Lecture 4 : Conditional Probability and Bayes' Theorem

The conditional sample space Motivating examples

1. Roll a fair die once

$$S = \boxed{\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{matrix}}$$

$$\begin{aligned} \text{Let } \ A &= 6 \text{ appears} \\ B &= \text{an even number appears} \end{aligned}$$

So

$$P(A) = \frac{1}{6}$$

$$P(B) = \frac{1}{2}$$

Now what about

$$P\left(\begin{array}{c}\text{6 appears } \underline{\text{given}} \text{ an even}\\ \text{number appears}\end{array}\right)$$

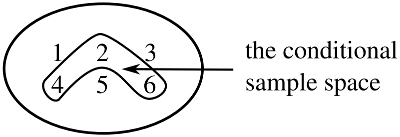
Philosophical Remark

(Ignore this remark unless you intend to be a scientist)
At present the above probability does not have a formal mathematical definition but we can still compute it. Soon we will give the formal definition and our computation "will be justified". This is the mysterious way mathematics works. Somehow there is a deeper reality underlying the formal theory.

Back to Stat 400

The above probability will be written given $P(A \text{ given } B)$ to he read $P(A \text{ given } B)$.

Now we know an even number occurred so the sample space changes



So there are only 3 possible outcomes given an even number occurred so

$$P(6 \text{ given an even number occurred}) = \frac{1}{3}$$

The new sample space is called the <u>conditional sample space</u>.

2. Very Important example

Suppose you deal two cards (in the usual way without replacement). What is $P(\heartsuit\heartsuit)$ i.e., P(two hearts in a row).

Well, $P(\text{first heart}) = \dfrac{13}{52}$.

Now <u>what about the second heart</u>?

Many of you will come up with ${}^{12}/_{51}$ and

$$P(\heartsuit\heartsuit) = \left({}^{13}/_{52}\right)\left({}^{12}/_{51}\right)$$

There are TWO theoretical points hidden in the formula.
Let's first look at

$$P(\underbrace{\heartsuit \text{ on } 2^{\text{nd}}}_{\text{this isn't really correct}}) = 12/51$$

What we really computed was the conditional probability

$$P(\heartsuit \text{ on } 2^{\text{nd}} \text{ deal} \mid \heartsuit \text{ on first deal}) = 12/51$$

Why Given we got a heart on the first deal the conditional sample space is the "new deck" with 51 cards and 12 hearts so we get

$$P(\heartsuit \text{ on } 2^{\text{nd}} \mid \heartsuit \text{ on } 1^{\text{st}}) = 12/51$$

The second theoretical point we used was the formula which we will justify formally later

$$\begin{aligned} P(\heartsuit\heartsuit) &= P(\heartsuit \text{ on } 1^{\text{st}})P(\heartsuit \text{ on } 2^{\text{nd}} \mid \heartsuit \text{ on } 1^{\text{st}}) \\ &= \binom{13}{52}\binom{12}{51} \end{aligned}$$

Basic Questions

These examples will occur repeatedly in today's lecture.

1. What is
$$P(\underbrace{\heartsuit \text{ on } 1^{\text{st}} \mid \heartsuit \text{ on } 2^{\text{nd}}}_{\text{reverse of pg. 5}})$$
and (easier).
2. What is $P(\heartsuit \text{ on } 2^{\text{nd}}$ with no information on what happened on the $1^{\text{st}})$.

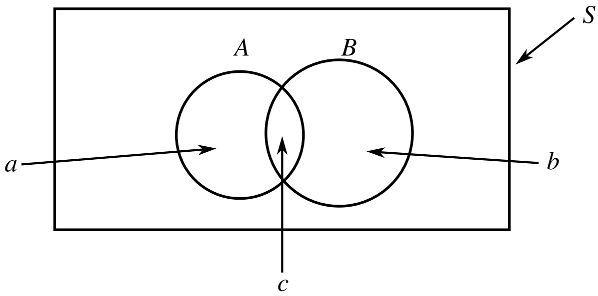
A Second Gold Star Problem

If you do 1- and 2. in the way described on this page you will get a gold star and learn a lot.

Don't use pg. 16 in Bayes Theorem

Let S = set of unordered pairs of distinct cards
Compute $\sharp(S)$.
Let A = subset of pairs of hearts = $\{\heartsuit\heartsuit\} \subset S$.
Compute $\sharp(A)$ and $P(A) = \dfrac{\sharp(A)}{\sharp(S)}$.
Let B = subset of pairs so that the second card is a heart.
Compute $\sharp(B)$ and $P(B)$ solving 2.
Now compute $P(\heartsuit \text{ on } 1^{\text{st}} \mid \heartsuit \text{ on } 2^{\text{nd}})$ by taking the ratio (pg. 11) or else computing the conditional sample space.

The Formal Mathematical Theory of Conditional Probability



$$\sharp(S) = n,\ \sharp(A) = a,\ \sharp(B) = b,\ \sharp(A \cap B) = C$$

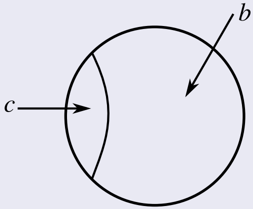
Problem

Let S be a finite set with the equally - likely probability measure and A and B be events with coordinalities shown in the picture.

Problem

Compute $P(A|B)$.

We are given B occurs so the conditional sample space is B



Only part of A is allowed since we know B occurred namely $A \cap B$

$$P(A|B) = \frac{\sharp(A \cap B)}{\sharp(B)}$$
$$= \frac{c}{b}$$

We can rewrite this as

$$P(A|B) = \frac{a}{b} = \frac{a/n}{b/n} = \frac{P(A \cap B)}{P(B)}$$

so

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad (*)$$

This intuitive formula for the equally likely probability measure leads to the following.

Formal Mathematical Definition

Let A and B be any two events in a sample space S with $P(B) \neq 0$. The conditional probability of A given B is written $P(A|B)$ and is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad (*)$$

so if $P(A) \neq 0$ then

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)} \qquad (**)$$

Since $A \cap B – B \cap A$.

We won't prove the next theorem but you could do it and it is useful.

Theorem

Fix B with $P(B) \neq 0$. $P(\cdot|B)$ satisfies the axioms (and theorems) of a probability measure - see Lecture 1.

For example

1. $P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B) - P(A_1 \cap A_2|B)$
2. $P(A'|B) = 1 - P(A|B)$

Z $\quad P(A|\cdot)$ *does not satisfy the axioms and theorems.*

The Multiplicative Rule for $P(A \cap B)$

Rewrite (**) as

$$P(A \cap B) = P(A)P(B|A)(\sharp)$$

($\sharp$) is very important, more important then (**).
It complement the formula

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Now we know how P interacts with the basic binary operations $\cup$ and $\cap$.

More generally

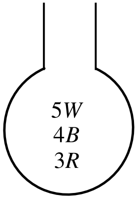
$$P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B)$$

Exercise

Write down $P(A \cap B \cap C \cap D)$.

Traditional Example

An urn contains 5 white chips, 4 black chips and 3 red chips.
Four chips are drawn sequentially without replacement. Find $P(WRWB)$.



12 chips

Solution

$$P(WRWB) = \left(\frac{5}{12}\right)\left(\frac{3}{11}\right)\left(\frac{4}{10}\right)\left(\frac{4}{9}\right)$$

What did we do formally

$$P(WRWB) = P(W) \cdot P(R|W)$$

$\underline{P(W|W \cap R) \cdot P(B|W \cap R \cap W)}$

Now we redo the gold star problem (not the gold star way) namely we compute

$$P(\underbrace{\heartsuit \;\; on \;\; 1^{st}}_{A} \mid \underbrace{\heartsuit \;\; on \;\; 2^{nd}}_{B})$$

By Definition

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\heartsuit\heartsuit)}{P\left(\begin{array}{c}\heartsuit \text{ on } 2^{\text{nd}} \text{ with the} \\ \text{other information}\end{array}\right)}$$

Now we know from pg. 5.

$$P(\heartsuit\heartsuit) = (13/52)(12/51)$$

Now we need

$$P(\heartsuit \text{ on } 2^{\text{nd}} \text{ with no other information}) = 13/52$$

We will prove this on pg. 18 - it also follows from the gold star approach on pg. 7. So

$$= P(\heartsuit \text{ on } 1^{\text{st}} \mid \heartsuit \text{ on } 2^{\text{nd}}) = \frac{\cancel{(13/52)}(12/\cancel{51})}{\cancel{(13/52)}} = \frac{12}{51}$$

$$= P(\underbrace{\heartsuit \text{ on } 2^{\text{nd}} \mid \heartsuit \text{ on } 1^{\text{st}}}_{\text{pg. 5}})!!$$

Bayes' Theorem (pg. 72)

Bayes' Theorem is a truly remarkable theorem. It tells you "how to compute $P(A|B)$ if you know $P(B|A)$ and a few other things".
For example - we will get a new way to compute are favorite probability $P(\heartsuit \text{ as } 1^{\text{st}} \mid \heartsuit \text{ on } 2^{\text{nd}})$ because we know $P(\heartsuit \text{ on } 2^{\text{nd}} \mid \heartsuit \text{ on } 1^{\text{st}})$.
First we will need on preliminary result.

The Law of Total Probability

Let $A_1, A_2, \ldots, A_k$ be mutually exclusive
($A_i \cap A_j = \emptyset$) and exhaustive.
($A_1 \cup A_2 \cup \ldots \cup A_k = S =$ the whole space)
Then for any event B

$$\begin{aligned} P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) \\ + \cdots + P(B|A_k)P(A_k) \end{aligned} \tag{b}$$

Special case $k = 2$ so we have A and A'

$$P(B) = P(B|A)P(A) + P(B|A')P(A') \tag{bb}$$

Now we can prove
$P(\heartsuit \text{ on } 2^{\text{nd}} \text{ with no other information}) = {}^{13}/_{52}$

$$\text{Put} \quad B = \heartsuit \text{ on } 2^{\text{nd}}$$
$$A = \text{heart on } 1^{\text{st}}$$
$$A' = \text{a nonheart on } 1^{\text{st}}$$

Lets write $\not\heartsuit$ for nonheart.
So,

$$P(\not\heartsuit \text{ on } 1^{\text{st}}) = {}^{39}/_{52}$$
$$P(\heartsuit \text{ on } 2^{\text{nd}} / \not\heartsuit \text{ on first}) = {}^{13}/_{51}$$

Now

$$
\begin{aligned}
P(B) &= P(B|A)P(A) + P(B|A')P(A') \\
&= P(\heartsuit \text{ on } 2^{\text{nd}} \,|\, \heartsuit \text{ on } 1^{\text{st}})P(\heartsuit \text{ on } 1^{\text{st}}) \\
&\quad + P(\heartsuit \text{ on } 2^{\text{nd}} \,|\, \cancel{\heartsuit} \text{ on } 1^{\text{st}})P(\cancel{\heartsuit} \text{ on } 1^{\text{st}}) \\
&= (12/51)(13/52) + (13/51)(39/52)
\end{aligned}
$$

add fractions

$$
= \frac{(12)(13) + (13)(39)}{(51)(52)}
$$

factor out 13 add to get 51

$$
= \frac{(13)(12 + 39)}{(51)(52)} = \frac{(13)(\cancel{51})}{(\cancel{51})(52)}
$$

$= (13)/(52)$ Done!

(b) is “very easy” to prove but we won’t do it.
Now we can state Bayes’ Theorem.

Bayes’ Theorem (pg. 73)

Let $A_1, A_2, \ldots, A_k$ be a collection of n mutually exclusive and exhaustive events with $P(A_i) > 0$
$i = 1, 2, \ldots, k$. Then for any event B with $P(B) > 0$

$$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum\limits_{i=1}^{k} P(B|A_i)P(A_i)}$$

Again we won’t give the proof.

Special Case $k = 2$

Suppose we have two events A and B with $P(A) > 0$, $P(A') > 0$ and $P(B > 0)$. Then

$$\sharp) P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A')P(A')}$$

Now we will compute (for the last time)

$$P(\heartsuit \text{ on } 1^{\text{st}} \mid \heartsuit \text{ on } 2^{\text{nd}})$$

Using Bayes' Theorem.
This is the obvious way to

do it since we know the probability "the other way around"

$$P(\heartsuit \text{ on } 2^{\text{nd}} \mid \heartsuit \text{ on } 1^{\text{st}}) = {}^{12}/_{51}$$

So let's do it.

$$\begin{aligned} \text{We put } A &= \heartsuit \text{ on } 1^{\text{st}} \\ \text{so } A' &= \not\heartsuit \text{ on } 1^{\text{st}} \\ \text{and } B &= \heartsuit \text{ on second} \end{aligned}$$

plugging into ($\sharp$) we get

$$P(\heartsuit \text{ on } 1^{\text{st}} \mid \heartsuit \text{ on } 2^{\text{nd}})$$
$$= \frac{P(\heartsuit \text{ on } 2^{\text{nd}} \mid \heartsuit \text{ on } 1^{\text{st}})P(\heartsuit \text{ on } 1^{\text{st}})}{P(\heartsuit \text{ on } 2^{\text{nd}} \mid \heartsuit \text{ on } 1^{\text{st}})P(\heartsuit \text{ on } 1^{\text{st}}) + P(\heartsuit \text{ on } 2^{\text{nd}} \mid \not\heartsuit \text{ on } 1^{\text{st}})P(\not\heartsuit \text{ on } 1^{\text{st}})}$$

$$= \frac{({}^{12}\!/_{51})({}^{13}\!/_{52})}{({}^{12}\!/_{51})({}^{13}\!/_{52}) + ({}^{13}\!/_{51})({}^{39}\!/_{52})}$$

swap
numbers
use

$$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \left(\frac{c}{b}\right)\left(\frac{a}{d}\right)$$

something
from
high school

$$= \frac{({}^{13}\!/_{51})({}^{12}\!/_{52})}{({}^{13}\!/_{51})({}^{12}\!/_{52}) + ({}^{13}\!/_{51})({}^{39}\!/_{52})}$$

factor this out

$$= \frac{({}^{13}\!/_{51})({}^{12}\!/_{52})}{({}^{13}\!/_{51})\left({}^{12}\!/_{52} + \dfrac{39}{52}\right)}$$

$$= \frac{\cancel{({}^{13}\!/_{51})}({}^{12}\!/_{52})}{\cancel{({}^{13}\!/_{51})}({}^{51}\!/_{52})} = \frac{({}^{12}\!/\cancel{_{52}})}{({}^{51}\!/\cancel{_{52}})}$$

$= {}^{12}\!/_{51}$ once again!

The algebra was hard but the approach was the most natural - a special case of

General Principle

If you know $P(B|A)$ and you want to compute $P(A|B)$ use Bayes' Theorem in the ($\sharp$) version, pg. 22.

Compulsory Reading (for your own heath)

In case you or someone you love tests positive for a rare (this is the point) disease, read Example 2.30, pg. 73. Misleading (and even bad) statistics is rampant in medicine.