Lecture 30: Confidence Intervals for σ^2

Today we will discuss the material in Section 7.4.
Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal population with mean μ and variance σ^2.

$$\boxed{X \sim N(\mu, \sigma^2)} \dashrightarrow X_1, X_2, \ldots, X_n$$

In this lecture we want to construct a $100(1-\alpha)\%$ confidence for σ^2. We recall that S^2 is a point estimator for σ^2.

What is new here is that we are going to note a "*multiplicative confidence interval*".
Here is the idea. *We want a random interval that has the point estimator* S^2 *in the interior*
Now given a number x there are two ways to make an interval $I(x)$ that has x in its interior.

1. The additive method

Choose two positive numbers c_1 and c_2. Put $I(x) = (x - c_1, x + c_2)$.

2. The multiplicative method

Choose a number $c_1 < 1$ and another number $c_1 > 1$. Put

$$I(x) = (c_1 x, c_2 x).$$

We will use the second method now. The clue to why we do this is that $S^2 > 0$. First we need to know the probability distribution of the point estimator S^2. We have already seen this

Theorem A (pg 278)

$$V = \left(\frac{n-1}{\sigma^2}\right) S^2 \sim \chi^2(n-1) \qquad (*)$$

Now we can give the confidence interval.

Theorem B

The random interval $\left(\dfrac{n-1}{\chi^2_{\alpha/2,n-1}} S^2, \dfrac{n-1}{\chi^2_{1-\alpha/2,n-1}} S^2\right)$ *is a* $100(1-\alpha)\%$ *confidence random interval for the population variance* σ^2 *from a normal population.*

Remark

It must be true (see page 2) that

$$c_1 = \frac{n-1}{\chi^2_{\alpha/2,n-1}} < 1 \quad \textit{and}$$
$$c_2 = \frac{n-1}{\chi^2_{1-\alpha/2,n-1}} > 1.$$

I have never checked this.
Now we prove Theorem B. We must prove

$$P\left(\sigma^2 \in \left(\frac{n-1}{\chi^2_{\alpha/2,n-1}} S^2, \frac{n-1}{\chi^2_{1-\alpha/2,n-1}} S^2\right)\right) = 1$$
$$LHS \;= P\left(\frac{n-1}{\chi^2_{\alpha/2,n-1}} S^2 < \sigma^2,\;\; \sigma^2 < \frac{n-1}{\chi^2_{1-\alpha/2,n-1}} S^2\right)$$

Remark

Now we manipulate the two resulting inequalities to get V so we can sue $(*)$

$$P\left(\frac{n-1}{\chi^2_{\alpha/2,n-1}} S^2 < \sigma^2,\quad \sigma^2 < \frac{n-1}{\chi^2_{1-\alpha/2,n-1}} S^2\right)$$

Swap and make a V

$$= P\left(\frac{n-1}{\sigma^2} S^2 < \chi^2_{\alpha/2,\, n-1},\quad \chi^2_{1-\alpha/2,\, n-1} < \frac{n-1}{\sigma^2} S^2\right)$$

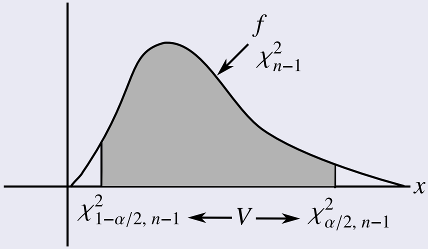
$$= P\left(V < \chi^2_{\alpha/2,\, n-1},\quad \chi^2_{1-\alpha/2,\, n-1} < V\right)$$

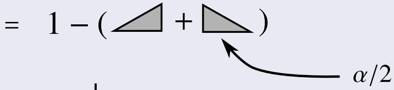
$$= P\left(\chi^2_{1-\alpha/2,\, n-1} < V < \chi^2_{\alpha/2,\, n-1}\right)$$

MAKE A PICTURE

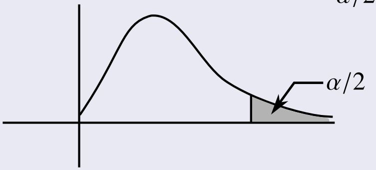
= *the shaded area*

Remark (Cont.)

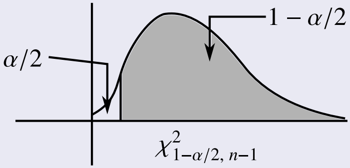




Now



and



$= 1 - (\alpha/2 + \alpha/2) = 1 - \alpha$

Question

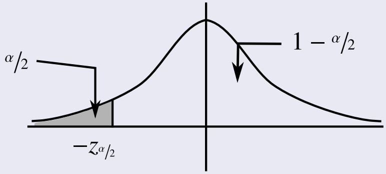
Why do we need the strange $\chi^2_{1-\alpha/2},\ n-1$? This is because the χ^2 density curve does not have the symmetry that the z-density and t-densities did. In all three coses we need something that cut off $\alpha/2$ on the *left* under the density curve so $1-\alpha/2$ on the *right*. For the z-curve $-z_{\alpha/2}$ did the job.

In other words

Lemma

$$z_{1-\alpha/2} = -z_{\alpha/2}$$

Proof.



so $-z_{\alpha/2}$ cots off $1 - \alpha/2$ to the right to $-z_{\alpha/2} = z_{1-\alpha/2}$ □

The Upper-Tailed $100(1-\alpha)\%$ Confidence Interrol for σ^2

Theorem

$\left(\dfrac{n-1}{\chi^2_{\alpha,\,n-1}}S^2, \infty\right)$ *is a* $100(1-\alpha)\%$ *confidence interrol for* σ^2

Proof.

If could be on the final - do it yourself. □

Remark

As used we took the lower limit from the two-sided interval and changed $\alpha/2$ *to* α.

The Lower-Tailed $100(1-\alpha)\%$ Confidence Interval for σ^2

Since S^2 is always positive $PCS^2 \in (-\infty, 01) = 0$ so the negative axis will not appear.

Lower tailed multiplication intervals go down *to 0 not* $-\infty$. Another (philosophical) way to look at it is.

$$\underbrace{\begin{array}{c}\text{additive group of } \mathbb{R} \\ (-\infty,\infty)\end{array}}_{\text{additive world}} \xrightarrow{e^x} \underbrace{\begin{array}{c}\text{multiplicative group of of positive} \\ \text{numbers, } (0,\infty)\end{array}}_{\text{multiplicative world}}$$

We are in the multiplicative world.

Theorem

$\left(0, \dfrac{n-1}{\chi^2_{1-\alpha,n-1}} S^2\right)$ *is a* $100(1-\alpha)\%$ *confidence interval for* σ^2*.*

Proof.

Do it yourself □

Remark

$\left(-\infty, \dfrac{n-1}{\chi^2_{1-\alpha,\, n-1}} S^2\right)$ *is also a* $100(1-\alpha)\%$ *confidence interval for* σ^2 *but the* $(-\infty, 0)$ *is "wasted space", Remember, small intervals or better.*

Confidence Intervals for the standard Deviation

Note that if $a > 0$, $b > 0$ and $x > 0$ then

$$a \leq x \leq b \leftrightarrow \sqrt{a} \leq \sqrt{x} \leq \sqrt{b}$$

so

$$\frac{n-1}{\chi^2_{\alpha/2,\, n-1}} S^2 \leq \sigma^2 \leq \frac{n-1}{\chi^2_{1-\alpha/2,\, n-1}} S^2$$

$$\Leftrightarrow \sqrt{\frac{n-1}{\chi^2_{\alpha/2,\, n-1}}} S \leq \sigma \leq \sqrt{\frac{n-1}{\chi^2_{1-\alpha/2,\, n-1}}} S$$

Hence

$$\left(\sqrt{\frac{n-1}{\chi^2_{\alpha/2, n-1}}} S < \sigma < \sqrt{\frac{n-1}{\chi^2_{1-\alpha/2,\, n-1}}} S \right) = P\left(\frac{n-1}{\chi^2_{\alpha/2,\, n-1}} S^2 \leq \sigma^2 \leq \frac{n-1}{\chi^2_{1-\alpha/2, n-1}} S^2 \right)$$

from pg3

$$= 1 - \alpha$$

In other words

$$P\left(\sigma \in \left(\sqrt{\frac{n-1}{\chi^2_{\alpha/2,\, n-1}}}S,\ \sqrt{\frac{n-1}{\chi^2_{1-\alpha/2,\, n-1}}}S\right)\right) = 1-\alpha$$

and we have

Theorem

The random interval

$$\left(\sqrt{\frac{n-1}{\chi^2_{\alpha/2,\, n-1}}}S,\ \sqrt{\frac{n-1}{\chi^2_{1-\alpha/2,\, n-1}}}S\right)$$

is a $100(1-\alpha)\%$ *confidence interval for the standard deviation* σ *in a normal population.*

Problem

Write down the upper and lower-tailed confidence intervals for σ.
(hint: just take the square notes of the end points of those for σ^2)