Lecture 27 : Random Intervals and Confidence IntervalsThe confidence interval formulas for the mean in an normal distribution when σ is known

1. Introduction

In this lecture we will derive the formulas for the symmetric two-sided confidence interval and the lower-tailed confidence intervals for the mean in a normal distribution *when the variance σ^2 is known*. At the end of the lecture I assign the problem of proving the formula for the upper-tailed confidence interval as HW 12. We will need the following theorem from probability theory that gives the distribution of the statistic $\overline{X}$ - the point estimator for μ.
Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from a normal distribution with mean μ and variance σ^2. We assume μ is unknown but σ^2 is known. We will need the following theorem from Probability Theory.

Theorem 1

$\overline{X}$ has normal distribution with mean μ and variance σ^2/n. Hence the random variable $Z = (\overline{X} - \mu)/\dfrac{\sigma}{\sqrt{n}}$ has standard normal distribution.

2 The two-sided confidence interval formula

Now we can prove the theorem from statistics giving the required confidence interval for μ. Note that it is symmetric around $\overline{X}$. There are also asymmetric two-sided confidence intervals. We will discuss them later. This is one of the basic theorems that you have to learn how to prove.

Theorem 2

The random interval $\left(\overline{X} - z_{\alpha/2}\dfrac{\sigma}{\sqrt{n}}, \overline{X} + z_{\alpha/2}\dfrac{\sigma}{\sqrt{n}}\right)$ *is a* $100(1-\alpha)\%-$ *confidence interval for* μ.

Proof.

We are required to prove

$$P\left(\mu \in \left(\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)\right) = 1 - \alpha.$$

We have

$$\begin{aligned}
\text{LHS} &= P\left(\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu, \mu < \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) \\
&= P\left(\overline{X} - \mu < z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, -z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \overline{X} - \mu\right) \\
&= P\left(\overline{X} - \mu < z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{X} - \mu > -z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) \\
&= P\left(\left(\overline{X} - \mu\right) / \frac{\sigma}{\sqrt{n}} < z_{\alpha/2}, (\overline{X} - \mu)/\frac{\sigma}{\sqrt{n}} > -z_{\alpha/2}\right) \\
&= P\left(Z < z_{\alpha/2}, Z > -z_{\alpha/2}\right) = P\left(-z_{\alpha/2} < Z < z_{\alpha/2}\right) = 1 - \alpha
\end{aligned}$$

To prove the last equality draw a picture. □

Once we have an actual sample $x_1, x_2, \ldots, x_n$ we obtain the observed value $\overline{x}$ for the random variable $\overline{X}$ and the observed value $\left(\overline{x} - z_{\alpha/2}\dfrac{\sigma}{\sqrt{n}}, \overline{x} + z_{\alpha/z}\dfrac{\sigma}{\sqrt{n}}\right)$ for the confidence (random) interval $\left(\overline{X} - z_{\alpha/2}\dfrac{\sigma}{\sqrt{n}}, \overline{X} + z_{\alpha/2}\dfrac{\sigma}{\sqrt{n}}\right)$. The observed value of the confidence (random) interval is also called the two-sided $100(1\alpha)\%$ confidence interval for μ.

3. The lower-tailed confidence interval

In this section we will give the formula for the lower-tailed confidence interval for μ.

Theorem 3

The random interval $\left(-\infty, \overline{X} + z_\alpha \frac{\sigma}{\sqrt{n}}\right)$ *is a* $100\,(1-\alpha)\%$*-confidence interval for* μ.

Proof.

We are required to prove

$$P\left(\mu \in \left(-\infty, \overline{X} + z_\alpha \frac{\sigma}{\sqrt{n}}\right)\right) = 1 - \alpha.$$

We have

$$\begin{aligned}
\text{LHS} &= P\left(\mu < \overline{X} + z_\alpha \frac{\sigma}{\sqrt{n}}\right) = P\left(-z_\alpha \frac{\sigma}{\sqrt{n}} < \overline{X} - \mu\right) \\
&= P\left(-z_\alpha < (\overline{X} - \mu)/\frac{\sigma}{\sqrt{n}}\right) \\
&= P\left(-z_\alpha < Z\right) \\
&= 1 - \alpha
\end{aligned}$$

To prove the last equality draw a picture - I want *you* to draw the picture on tests and the homework. □

Once we have an actual sample $x_1, x_2, \ldots, x_n$ we obtain the observed value $\overline{x}$ for the random variable $\overline{X}$ and the observed value $\left(-\infty, \overline{x} + z_\alpha \dfrac{\sigma}{\sqrt{n}}\right)$ for the confidence (random) interval $\left(-\infty, \overline{X} + z_\alpha \dfrac{\sigma}{\sqrt{n}}\right)$. The observed value of the confidence (random) interval is also called the lower-tailed $100(1-\alpha)\%$ confidence interval for μ.
The number random variable $\overline{X} + z_\alpha \dfrac{\sigma}{\sqrt{n}}$ or its observed value $\overline{x} + z_\alpha \dfrac{\sigma}{\sqrt{n}}$ is often called a confidence *upper bound* for μ because

$$P\left(\mu < \overline{X} + z_\alpha \frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha.$$

4. The upper-tailed confidence interval for μ

Homework 12 (to be handed in on Monday, Nov.28) is to prove the following theorem.

Theorem 4

The random interval $\left(\overline{X} z_\alpha \dfrac{\sigma}{\sqrt{n}}, \infty\right)$ *is a* $100(1-\alpha)\%$ *confidence interval for* μ.