cont.)

*gamma function is called the <u>incomplete gamma function</u> (divided by $\Gamma(\alpha)$)*

$$F(x) = \frac{1}{\Gamma(\alpha)} \int_0^x x^{\alpha-1} e^{-x} dx$$

*(see page 13 for the actual gamma function)*
*It is tabulated in the text Table A.4 for some (integral values of $\alpha$)*

## Proposition

*Suppose $X$ has gamma distribution with parameters $\alpha$ and $\beta$. Then $Y = \dfrac{X}{\beta}$ has standard gamma distribution.*

## Proof.

We can prove this, $Y = \dfrac{X}{\beta}$ so $X = \beta y$.

Now $f_X(x)dx = \dfrac{1}{\beta^\alpha}\dfrac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta}dx$.

Now substitute $x = \beta y$ to get

$$\begin{aligned}
f_Y(y)dy &= \frac{1}{\beta^\alpha}\frac{1}{\Gamma(\alpha)}(\beta y)^{\alpha-1}e^{-\frac{\beta y}{\beta}}d(\beta y) \\
&= \frac{1}{\cancel{\beta^\alpha}}\frac{1}{\Gamma(\alpha)}\cancel{\beta^{\alpha-1}}y^{\alpha-1}e^{-y}\cancel{\beta}dy \\
&= \underbrace{\frac{1}{\Gamma(\alpha)}y^{\alpha-1}e^{-t}}_{\text{standard gamma}}dy
\end{aligned}$$

$\square$

## Example 1 (4.24 (cut down))

*Suppose $X$ has gamma distribution with parameters $\alpha = 8$ and $\beta = 15$. Compute*

$$P(60 \le X \le 120)$$

## Solution

*Standardize, divide EVERYTHING by $\beta = 15$.*

$$\begin{aligned} P(60 \le X \le 120) &= P\left(\frac{60}{15} \le \frac{X}{15} \le \frac{120}{15}\right) \\ &= P(4 \le Y \le 8) = F(8) - F(4) \\ \textit{from table A.4} \\ &= .547 - .051 = .496 \end{aligned}$$