Lecture 11 : The Basic Numerical Quantities Associated to a Continuous X

In this lecture we will introduce four basic numerical quantities associated to a continuous random variable X. You will be asked to calculate these (and the *cdf* of X) given $f(x)$ on the midterms and the final.

These quantities are

1. The p-th percentile $\eta(P)$.
2. The α-th critical value X_α.
3. The expected value $E(X)$ or μ.
4. The variance $V(X)$ or σ^2.

I will compute all these for $\cup(a, b)$ the linear distribution and $\cup(a, b)$.

Percentiles and Critical Values of Continuous Random Variables

Percentiles

Let P be a number between 0 and 1. The $100p$-th percentile, denoted $\eta(P)$, of a continuous random variable X is the unique number satisfying

$$P(X \leq \eta(P)) = P \tag{$\sharp$}$$

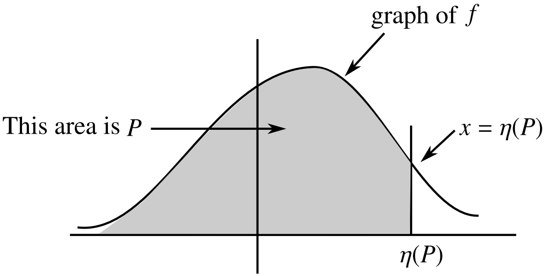
or

$$F(\eta(P)) = P \tag{$\sharp\sharp$}$$

So if you know F you can find $\eta(P)$. Roughly

$$\eta(P) = F^{-1}(P)$$

The geometric interpretation of $\eta(P)$ is very important



The geometric interpretation of ($\sharp$)

$\eta(P)$ is the number such that the vertical line $x = \eta(P)$ cuts off area P to the left under the graph of $f(x)$.
(this is the picture above)

Special Case The median $\widetilde{\mu}$

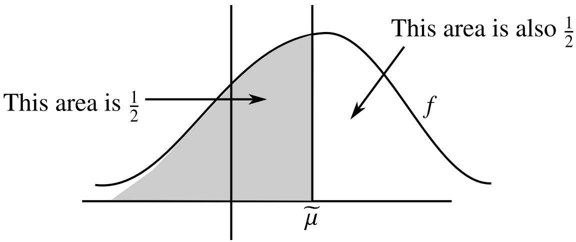
The median $\widetilde{\mu}$ is the unique number so that

$$P(X \leq \widetilde{\mu}) = \frac{1}{2}$$

$$\text{or} \qquad F(\widetilde{\mu}) = \frac{1}{2}$$

so the median is the 50-th percentile.

The picture



Since the total area is 1, the area to the right of the vertical line $x = \widetilde{\mu}$ also $\dfrac{1}{2}$. So $x = \widetilde{\mu}$ bisects the area.

Critical Values

Roughly speaking if you switch left to right in the definition of percentile you get the definition of the critical value. Critical values play a key role in the formulas for *confidence intervals* (later).

Definition

Let α be a real number between 0 and 1. Then the α-th critical value, denoted x_α, is the unique number satisfying

$$P(X \geq x_\alpha) = \alpha \qquad \text{(b)}$$

Let's rewrite (b) in terms of F. We have

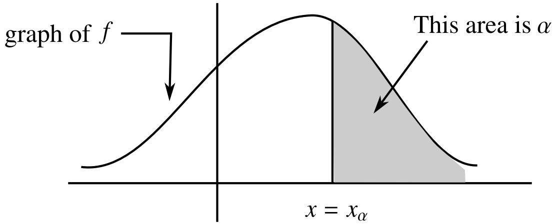
$$P(X \geq x_\alpha) = 1 - P(X \leq x_\alpha)$$
$$= 1 - F(x_\alpha)$$

So (b) becomes

$$1 - F(x_\alpha) = \alpha$$
$$F(x_\alpha) = 1 - \alpha$$
$$x_\alpha = F^{-1}(1 - \alpha) \qquad \text{(bb)}$$

What about the geometric interpretation?

The geometric interpretation



x_α is the number so that the vertical line $x = x_\alpha$ cuts off area α to the *right* under the graph of $f(x)$.

Relation between critical values and percentiles

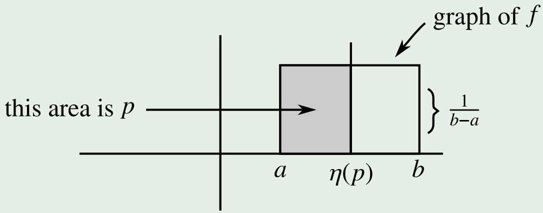
$x = x_\alpha$ cuts off area $1 - \alpha$ to the *left* since the total area is 1. But $n(1 - \alpha)$ is the number such that $x = \eta(1 - \alpha)$ cuts off area $1 - \alpha$ to the left.
So

$$\underline{x_\alpha = \eta(1 - \alpha)}$$

Computation of Examples

Example 1 ($X \sim \bigcup(a, b)$)

Lets compute the $\eta(p)$-th percentile for $X \sim \bigcup(a, b)$



So the point $\eta(p)$ between a and b must have the property that the area of the shaded box is p. But the base of the box is $\eta(p) - a$ and the ????? is $\dfrac{1}{h-a}$ so

$$\text{Area} = bh = (\eta(p) - a)\left(\frac{1}{b-a}\right) \quad \text{so}$$

$$(n(p) - a)\left(\frac{1}{b-a}\right) = p \quad \text{or}$$

$$\eta(p) = a + p(b - a) = (1 - p)a + pb \tag{*}$$

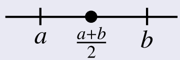
Example 1 (Cont.)

How about the median $\widetilde{\mu}$.
So we want $\eta(\frac{1}{2})$. By (*) we have

$$\widetilde{\mu} = \eta\left(\frac{1}{2}\right) - a + \frac{b-a}{2} = \frac{a+b}{2}$$

Remark

$\dfrac{a+b}{2}$ is the midpoint of the interval $[a, b]$.



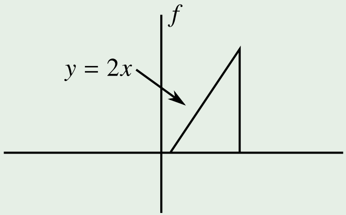
Critical Values for $\bigcup(a, b)$

$$
\begin{aligned}
x_\alpha &= \eta(1-\alpha) = a + (1-\alpha)(b-\alpha) \\
&= a + b - a - \alpha b + \alpha a \\
\text{So} \quad x_\alpha &= \alpha a + (1-\alpha) b.
\end{aligned}
$$

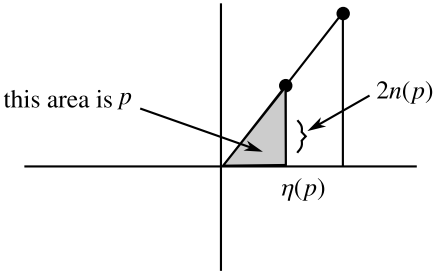
Example 2 (The linear distribution)

Recall the linear distribution has density

$$
f(x) = \begin{matrix} 0, & x < 0 \\ 2x, & 0 \leq x \leq 1 \\ 0, & x > 1 \end{matrix}
$$



The 100p-th percentile



We want the area of the triangle to be p. But the box is $\eta(p)$ and the height is $Z\eta(p)$ so

$$A = \frac{1}{2}bh = \frac{1}{2}\eta(p)(2n(p))$$
$$= \eta(p)^2$$

We have to solve

$$\eta(p)^2 = p$$
$$\text{So} \quad \eta(p) = \sqrt{p}$$

In particular

$$\widetilde{\mu} = \eta\left(\frac{1}{2}\right) = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$$

This will be important ????.

Expected Value

Definition

The expected value or mean $E(X)$ or μ of a continuous random variable is defined by

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

We will compute some examples.

Example 1 ($X \sim \bigcup(a, b)$)

$$E(X) = \int_{-\infty}^{\infty} f(x)dx = \int_{a}^{b} \frac{1}{b-a} x \, dx$$

$$= \frac{1}{b-a} \left(\frac{x^2}{2} \right) \Bigg|_{x=a}^{x-b} = \frac{1}{2} \frac{(b^2 - a^2)}{b-a} = \frac{b+a}{2}$$

Example 1 (Cont.)

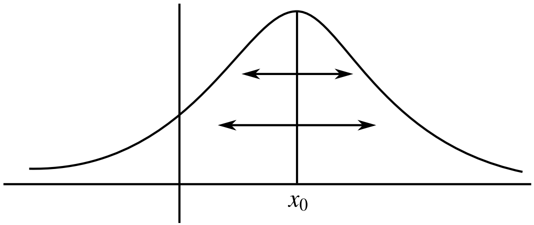
Now we showed on page 9 that if $X \sim \bigcup(a, b)$ then the median $\widetilde{\mu}$ was given by $\widetilde{\mu} = \dfrac{a+b}{2}$.

Hence in this *the mean is equal to the median*

$$\mu = \widetilde{\mu} = \frac{a+b}{2}$$

Z This is not always the case as we will see shortly.

The “reason” $\mu = \widetilde{\mu}$ is that $f(x)$ has a point of symmetry i.e. a point x_0 so that $f(x_0 f y) = f(x_0 - y)$



This means that the graph is symmetrical about the vertical line (mirror) $x = x_0$.

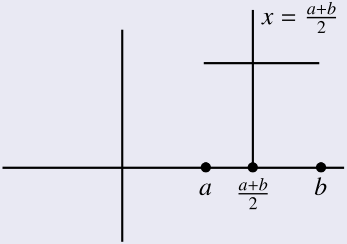
Proposition (Useful fact)

If x_0 is a point of symmetry for $f(x)$ then

$$\mu = \widetilde{\mu} = x_0$$

Proposition (Cont.)

Now if $X \sim \bigcup(a, b)$ *then* $x_0 = \dfrac{a+b}{2}$ *is a point of symmetry for* $f(x)$

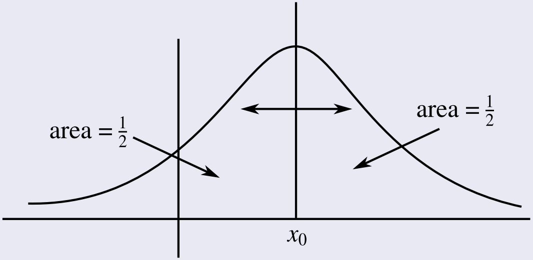


For a change we will prove the proposition

Proof

$\widetilde{\mu} = x_0$ *is immediate because by symmetry there is equal area to the left and right of* x_0.

Proof (Cont.)



Since the total area is 1, the area to the left of x_0 is $\frac{1}{2}$.
Hence $\widetilde{\mu} = x_0$.
It is harder to prove

$$E(X) = \int_{-\infty}^{\infty} x f(x) = x_0$$

Trick : Since x_0 is a constant and $\int_{-\infty}^{\infty} f(x)dx = 1$ we have

$$\int_{-\infty}^{\infty} x_0 f(x) dx = x_0$$

Proof (Cont.)

Thus to show

$$\int_{-\infty}^{\infty} xf(x)dx = x_0$$

It suffices to show

$$\int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{\infty} x_0 f(x)dx$$

or

$$\int_{-\infty}^{\infty} (x - x_0)f(x)dx = 0$$

But if we put

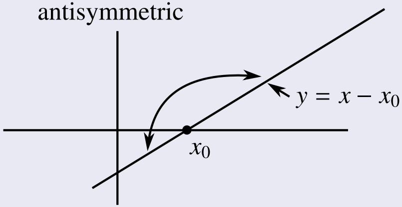
$$g(x) = (x - x_0)f(x) \quad \textit{then}$$

$g(x)$ *is antisymmetric or "odd" about* x_0

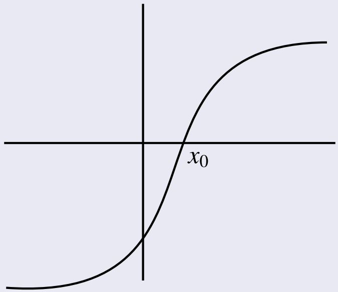
$$g(x_0 + y) = -g(x_0 + g)$$

Proof (Cont.)

This is because $x - x_0$ is



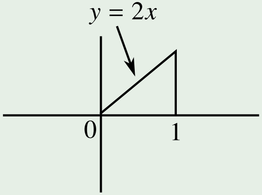
But antisymmetric symmetric = antisymmetric (or odd-even = odd).
Finally the integral of on antisymmetric (or "odd") function from $-\infty$ to ∞ is zero.



The integral to the left of x_0 cancels the area to the right. □

This fact can save a lot of painful computation of expected values.

Example 2 (The linear distribution)



We have seen $\widetilde{\mu} = \dfrac{\sqrt{2}}{2}$, page 12, $f(x)$ is certainly not symmetric so it is possible $\mu = \widetilde{\mu}$ and we will see that it is the case.

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f(x) dx \\
&= \int_0^1 x(2x) dx \\
&= 2 \int_0^1 x^2 \, dx \\
&= 2\left(\frac{1}{3}\right) = \frac{2}{3}
\end{aligned}
$$

Handy fact $\int_0^1 x^n = \frac{1}{n}$.
So $\mu = \frac{2}{3}$ and $\widetilde{\mu} = \frac{2}{\sqrt{2}}$.
They aren't equal, which one is bigger?

Variance

The variance $V(X)$ or σ^2 of a continuous random variable is defined by

$$V(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\,dx$$

Remark

Once we learn about change of continuous random variable we will see this is

$$E\left((X - \mu)^2\right)$$

new random variable obtains from X using $h(x) = (x - \mu)^2$.

Once again there is a shortcut formula for $V(X)$.

Proposition (Shortcut Formula)

$$\begin{aligned} V(X) &= E(X^2) - (E(X))^2 \\ &= E(X^2) - \mu^2 \end{aligned}$$

This is the formula to use

Example 1 ($X \sim \bigcup(a, b)$)

We know $\mu = \dfrac{a+b}{2}$. We have to compute $E(X^2)$

Example 1 (Cont.)

$$
\begin{aligned}
E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)\,dx \\
&= \int_a^b x^2 \tfrac{1}{b-a}\,dx \\
&= \frac{1}{b-a}\left(\frac{x^3}{3}\right)\Bigg|_{x=a}^{x=b} \\
&= \frac{1}{3}\,\frac{b^3-a^3}{b-a} = \frac{1}{3}(b^2+ab+a^2)
\end{aligned}
$$

So
$$
\begin{aligned}
V(X) &= \frac{1}{3}(a^2+ab+b^2) - \overbrace{\left(\frac{a+b}{2}\right)^2}^{\mu^2} \\
&= \frac{a^2+ab+b^2}{3} - \frac{a^2+2ab+b^2}{4} \\
&= \frac{a^2-2ab+b^2}{12} = \frac{(b-a)^2}{12}
\end{aligned}
$$

Example 2 (The linear distribution)

We have seen (pg. 21)

$$\mu = \frac{2}{3}$$

We need $E(X^2)$

$$E(X^2) = \int_{-\infty}^{\infty} X^2 f(x)dx$$

$$= \int_0^1 x^2(2x)dx$$

$$= 2\int_0^1 x^3 dx = 2\left(\frac{1}{4}\right) = \frac{1}{2}$$

SO $\quad V(X) = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9}$

$$= \frac{9}{18} - \frac{8}{18} = \frac{1}{18}$$