

Lecture 27 : Random Intervals and Confidence Intervals

The confidence interval formulas for the mean in an normal distribution when σ is known

1. Introduction

In this lecture we will derive the formulas for the symmetric two-sided confidence interval and the lower-tailed confidence intervals for the mean in a normal distribution *when the variance σ^2 is known*. At the end of the lecture I assign the problem of proving the formula for the upper-tailed confidence interval as HW 12. We will need the following theorem from probability theory that gives the distribution of the statistic \bar{X} - the point estimator for μ .

Suppose that X_1, X_2, \dots, X_n is a random sample from a normal distribution with mean μ and variance σ^2 . We assume μ is unknown but σ^2 is known. We will need the following theorem from Probability Theory.

Theorem 1

\bar{X} has normal distribution with mean μ and variance σ^2/n . Hence the random variable $Z = (\bar{X} - \mu) / \frac{\sigma}{\sqrt{n}}$ has standard normal distribution.

2 The two-sided confidence interval formula

Now we can prove the theorem from statistics giving the required confidence interval for μ . Note that it is symmetric around \bar{X} . There are also asymmetric two-sided confidence intervals. We will discuss them later. This is one of the basic theorems that you have to learn how to prove.

Theorem 2

The random interval $\left(\bar{X} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\right)$ is a $100(1 - \alpha)\%$ - confidence interval for μ .

Proof.

We are required to prove

$$P\left(\mu \in \left(\bar{X} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\right)\right) = 1 - \alpha.$$

We have

$$\begin{aligned}\text{LHS} &= P\left(\bar{X} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} < \mu, \mu < \bar{X} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\right) \\&= P\left(\bar{X} - \mu < z_{\alpha/2} \frac{\sigma}{\sqrt{n}}, -z_{\alpha/2} \frac{\sigma}{\sqrt{n}} < \bar{X} - \mu\right) \\&= P\left(\bar{X} - \mu < z_{\alpha/2} \frac{\sigma}{\sqrt{n}}, \bar{X} - \mu > -z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\right) \\&= P\left((\bar{X} - \mu) / \frac{\sigma}{\sqrt{n}} < z_{\alpha/2}, (\bar{X} - \mu) / \frac{\sigma}{\sqrt{n}} > -z_{\alpha/2}\right) \\&= P(Z < z_{\alpha/2}, Z > -z_{\alpha/2}) = P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha\end{aligned}$$

To prove the last equality draw a picture.

□

Once we have an actual sample x_1, x_2, \dots, x_n we obtain the observed value \bar{x} for the random variable \bar{X} and the observed value $\left(\bar{x} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\right)$ for the confidence (random) interval $\left(\bar{X} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\right)$. The observed value of the confidence (random) interval is also called the two-sided $100(1-\alpha)\%$ confidence interval for μ .

3. The lower-tailed confidence interval

In this section we will give the formula for the lower-tailed confidence interval for μ .

Theorem 3

The random interval $\left(-\infty, \bar{X} + z_{\alpha} \frac{\sigma}{\sqrt{n}}\right)$ is a $100(1-\alpha)\%$ -confidence interval for μ .

Proof.

We are required to prove

$$P\left(\mu \in \left(-\infty, \bar{X} + z_{\alpha} \frac{\sigma}{\sqrt{n}}\right)\right) = 1 - \alpha.$$

We have

$$\begin{aligned}\text{LHS} &= P\left(\mu < \bar{X} + z_{\alpha} \frac{\sigma}{\sqrt{n}}\right) = P\left(-z_{\alpha} \frac{\sigma}{\sqrt{n}} < \bar{X} - \mu\right) \\ &= P\left(-z_{\alpha} < (\bar{X} - \mu) / \frac{\sigma}{\sqrt{n}}\right) \\ &= P(-z_{\alpha} < Z) \\ &= 1 - \alpha\end{aligned}$$

To prove the last equality draw a picture - I want *you* to draw the picture on tests and the homework. □

Once we have an actual sample x_1, x_2, \dots, x_n we obtain the observed value \bar{x} for the random variable \bar{X} and the observed value $\left(-\infty, \bar{x} + z_\alpha \frac{\sigma}{\sqrt{n}}\right)$ for the confidence (random) interval $\left(-\infty, \bar{X} + z_\alpha \frac{\sigma}{\sqrt{n}}\right)$. The observed value of the confidence (random) interval is also called the lower-tailed $100(1 - \alpha)\%$ confidence interval for μ .

The number random variable $\bar{X} + z_\alpha \frac{\sigma}{\sqrt{n}}$ or its observed value $\bar{x} + z_\alpha \frac{\sigma}{\sqrt{n}}$ is often called a confidence *upper bound* for μ because

$$P\left(\mu < \bar{X} + z_\alpha \frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha.$$

4. The upper-tailed confidence interval for μ

Homework 12 (to be handed in on Monday, Nov.28) is to prove the following theorem.

Theorem 4

The random interval $\left(\bar{X}z_{\alpha}\frac{\sigma}{\sqrt{n}}, \infty\right)$ is a $100(1 - \alpha)\%$ confidence interval for μ .