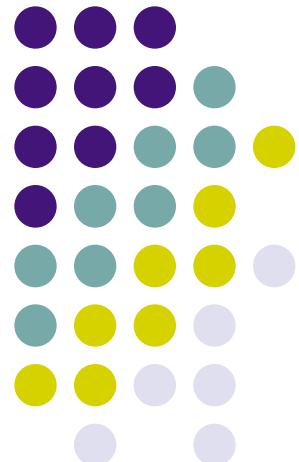
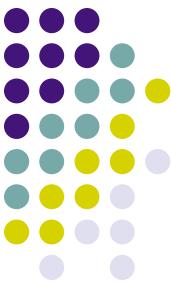


数字图像处理

第五讲
频域图像增强 (Part I)
傅里叶变换基础





频域vs空域

- 共性：都为了图像增强
- 两大类方法：
 - **空间域方法**：图像平面本身，对图像的像素直接处理；
 - **频域方法**：修改图像的频谱例如傅里叶变换为基础；

离散

连续



提纲

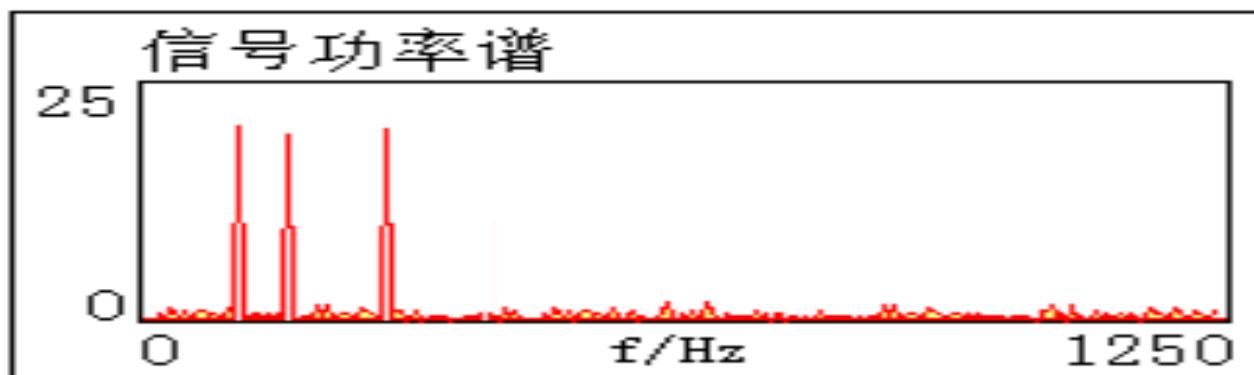
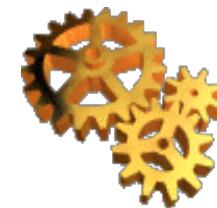
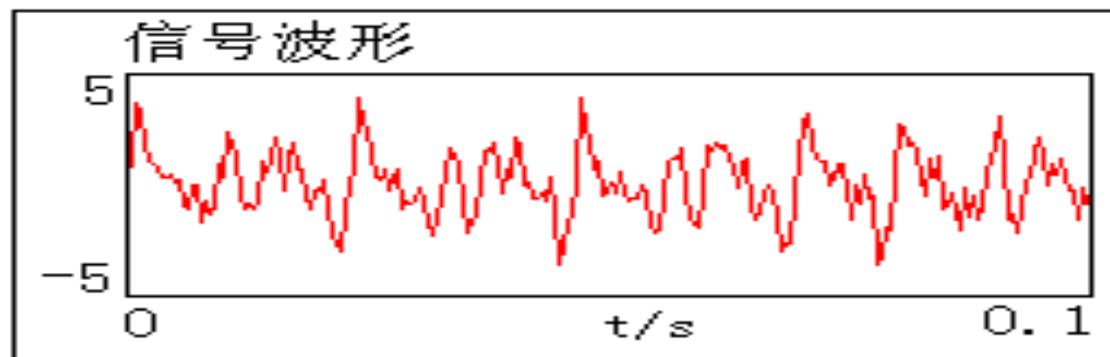
- 一维连续傅里叶变换
- 一维离散傅里叶变换 (及其计算)
- 二维连续傅里叶变换
- 二维离散傅里叶变换
- 傅里叶变换的幅值谱
- 傅里叶变换的性质

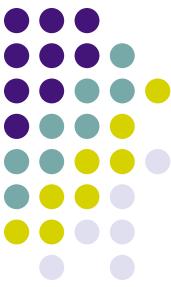


图像是连续信号的量化采样

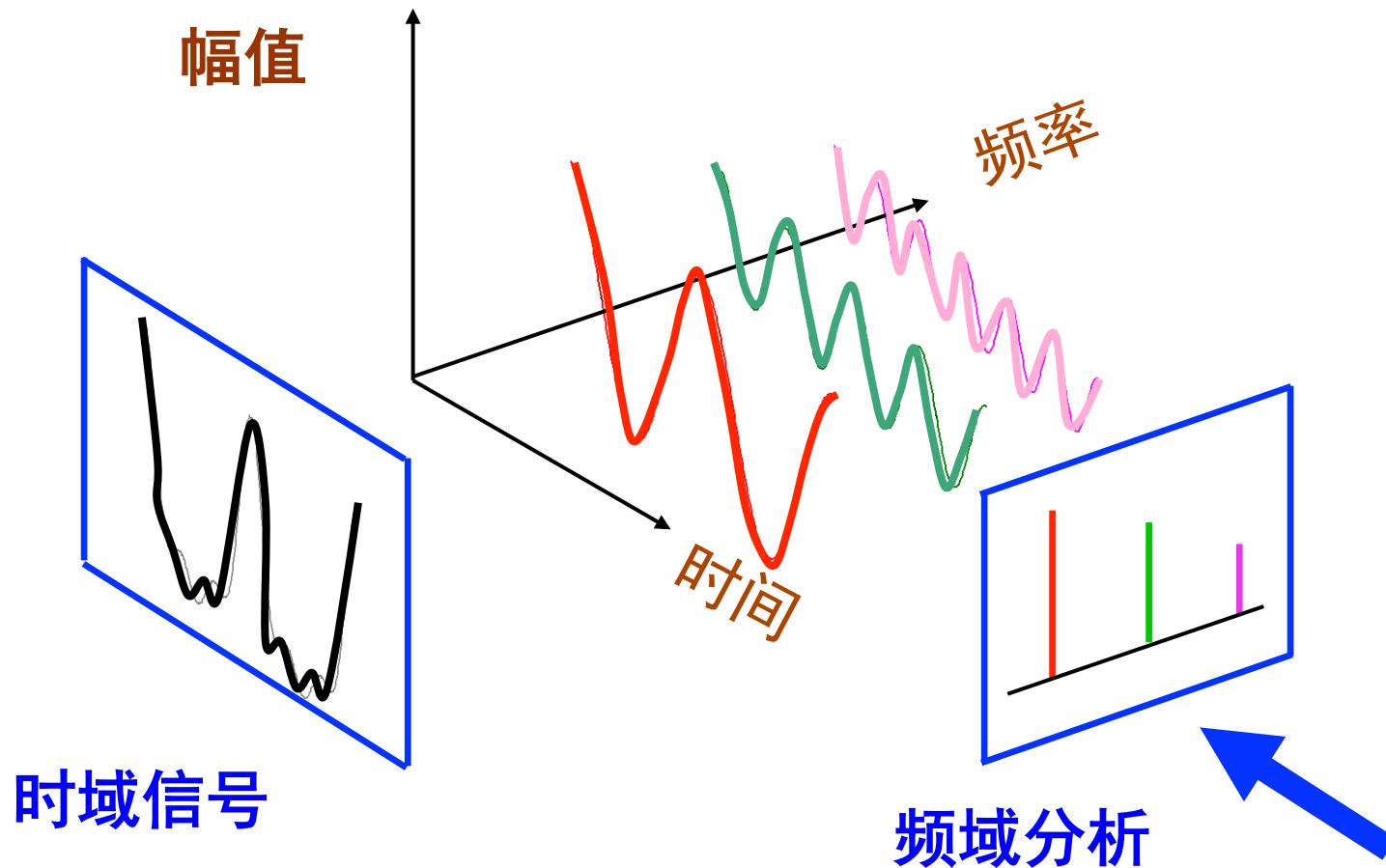
信号通常包括丰富的频域信息

图例：受噪声干扰的多频率成分信号





信号频谱代表了信号在不同频率分量成分的大小，能够提供比时域信号波形更直观，丰富的信息。





怎么把信号投影到频域空间？

- 傅里叶，法国数学家、物理学家（1768-1830）
- 《热分析理论》



傅里叶变换

任何周期函数都可以表示为不同频率的正弦函数和/余弦函数之和



傅里叶变换的意义

- 解决了频域信息如何表示这个基本问题
- 带来了“信号处理领域”的一场革命



一维连续傅里叶及其反变换

- 连续函数 $f(x)$ 的傅立叶变换 $F(u)$:

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx$$

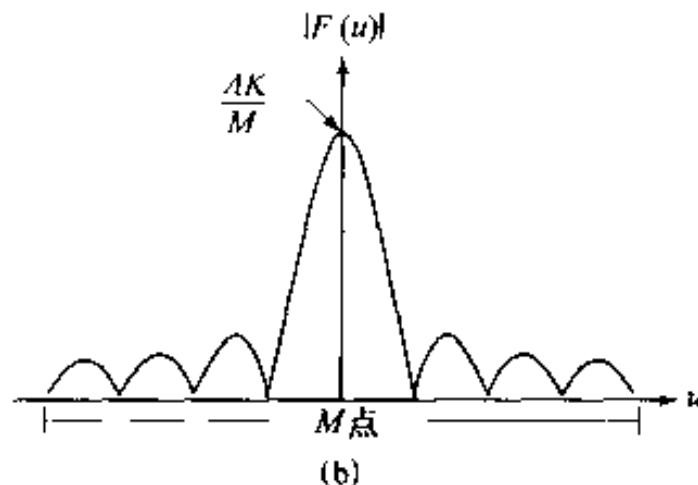
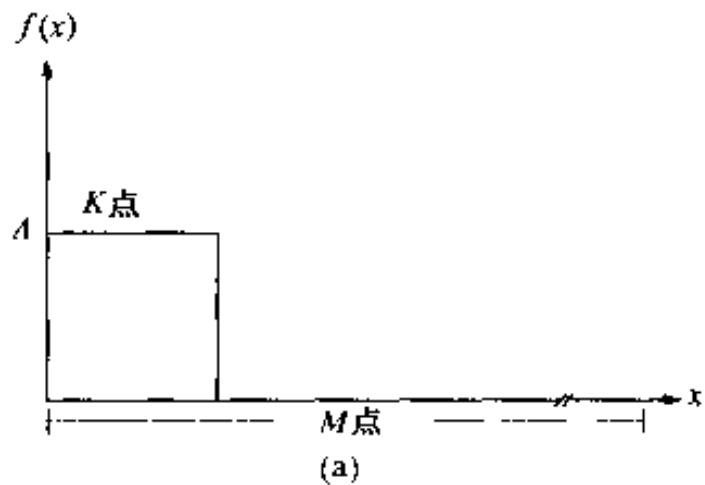
- 傅立叶变换 $F(u)$ 的反变换:

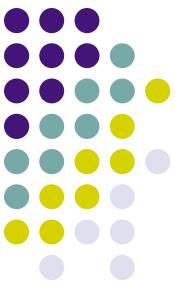
$$f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du$$



一维连续傅立叶变换

- 矩形函数 $f(x)$
- 傅里叶谱函数 $|F(u)|$





一维连续傅立叶变换

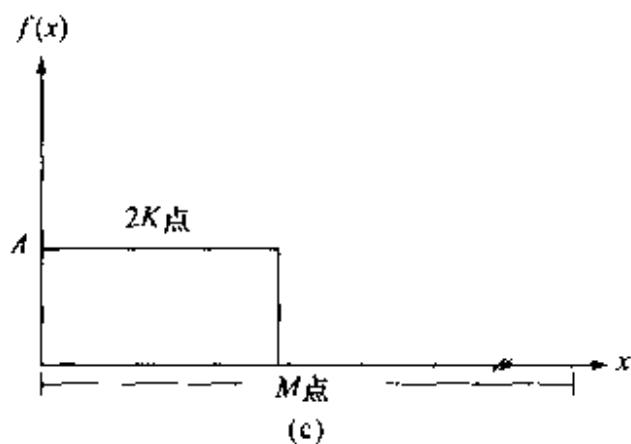
- 计算公式

$$\begin{aligned} \text{解: } F(u) &= \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx \\ &= \int_0^X A e^{-j2\pi ux} dx = \frac{-A}{j2\pi u} [e^{-j2\pi ux}]_0^X \\ &= \frac{-A}{j2\pi u} [e^{-j2\pi uX} - 1] = \frac{-A}{j2\pi u} [e^{-j\pi uX} - e^{j\pi uX}] e^{-j\pi uX} \\ &= \frac{A}{\pi u} \sin(\pi uX) e^{-j\pi uX} \\ |F(u)| &= AX \left| \frac{\sin(\pi uX)}{\pi uX} \right| \end{aligned}$$

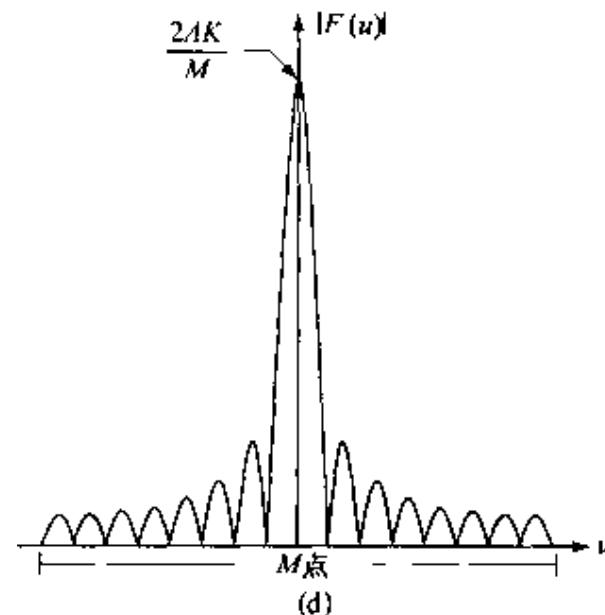


一维连续傅立叶变换

- 矩形函数 $f(x)$



- 傅里叶谱函数 $F(u)$



信号点数增加，傅里叶谱变窄



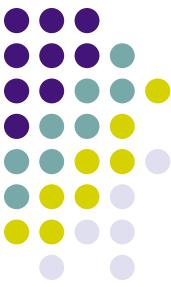
一维离散傅里叶变换DFT及其反变换

- 离散函数 $f(x)$ (其中 $x, u=0,1,2,\dots,N-1$)的傅立叶变换:

$$F(u) = \sum_{x=0}^{N-1} f(x) e^{-j2\pi ux/N}$$

- $F(u)$ 的反变换:

$$f(x) = \frac{1}{N} \sum_{u=0}^{N-1} F(u) e^{j2\pi ux/N}$$



一维离散傅里叶变换DFT及其反变换

- 离散傅里叶变换及其反变换总存在。
- 用欧拉公式得 $e^{j\theta} = \cos\theta + j\sin\theta$

$$F(u) = \sum_{x=0}^{N-1} f(x) [\cos 2\pi ux / N - j \sin 2\pi ux / N]$$

u 值决定了变换的频率成份

$F(u)$ 覆盖的域 (u 值) 称为频率域

其中每一项都被称为FT 的频率分量



一些常用的傅立叶变换函数

函数	$f(t)$	$F(u)$
高斯	$e^{-\pi t^2}$	$e^{-\pi u^2}$
矩形脉冲	$\Pi(t)$	$\sin(\pi u)/\pi u$
三角脉冲	$\Lambda(t)$	$\sin^2(\pi u)/(\pi u)^2$
冲激	$\delta(t)$	1
单位阶跃	$u(t)$	$[\delta(u) - j/\pi u]/2$
余弦	$\cos(2\pi ft)$	$[\delta(u+f) + \delta(u-f)]/2$
正弦	$\sin(2\pi ft)$	$j[\delta(u+f) - \delta(u-f)]/2$
复指数	$e^{2\pi ft}$	$\delta(u-f)$



二维连续傅里叶变换

1) 定义

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx \quad F(u, v) = \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(ux+vy)} dx dy$$

2) 傅里叶反变换

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du \quad f(x, y) = \int_{-\infty}^{\infty} F(u, v) e^{j2\pi(ux+vy)} du dv$$



二维连续傅里叶变换

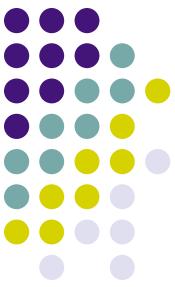
3) 傅里叶变换特征参数

$$F(u, v) = R(u, v) + jI(u, v)$$

频谱/幅度谱/模 $|F(u, v)| = \sqrt{R^2(u, v) + I^2(u, v)}$

能量谱/功率谱 $P(u, v) = |F(u, v)|^2 = R^2(u, v) + I^2(u, v)$

相位谱 $\phi(u, v) = \arctan \frac{I(u, v)}{R(u, v)}$



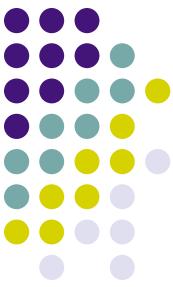
二维DFT傅里叶变换

- 一个图像尺寸为 $M \times N$ 的函数 $f(x,y)$ 的离散傅立叶变换 $F(u,v)$:

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$$

- $F(u, v)$ 的反变换:

$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}$$



F(0,0)

- $(u,v)=(0,0)$ 位置的傅里叶变换值为

$$F(0,0) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) = \bar{f}(x, y)$$

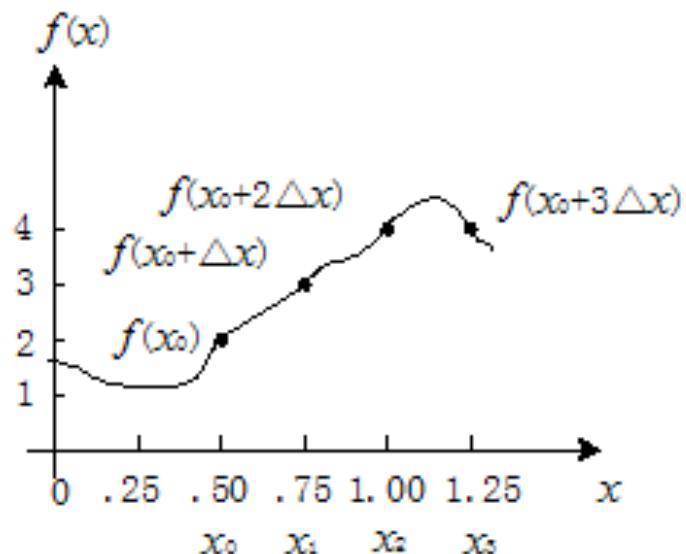
即 $f(x,y)$ 的均值，原点 $(0,0)$ 的傅里叶变换是图像的平均灰度。

$F(0,0)$ 称为频率谱的直流分量(系数)，
其它 $F(u,v)$ 值称为交流分量(交流系数)。

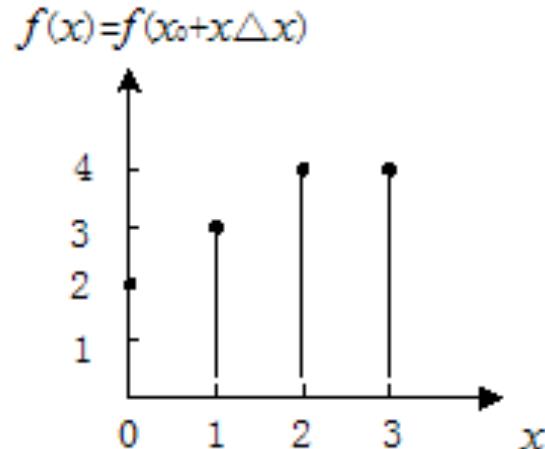


例 设一函数如图 (a) 所示, 如果将此函数在自变量 $x_0 = 0.5, x_1 = 0.75, x_2 = 1.00, x_3 = 1.25$ 取样

并重新定义为图 (b) 离散函数, 求其傅里叶变换。



(a)



(b)



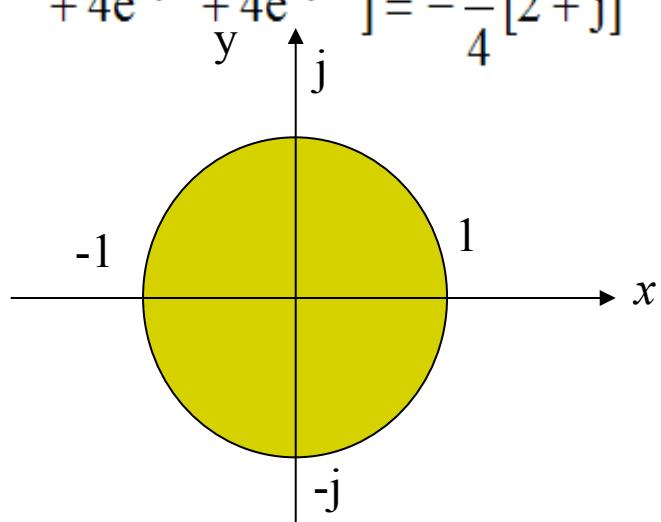
$$\begin{aligned}
 F(0) &= \frac{1}{4} \sum_{x=0}^3 f(x) \exp\{0\} \\
 &= (1/4)[f(0) + f(1) + f(2) + f(3)] = (1/4)[2 + 3 + 4 + 4] = 3.25
 \end{aligned}$$

$$F(1) = \frac{1}{4} \sum_{x=0}^3 f(x) \exp\{-j2\pi x/4\} = \frac{1}{4} [2e^0 + 3e^{-j\pi/2} + 4e^{-j\pi} + 4e^{-j3\pi/2}] = \frac{1}{4} [-2 + j]$$

$$F(2) = \frac{1}{4} \sum_{x=0}^3 f(x) \exp\{-j4\pi x/N\} = \frac{1}{4} [2e^0 + 3e^{-j\pi} + 4e^{-j2\pi} + 4e^{-j3\pi}] = -\frac{1}{4}[1 + j0]$$

$$F(3) = \frac{1}{4} \sum_{x=0}^3 f(x) \exp\{-j6\pi x/4\} = \frac{1}{4} [2e^0 + 3e^{-j3\pi/2} + 4e^{-j3\pi} + 4e^{-j9\pi/2}] = -\frac{1}{4}[2 + j]$$

$$F(u) = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix}$$

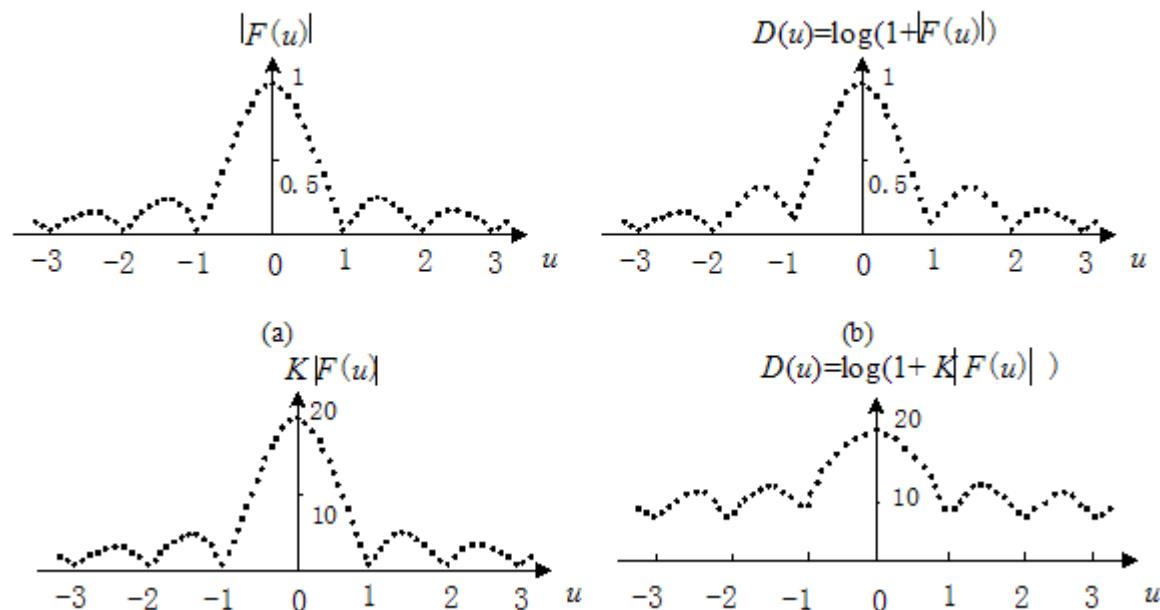




可视化小技巧：对数化

许多图像的傅里叶频谱的幅度随着频率的增大而迅速减小，这使得在显示与观察一副图像的频谱时遇到困难。但以图像的形式显示它们时，其高频项变得越来越不清楚。

解决办法：
对数化



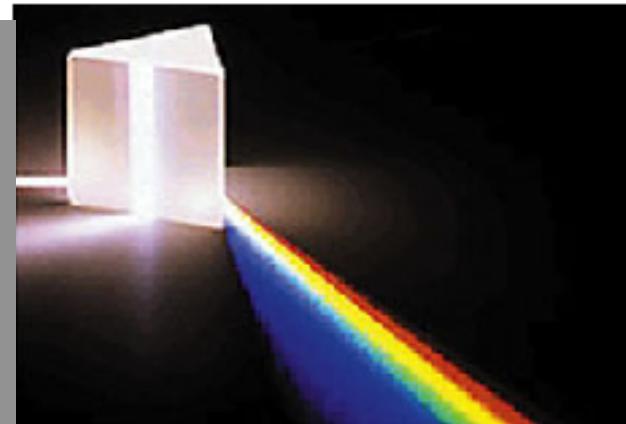


傅里叶变换的直观意义

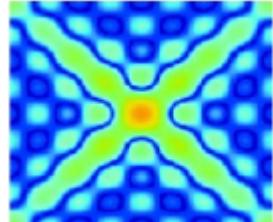
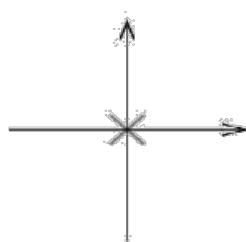
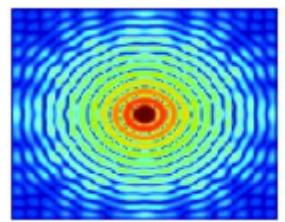
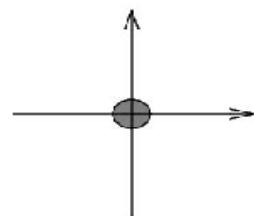
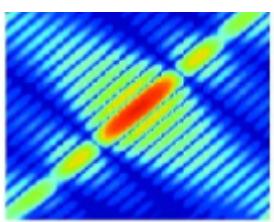
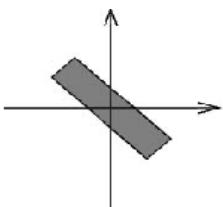
傅里叶变换好比一个玻璃棱镜

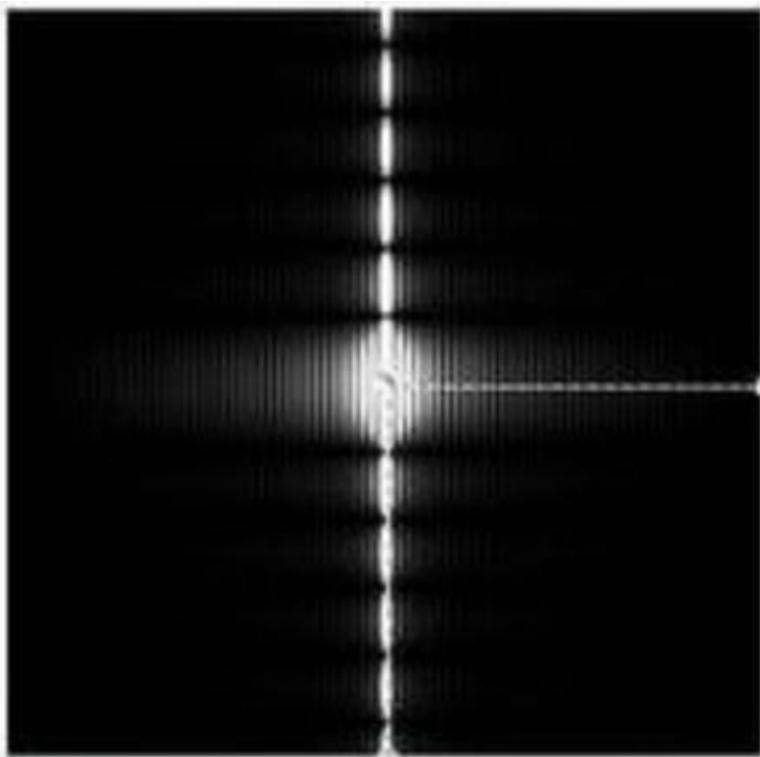
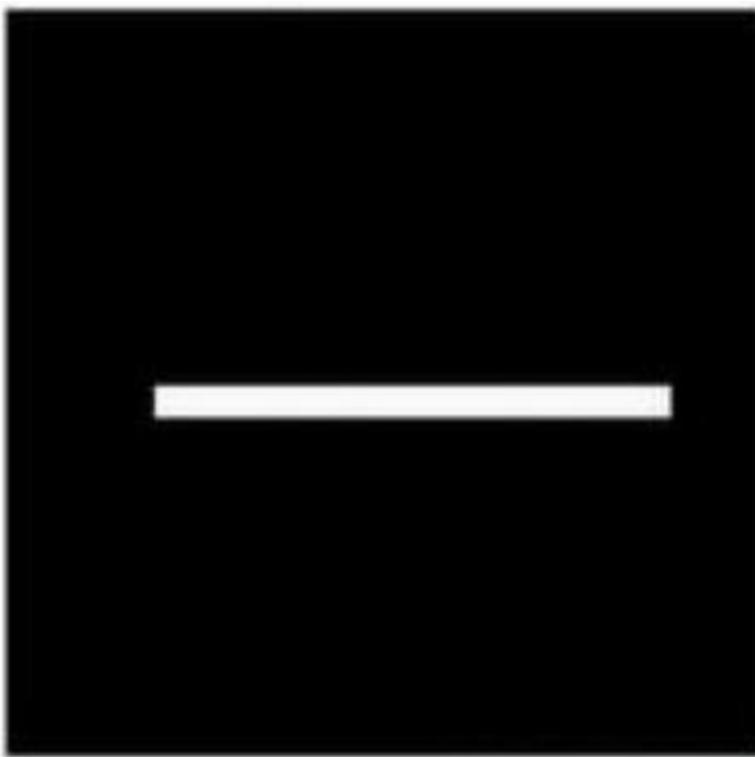
棱镜是可以将光分成不同颜色的物理仪器，每个成分的颜色由波长决定。

傅里叶变换可看做是“数学中的棱镜”，将函数基于频率分成不同的成分。

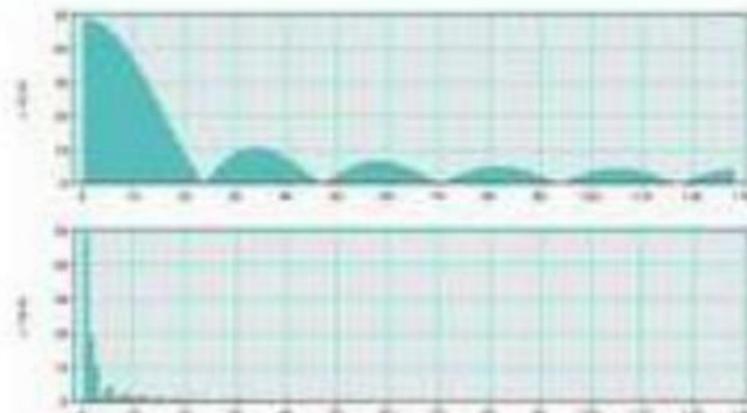


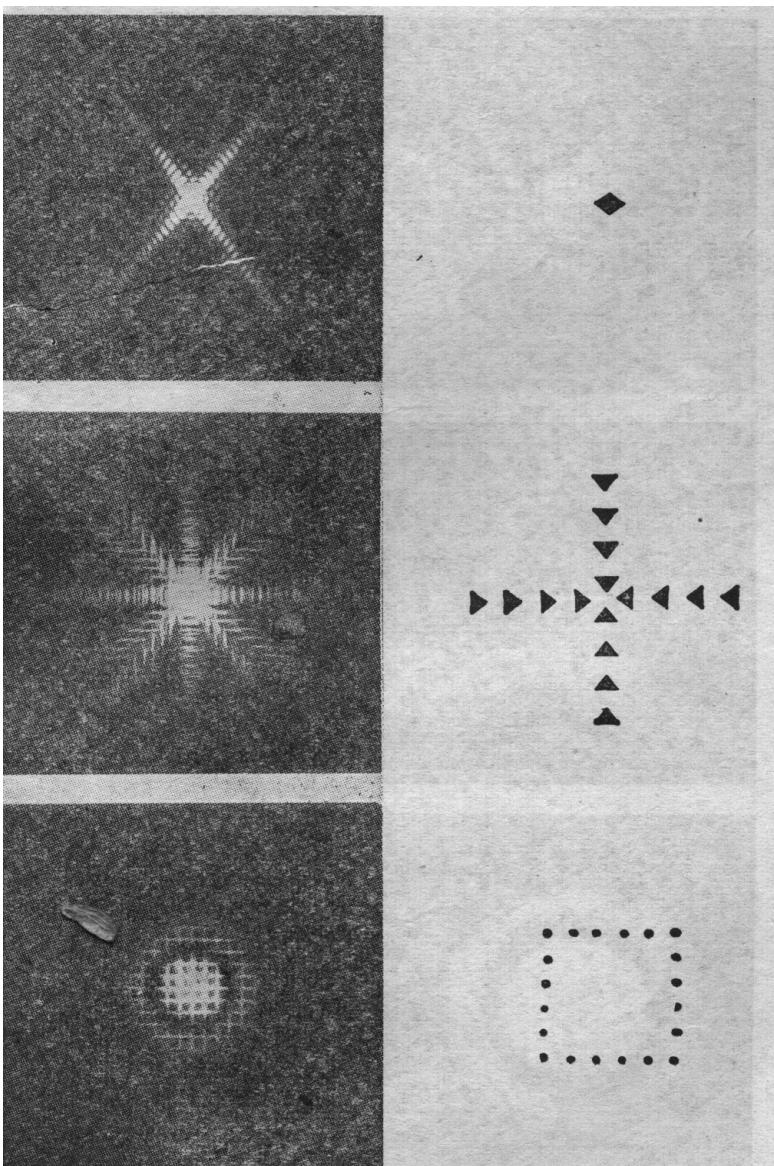
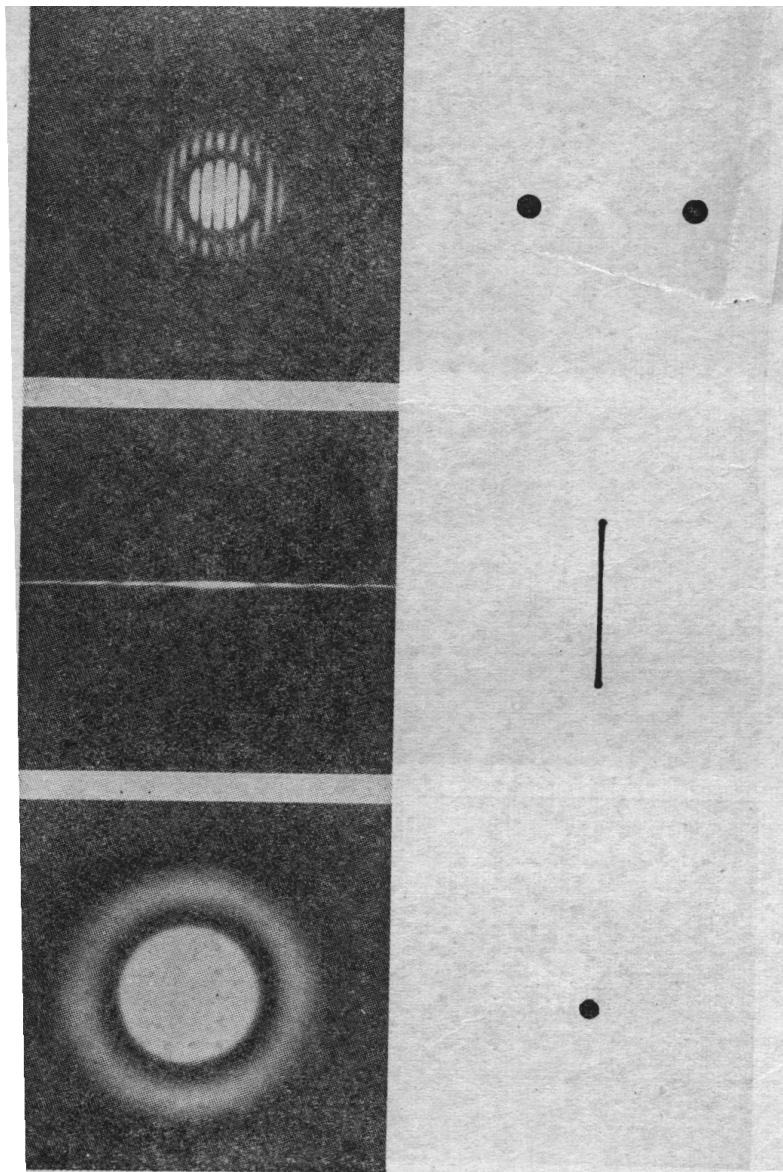
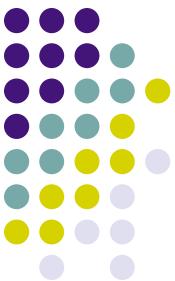
一些图像的傅里叶变换



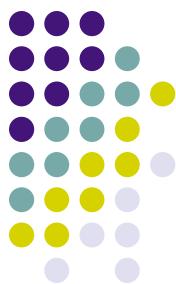


在垂直方向上需要更多的频率分量，所以它的波峰比较宽，而水平方向上的波峰比较窄



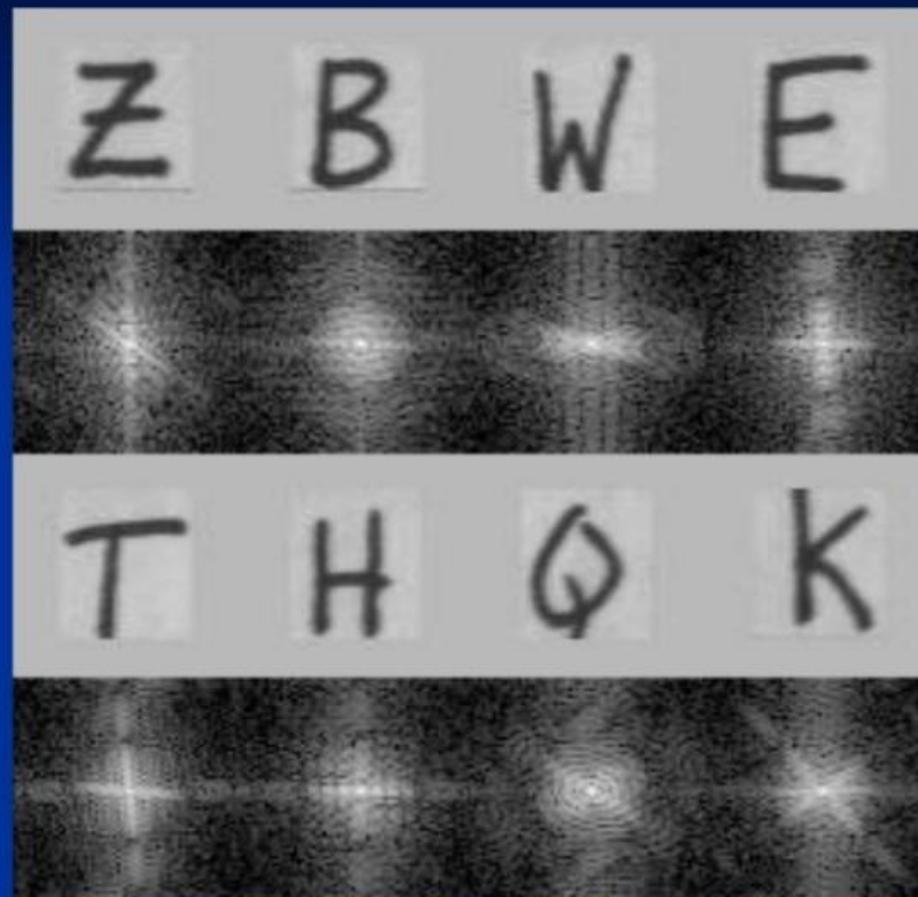


有趣的幅度谱

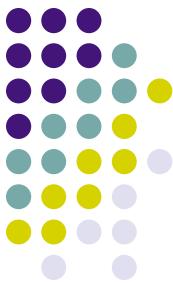


幅度谱和相位谱

- 从幅度谱中我们可以看出明亮线和原始图像中对应的轮廓线是垂直的。如果原始图像中有圆形区域那么幅度谱中也呈圆形分布

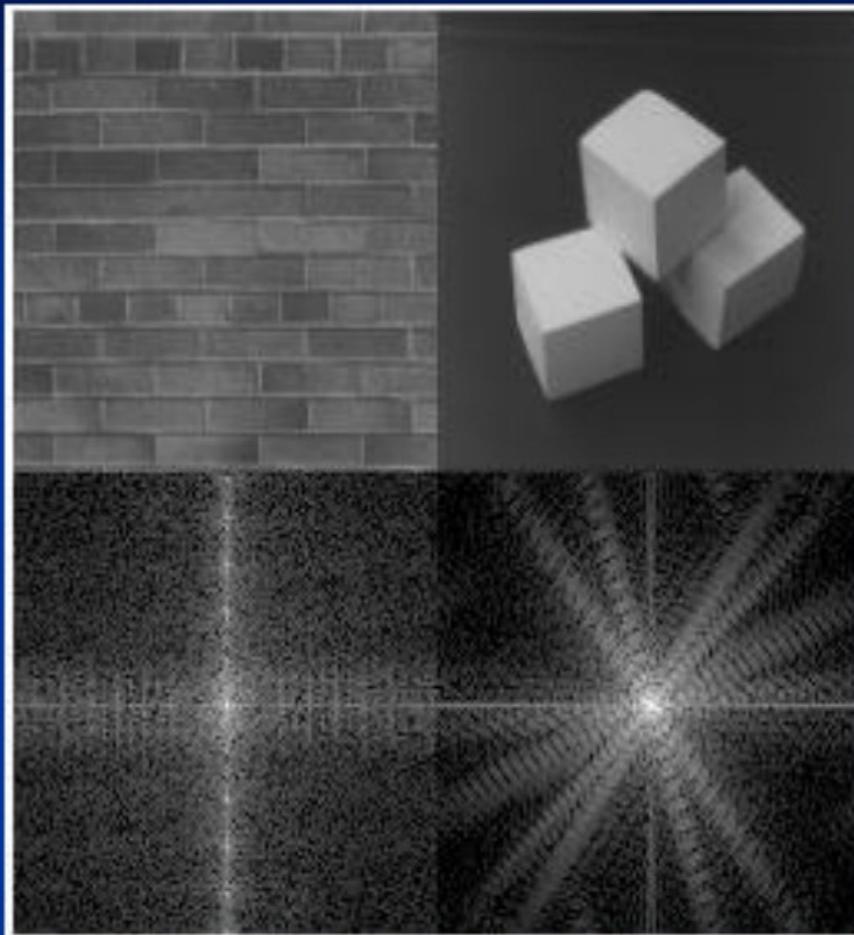


有趣的幅度谱



幅度谱和相位谱

- 从幅度谱中我们可以看出明亮线反映出原始图像的灰度级变化，这正是图像的轮廓边

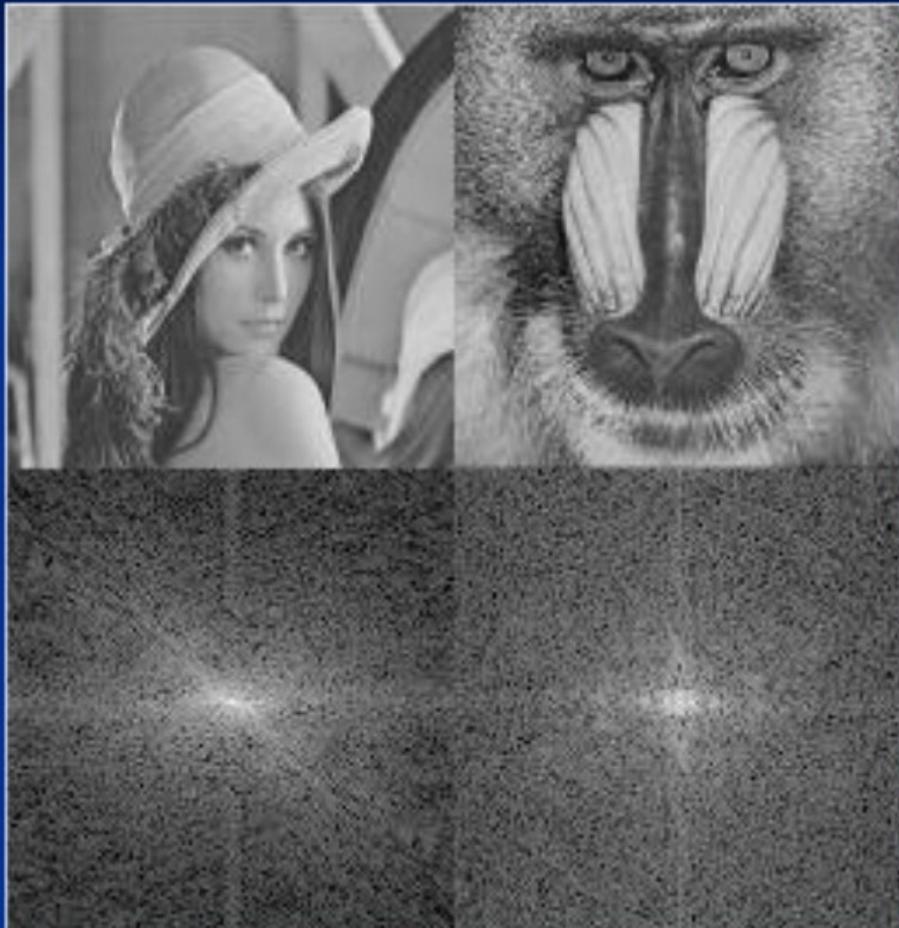


有趣的幅度谱



幅度谱和相位谱

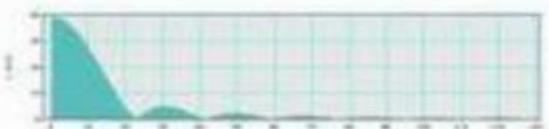
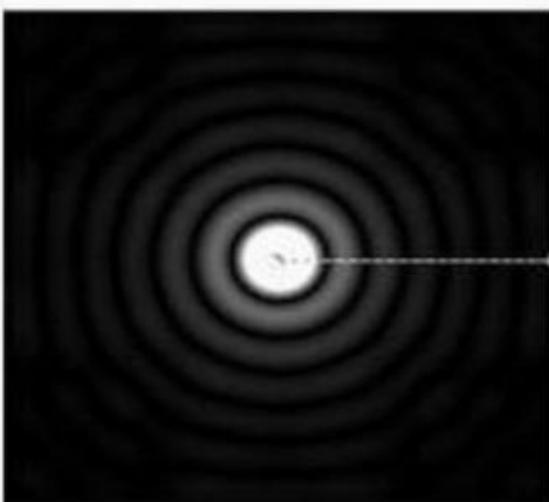
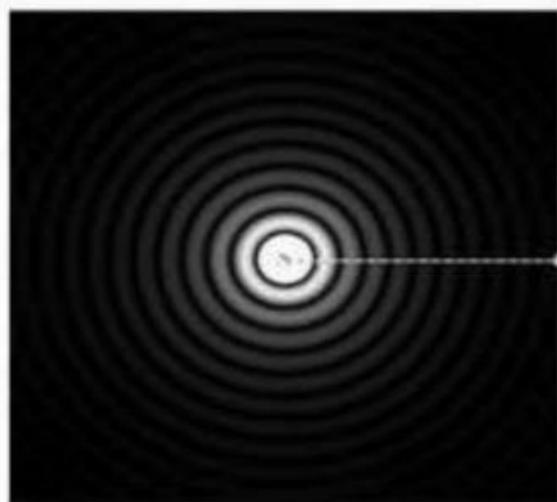
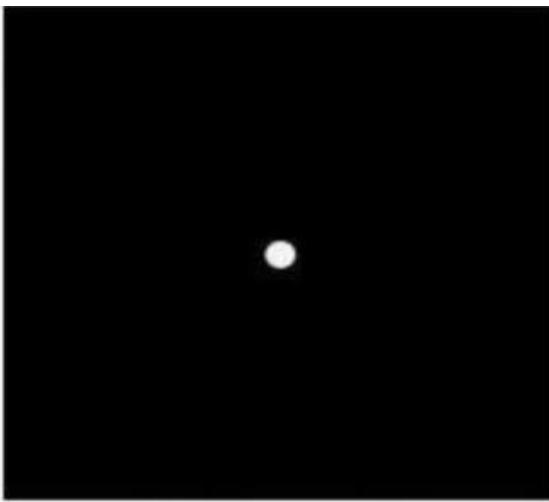
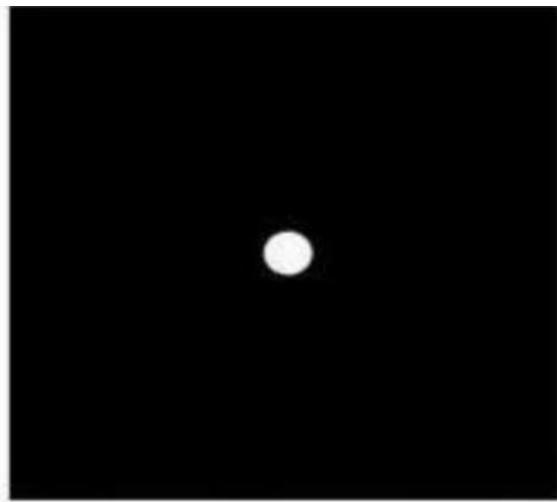
- 这些图像没有特定的结构，左上角到右下角有一条斜线，它可能是由帽子和头发之间的边线产生的
- 两个图像都存在一些小边界





频谱图中暗的点数更多，那么实际图像是比较柔和的，反之，如果频谱图中亮的点数多，那么实际图像一定是尖锐的，边界分明且边界两边像素差异较大的



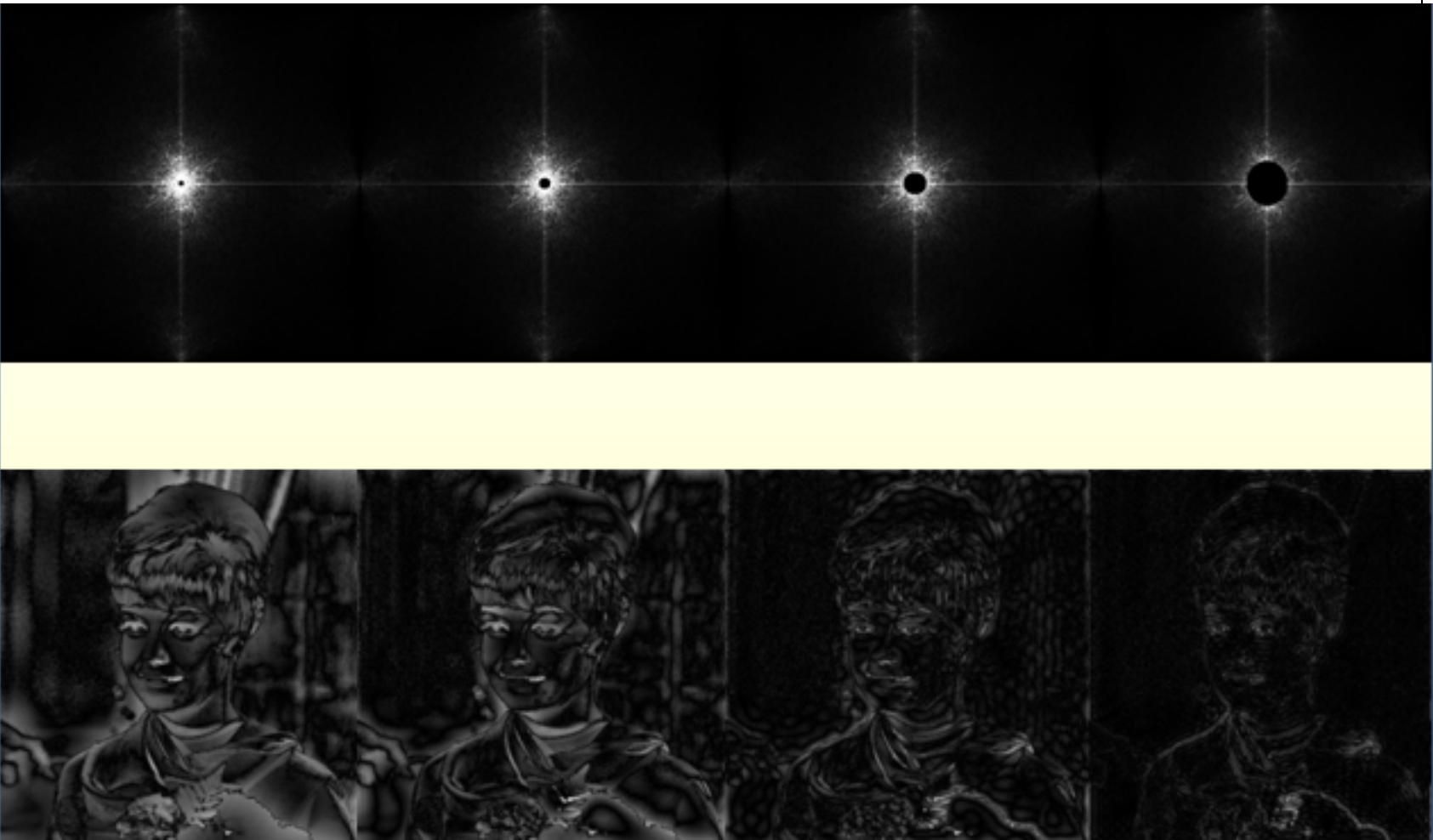
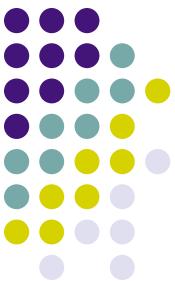


可以看到大圆经傅立叶变换之后，其圆环小；而小圆经傅立叶变换之后，其圆环反而大。

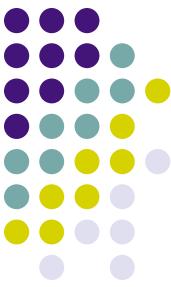
因为越尖锐变换越剧烈的信号总包含着更多的频率成分。



高频部分（距离中心较远区域）越多，图像细节越丰富。反之图像越平滑，显得模糊



高频部分（距离中心较远区域）对应于图像的细节（边缘，线条等）。低频部分（中心部分）对应于图像的整体信息。



傅里叶变换的作用

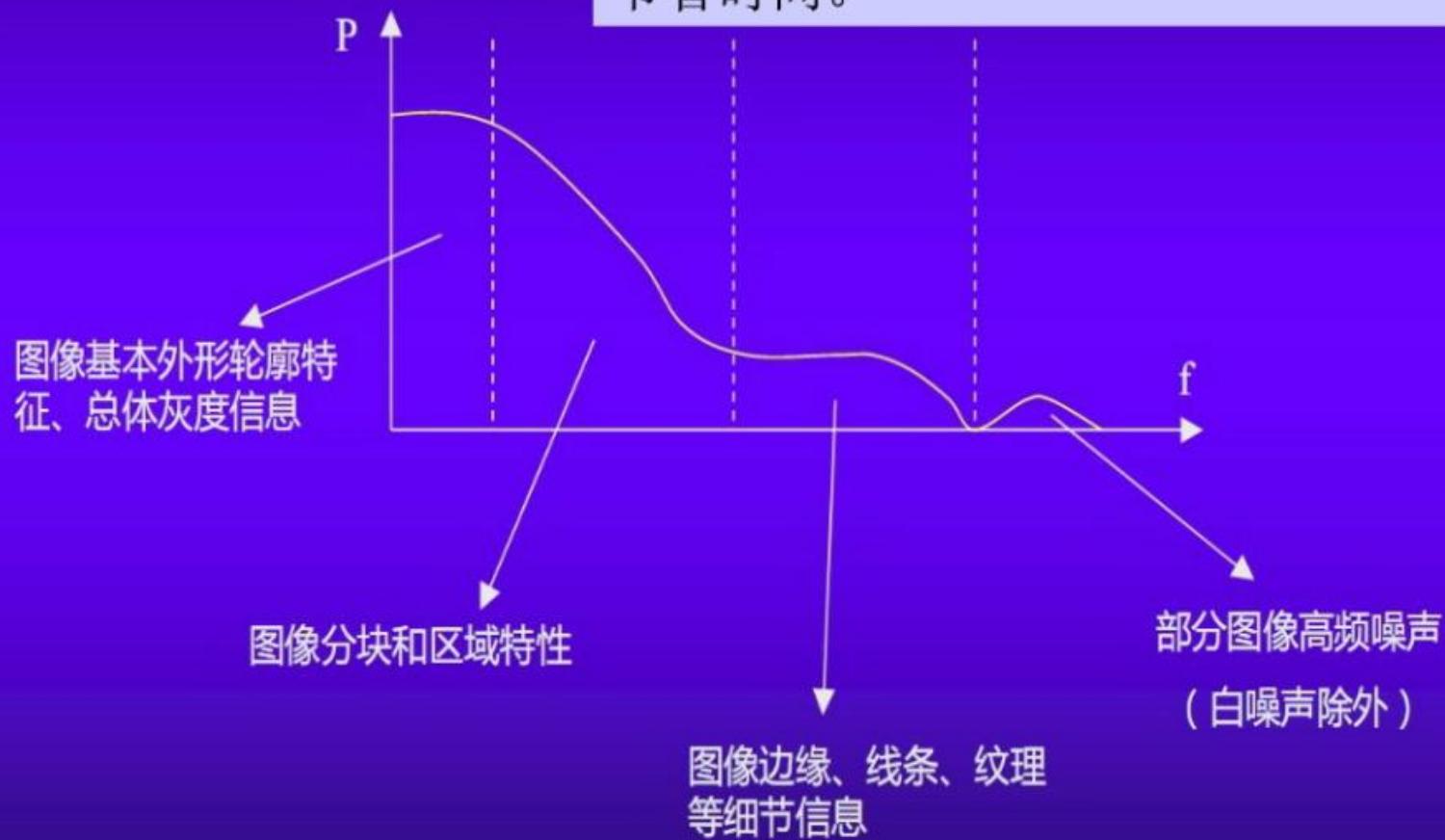
- * 傅里叶变换将信号分成不同频率成份。类似光学中的分色棱镜把白光按波长(频率)分成不同颜色，称数学棱镜。
- * 傅里叶变换的成份：直流分量和交流分量
- * 信号变化的快慢与频率域的频率有关。噪声、边缘、跳跃部分代表图像的高频分量；背景区域和慢变部分代表图像的低频分量

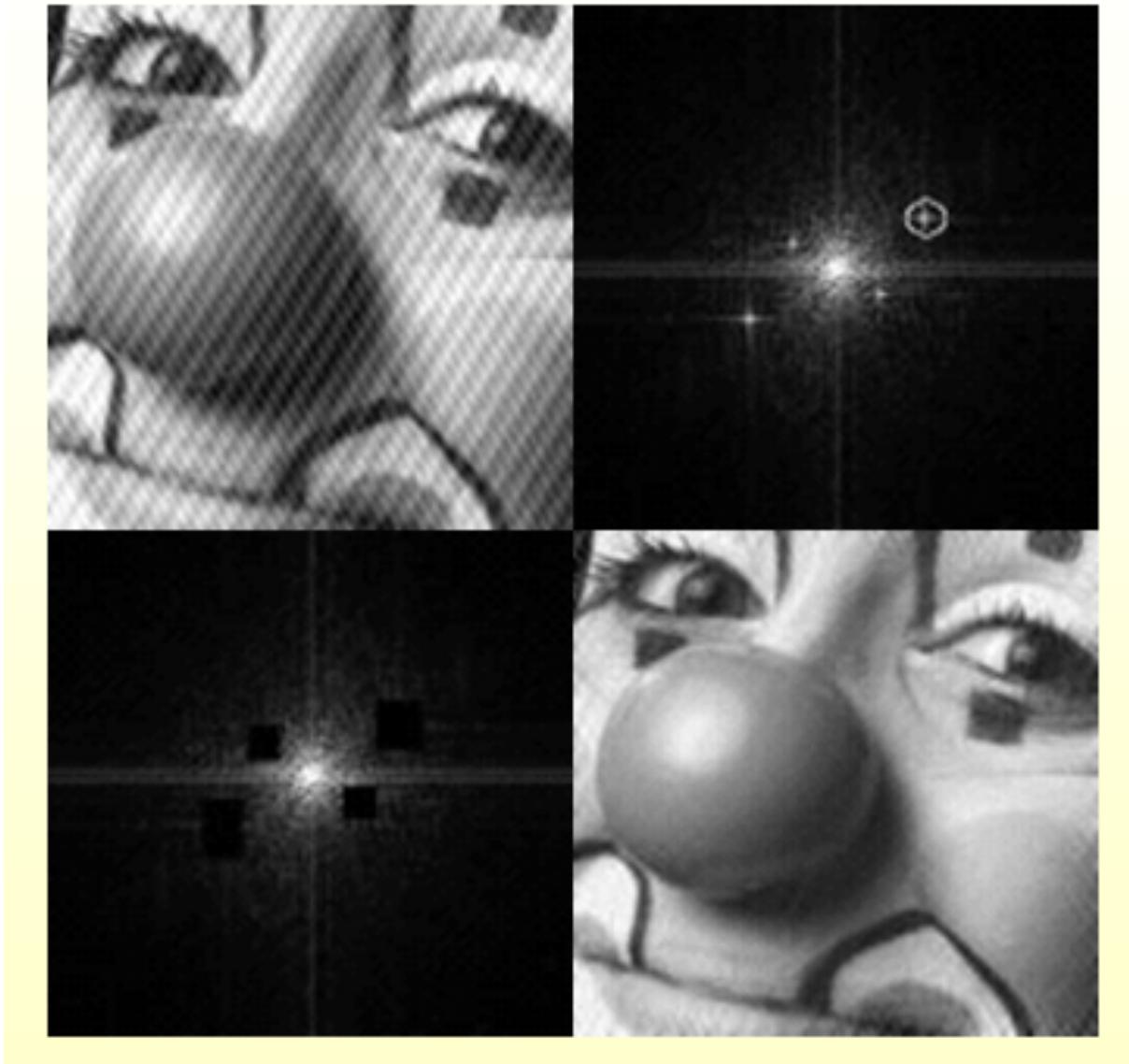


为什么要做图像变换？

图像信号的频域模型

◆ 变换后的图象，大部分**能量**都分布于**低频谱段**，这对以后图象的**压缩、传输**都比较有利。使得运算次数减少，节省时间。





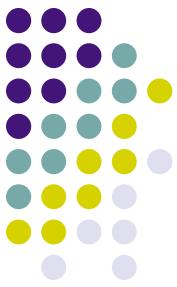
通过去除图像频域谱的噪声点，增强图像



例题：对一幅图像实施二维DFT，显示并观察其频谱。

解：源程序及运行结果如下：

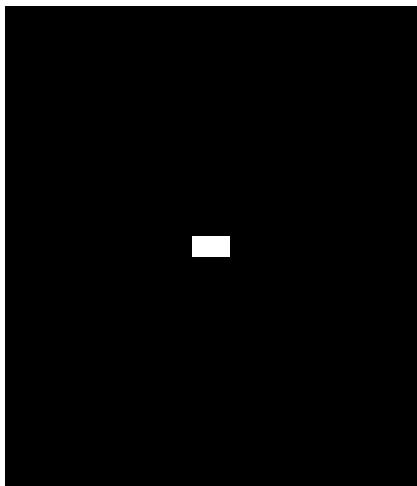
```
%对单缝进行快速傅里叶变换，以三种方式显示频谱，  
%即：直接显示（坐标原点在左上角）；把坐标原点平  
%移至中心后显示；以对数方式显示。  
f=zeros(512,512);  
f(246:266,230:276)=1;  
subplot(221),imshow(f,[]),title('单狭缝图像')  
F=fft2(f); %对图像进行快速傅里叶变换  
S=abs(F);  
subplot(222)  
imshow(S,[]) %显示幅度谱  
title('幅度谱（频谱坐标原点在坐上角）')
```



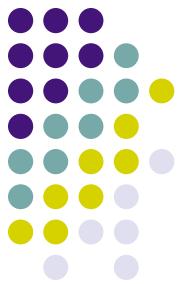
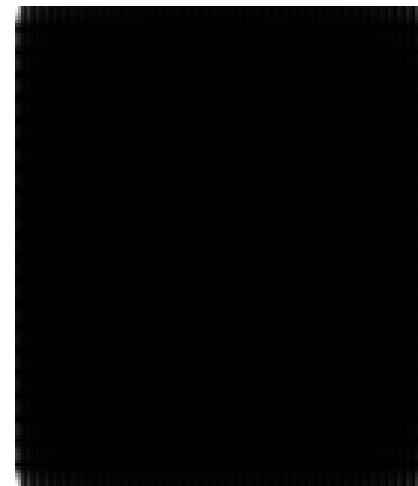
```
Fc=fftshift(F); %把频谱坐标原点由左上角移至屏幕中央
subplot(223)
Fd=abs(Fc);
imshow(Fd,[])
ratio=max(Fd(:))/min(Fd(:))
%ratio = 2.3306e+007,动态范围太大,显示器无法正常显示
title('幅度谱（频谱坐标原点在屏幕中央）')
S2=log(1+abs(Fc));
subplot(224)
imshow(S2,[])
title('以对数方式显示频谱')
```

运行上面程序后，结果如下：

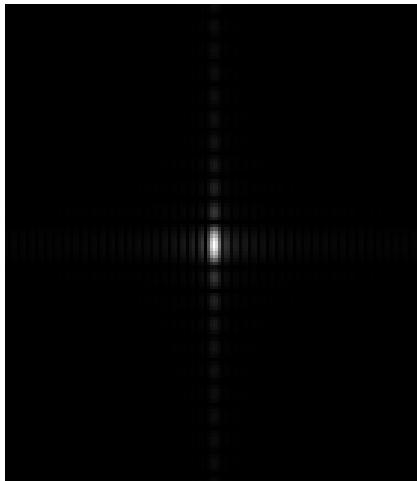
单狭缝图像



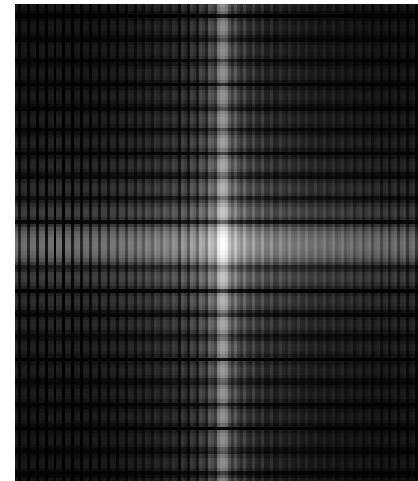
幅度谱 (频谱坐标原点在坐上角)

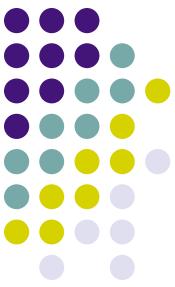


幅度谱 (频谱坐标原点在屏幕中央)



以对数方式显示频谱





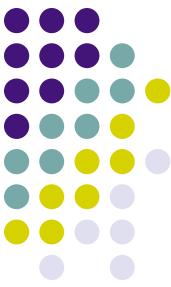
二维离散傅里叶变换的性质

➤ 线性组合律

$$\begin{cases} f_1(x, y) \leftrightarrow F_1(u, v) \\ f_2(x, y) \leftrightarrow F_2(u, v) \end{cases} \Rightarrow c_1 f_1(x, y) + c_2 f_2(x, y) \leftrightarrow c_1 F_1(u, v) + c_2 F_2(u, v)$$

证明：

$$\begin{aligned} & DFT [c_1 f_1(x, y) + c_2 f_2(x, y)] \\ &= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} [c_1 f_1(x, y) + c_2 f_2(x, y)] \cdot e^{-j2\pi \left(\frac{ux}{M} + \frac{vy}{N} \right)} \\ &= c_1 \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f_1(x, y) e^{-j2\pi \left(\frac{ux}{M} + \frac{vy}{N} \right)} + c_2 \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f_2(x, y) e^{-j2\pi \left(\frac{ux}{M} + \frac{vy}{N} \right)} \\ &= c_1 F_1(u, v) + c_2 F_2(u, v) \end{aligned}$$

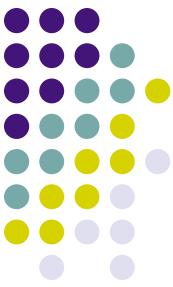


%imagelinear.m
%该程序验证了二维DFT的线性性质

```
f=imread('D:\chenpc\data\thry\chpt4\Fig4.04(a).jpg');  
g=imread('D:\chenpc\data\thry\chpt4\Fig4.30(a).jpg');  
[m,n]=size(g);  
f(m,n)=0;  
f=im2double(f);  
g=im2double(g);  
subplot(221)  
imshow(f,[])  
title('f')  
subplot(222)  
imshow(g,[])  
title('g')
```



```
F=fftshift(fft2(f));
G=fftshift(fft2(g));
subplot(223)
imshow(log(abs(F+G)),[])
FG=fftshift(fft2(f+g));
title('DFT(f)+DFT(g)')
subplot(224)
imshow(log(abs(FG)),[])
title('DFT(f+g)')
```



f



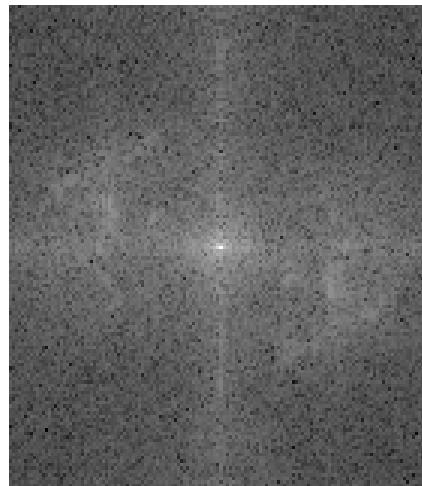
g



DFT(f)+DFT(g)



DFT(f+g)





➤ 可分离性

二维DFT可视为由沿x,y方向的两个一维DFT所构成。

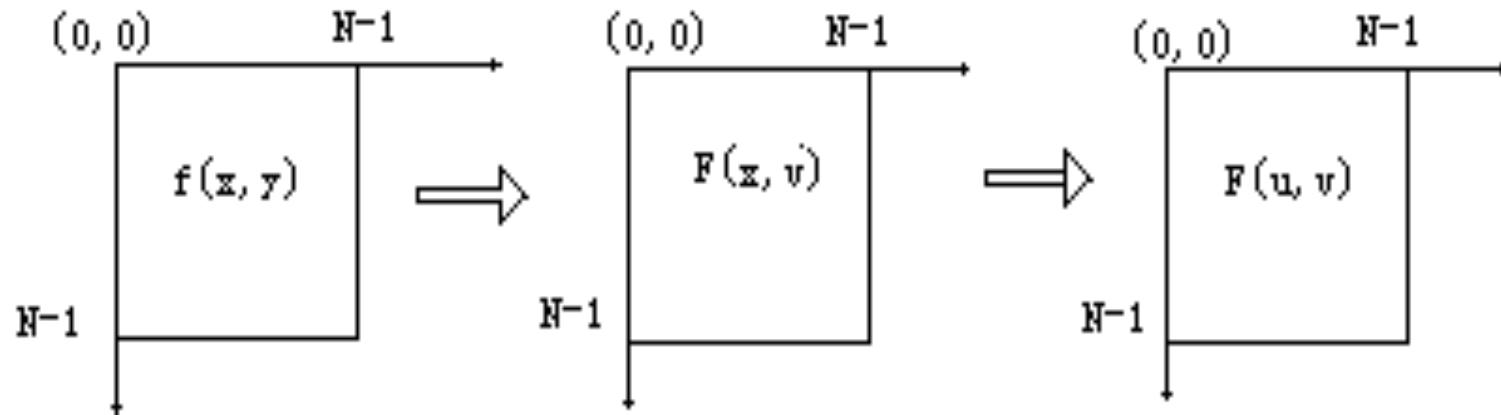
$$\begin{aligned}
 F(u, v) &= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \cdot e^{-j2\pi \left(\frac{ux}{M} + \frac{vy}{N} \right)} \\
 &= \sum_{x=0}^{M-1} \left[\sum_{y=0}^{N-1} f(x, y) \cdot e^{-j2\pi \frac{vy}{N}} \right] \cdot e^{-j2\pi \frac{ux}{M}} \\
 &= \sum_{x=0}^{M-1} F(x, v) \cdot e^{-j2\pi \frac{ux}{M}}
 \end{aligned}$$

$$\begin{aligned}
 f(x, y) &= \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) \cdot e^{j2\pi \left(\frac{ux}{M} + \frac{vy}{N} \right)} \\
 &= \frac{1}{M} \sum_{u=0}^{M-1} \left[\frac{1}{N} \sum_{v=0}^{N-1} F(u, v) \cdot e^{j2\pi \frac{vy}{N}} \right] \cdot e^{j2\pi \frac{ux}{M}} \\
 &= \frac{1}{M} \sum_{u=0}^{M-1} F(u, y) \cdot e^{j2\pi \frac{uy}{M}}
 \end{aligned}$$

其中：

$$\begin{cases} F(x, v) = \sum_{y=0}^{N-1} f(x, y) \cdot e^{-j2\pi \frac{vy}{N}} \sim y \text{ 方向的 DFT} \\ F(u, v) = \sum_{x=0}^{M-1} f(x, v) \cdot e^{-j2\pi \frac{ux}{M}} \sim x \text{ 方向的 DFT} \end{cases}$$

$$\begin{cases} F(u, y) = \frac{1}{N} \sum_{v=0}^{N-1} F(u, v) \cdot e^{j2\pi \frac{vy}{N}} \sim y \text{ 方向的 IDFT} \\ f(x, y) = \frac{1}{M} \sum_{u=0}^{M-1} F(u, y) \cdot e^{j2\pi \frac{uy}{M}} \sim x \text{ 方向的 IDFT} \end{cases}$$



例题：编程验证二维离散傅里叶变换可分离为两个一维离散傅里叶变换。

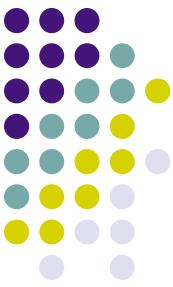
解：

%myseparable.m

%该程序验证了二维DFT的可分离性质

%该程序产生了冈萨雷斯《数字图像处理》（第二版）

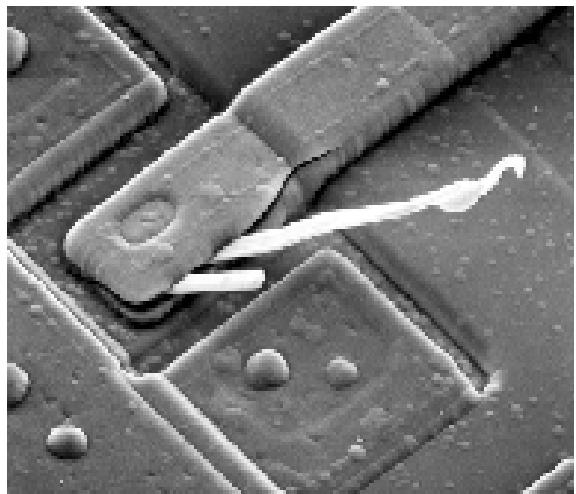
%P125 图4.4



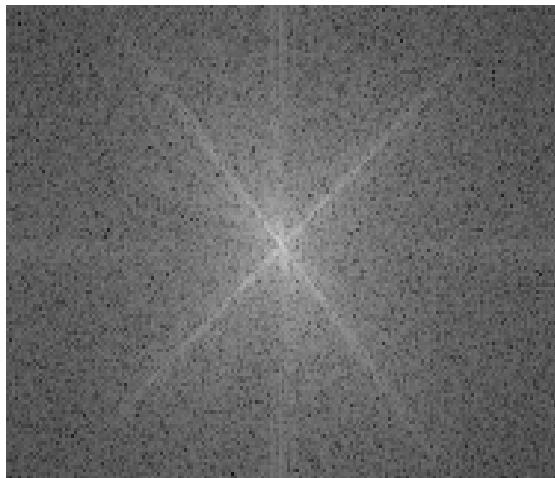
```
f=imread('D:\chenpc\data\thry\chpt4\Fig4.04(a).jpg');
subplot(211)
imshow(f,[])
title('原图')
F=fftshift(fft2(f));
subplot(223)
imshow(log(1+abs(F)),[])
title('用fft2实现二维离散傅里叶变换')
[m,n]=size(f);
F=fft(f);      %沿x方向求离散傅里叶变换
G=fft(F')';    %沿y方向求离散傅里叶变换
F=fftshift(G);
subplot(224)
imshow(log(1+abs(F)),[])
title('用fft实现二维离散傅里叶变换')
```



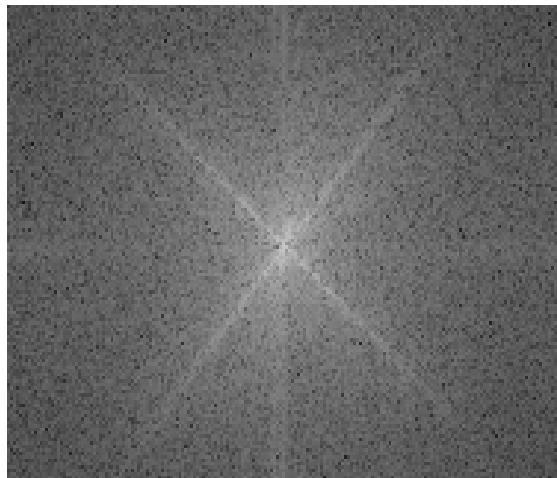
原图



用 fft2 实现二维离散傅里叶变换



用 fft 实现二维离散傅里叶变换





➤ 平移性

$$f(x, y) \Leftrightarrow F(u, v) \Rightarrow \begin{cases} f(x, y) \cdot e^{j2\pi\left(\frac{u_0x}{M} + \frac{v_0y}{N}\right)} \Leftrightarrow F(u - u_0, v - v_0) \\ f(x - x_0, y - y_0) \Leftrightarrow F(u, v) \cdot e^{-j2\pi\left(\frac{ux_0}{M} + \frac{vy_0}{N}\right)} \end{cases}$$

证明：

(1) 频域移位

$$\begin{aligned} & DFT \left[f(x, y) \cdot e^{j2\pi\left(\frac{u_0x}{M} + \frac{v_0y}{N}\right)} \right] \\ &= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \cdot e^{j2\pi\left(\frac{u_0x}{M} + \frac{v_0y}{N}\right)} \cdot e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)} \\ &= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \cdot e^{-j2\pi\left(\frac{(u-u_0)x}{M} + \frac{(v-v_0)y}{N}\right)} \\ &= F(u - u_0, v - v_0) \end{aligned}$$



结论：

$$f(x, y) \cdot e^{j2\pi\left(\frac{u_0x}{M} + \frac{v_0y}{N}\right)} \Leftrightarrow F(u - u_0, v - v_0)$$

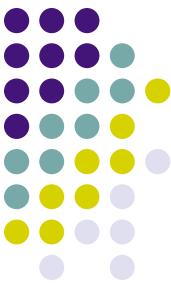
当 $u_0 = \frac{M}{2}, v_0 = \frac{N}{2}$

$$e^{j2\pi(u_0x/M + v_0y/N)} = e^{j\pi(x+y)} = (-1)^{x+y}$$

$$\Rightarrow f(x, y) \cdot (-1)^{x+y} \Leftrightarrow F\left(u - \frac{M}{2}, v - \frac{N}{2}\right)$$

即如果需要将频域的坐标原点从显示屏起始点（0, 0）移至显示屏的中心点只要将 $f(x,y)$ 乘以 $(-1)^{x+y}$ 因子再进行傅里叶变换即可实现。

例题：利用 $(-1)^{x+y}$ 对单缝图像 $f(x,y)$ 进行调制，实现把频谱坐标原点移至屏幕正中央的目标。

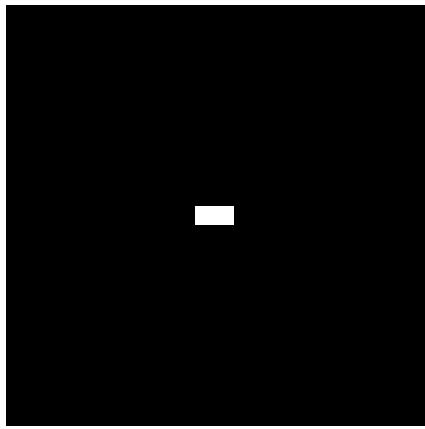


解：完成本题的源程序为：

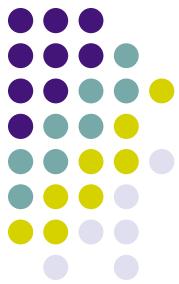
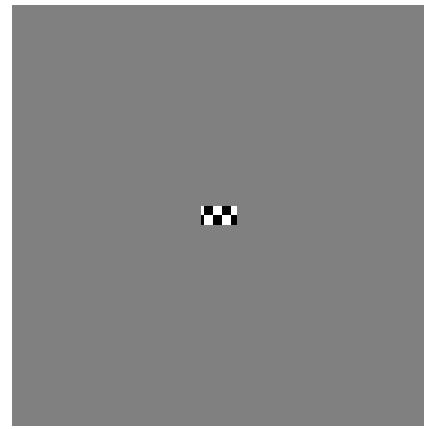
%在傅里叶变换之前，把函数乘以 $(-1)^{x+y}$ ，相当于把频谱
%坐标原点移至屏幕窗口正中央。

```
f(512,512)=0;  
f=mat2gray(f);  
[Y,X]=meshgrid(1:512,1:512);  
f(246:266,230:276)=1;  
g=f.*(-1).^ (X+Y);  
subplot(221),imshow(f,[]),title('原图像f(x,y)')  
subplot(222),imshow(g,[]),title('空域调制图像  
g(x,y)=f(x,y)*(-1)^(x+y)')  
F=fft2(f);  
subplot(223),imshow(log(1+abs(F)),[]),title('f(x,y)的傅里叶  
频谱')  
G=fft2(g);  
subplot(224),imshow(log(1+abs(G)),[]),title('g(x,y)的傅里叶  
频谱')
```

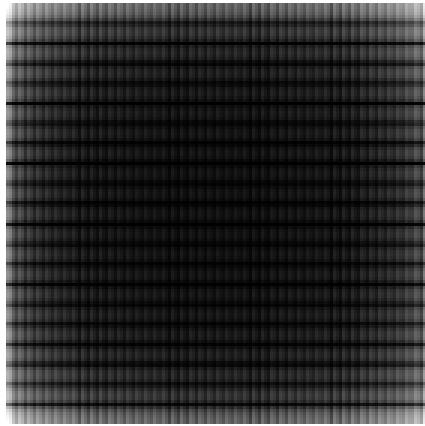
原图 像 $f(x,y)$



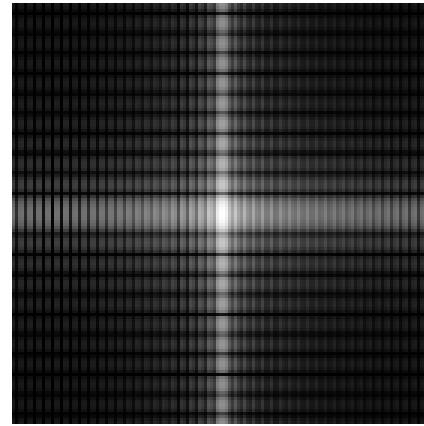
空域调制图像 $g(x,y)=f(x,y)^*(-1)^{x+y}$



$f(x,y)$ 的傅里叶频谱



$g(x,y)$ 的傅里叶频谱





(2) 空域移位:

$$DFT[f(x - x^0, y - y^0)]$$

$$\begin{aligned} &= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x - x_0, y - y_0) \cdot e^{-j2\pi \left(\frac{ux}{M} + \frac{vy}{N} \right)} \\ &= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x - x_0, y - y_0) \cdot e^{-j2\pi \left(\frac{u(x-x_0+x_0)}{M} + \frac{v(y-y_0+y_0)}{N} \right)} \\ &= e^{-j \left(\frac{ux_0}{M} + \frac{vy_0}{N} \right) \cdot 2\pi} \cdot \sum_{x=-x_0}^{M-1-x_0} \sum_{y=-y_0}^{N-1-y_0} f(x, y) \cdot e^{-j2\pi \left(\frac{ux}{M} + \frac{vy}{N} \right)} \\ &= e^{-j \left(\frac{ux_0}{M} + \frac{vy_0}{N} \right) \cdot 2\pi} \cdot \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \cdot e^{-j2\pi \left(\frac{ux}{M} + \frac{vy}{N} \right)} \\ &= e^{-j \left(\frac{ux_0}{M} + \frac{vy_0}{N} \right) \cdot 2\pi} \cdot F(u, v) \end{aligned}$$



➤ 周期性和共轭对称性

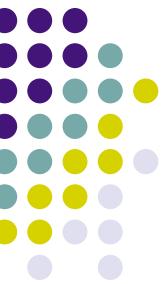
周期性:

$$\begin{cases} F(u, v) = F(u + mM, v + nN) \\ f(x, y) = f(x + mM, y + nN) \end{cases}$$

$$(m, n = 0, \pm 1, \pm 2, \dots)$$

共轭对称性:

$$\begin{aligned} F(u, v) &= F^*(-u, -v) \\ |F(u, v)| &= |F(-u, -v)| \end{aligned}$$



证明： (1) 周期性：

$$\begin{cases} F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \cdot e^{-j2\pi \left(\frac{ux}{M} + \frac{vy}{N} \right)} \\ f(x, y) = \frac{1}{MN} \cdot \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) \cdot e^{j2\pi \left(\frac{ux}{M} + \frac{vy}{N} \right)} \end{cases}$$

$$e^{-j2\pi m} = 1 \quad \xrightarrow{\hspace{1cm}} \quad \begin{cases} F(u + mM, v + nN) = F(u, v) \\ f(x + mM, y + nN) = f(x, y) \end{cases}$$

(2) 共轭对称性：

$$\begin{aligned} F(u, v) &= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \cdot e^{-j2\pi \left(\frac{ux}{M} + \frac{vy}{N} \right)} = \left\{ \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \cdot e^{-j2\pi \left[\frac{(-u)x}{M} + \frac{(-v)y}{N} \right]} \right\}^* \\ &= F^*(-u, -v) \end{aligned}$$

$|F(u, v)| = |F(-u, -v)|$, 即 $F(u, v)$ 关于原点对称



➤ 旋转不变性

$$f(x, y) \Leftrightarrow F(u, v)$$

$\xrightarrow{\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ u = \omega \cos \varphi \\ v = \omega \sin \varphi \end{cases}}$

$$f(r, \theta) \Leftrightarrow F(\omega, \varphi)$$

$\xrightarrow{} f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$

证明：

$$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \cdot e^{-j2\pi(ux+vy)} \cdot dx dy$$

若 $\begin{cases} x = r \cos \theta \\ y = r \sin \theta, \end{cases}$ 则 $\begin{cases} u = \omega \cos \varphi \\ v = \omega \sin \varphi, \end{cases}$ 则：

$$F(\omega, \varphi) = \int_0^{\infty} \int_0^{2\pi} f(r, \theta) \cdot e^{-j2\pi\omega r \cos(\varphi - \theta)} \cdot r \cdot dr \cdot d\theta$$

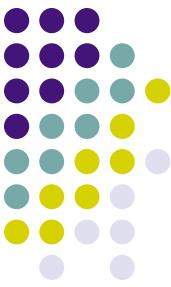


$$F(\omega, \varphi + \theta_0) = \int_0^{\infty} \int_0^{2\pi} f(r, \theta) \cdot e^{-j2\pi r \omega \cos[\varphi - (\theta - \theta_0)]} \cdot r dr d\theta$$

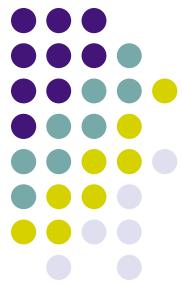
$$= \int_0^{\infty} \int_{-\theta_0}^{2\pi - \theta_0} f(r, \theta + \theta_0) \cdot e^{-j2\pi r \omega \cos(\varphi - \theta)} \cdot r dr d\theta$$

$$\begin{aligned} f(r, \theta) &= f(r, \theta + 2\pi) \\ &= \int_0^{\infty} \int_0^{2\pi} f(r, \theta + \theta_0) \cdot e^{-j2\pi r \omega \cos(\varphi - \theta)} \cdot r dr d\theta \end{aligned}$$

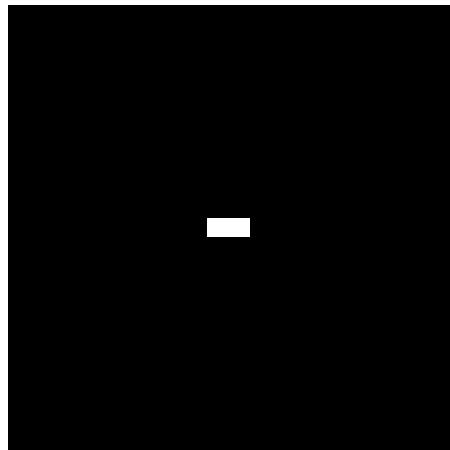
注：为看清问题的实质、简化旋转不变性的证明，以上用二维连续傅里叶变换进行证明。实际上，由连续积分公式进行离散化处理，即可得到离散公式，证明可参照连续情况进行。



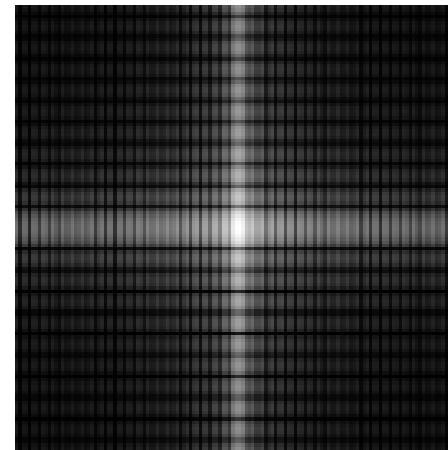
```
f=zeros(512,512);
f(246:266,230:276)=1;
subplot(221);
imshow(f[])
title('原图')
F=fftshift(fft2(f));
subplot(222);
imshow(log(1+abs(F)),[])
title('原图的频谱')
f=imrotate(f,45,'bilinear','crop');
subplot(223)
imshow(f[])
title('旋转45^0图')
Fc=fftshift(fft2(f));
subplot(224);
imshow(log(1+abs(Fc)),[])
title('旋转图的频谱')
```



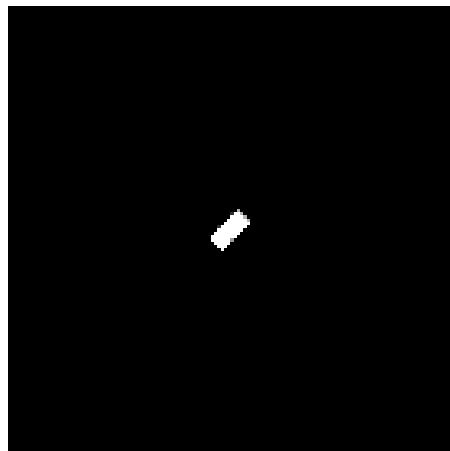
原图



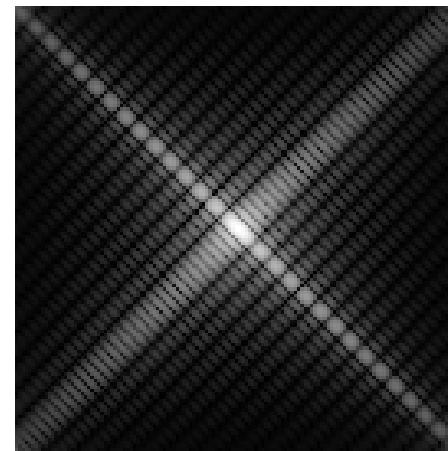
原图的频谱

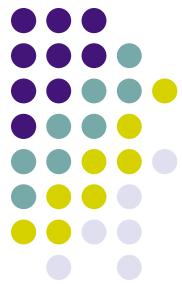


旋转 45^0 图



旋转图的频谱





下一讲

