Lecture 1

The Mathematical Theory of Probability

1. Introduction

Today we will do §2.1 and 2.2. We will skip Chapter 1.

We all have an intuitive notion of probability.

Let's see.

What is the probability of tossing two heads in a row with a fair coin?

Method 1.

List <u>all</u> possible outcomes

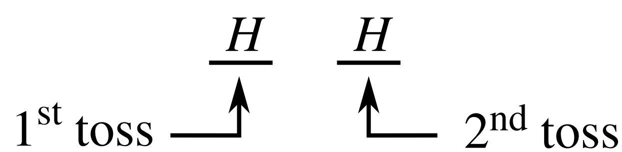
$$\{HH, HT, TH, TT\}$$

so $P = ?$.

Question

What did we just assume to arrive at that answer?

Another way



However it is important to put probability into a formal mathematic framework for many reasons.

1. Even “elementary”

Problems become too hard unless we can break them down into simpler problems using the rule of Set Theory.

Examples

Let’s see how you can deal with these now and later.

(there is another reason which we will run into later - we often have infinite sets and need calculus e.g. financial math)

Problems

1. What is the probability of getting <u>one</u> head in one hundred tosses of a fair coin?

2. What is the probability of getting 27 heads in one hundred tosses of a fair coin?

Prediction - nobody will get this one now.

In two weeks everybody will.

2. Transition from the naive theory to the formal mathematical theory

To make the transition we introduce the word “experiment” which will be taken to mean “any action or process whose outcome is subject to uncertainty”

text - pg. 47.

Examples

Tossing a fair coin 100 times.

Dealing 5 cards from a 52 card deck - a poker hand.

Dealing 13 cards from a 52 card deck - a bridge hand.

Definition

The set of all possible outcomes of on experiment will be called the sample space of that experiment and denote s.

Experiment

3 tosses of a fair coin.

$$s = \left\{ \begin{array}{c} HHH,\ HHT,\ HTH,\ HTT, \\ THH,\ THT,\ TTH,\ TTT \end{array} \right\}$$

Definition

A subset A of s is called an event.

Problem

Find P (at least one head in 3 tosses of a fair coin)

We are looking for $P(A)$ where A is a subset of the previous s.

$$s = \{HHH,\ HHT,\ HTH,\ HTT,\ THH,\ THT,\ TTH,\ TTT\}$$

We will call this “our favorite sample space” from now on.

3. The Formal Mathematical Theory

Let s be a set (the sample space). A probability measure P on s is a rule (function) which assigns a real number $P(A)$ to any subset A of s (i.e., to any event) such that the following axioms are satisfied

1. For any event $A \subset s$ we have $P(A) \geq 0$

2. $P(s) = 1$

3. If $A_1, A_z, \quad , A_n, \ldots$

is a possibly infinite collection of pairwise disjoint (mutually exclusive) events then

$$P\left(A, \cup A_z \cup \ldots \cup A_n \cup \ldots\right) = \underbrace{\sum_{n=1}^{\infty} P(A_n)}_{\substack{\text{sum of an infinite series} \\ \text{not just ordinary sum.}}}$$

mutually exclusive means $A_i \cap A_j = \phi$ for any pair $i,\ j$ with $i \neq j$.

Special cases

1. Two mutually-exclusive events A_1 and A_2 (so $A_1 \cap A_2 = \phi$)

$$P(A_1 \cup A_2) = P(A_1) + P(A_2)$$

2. n mutually-exclusive events $A_1, A_2, \ldots, A_n$

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

A Class of Examples

Let s be a set with n elements. Let $A \subset s$ be any subset. Define

$$P(A) = \frac{\sharp(A)}{\sharp(s)} = \frac{\sharp(A)}{n}$$

Then P satisfies the axioms $1.$, $2.$ and $3.$

Here $\sharp(A)$ means the number elements in A. This is called the "equally likely probability measure".

An example in the above class

Take our favorite sample space

$$s = \left\{ \begin{array}{c} HHH,\ HHT,\ HTH,\ HTT \\ THH,\ THT,\ TTH,\ TTT \end{array} \right\}$$

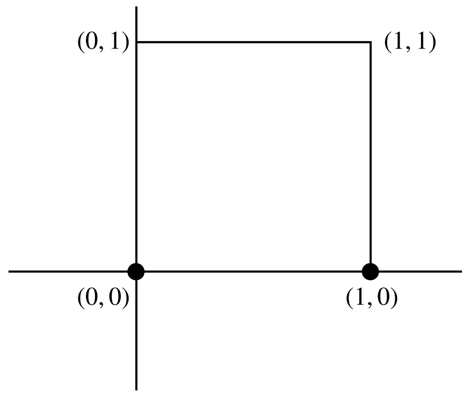
Let A be the subset (event) of outcomes with at least one head and one tail.

All the outcomes are equally likely (because the coin is fair) so

$$P(A) = \frac{\sharp(A)}{\sharp(s)} = \frac{P}{8}$$

A continuous Example 15

Consider the unit square s in the plane



Let $A \subset s$ be any subset.

Define

$$P(A) = \text{Area of } A$$

Then P satisfies the axioms $1.$, $2.$ and $3.$