Lecture 24: The Sample Variance S^2 The squared variation

Suppose we have n numbers $x_1, x_2, \dots, x_n$. Then their *squared variation*
$sv = sv(x_1, x_2, \dots, x_n)$ $sv(x_1, x_2, \dots, x_n) = \sum\limits_{i=1}^{n} (x_i - \overline{x})^2$
Their *mean* (average) squared variation msv or σ_n^2 (denoted σ^2 and called the "population variance on page 33 of our text) is given by

$$msv = \sigma_n^2 = \frac{1}{n} sv = \frac{1}{n} \sum_{i=1}^{n} (x_i - \overline{x})^2$$

Here $\overline{x}$ is the average $\frac{1}{n} \sum\limits_{i=1}^{n} x_i$.

The msv measure how much the numbes $x_1, x_2, \ldots, x_n$ vary (precisely how much they vary from their average $\overline{x}$). For example if they are all equal then they will be all equal to their average $\overline{x}$ so

$$sv = 0 \quad \text{and} \quad msv = 0$$

We also define the sample variance s^2 by

$$S^2 = \frac{1}{n-1} sv = \frac{n}{n-1} msv$$

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \overline{x})^2$$

Amazingly, s^2 is more important then msv in statistics

The Shortcut Formula for the Squared Variation

Theorem

$$sv(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i^2 - \frac{1}{n}(\sum_{i=1}^{n} x_i)^2 \qquad (*)$$

Proof

Note since $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ *we have* $\sum_{i=1}^{n} x_i = n\overline{x}$

Now

$$\sum_{i=1}^{n} (x - i - \overline{x})^2 = \sum_{i=1}^{n} (x_i^2 - 2x_1\overline{x} + \overline{x})^2$$

$$= \sum_{i=1}^{n} x_1^2 - \sum_{i=1}^{n} 2\, x_i\, \overline{x} + \sum_{i=1}^{n} \overline{x}^2$$

$$= \sum_{i=1}^{n} x_1^2 - 2\overline{x}\sum_{i=1}^{n} x_i + \overline{x}^2 \sum_{i=1}^{n} 1$$

Proof (Cont.)

$$
\begin{aligned}
&= \sum_{i=1}^{n} x_i^2 - 2\overline{x}(n\overline{x}) + n\overline{x}^2 \\
&= \sum_{i=1}^{n} x_i^2 - 2n\overline{x}^2 + n\overline{x}^2 \\
&= \sum_{i=1}^{n} x_i^2 - n\overline{x}^2 \\
&= \sum_{i=1}^{n} x_i^2 - n\left(\frac{\sum\limits_{i=1}^{n} x_i}{n}\right)^2 \\
&= \sum_{i=1}^{n} x_i^2 - \cancel{n}\frac{\left(\sum\limits_{i=1}^{n} x_i\right)^2}{\cancel{n^2}} \\
&= \sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2
\end{aligned}
$$

$\square$

Corollary 1

Divide both sides of $(*)$ *by* n *to get*

$$msv = \frac{1}{n}\sum_{i=1}^{n} x_1^2 - \frac{1}{n^2}\left(\sum_{i=1}^{n} x_i\right)^2$$

Corollary 2 ((Shortcut formula for s^2))

Divide both sides of $(*)$ *by* $n-1$ *to get*

$$S^2 = -\frac{1}{n-1}\sum_{i=1}^{n} x_i^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n} x_i\right)^2$$

It is this last formula that we will need.

Let met give a conceptual proof of the theorem the way a professorial mathematician would prove the theorem.

Definition

A polynomial $p(x_1, x_2, \dots, x_n)$ is symmetric, if it is unchanged by permuting the variables.

Examples 3

$$p(x, y, z) = x^2 + y^2 + z^2 \quad \text{is symmetric}$$
$$p(x, y, z) = xy + z^2 \quad \text{is not symmetric}$$

Theorem

Any symmetric polynomial pin $x_1, x_2, \dots, x_n$ can be rewritten as a polynomial in the power sums $\sum\limits_{i=1}^{n} x_i^k$ that is

$$p(x_1, \dots, x_n) = q\left(\sum x_i, \sum x_1^2, \dots, \sum x_i^\ell\right)$$

if deg $p = \ell$.

Bottom Line

$sv = \sum_{i=1}^{n}(x_i - \overline{x})^2$ is a symmetric polynomial in $x_1, x_2, \ldots, x_n$ so there exist a and b with

$$sv(x_1, x_2, \ldots, X_n) = a\sum_{i=1}^{n} x_i^2 + b\left(\sum_{i=1}^{n} x_i\right)^2 \qquad (**)$$

This is true for all $x_1, \ldots, x_n$ (an "identify") so we just choose $x_1, \ldots, x_n$ cleverly to get a and b.

First choose $x_1 = 1$, $x_2 = -1$, $x_3 = \ldots = x_n = 0$ so $\sum_{i=1}^{n} x_i = 0$ and $\sum_{i=1}^{n} x_i^2 = 2$ since $\overline{x} = 0$

$$\text{and} \quad sv(1, -1, 0, \ldots, 0) = \sum_{i=1}^{n}(x_i - \overline{x})^2 = \sum_{i=1}^{n} x_i^2$$

$(**)$ becomes

$$2 = a2 + b(0) \quad \text{so} \quad a = 1$$

To find b take all the x's to be 1. so $\overline{x} = 1$ and $sv(1, 1 : 1) = 0$ (there is no variation in the x's)

$$\sum_{i=1}^{n} x_1^2 = n, \quad \sum_{i=1}^{n} x_i = n \text{ so}$$

$$sv(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i^2 + b(\sum x_i)^2$$

gives as

$$0 = h + bn^2 \quad \text{so} \quad b = -\frac{1}{n}$$

and

$$sv(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} x_i^2 - \frac{1}{n}(\sum x_i)^2$$

as before.

Remark 1

Any symmetric quadratic function $q(x_1, x_2, \dots, x_n)$ is a linear combination of $\sum\limits_{i=1}^{n} x_1^2$ and $(\sum\limits_{i=1}^{n} x_i)^2$ that is

$$q(x_1, \dots, x_n) = a \sum_{i=1}^{n} x_i^2 + b\left(\sum_{i=1}^{n} x_i\right)^2$$

In Which We Return to Statistics

Estimating the Population Variance We have seen that $\overline{X}$ is a good (the best) estimator of the population mean-μ, in particular it was an unbiased estimator.

$$E(\overline{X}) = \mu$$

sample mean
random variable

papulation mean

How do we estimate the population variance?

$$\boxed{\begin{array}{c} X \\ \\ V(x) = \sigma^2 \end{array}} \longrightarrow x_1, x_2, \ldots, x_n \to s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2$$

Answer - use the *Sample* variance s^2 to estimate the *population* variance σ^2
The reason is that if we take the associated sample variance random variable

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n-1}(X_i - \overline{X})^2$$

then we have

Amazing Theorem

$$E(S^2) = \sigma^2$$

Sample Variance

Population Variance

Why do you need $\dfrac{1}{n-1}$? We will see.

Before starting the proof we first note the Corollary 2, page 2 implies

Proposition (Shortcut formula for the sample variance random variable's)

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n} X_i^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n} X_i\right)^2 \tag{b}$$

Why does this follow from the formula for s^2? We will also need the following

Proposition

Suppose Y is a random variable then

$$E(Y^2) = E(Y)^2 + V(Y) \tag{\#}$$

Proof.

$$V(Y) = E(Y^2) - (E(Y))^2$$

(Shortcut formula for $V(Y)$ □

Corollary

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a population of mean μ and variance σ^2. Then

(i) $E(X_i^2) = \mu^2 + \sigma^2$

(ii) $E(T_0) = n^2\mu^2 + n\sigma^2$

Proof.

(i) $E(X_i) = \mu$ and $V(Y) = \sigma^2$ so plug into (#)

(ii) $E(T_0) = n\mu$ and $V(T_0) = n\sigma^2$
so plug into (#)

$\square$

We can now prove (b)

$$E(S^2) = E\left(\frac{1}{n-1}\sum_{i=1}^{n} X_i^2 - \frac{1}{n(n-1)}(\sum X_i)^2\right)$$

since E is linear

$$= \frac{1}{n-1}\sum_{i=1}^{n} E(X_i^2) - \frac{1}{n(n-1)}E(T_0^2)$$

by (i) and (ii)

$$\begin{aligned}
&= \frac{1}{n-1}\sum_{i=1}^{n}(\mu^2+\sigma^2) - \frac{1}{n-1}\frac{1}{n}(n^2\mu^2+n\sigma^2) \\
&= \frac{1}{n-1}\left[n\mu^2+n\sigma^2-\frac{1}{n}(n^2\mu^2+n\sigma^2)\right] \\
&= \frac{1}{n-1}\left[\cancel{n\mu^2}+n\sigma^2-\cancel{n\mu^2}-\sigma^2\right] \\
&= \frac{1}{n-1}\left[(n-1)\sigma^2\right] \\
&= \sigma^2
\end{aligned}$$

Amazing - you need $\dfrac{1}{n-1}$ not $\dfrac{1}{n}$.