Lecture 11 : The Basic Numerical Quantities Associated to a Continuous X

In this lecture we will introduce four basic numerical quantities associated to a continuous random variable X. You will be asked to calculate these (and the *cdf* of X) given $f(x)$ on the midterms and the final.
These quantities are

1. The p-th percentile $\eta(P)$.
2. The α-th critical value X_α.
3. The expected value $E(X)$ or μ.
4. The variance $V(X)$ or σ^2.

I will compute all these for $\cup(a, b)$ the linear distribution and $\cup(a, b)$.

Percentiles and Critical Values of Continuous Random Variables

Percentiles

Let P be a number between 0 and 1. The $100p$-th percentile, denoted $\eta(P)$, of a continuous random variable X is the unique number satisfying

$$P(X \leq \eta(P)) = P \qquad (\sharp)$$

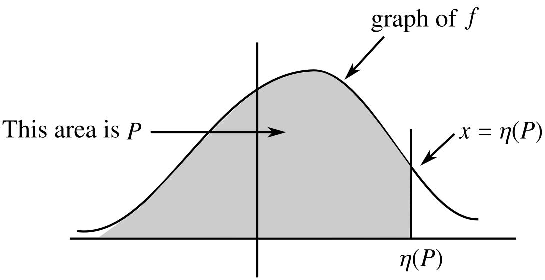
or

$$F(\eta(P)) = P \qquad (\sharp\sharp)$$

So if you know F you can find $\eta(P)$. Roughly

$$\eta(P) = F^{-1}(P)$$

The geometric interpretation of $\eta(P)$ is very important



The geometric interpretation of ($\sharp$)

$\eta(P)$ is the number such that the vertical line $x = \eta(P)$ cuts off area P to the left under the graph of $f(x)$.
(this is the picture above)

Special Case The median $\widetilde{\mu}$

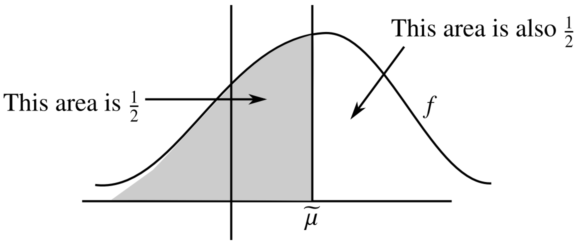
The median $\widetilde{\mu}$ is the unique number so that

$$P(X \leq \widetilde{\mu}) = \frac{1}{2}$$

$$\text{or} \qquad F(\widetilde{\mu}) = \frac{1}{2}$$

so the median is the 50-th percentile.

The picture



Since the total area is 1, the area to the right of the vertical line $x = \widetilde{\mu}$ also $\dfrac{1}{2}$. So $x = \widetilde{\mu}$ bisects the area.

Critical Values

Roughly speaking if you switch left to right in the definition of percentile you get the definition of the critical value. Critical values play a key role in the formulas for confidence intervals (later).

Definition

Let α be a real number between 0 and 1. Then the α-th critical value, denoted x_α, is the unique number satisfying

$$P(X \geq x_\alpha) = \alpha \tag{b}$$

Let's rewrite (b) in terms of F. We have

$$\begin{aligned} P(X \geq x_\alpha) &= 1 - P(X \leq x_\alpha) \\ &= 1 - F(x_\alpha) \end{aligned}$$

So (b) becomes

$$\begin{aligned} 1 - F(x_\alpha) &= \alpha \\ F(x_\alpha) &= 1 - \alpha \\ x_\alpha &= F^{-1}(1 - \alpha) \end{aligned} \tag{bb}$$

What about the geometric interpretation?

The geometric interpretation