Numerical Analysis of Sakiadis Flow Using Python
Seyyed Mohammad Amin Hosseini

Amirkabir University of Technology, Faculty of Mechanical Engineering
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Abstract

This paper presents a numerical analysis of Sakiadis flow, a fundamental problem in fluid mechanics, using the Python programming language. Sakiadis flow, first studied by Sakiadis[1] in 1961, describes a boundary layer flow model in a Newtonian or non-Newtonian fluid over a flat plate that starts moving from rest. In this study, we used the scipy.integrate.solve_bvp function to solve the boundary value problem (BVP) that describes Sakiadis flow. This function is a numerical method for solving ordinary differential boundary value problems. The results obtained from solving the equations are visualized using the Matplotlib graphics library. These graphs provide important insights into flow behavior and the impact of various parameters on it. This numerical analysis demonstrates the feasibility of examining Sakiadis flow under different conditions and with variable parameters, which can help in better understanding the physics of this flow and its applications in industry.

1 Introduction

In 1904, Prandtl [2] introduced the initial concept of the boundary layer. Since then, researchers have found many theoretical solutions for boundary layer flows. To this day, Blasius and Falkner-Skan flows have been extensively studied in relation to boundary layers. [3] All these solutions describe moving flows over a stationary boundary. Only Sakiadis first examined the boundary layer flow over a moving boundary.

Sakiadis considered that a continuous boundary layer exists on a flat plate where a stationary fluid experiences a large shear when it encounters a moving boundary and forms a boundary layer; the Sakiadis boundary layer, a vital element in fluid dynamics, has attracted more attention due to its unique properties and wide applications. This flow regime originates from

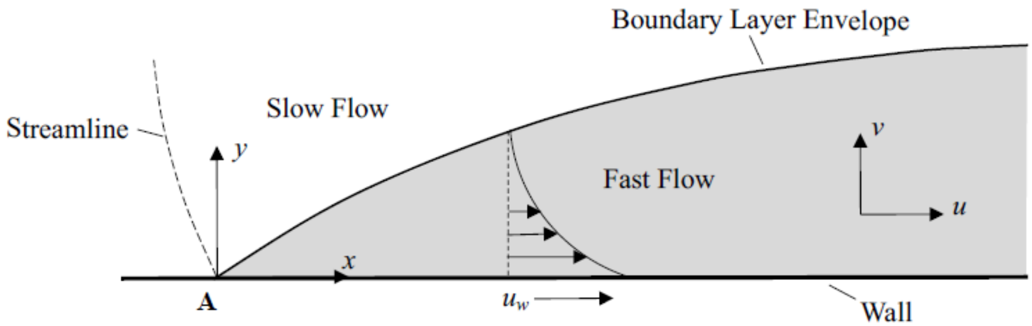

Figure 1: Configuration of the Sakiadis boundary layer problem with X-Y coordinate system.

the interaction between a solid surface and a fluid, exhibiting unique characteristics that make it an attractive subject for research. In this paper, we present a comprehensive numerical analysis of Sakiadis flow using the Python programming language. Using Python scientific libraries such as SciPy and Matplotlib, we have solved the differential equations that describe Sakiadis flow and visualized the results.

In Figure 1, a flat wall moving at speed U_W in an incompressible Newtonian fluid with density ρ and viscosity μ is shown. This wall can be considered as a moving boundary that causes the fluid to flow around it. In this paper, we examine the results related to velocity profile, displacement thickness, vertical fluid velocity, shear stress, and momentum thickness. In this research, we have set the fluid velocity at the wall to $1m/s$ and the wall length similarly to 1 meter. These results show how Sakiadis flow is affected by various variables and how these effects can be used in the design and optimization of mechanical systems.

2 Governing Equations

In this section, we examine the governing equations for Sakiadis flow. These equations include the equations of motion (Navier-Stokes equations). In Sakiadis flow, like Blasius flow, we have zero pressure gradient in the flow over the boundary. The boundary layer equations are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

Equation 1 is the continuity equation that results from the principle of momentum conservation for incompressible fluids.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

Equations 1 and 2 are from the Navier-Stokes equations for incompressible fluids as presented by Blasius. The boundary conditions for these equations are as follows:

@y=0 : u=U$_w$, $\quad v = 0$,
$@y \to \infty : u \to 0$,
$@x = 0 : u = 0$.

In this problem, we assume that the surface length is 1 meter and the wall velocity U_w is 1 m/s. Additionally, we assume that the fluid at 15 degrees Celsius has a viscosity of $\nu = 1.5 \times 10^{-5}\, m^2/s$ and a density of $\rho = 1.225\, kg/m^3$, which both affect fluid flow.

Using the definition of the stream function ψ as $u = \frac{\partial\psi}{\partial y}$ and $v = -\frac{\partial\psi}{\partial x}$ and defining the similarity variable η as $\eta = y\sqrt{\frac{U_w}{\nu x}}$ and $\psi(x,y)/(U_w\delta(x)) = f(\eta)$, the above equations are transformed as follows:

$$f''' + 0.5ff'' = 0, \tag{3}$$

The boundary conditions for the Sakiadis equation are as follows:

$@\eta = 0 : f = 0, \quad f' = 1,$
$@\eta \to \infty : f' \to 0.$

The velocity profile and transverse velocity are calculated using the following formulas:

u/U$_w$ $= f'$,
$v = \frac{1}{2}\sqrt{\frac{\nu U_w}{x}}(\eta f' - f).$

Using equations 2 and 2, we can arrive at the following results: $\frac{\partial u}{\partial x=-\frac{U_w\eta}{x}f'', \frac{\partial v}{\partial y}=\frac{U_w}{2}\frac{\eta}{x}f''.}$

For Sakiadis, δ_{99} is defined as follows: $u(y = \delta_{99}) = 0.01U_w$. The displacement thickness, momentum thickness, and wall shear stress are defined as follows:

$\delta^* = \int_{y=0}^{\infty}\left(\frac{u}{U_w}\right)dy = f(\eta_1)\sqrt{\frac{\nu x}{U_w}},$
$\theta = \int_{y=0}^{\infty}\left(\frac{u}{U_w}\right)^2 dy = \sqrt{\frac{\nu x}{U_w}}\int_{\eta=0}^{\eta=\infty} f'^2 d\eta,$

$\tau_{xy} = \eta \frac{\partial u}{\partial y} = \eta U_w \sqrt{\frac{U_w}{\nu x}} f''(0)$

The exact solution for displacement thickness and momentum thickness was provided by Mr. Frondelius et al. [4], and the shear stress is obtained by calculation in the written code where the value of $f''(0)$ is -0.4437.

$\delta^* = 1.62\sqrt{\frac{\nu x}{U_w}}$,

$\theta = 0.887\sqrt{\frac{\nu x}{U_w}}$,

$\tau_w = -0.4437\rho U_w^2 \sqrt{\frac{\nu}{U_w x}}$.

3 Method and Solution Approach

First, we import the necessary libraries for running the code. These libraries include numpy for mathematical computation operations, matplotlib.pyplot for plotting graphs, and scipy.integrate.solve_bvp for solving the boundary value equation.

Then, we define the sakiadis_flow function that represents the third-degree differential equation. This function has three inputs: eta which is the similarity variable and f which is a three-element array whose elements are f, f', and f'', respectively. The output of this function is a three-element array whose elements are $\frac{df}{d\eta}$, $\frac{df'}{d\eta}$, and $\frac{df''}{d\eta}$, respectively.

We define the bc function that represents the boundary conditions. This function has two inputs: ya and yb which are the values of f, f', and f'' at $\eta = 0$ and $\eta \to \infty$, respectively. The output of this function is a three-element array whose elements are the boundary conditions for f, f', and f'', respectively.

3.1 Solving the Equation

To solve the equation, we first define an array for η that is symmetrical from 0 to 10 with 100 points. Then we define an initial guess for f, f', and f'' as zero arrays. Using this initial guess, we call the solve_bvp function to solve the equation.

3.2 Calculating and Displaying Results

We then use the results from solving the equation to calculate and display the velocity profile, displacement thickness, and momentum thickness. These results allow us to understand how velocity and its changes vary along the boundary layer.

4 Discussion and Conclusion

In this section, we examine the results obtained from solving the governing equations for Sakiadis flow. These results include velocity profile, displacement thickness, momentum thickness, and shear stress.

4.1 Velocity Profile

The velocity profile, calculated using equation 3, shows how fluid velocity changes along the boundary layer. Near the wall (which is equivalent to $\eta = 0$), the fluid velocity is equal to the wall velocity. This means that the fluid at the boundary moves at the same speed as the wall, satisfying the no-slip condition.

As η increases toward infinity, the fluid velocity approaches zero. This means that at greater distances from the wall, the fluid velocity decreases until it eventually reaches zero. This behavior shows how fluid velocity in the boundary layer decreases with increasing distance from the wall.

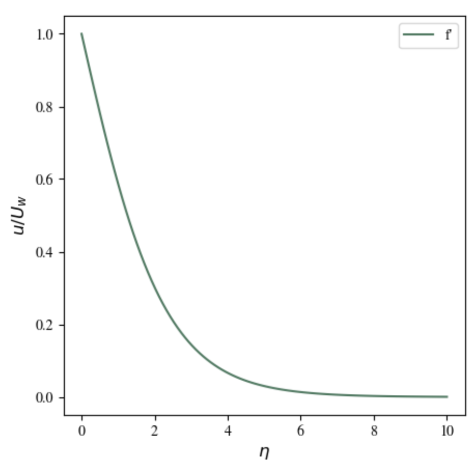

Figure 2: Velocity profile for Sakiadis flow

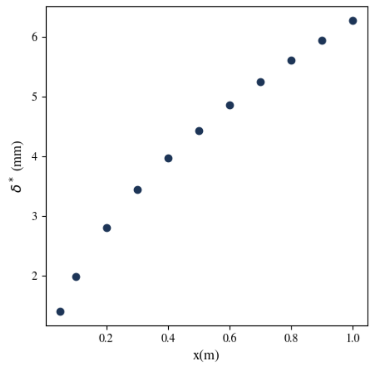

Figure 3: Displacement thickness versus x

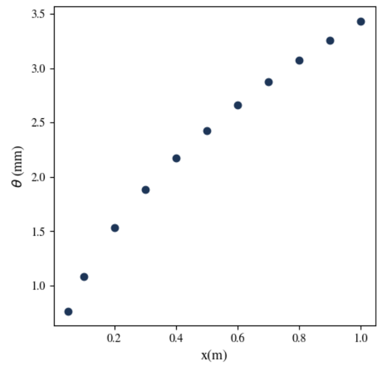

Figure 4: Momentum thickness versus x

As you can see in Figure 6, the velocity profile shows that fluid velocity gradually decreases with increasing η. This result matches the boundary layer flow result presented by Sakiadis[1].

Finally, the velocity profile is a powerful tool for understanding the behavior of fluids in the boundary layer. Using the velocity profile, we can understand how fluid velocity changes with changing distance from the wall, and how boundary layer effects diminish with increasing distance from the wall.

4.2 Displacement Thickness and Momentum Thickness

Displacement thickness and momentum thickness, calculated using formulas 2 and 2, show how boundary layer thickness changes with increasing distance from the leading edge.

Displacement thickness, δ^*, is a measure of the "width" of the boundary layer. This value indicates the distance from the wall at which the fluid velocity reaches 99% of the infinite velocity (or velocity outside the boundary layer). As you can see in Figure 3, with increasing x, i.e., with increasing distance from the leading edge, the displacement thickness increases. This shows that the boundary layer expands with increasing distance from the leading edge.

Momentum thickness, θ, is a measure of velocity distribution in the boundary layer. This value is equal to the vertical distance from the wall that if all fluid velocity were concentrated at that distance, it would maintain the same amount of momentum. As you can see in Figure 4, like displacement thickness, momentum thickness also increases with increasing x.

Both of these values, displacement thickness and momentum thickness, are important tools for describing and analyzing boundary layer flow. Using these values, we can understand how the boundary layer changes with increasing distance from the leading edge.

4.3 Shear Stress

Shear stress, calculated using formula 2, shows how wall shear stress changes with increasing distance from the leading edge.

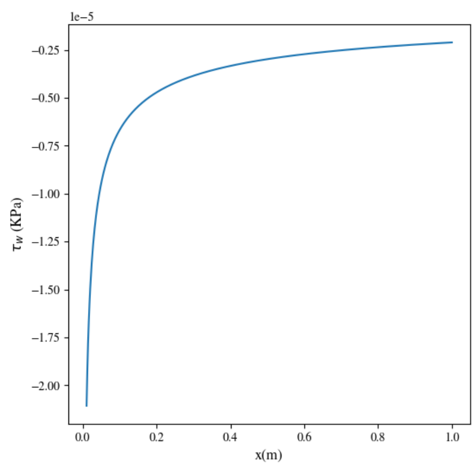

Figure 5: Shear stress versus x

Shear stress is a measure of boundary layer effects on the wall. This value indicates the shear force exerted on the wall by the fluid. As you can see in Figure 5, with increasing x, i.e., with increasing distance from the leading edge, the shear stress decreases. This shows that boundary layer effects diminish with increasing distance from the leading edge.

4.4 Velocity Profile

As mentioned, the velocity profile shows how fluid velocity changes along the boundary layer. Near the wall (which is equivalent to $\eta = 0$), the fluid velocity equals the wall velocity. As η increases toward infinity, the fluid velocity approaches zero.

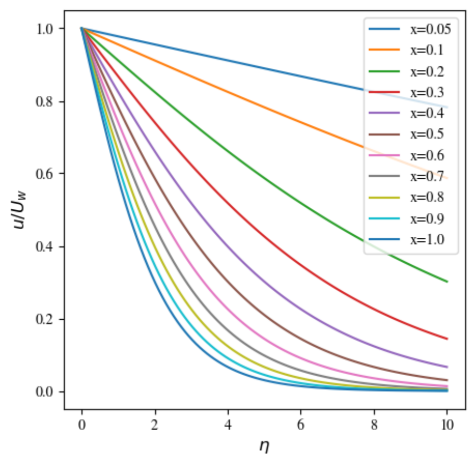

Figure 6: Velocity profile for different x

Figure 6 shows how the velocity profile changes for each section of x. Each colored line corresponds to a specific value of x and shows how the velocity profile changes with changing x.

4.5 Transverse Velocity

Transverse velocity, which is in the direction perpendicular to the main flow and shown in Figure 7, shows how the fluid moves in the vertical direction. In Sakiadis flow, the transverse velocity is usually negligible but can have a significant effect under certain conditions. For example, in cases where the main flow has severe variations or when the pressure gradient in the transverse direction is significant, the transverse velocity can have a significant effect on the overall behavior of the flow.

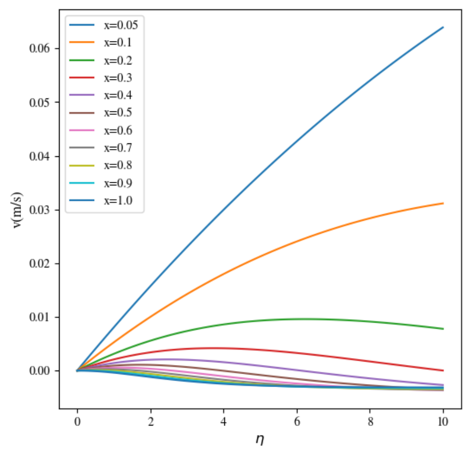

Figure 7: Transverse velocity versus x

4.6 Conclusion

In this study, we examined the Sakiadis boundary layer flow. Using the governing equations for this flow and appropriate boundary conditions, we were able to calculate the velocity profile, transverse velocity, displacement thickness, mass thickness, and shear stress.

The results showed that the numerical method used to solve the governing equations for Sakiadis flow is capable of producing accurate and valid results and also how these variables change with increasing distance from the leading edge. Specifically, we saw how fluid velocity in the boundary layer decreases with increasing distance from the wall, and how boundary layer thickness increases with increasing distance from the leading edge.

References

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