

Lecture 8 : The Geometric Distribution

The geometric distribution is a special case of negative binomial, it is the case $r = 1$. It is so important we give it special treatment.

Motivating example

Suppose a couple decides to have children until they have a girl. Suppose the probability of having a girl is P . Let

X = the number of boys that precede the first girl

Find the probability distribution of X . First X could have any possible whole number value (although $X = 1,000,000$ is very unlikely)

$$P(X = k) = P(\underbrace{B \ B \ B \ \dots \ B}_k \ G) \quad \uparrow \quad p$$

$$= q^k p \quad (\text{where } q = 1 - p)$$

We have suppose birth are independent.

We have motivated.

Definition

Suppose a discrete random variable X has the following pmf

$$P(X = k) = q^k P, \quad 0 \leq k < \infty$$

The X is said to have geometric distribution with parameter P .

Remark

Usually this is developed by replacing “having a child” by a Bernoulli experiment and having a girl by a “success” (PC). I could have used coin flips.

Proposition

Suppose X has geometric distribution with parameter p .

Then

(i) $E(X) = \frac{q}{p}$

(ii) $V(X) = \frac{q}{p^2}$

Proof of (i) (you are not responsible for this).

$$\begin{aligned} E(X) &= (0)(p) + (1)(qp) + (2)(q^2p) + \cdots + (k)(q^k p) + \cdots \\ &= p(q + 2q + \cdots + kq^k + \cdots) \end{aligned}$$

Now

$$\frac{X}{(1-x)^2} = x + 2x^2 + 3x^3 + \cdots + kx^k + \cdots$$

↑
— why?

So

$$EX() = p \left(\frac{q}{(1-q)^2} \right) = p \left(\frac{q}{p^2} \right) = \frac{q}{p}$$



The Negative Binomial Distribution

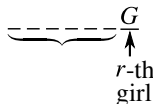
Now suppose the couple decides they want more girls - say r girls, so they keep having children until the r -th girl appears. Let X = the number of boys that precede the r -th girl.

Find the probability distribution of X .

Remark

Sometimes (eg. pg. 13-14) it is better to write X_r instead of X .

Let's compute $P(X = k)$



What do we have preceding the r -th girl. Of course we must have $r - 1$ girls and since we are assuming $X = k$ we have k boys so $k + r - 1$ children.

All orderings of boys and girls have the same probability so

$$P(X = k) = (?) P(\underbrace{B \dots B}_{k-1} \underbrace{G \dots G}_{r-1} G)$$

or

$$P(X = k) = (?)q^k \cdot p^{r-1} \cdot q = (?)q^k p^r$$

(?) is the number of words of length $k + r - 1$ in B and G using k B 's (where $r - 1$ G 's).

Such a word is determined by choosing the slots occupied by the boys so there are $\binom{k+r-1}{k}$ words so

$$P(X = k) = \binom{k+r-1}{k} p^r q^k$$

So we have motivated the following.

Definition

A discrete random variable X is said to have negative binomial distribution with parameters r and p if

$$P(X = k) = \binom{k + r - 1}{k} p^r q^k, \quad 0 \leq k < \infty$$

The text denotes this pmf by $nb(x; r, p)$ so

$$nb(x; r, p) = \binom{x + r - 1}{x} p^r q^x, \quad 0 \leq x < \infty.$$

Proposition

Suppose X has negative binomial distribution with parameters r and p . Then

(i) $E(X) = r \frac{q}{p}$

(ii) $V(X) = \frac{rq}{p^2}$

Waiting Times

The binomial, geometric and negative binomial distributions are all tied to repeating a given Bernoulli experiment (flipping a coin, having a child) infinitely many times.

Think of discrete time $0, 1, 2, 3, \dots$ and we repeat the experiment at each of these discrete times. - Eg., flip a coin every minute.

Now you can do the following things

- 1 Fix a time say n and let $X = \#$ of successes in that time period. Then $X \sim \text{Bin}(n, p)$. We should write X_n and think of the family of random variable parametrized by the discrete time n as the “binomial process”. (see page. 18 - the Poisson process).
- 2 ((discrete) waiting time for the first success)
Let Y be the amount of time up to the time the first success occurs.

This is the geometric random variable. Why?

Suppose we have in our boy/girl example

$$\underbrace{\begin{matrix} B & B & B & & B & G \\ \hline 0 & 1 & 2 & 3 & & k \end{matrix}}_k$$

So in this case

$$X = \# \text{ of boys} = k$$
$$Y = \text{waiting time} = k$$

so $Y = X$.

Waiting time for r -th success

Now let Y_n = the waiting time up to the r -th success then there is a difference between X_r and Y_r .

Suppose $X_r = k$ so there are k boys before the r -th girl arrives.

$$\overbrace{0 \ 1 \ 2}^B \quad \overbrace{k+r-2}^{\quad} \quad \overbrace{k+r-1}^G$$

k B 's $r - 1$ G 's so $ktr - 1$ slots.

But start at 0 so the last slot is $k + r - 2$ so

$$Y_r = X_r + r - 1$$

The Poisson Distribution

For a change we won't start with a motivating example but will start with the definition.

Definition

A discrete random variable X is said to have Poisson distribution with parameter λ .

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad 0 \leq k < \infty$$

We will abbreviate this to $X \sim P(\lambda)$.

I will now try to motivate the formula which looks complicated.

Why is the factor of $e^{-\lambda}$ there? It is there to make the total probability equal to 1.

$$\text{Total Probability} = \sum_{k=0}^{\infty} P(X = k)$$

$$= \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

But from calculus

$$e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}$$

Total probability = $e^{-\alpha} \cdot e^{\alpha} = 1$ as it has to be.

Proposition

Suppose $X \sim P(\lambda)$. Then

(i) $E(X) = \lambda$

(ii) $V(X) = \lambda$

Remark

It is remarkable that $E(X) = V(X)$.

Example (3.39)

Let X denote the number of creatures of a particular type captured during a given time period. Suppose $X \sim P(4.5)$. Find $P(X = 5)$ and $P(X \leq 5)$.

Solution

$$P(X = 5) = e^{-4.5} \frac{(4.5)^5}{5!}$$

(just plug into the formula using $\lambda = 4.5$)

$$\begin{aligned} P(X \leq 5) &= P(X = 0) + P(X = 1) + P(X = 2) \\ &\quad + P(X = 3) + P(X = 4) + P(X = 5) \\ &= e^{-\lambda} + e^{-\lambda} \lambda + e^{-\lambda} \frac{\lambda^2}{2} \\ &\quad + \underbrace{e^{-\lambda} \frac{\lambda^3}{3!} + e^{-\lambda} \frac{\lambda^4}{4!} + e^{-\lambda} \frac{\lambda^5}{5!}}_{\text{don't try to evaluate this}} \end{aligned}$$

The Poisson Process

A very important application of the Poisson distribution arises in counting the number of occurrences of a certain event in time t

- 1 Animals in a trap.
- 2 Calls coming into a telephone switch board.

Now we could let t vary so we get a one-parameter family of Poisson random variable X_t , $0 \leq t < \infty$.

Now a Poisson process is completely determined once we know its mean λ .

So far each t , X_t is a Poisson random variable. So $X_t \sim P(\lambda(t))$.

So the Poisson parameter λ is a function of t .

In the Poisson process one assume that $\lambda(t)$ is the simplest possible function of t (aside from a constant function) namely a linear function

$$\lambda(t) = \alpha t.$$

Necessarily

$\alpha = \lambda(1)$ = the average number of observations in unit time.

Remark

In the text, page 124, the author proposes 3 axioms on a one parameter family of random variables X_t . So that X_t is a Poisson process i.e.,

$$X_t \sim P(\alpha t)$$

Example

(from an earlier version of the text)

The number of tickets issued by a meter reader can be modelled by a Poisson process with a rate of 10 ticket every two pairs.

(a) What is the probability that exactly 10 tickets are given out during a particular 12 hour period.

Solution

We want $P(X_{12} = 10)$.

First find α = average # of tickets by unit time.

$$\text{So } \alpha = \frac{10}{2} = 5$$

$$\text{So } X_t \sim P(5t)$$

Solution (Cont.)

So $X_{12} \sim P((5)(12)) = P(60)$

$$\begin{aligned} P(X_{12} = 10) &= e^{-\lambda} \frac{\lambda^{10}}{(10)!} \\ &= e^{-60} \frac{(60)^{10}}{(10)!} \end{aligned}$$

(b) What is the probability that at least 10 tickets are given out during a 12 hour time period.

We wait

$$P(X_{12} \geq 10) = 1 - P(X \leq 9)$$

$$= 1 - \sum_{k=0}^9 e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= 1 - \underbrace{\sum_{k=0}^9 e^{-60} \frac{(60)^k}{k!}}$$

not something you
want to try to
evaluate by hand.

Waiting Times

Again there are waiting time random variables associated to the Poisson process.

Let Y = waiting time until the first animal is caught in the trap.

and Y_r = waiting time until the r -th animal is caught in the trap.

Now Y and Y_r are continuous random variables which we are about to study. Y is exponential and Y_r has a special kind gamma distribution.