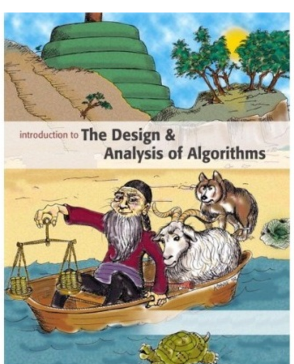
Introduction to

Algorithm Design and Analysis

[2] Asymptotics



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In the Last Class…

- **Algorithm – the spirit of computing**
  - Model of computation
- **Algorithm design and analysis**
  - Design
    - Correctness proof by induction
  - Analysis
    - Worst-case / average-case complexity

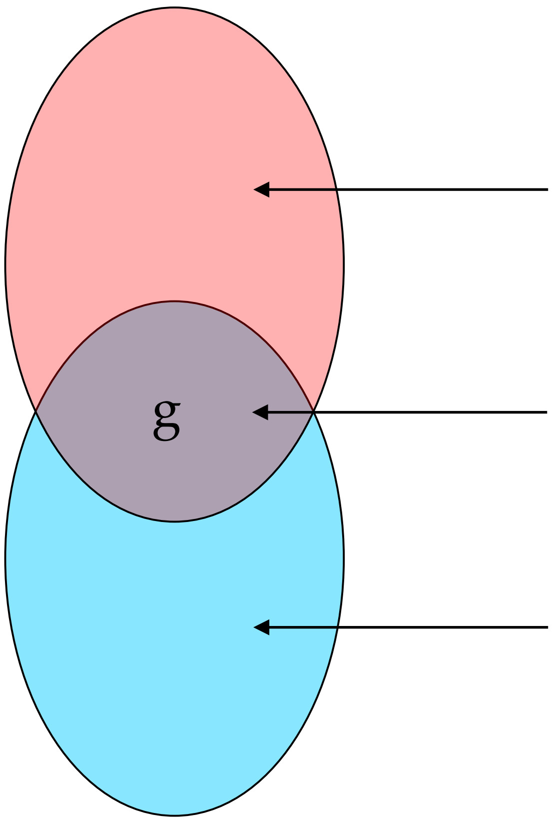
Asymptotic Behavior

- **Asymptotic growth rate of functions**
  - Basic idea
- **Key notations**
  - O, Ω, Θ
  - o, ω
- **Brute force enumeration**
  - By iteration
  - By recursion

How to Compare Two Algorithms

- **Algorithm analysis, with *simplifications***
  - Measuring the cost by the number of critical operations
  - Large input size only
    - Only the leading term in $f(n)$ is considered
    - Constant coefficients are ignored
- **Capturing the essential part in the cost in a mathematical way**
  - Asymptotic growth rate of $f(n)$

Relative Growth Rate



Ω(g):functions that grow at least as fast as g

Θ(g):functions that grow at the same rate as g

O(g):functions that grow no faster as g

"Big Oh"

- **Basic idea** $f(n) \in O(g(n))$
  - For sufficiently large input size, g(n) is an upper bound for f(n)
- **Definition – "$\varepsilon - N$"**
  - Giving g: $\mathrm{N} \rightarrow \mathrm{R}^+$, then O($g$) is the set of f:$\mathrm{N} \rightarrow \mathrm{R}^+$, such that for some $c \in \mathrm{R}^+$ and some $n_0 \in \mathrm{N}$, $f(n) \leq cg(n)$ for all $n \geq n_0$
- **Definition – "$\boldsymbol{lim_{n\to\infty}}$"**
  - $f \in O(g)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c < \infty$

The limit may not exist, though it usually does.

Example

- **Let f(n)=n², g(n)=nlogn, then:**

L'Hospital's rule

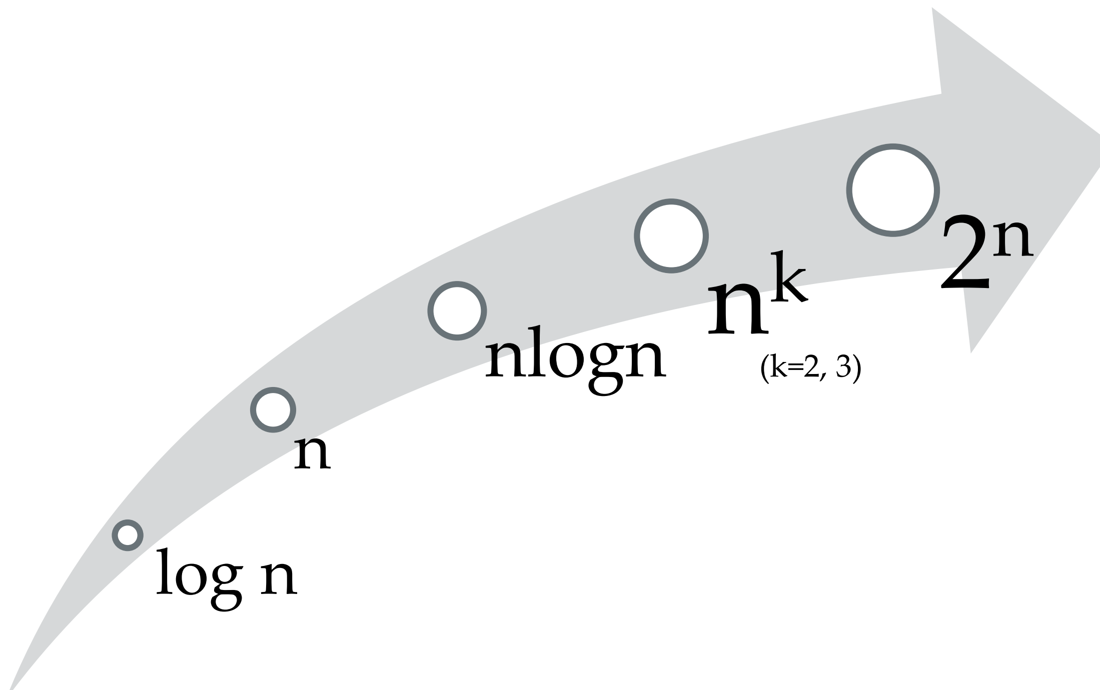
  - o $f \notin O(g)$, since

$$\lim_{n\to\infty} \frac{n^2}{n\log n} = \lim_{n\to\infty} \frac{n}{\log n} = \lim_{n\to\infty} \frac{1}{\frac{1}{n\ln 2}} = +\infty$$

  - o $g \in O(f)$, since

$$\lim_{n\to\infty} \frac{n\log n}{n^2} = \lim_{n\to\infty} \frac{\log n}{n} = \lim_{n\to\infty} \frac{1}{n\ln 2} = 0$$

Asymptotic Growth Rate



Asymptotic Order

- Logarithm *log n*

$$\log n \in O(n^{\alpha}) \text{ for } \textbf{any } \alpha>0$$

- Power n^k

$$n^{k} \in O(c^{n}) \text{ for } \textbf{any } c>1$$

- Factorial *n!*

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n} \quad \text{(Stirling's formula)}$$

"Big Ω"

- **Basic idea of $f(n) \in \Omega(g(n))$**
  - o Dual of "O"
- **Definition – "$\boldsymbol{\varepsilon - N}$"**
  - o Giving g: N→R$^+$, then $\Omega(g)$ is the set of f:N→R$^+$, such that for some $c \in R^+$ and some $n_0 \in N$, $f(n) \geq cg(n)$ for all $n \geq n_0$
- **Definition – "$\boldsymbol{lim_{n\to\infty}}$"**
  - o $f \in \Omega(g)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c > 0$ (the limit may be ∞)

The Set Θ

- **Basic idea of $f(n) \in \Theta(g(n))$**
  - Roughly the same
  - $\Theta(g) = O(g) \cap \Omega(g)$
- **Definition – "$\boldsymbol{\varepsilon - N}$"**
  - Giving g:N→R⁺, then $\Theta(g)$ is the set of f:N→R⁺, such that for some $c_1, c_2 \in R^+$ and some $n_0 \in N$,

$$0 \le c_1 g(n) \le f(n) \le c_2 g(n), \text{ for all } n \ge n_0$$

- **Definition – "$\boldsymbol{lim_{n\to\infty}}$"**
  - $f \in \Theta(g)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c \ (0 < c < \infty)$

Some Empirical Data

algorithm		1	2	3	4
Run time in *ns*		$1.3n^3$	$10n^2$	$47n\log n$	$48n$
time for size	10^3	1.3*s*	10*ms*	0.4*ms*	0.05*ms*
	10^4	22*m*	1*s*	6*ms*	0.5*ms*
	10^5	15*d*	1.7*m*	78*ms*	5*ms*
	10^6	41*yrs*	2.8*hrs*	0.94*s*	48ms
	10^7	41*mill*	1.7*wks*	11*s*	0.48*s*
max Size in time	sec	920	10,000	1.0×10^6	2.1×10^7
	min	3,600	77,000	4.9×10^7	1.3×10^9
	hr	14,000	6.0×10^5	2.4×10^9	7.6×10^{10}
	day	41,000	2.9×10^6	5.0×10^{10}	1.8×10^{12}
time for 10 times size		$\times1000$	$\times100$	$\times10+$	$\times10$

on 400Mhz Pentium II, in C

from: Jon Bentley: *Programming Pearls*

Properties of O, Ω and Θ

- **Transitive property:**
  - If $f \in O(g)$ and $g \in O(h)$, then $f \in O(h)$
- **Symmetric properties**
  - $f \in O(g)$ if and only if $g \in \Omega(f)$
  - $f \in \Theta(g)$ if and only if $g \in \Theta(f)$
- **Order of sum function**
  - $O(f+g)=O(\max(f,g))$

"Little Oh"

- **Basic idea of $f(n) \in o(g(n))$**
  - Non-ignorable gap between f and its upper bound g
- **Definition –"$\boldsymbol{\varepsilon - N}$"**
  - Giving g:N→R$^+$, then o(g) is the set of f:N→R$^+$, such that for any c∈R$^+$, there exists some n_0∈N,

    $$0 \leq f(n) < cg(n), \text{ for all } n \geq n_0$$
- **Definition – "$\boldsymbol{lim_{n\to\infty}}$"**
  - $f \in o(g)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$

"Little ω"

- **Basic idea of $f(n) \in \omega(g(n))$**
  - Dual of "o"
- **Definition – "$\varepsilon - N$"**
  - Giving g:N→R$^+$, then ω(g) is the set of f:N→R$^+$, such that for any c∈R$^+$, there exists some n_0∈N,

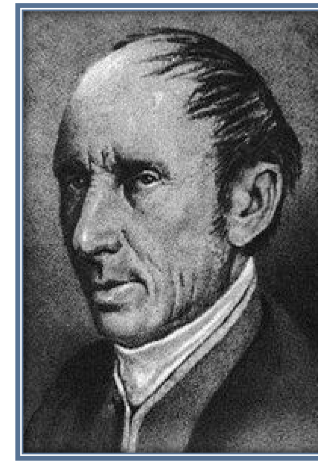
$$0 \le cg(n) < f(n), \text{ for all } n \ge n_0$$

- **Definition – "$\lim_{n\to\infty}$"**
  - $f \in \omega(g)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$

Do You Know Infinity

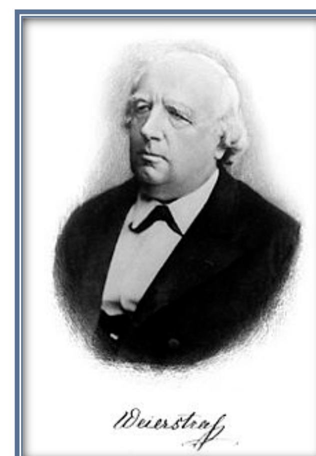
- **Mathematical analysis**
  (differentiation / integration)
  - Firm foundation

Cauchy



- **How to talk about *infinity*?**
  - $(\varepsilon - N)$-definition
  - $(\varepsilon - \delta)$-definition

Weierstrass



Brute Force Enumeration by Iteration

- **Swapping array elements**
  - <time, space>
    - From <$O(n^2)$, $O(1)$>
    - To <$O(n)$, $O(n)$>
    - To <$O(n)$, $O(1)$>
- **Maximum subsequence sum**
  - Time
    - From $O(n^3)$
    - To $O(n^2)$
    - To $O(n \log n)$
    - To $O(n)$

Swapping Array Elements

- E.g., 1,2,3,4 | 5,6,7 => 5,6,7,1,2,3,4
- Brute force

Space sensitive

	Time	Space
BF1	O(n^2)	O(1)
BF2	O(n)	O(n)
Your Task	O(n)	O(1)

Time sensitive

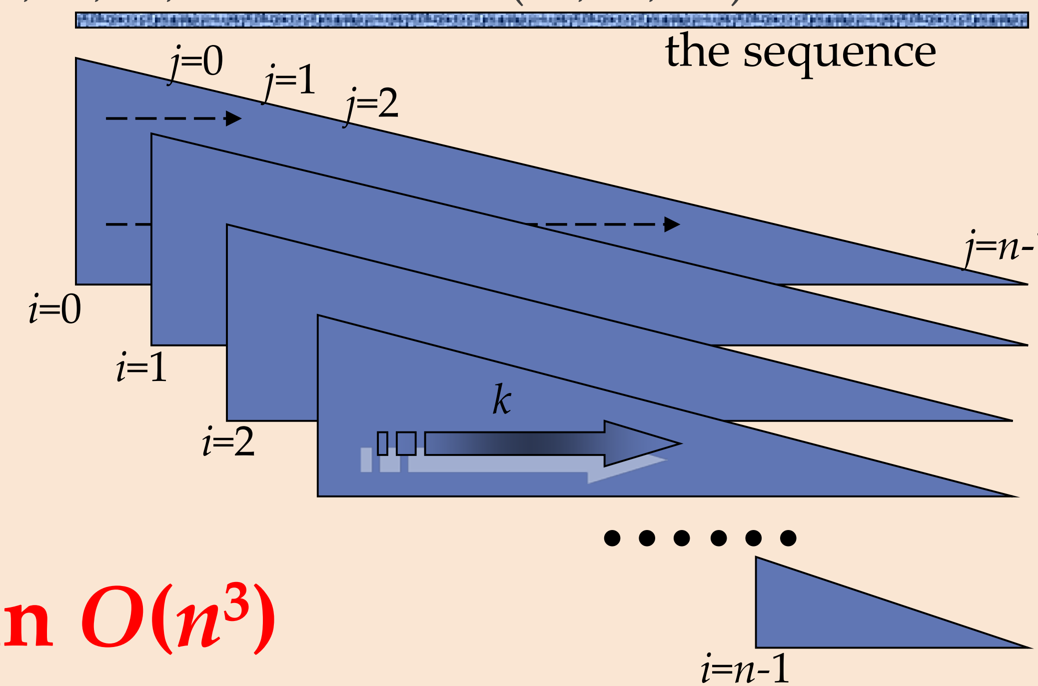
- Your task
  - Both time and space efficient

Max-sum Subsequence

- **The problem: Given a sequence *S* of integer, find the largest sum of a consecutive subsequence of *S*. (0, if all negative items)**
    - An example: -2, 11, -4, 13, -5, -2; the result 20: (11, -4, 13)

A brute-force algorithm:

```
MaxSum = 0;
 for (i = 0; i < N; i++)
  for (j = i; j < N; j++)
  {
    ThisSum = 0;
    for (k = i; k <= j; k++)
    ThisSum += A[k];
    if (ThisSum > MaxSum)
      MaxSum = ThisSum;
  }
 return MaxSum;
```



in $O(n^3)$

More Precise Complexity

The total cost is : $\sum_{i=0}^{n-1}\sum_{j=i}^{n-1}\sum_{k=i}^{j}1$

$$\sum_{k=i}^{j}1 = j-i+1$$

$$\sum_{j=i}^{n-1}(j-i+1) = 1+2+\ldots+(n-i) = \frac{(n-i+1)(n-i)}{2}$$

$$\sum_{i=0}^{n-1}\frac{(n-i+1)(n-i)}{2} = \sum_{i=1}^{n}\frac{(n-i+2)(n-i+1)}{2}$$

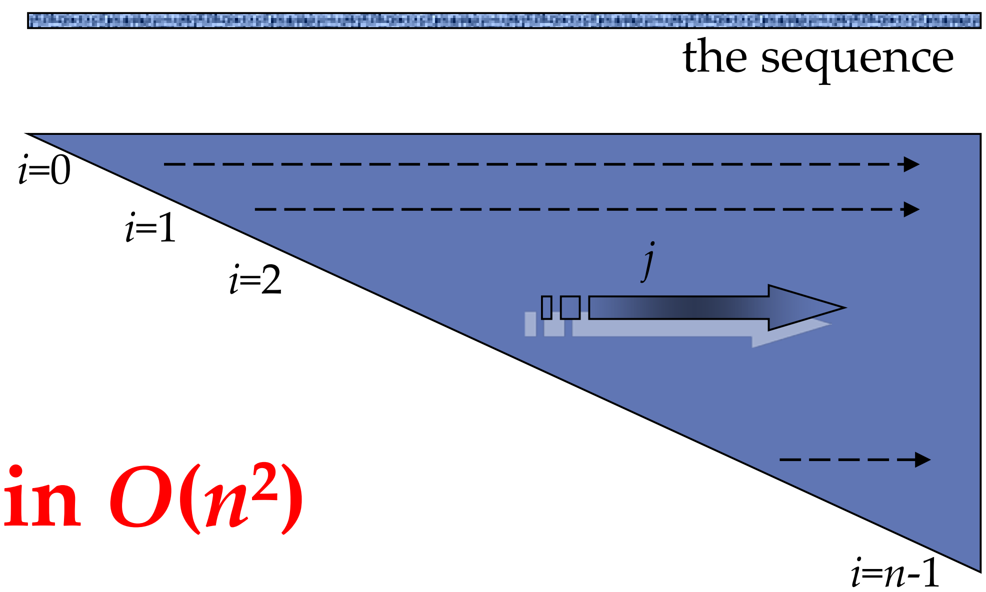
$$= \frac{1}{2}\sum_{i=1}^{n}i^2 - (n+\frac{3}{2})\sum_{i=1}^{n}i + \frac{1}{2}(n^2+3n+2)\sum_{i=1}^{n}1$$

$$= \frac{n^3+3n^2+2n}{6}$$

Decreasing the Number of Loops

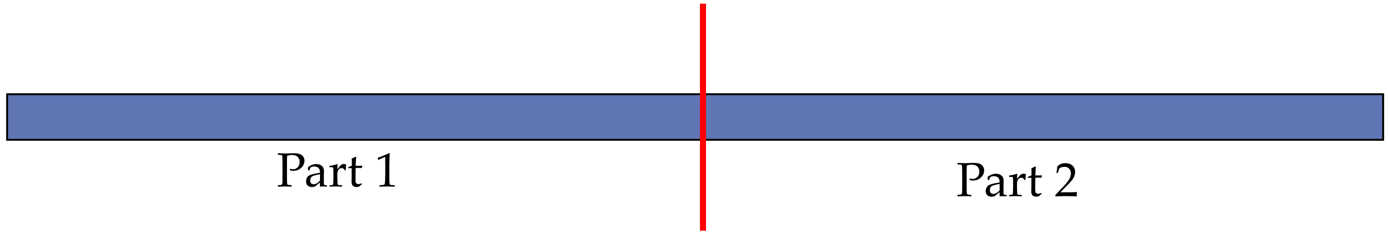
An improved algorithm

```
MaxSum = 0;
 for (i = 0; i < N; i++)
 {
  ThisSum = 0;
  for (j = i; j < N; j++)
  {
   ThisSum += A[j];
   if (ThisSum > MaxSum)
    MaxSum = ThisSum;
  }
 }
 return MaxSum;
```

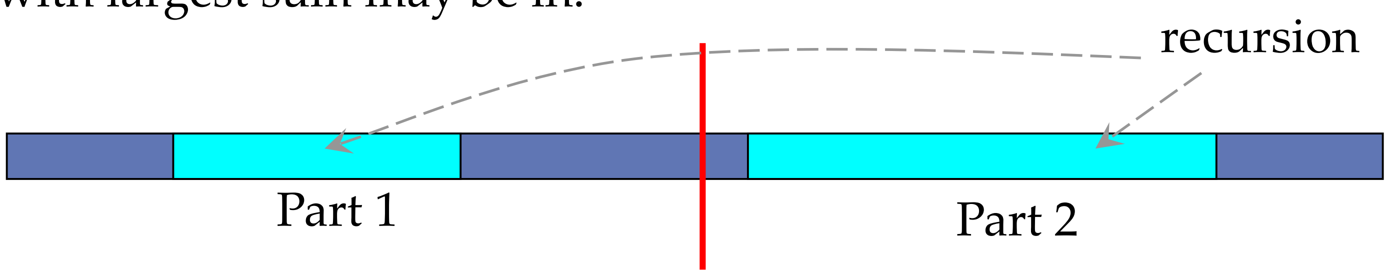


in $O(n^2)$

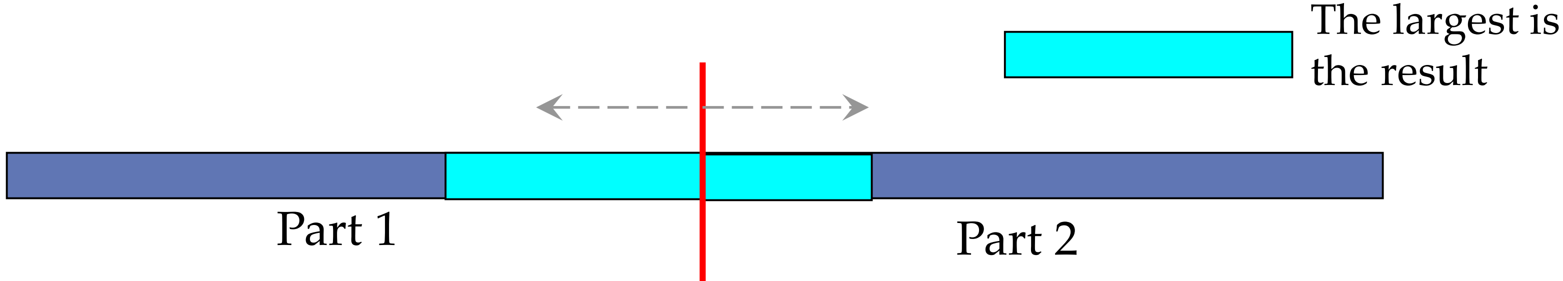
Power of Divide and Conquer



the sub with largest sum may be in:



or:



in $O(n\log n)$

Power of
Divide and Conquer

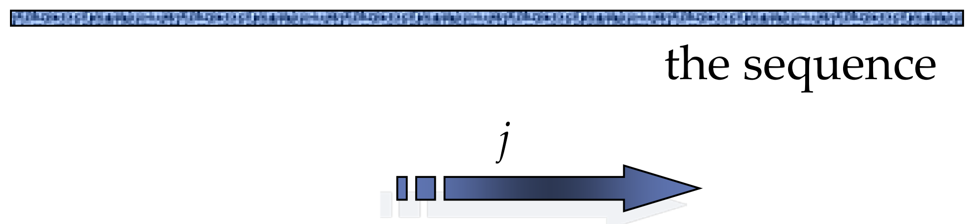
```
Center = (Left + Right) / 2;
  MaxLeftSum = MaxSubSum(A, Left, Center);  MaxRightSum = MaxSubSum(A, Center + 1, 
Right);

  MaxLeftBorderSum = 0; LeftBorderSum = 0;
  for (i = Center; i >= Left; i--)
  {
    LeftBorderSum += A[i];
    if (LeftBorderSum > MaxLeftBorderSum)    MaxLeftBorderSum = LeftBorderSum;
  }

  MaxRightBorderSum = 0; RightBorderSum = 0;
  for (i = Center + 1; i <= Right; i++)
  {
    RightBorderSum += A[i];
    if (RightBorderSum > MaxRightBorderSum)  MaxRightBorderSum = RightBorderSum;
  }
  return Max3(MaxLeftSum, MaxRightSum,
        MaxLeftBorderSum + MaxRightBorderSum);
```

Note: this is the core part of the procedure, with base case and wrap omitted.

A Linear Algorithm

First scan the array to eliminate the case of "all negative integers"

```
ThisSum = MaxSum = 0;
 for (j = 0; j < N; j++)
 {
  ThisSum += A[j];
  if (ThisSum > MaxSum)
   MaxSum = ThisSum;
  else if (ThisSum < 0)
   ThisSum = 0;
 }
 return MaxSum;
```



This is an example of "online algorithm"

Negative item or subsequence cannot be a prefix of the subsequence we want.

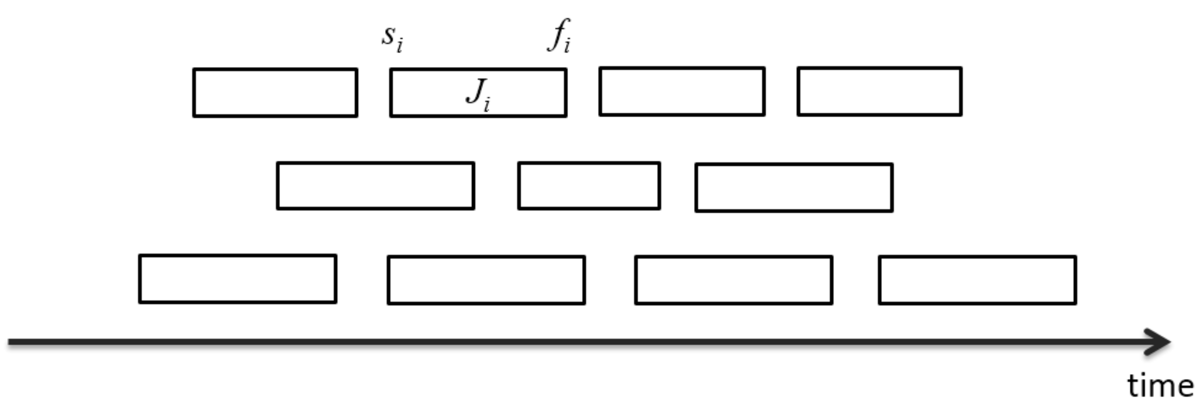
in $O(n)$

Brute Force Enumeration by Recursion

- **Job scheduling**
  - Problem definition
  - Brute force recursion
  - Further improvements
- **Matrix chain multiplication**
  - Problem definition
  - Brute force recursion(s)
  - Further improvements

Job Scheduling

- Jobs: J_i=[s_i, j_i)
- Max number of **compatible** jobs
- Further improvements
  - o Dynamic programming (L16)
  - o Greedy algorithms (L14)



Matrix Chain Multiplication

- **The task:**

  Find the product: $A_1 \times A_2 \times \ldots \times A_{n-1} \times A_n$

  A_i is 2-dimentional array of different legal size

- **The Challenge:**
  - Matrix multiplication is associative
  - Different computing order results in great difference in the number of operations
- **The problem:**
  - Which is the best computing order

Cost of Matrix Multiplication

Let $C = A_{p\times q}\times$

An example: $A_1 \times A_2 \times A_3 \times A_4$

30×1 1×40 40×10 10×25

$((A_1\times A_2)\times A_3)\times A_4$: 20700 multiplications

$A_1\times(A_2\times(A_3\times A_4))$: 11750

$(A_1\times A_2)\times(A_3\times A_4)$: 41200

$A_1\times((A_2\times A_3)\times A_4)$: 1400

$$c_{i,j} = \sum_{k=1}^{q} a_{ik} b_{kj}$$

There are q multiplication

C has $p\times r$ elements as $c_{i,j}$

So, pqr multiplications altogether

Solutions

- **Brute force recursion (L16)**
  - BF1
  - BF2

- **Dynamic programming (L16)**
  - Based on brute force recursion 2

Thank you!

Q & A

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