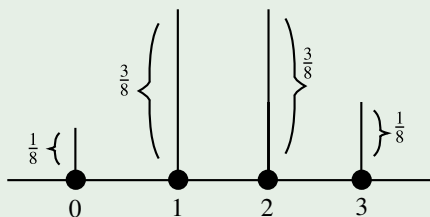# Lecture 9 : Change of discrete random variable

You have already seen (I hope) that whenever you have "variables" you need to consider change of variables. Random variables are no different.

The notion of "change of random variable" is handle too briefly on page 103 of the text. This is something I will test you on.

Example 1

Suppose $X \sim Bm\left(3, \dfrac{1}{2}\right)$.

line graph

table

X	0	1	2	3
$P(X = x)$	$\dfrac{1}{8}$	$\dfrac{3}{8}$	$\dfrac{3}{8}$	$\dfrac{1}{8}$

(b)

Suppose what to define a new random variable $Y = 2X - 1$.

How do we do it?

So how do we define $P(Y = k)$?

Answer - express Y in terms of X and compute so

$$P(Y = k) = P(2X - 1 = k)$$
$$= P\left(X = \frac{k + 1}{2}\right) \qquad (*)$$

The right-hand site is the logical <u>definition</u> of the left-hand side.

But as is often the case in probability it is easier to pretend we know what $P(Y = k)$ means already and then the next two steps are a computation.

So let's compute the *pmf* of *Y*.

What are the possible values of *Y*?

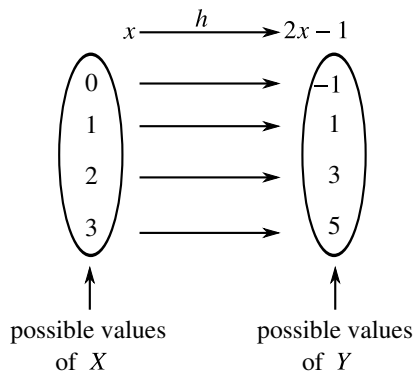From (*) k is a possible value of *Y* $\Leftrightarrow \dfrac{k+1}{2}$ is a possible values of *X*.

$$\iff \frac{k+1}{2} = \left\{ \begin{array}{l} 0 \\ 1 \\ 2 \\ 3 \end{array} \right. \iff Y = \left\{ \begin{array}{l} -1 \\ 1 \\ 3 \\ 5 \end{array} \right.$$

Note

$$\frac{k+1}{2} = x \Leftrightarrow k = 2x - 1$$

possible value of *X* ↑   ↑ possible value of *Y*

So the possible values of Y are obtained by applying the function $h(x) = 2x - 1$ to the possible values of X.

(note $Y = f(X)$).

Just "push forward" the values of X.

Example (Cont.)

Now we have computed the possible values of Y we need to compute their probabilities. Just repeat what we did

$$P(Y = -1) = P(2X - 1 = -1)$$
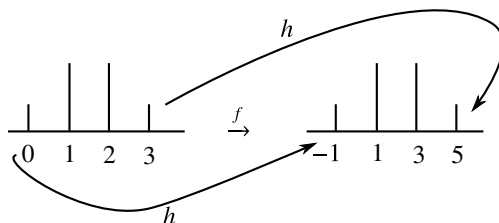$$= P(X = 0) = \frac{1}{8}$$
$$P(Y = 1) = P(2X - 1 = 1)$$
$$= P(X = 1) = \frac{3}{8}$$

Similarly

$$P(Y = 3) = \frac{3}{8} \quad and \quad P(Y = 5) = \frac{1}{8}$$

X	-1	1	3	5
$P(Y = y)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

So we have the "same probabilities" as before namely $\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$ it is just then are pushed-forward to new locations

Example 2 (Probabilities can "coalesce")

There is one tricky point. Several different possible valves of *X* can push-forward to the same values of *Y*. We now give an example.

Suppose *X* has proof

Problem (from on old midterm)

Find a linear change of variable $Y = aX + b$ so that $Y \sim \operatorname{Bin}\left(2, \dfrac{1}{2}\right)$

Problem (Cont.)

We will make the change of variable $Y = X^2$. So what happens when we push forward the three values $-1, 0, 1$ by $h(x) = x^2$.
We get only the two values 0 and 1.

$$
\begin{array}{ccc}
-1 & \xrightarrow{\;h(x)\;} & 1 \\
0 & \longrightarrow & 0 \\
1 & \longrightarrow & 1
\end{array}
$$

What happens with the corresponding probabilities

$$P(Y = 0) = P(X^2 = 0) = P(X = 0) = \frac{1}{2}$$

But

$$
\begin{aligned}
P(Y = 1) = P(X^2 = 1) &= P(X = 1 \ \text{or} \ X = -1) \\
&= P((X = 1) \cup (X - 1)) \\
&= P(X = 1) + P(X = -1) \\
&= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}
\end{aligned}
$$

Lecture 9 : Change of discrete random variable

Problem (Cont.)

So we set

g	0	1
$P(y = 9)$	$\dfrac{1}{2}$	$\dfrac{1}{2}$

So,

Think of two masses (probabilities) of mass $\dfrac{1}{4}$ coalesing into a combined mass of $\dfrac{1}{2}$.

Lecture 9 : Change of discrete random variable

The Expected Value Formula

If $h(x)$ in the transformation law $Y = h(X)$ is complicated it can be very hard to explicitly compute the *pmf* of Y Amazingly we can compute the expected value $E(Y)$ using the old proof $P_X^{(x)}$ of X according to <u>Theorem</u>

$$E(h(X)) = \sum_{\substack{\text{possible} \\ \text{values of } X}} h(x)P_X^{(x)}$$

$$= \sum_{\substack{\text{possible values} \\ \text{of } X}} h(x)P(X = x)$$

Lecture 9 : Change of discrete random variable

We will illustrate this with the *pmf*'s of Example 1.

First we compute $E(Y)$ using the definition of $E(Y)$.

Y	-1	1	3	5
$P(Y = y)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

($\sharp$)

$$
\begin{aligned}
E(Y) &= \sum_{\substack{\text{possible value} \\ \text{of } Y}} y \, P(Y = y) \\
&= (-1)\left(\frac{1}{8}\right) + (1)\left(\frac{3}{8}\right) + (3)\left(\frac{3}{8}\right) + (5)\left(\frac{1}{8}\right) \\
&= \frac{-1 + 3 + 9 + 5}{8} \\
&= \frac{16}{8} = 2
\end{aligned}
$$

Notice to do the previous computations we needed the table ($\sharp$) which we computed on page 5.

Now we use the Theorem.

So now we use that Y is a function of the random variable X and use the proof of X from the table on page 1.

$$
\begin{array}{c|c|c|c|c}
x & 0 & 1 & 2 & 3 \\
\hline
P(X = x) & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8}
\end{array}
\tag{b}
$$

$$
\begin{aligned}
E(X) &= \sum_{\substack{\text{possible values} \\ \text{of } X}} h(x) P(X = x) \\
&= \sum_{x=0,1,2,3} (2x - 1) P(X = x) \\
&= (-1)\left(\frac{1}{8}\right) + (1)\left(\frac{3}{8}\right) + (3)\left(\frac{5}{8}\right) + (5)\left(\frac{3}{8}\right) = 2
\end{aligned}
$$

The most common change of variable is linear $Y = X + b$ so we will give formulas to show how expected value and variance behave under such a change.

Theorem

  (i) $E(aX + b) = aE(X) + b$

  (ii) $V(aX + b) = a^2 V(X)$

(so $V(-X) = V(X)$)