Lecture 25: Stat 400, Lecture 25 Sampling from $N(\mu, \sigma^2)$ and the CLT

$$\boxed{X \sim N(\mu, \sigma^2)} \dashrightarrow X_1, X_2, \ldots, X_n$$

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a normal population. We have seen that we should use the *sample* mean $\overline{X}$ to estimate the population mean μ and the *sample* variance S^2 to estimate the population variance σ^2.

$\overline{X}$ and S^2 are random variables.

\$64,000 question

How are $\overline{X}$ and S^2 distributed ? The answer is given by the following considerations
Any linear combination of Independent normal random variables is again normal so $\overline{X}$ is normal. Since $E(\overline{X}) = \mu$ and $V(\overline{X}) = \dfrac{\sigma^2}{n}$ we have

$$\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

Suppose $Z, Z_2, \ldots, Z_n$ are independent standard normal random variables. Then

$$Z_1^2 + \ldots + Z_b^2 \sim \chi^2(n) \qquad (*)$$

Chi-squared with n degrees of

$$\text{Now} \qquad Z_i = \frac{X_i - \mu}{\sigma} \sim N(0,1)$$

So

$$\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} \right)^2 = \frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2 \sim \chi^2(n)$$

Now replace μ by its estimation $\overline{X}$ Rule of thumb - *every time you replace a quantity by its estimator you lose one degree of freedom in the chi-squared distribution*

So by the "rule of thumb"

$$Y = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \overline{X})^2 \sim \chi^2(n-1)$$

Now $S^2 = \dfrac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$

So $Y = \dfrac{n-1}{\sigma^2}S^2$ and we obtain the critical

$$\frac{n-1}{\sigma^2}S^2\sigma\chi^2(n-1) \qquad (**)$$

Remark

This isn't a proof because we used "the rule of thumb" but the result is true

Bottom Line

Theorem

Let $X_1, X_2, \ldots, X_n$ be a random Sample from a normal population with mean μ and variance σ^2. Then

(i) $\overline{X} \sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$

(ii) $\dfrac{n-1}{\sigma^2} S^2 \sim \chi^2(n-1)$

(iii) $\overline{X}$ *and* S^2 *are independent.*

The above statement is on exact statement but if we take a large sample $(n > 30)$ from any population with mean μ and variance σ^2 we may assume

Theorem (Cont.)

to a good approximation that the population has N(μ, σ^2) *distribution and we have by CLT*

Theorem

If $X_1, X_2, \ldots, X_n$ *is a large* $(n > 30)$ *random sample from any population with mean* μ *and variance* σ^2 *then*

(i) $\overline{X} \approx N(\mu, \frac{\sigma^2}{n})$

(ii) $S^2 \approx \chi^2(n-1)$

(iii) $\overline{X}$ *and* S^2 *are approximately independent. (then are not independent unless the population is normal.)*