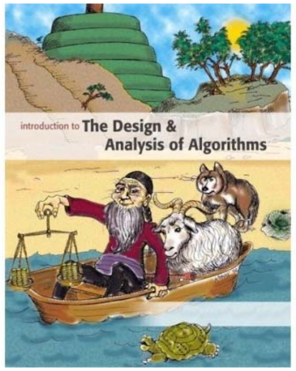
Introduction to

# *Algorithm Design and Analysis*

## [10] Union-Find



*Yu Huang*

http://cs.nju.edu.cn/yuhuang
Institute of Computer Software
Nanjing University

## In the Last Class

- **Hashing**
  - Basic idea
- **Collision handling for hashing**
  - Closed address
  - Open address
- **Amortized analysis**
  - Array doubling
  - Stack operations
  - Binary counter

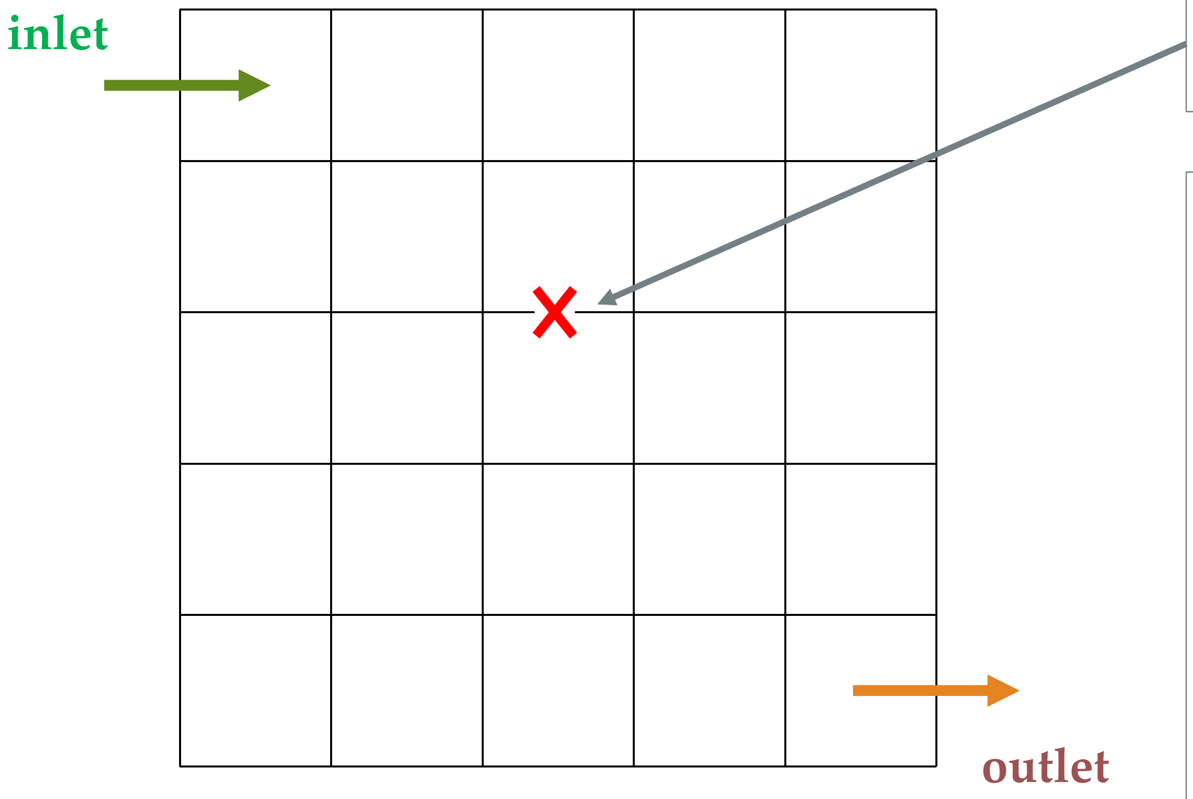
## Union-Find

- **Dynamic Equivalence Relation**
    - Examples
    - Definitions
    - Brute force implementations
- **Disjoint Set**
    - Straightforward Union-Find
    - Weighted Union + Straightforward Find
    - Weighted Union + Path-compressing Find

# Minimum Spanning Tree

- **Kruskal's algorithm, greedy strategy:**
  - Select one edge
    - With the minimum weight
    - Not in the tree
  - Evaluate this edge
    - This edge will **NOT** result in a cycle
- **Critical issue:**
  - How to know "NO CYCLE"?

# Maze Generation



Select a wall to pull down randomly

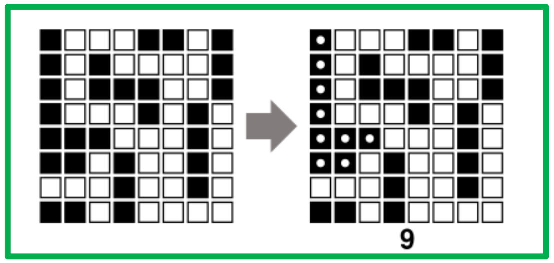
If *i* and *j* are in same *equivalence class*, then select another wall to pull down.

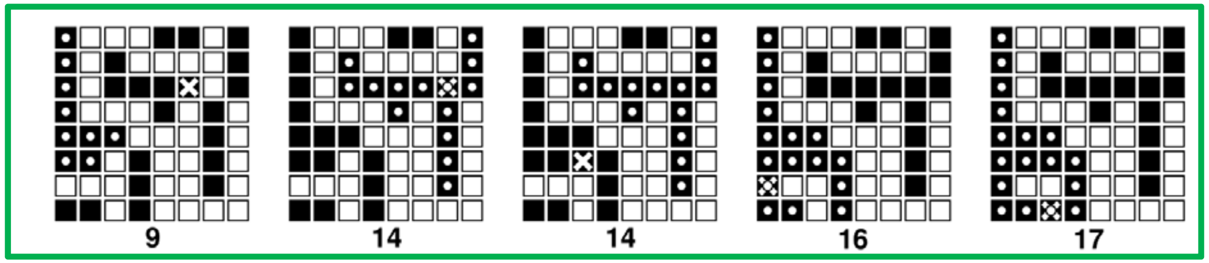
Otherwise, joint the two classes into one.

The maze is complete when the inlet and outlet are in one equivalence class.

## Black Pixels

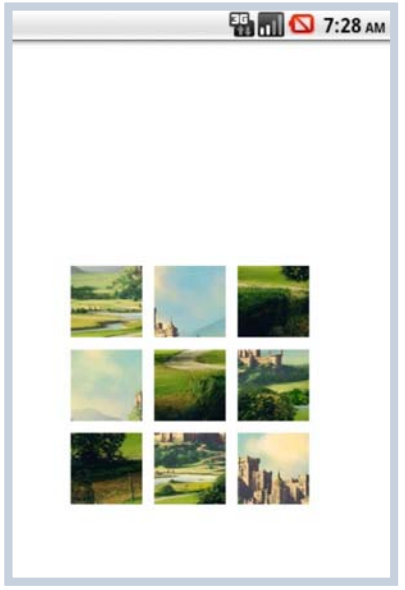
- **Maximum black pixel component**
  - Let $\alpha$ be the size of the component
- **Color one pixel black**
  - How $\alpha$ changes?
  - How to choose the pixel, to accelerate the change in $\alpha$

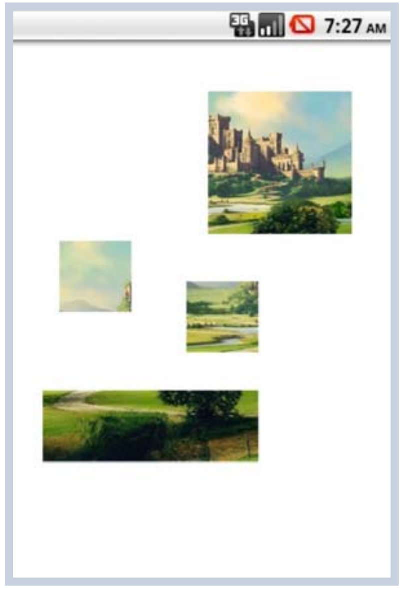




# Jigsaw Puzzle

- **Multiple pieces may be glued together**
- **From "one player" to "two players"**
  - Each group can only be moved in a mutual exclusive way
  - How to decide the relation of "in the same group"





# Dynamic Equivalence Relations

- **Equivalence**
  - Reflexive, symmetric, transitive
  - Equivalent classes forming a partition
- **Dynamic equivalence relation**
  - Changing in the process of computation
  - **IS** instruction: *yes* or *no* (in the same equivalence class)
  - **MAKE** instruction: combining two equivalent classes, by relating two unrelated elements, and influencing the results of subsequent IS instructions.
  - Starting as equality relation

# Implementation: How to Measure

- **The number of basic operations for processing a sequence of $m$ MAKE and/or IS instructions on a set $S$ with $n$ elements.**
- **An Example: $S$={1,2,3,4,5}**
  - 0. [create] {{1}, {2}, {3}, {4}, {5}}
  - 1. **IS** 2≡4? No
  - 2. **IS** 3≡5? No
  - 3. **MAKE** 3≡5. {{1}, {2}, {3,5}, {4}}
  - 4. **MAKE** 2≡5. {{1}, {2,3,5}, {4}}
  - 5. **IS** 2≡3? Yes
  - 6. **MAKE** 4≡1. {{1,4}, {2,3,5}}
  - 7. **IS** 2≡4? No

## Union-Find based Implementation

- **The maze problem**
  - Randomly delete a wall and union two cells
  - Loop until you find the inlet and outlet are in one equivalent class
- **The Kruskal algorithm**
  - Find whether u and v are in the same equivalent class
  - If not, add the edge and union the two nodes
- **The black pixels problem**
  - Find two black pixels not in the same group
  - How the union will increase $\alpha$

## Implementation: Choices

- **Matrix (relation matrix)**
  - Space in $\Theta(n^2)$, and worst-case cost in $\Omega(mn)$ (mainly for row copying for MAKE)
- **Array (for equivalence class ID)**
  - Space in $\Theta(n)$, and worst-case cost in $\Omega(mn)$ (mainly for search and change for MAKE)
- **Forest of rooted trees**
  - A collection of disjoint sets, supporting *Union* and *Find* operations
  - Not necessary to traverse all the elements in one set

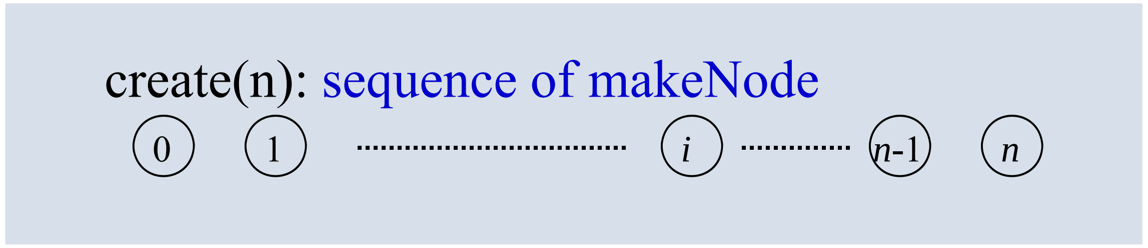
## Union-Find ADT

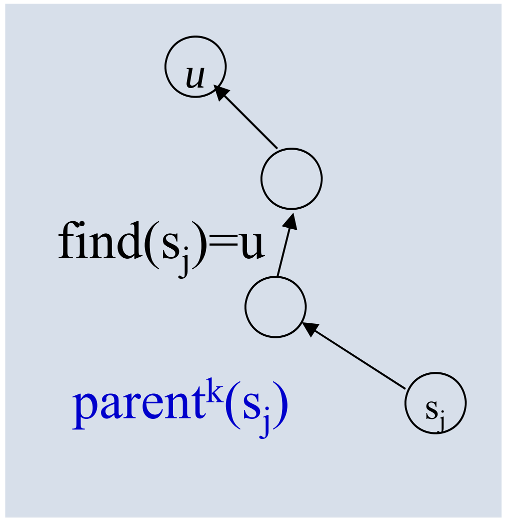
- **Constructor: Union-Find** create(int n)
  - sets=create($n$) refers to a newly created group of sets $\{1\}, \{2\}, \ldots, \{n\}$ ($n$ singletons)
- **Access Function: int** find(UnionFind sets, $e$)
  - find(sets, $e$)=<$e$>
- **Manipulation Procedures**
  - **void** makeSet(UnionFind sets, **int** $e$)
  - **void** union(UnionFind sets, **int** $s$, **int** $t$)

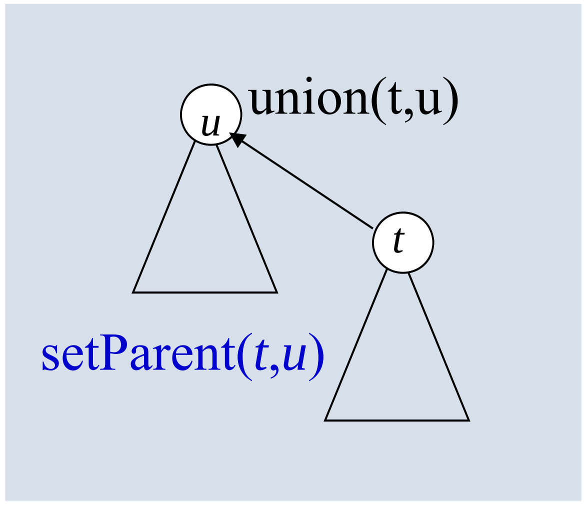
# Using Rooted Tree

- **IS** $s_i \equiv s_j$ :
  - o $t$=find($s_i$);
  - o $u$=find($s_j$);
  - o ($t$==$u$)?
- **MAKE** $s_i \equiv s_j$ :
  - o $t$=find($s_i$);
  - o $u$=find($s_j$);
  - o union($t$,$u$);

**implementation by inTree**







# Union-Find Program

- **A union-find program of length *m***
  - is (a *create*($n$) operation followed by) a sequence of $m$ union and/or find operations in any order
- **A union-find program is considered an input**
  - The object on which the analysis is conducted
- **The measure: number of accesses to the *parent***
  - assignments: for union operations
  - lookups: for find operations

$$\left.\begin{array}{l}\text{assignments: for union operations}\\ \text{lookups: for find operations}\end{array}\right\}\ \textbf{link operation}$$

## Worst-case Analysis for Union-Find Program

- Assuming each lookup/assignment take O(1).
- Each makeSet or union does one assignment, and each find does $d$+1 lookups, where $d$ is the depth of the node.

```
1.    Union(1,2)
2.    Union(2,3)
        ⋮
n-1.  Union(n-1,n)
n.    Find(1)
        ⋮
m.    Find(1)
```

Example

The sequence of *Union* makes a chain of length $n$-1, which is the tree with the largest height

*Find*(1) needs $n$ array lookups

operations done:
$n+(n-1)+(m-n+1)n$ $\Theta(mn)$

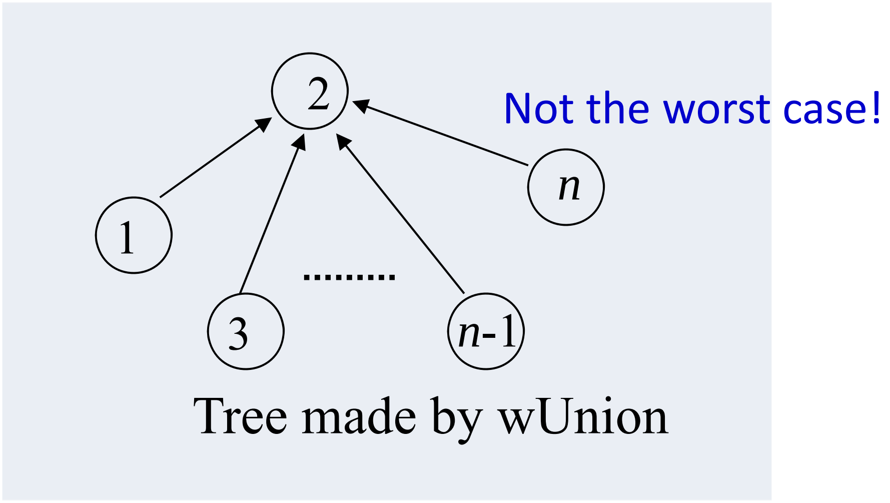
# Weighted Union: for Short Trees

- **Weighted union (*wUnion*)**
  - always have the tree with **fewer nodes** as subtree

To keep the *Union* valid, each *Union* operation is replaced by:

$t$=find($i$);
$u$=find($j$);
union($t$,$u$)

The order of ($t$,$u$) satisfying the requirement

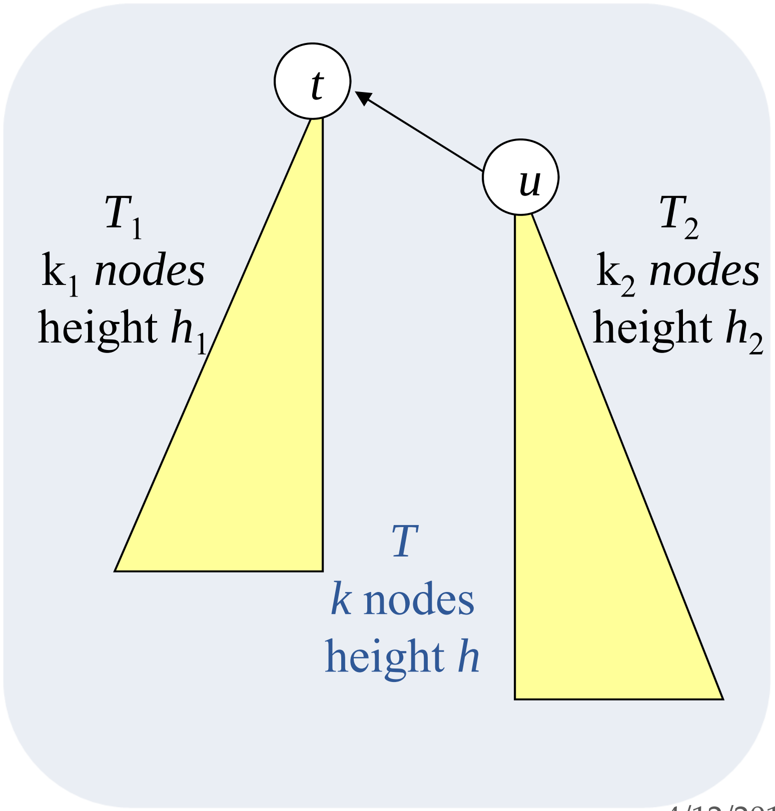


Tree made by wUnion

Cost for the program:
$n+3(n-1)+2(m-n+1)$

# Upper Bound of Tree Height

- **After any sequence of *Union* instructions, implemented by *wUnion*, any tree that has *k* nodes will have height at most $\lfloor \log k \rfloor$**
- **Proof by induction on *k*:**
  - o base case: $k$=1, the height is 0.
  - o by inductive hypothesis:
    - $h_1 \le \lfloor \lg k_1 \rfloor$, $h_2 \le \lfloor \lg k_2 \rfloor$
  - o h=max(h1, h2+1), k=k1+k2
    - if $h=h_1$, $h \le \lfloor \lg k_1 \rfloor \le \lfloor \lg k \rfloor$
    - if $h=h_2+1$, note: $k_2 \le k/2$
      so, $h_2+1 \le \lfloor \lg k_2 \rfloor + 1 \le \lfloor \lg k \rfloor$

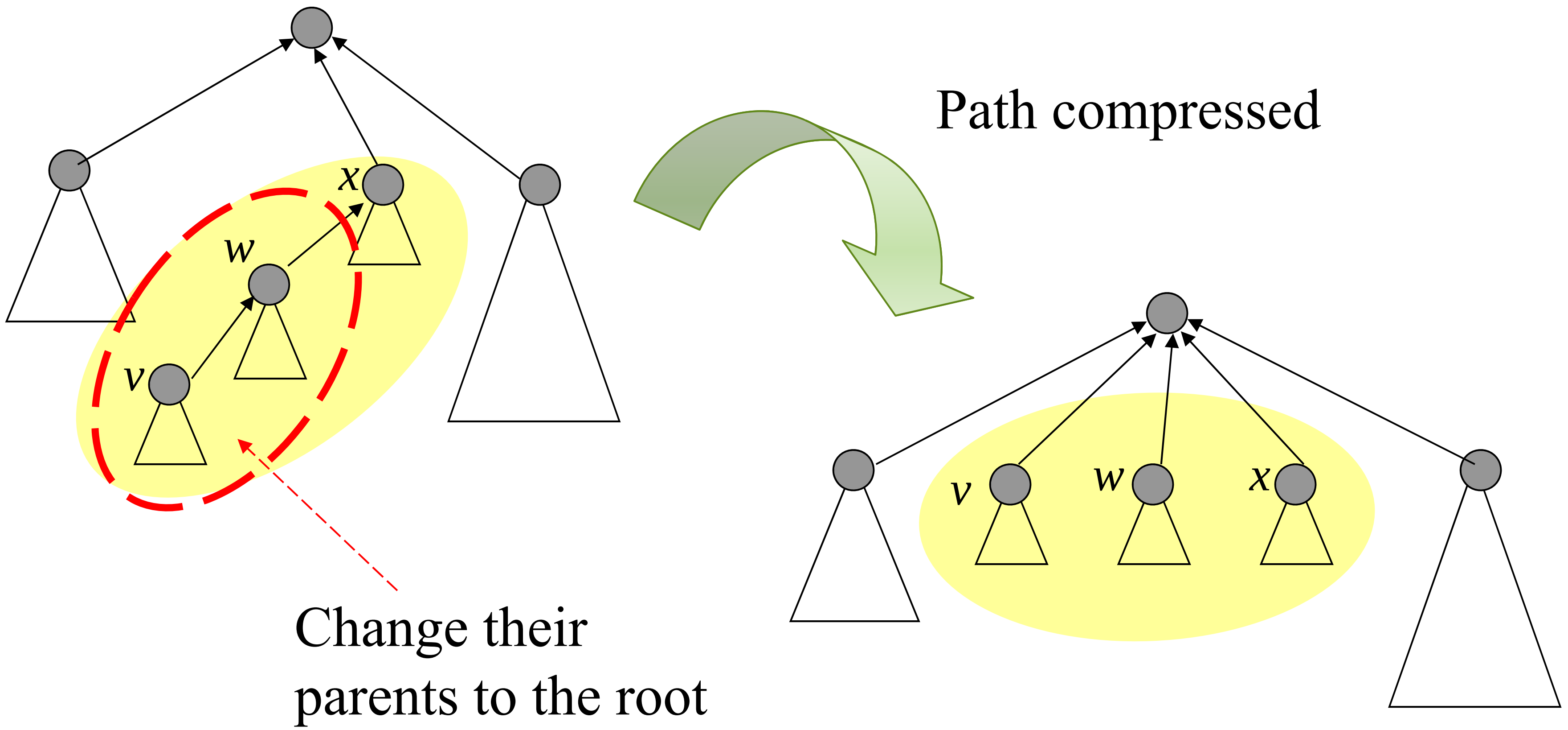


# Upper Bound for Union-Find Program

- **A Union-Find program of size *m*, on a set of *n* elements, performs O(*n*+*m*log*n*) link operations in the worst case if *wUnion* and straight *find* are used**
- **Proof:**
  - o At most *n*-1 *wUnion* can be done, building a tree with height at most $\lfloor \log n \rfloor$,
  - o Then, each *find* costs at most $\lfloor \log n \rfloor+1$.
  - o Each *wUnion* costs in *O*(1), so, the upper bound on the cost of any combination of *m wUnion*/*find* operations is the cost of *m find* operations, that is $m(\lfloor \log n \rfloor+1) \in O(n+m\log n)$

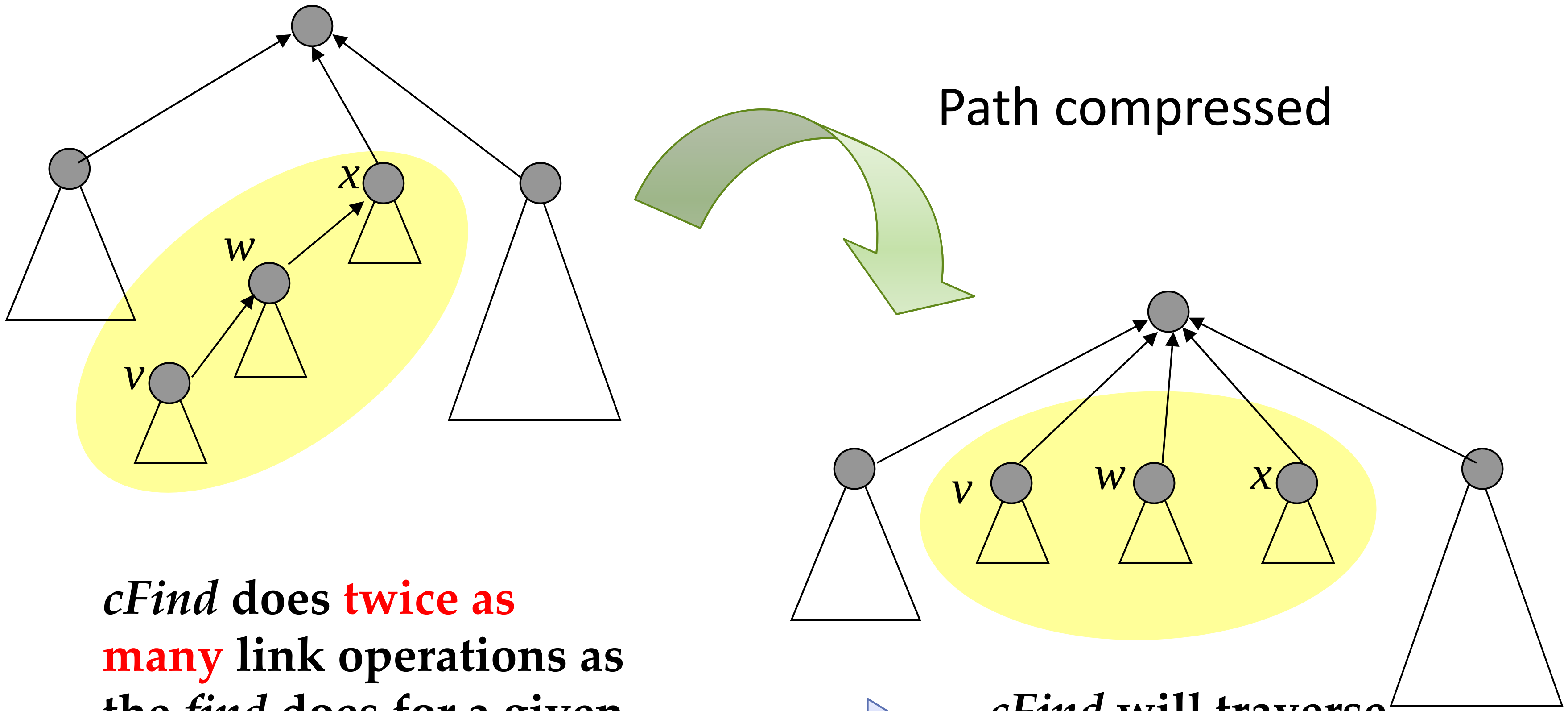
*There do exist programs requiring Ω(n+(m-n)logn) steps.*

# Path Compression



Change their
parents to the root

# Challenges for the Analysis



*cFind* does twice as many link operations as the *find* does for a given node in a given tree.

but…

*cFind* will traverse shorter paths

## Analysis: the Basic Idea

- ***c*Find may be an expensive operation**
  - in the case that find($i$) is executed and the node $i$ has great depth.
- **However, such *c*Find can be executed only for limited times**
  - Path compressions depends on previous unions
- **So, *amortized analysis* applies**

# Co-Strength of *wUnion* and *cFind*

- $O((n+m)\log^*(n))$
  - Link operations for a *Union-Find* program of length $m$ on a set of $n$ elements is in the worst case.
  - Implemented with *wUnion* and *cFind*

**What's log*($n$)?**

- Define the function $H$ as following:

$$\begin{cases} H(0)=1 \\ H(i)=2^{H(i-1)} \quad for\ i>0 \end{cases}$$

- Then, $\log^*(j)$ for $j\geq 1$ is defined as:

$$\log^*(j)=\min\{\, k \mid H(k)\geq j \,\}$$

# Definitions with a *Union-Find* Program *P*

- **Forest *F***: the forest constructed by the sequence of *union* instructions in *P*, assuming:
  - o *wUnion* is used;
  - o the *finds* in the *P* are ignored
- **Height** of a node *v* in any tree: the height of the subtree rooted at *v*
- **Rank** of *v*: the height of *v* ***in F***

Note: *cFind* changes the height of a node, but the rank for any node is invariable.

# Constraints on Ranks in *F*

- **The upper bound of the number of nodes with rank $r$ ($r \geq 0$) is $\frac{n}{2^r}$**
  - Remember that the height of the tree built by *wUnion* is at most $\lfloor \lg n \rfloor$, which means the subtree of height $r$ has at least $2^r$ nodes.
  - The subtrees with root at rank $r$ are disjoint.
- **There are at most $\lfloor \log n \rfloor$ different ranks.**
  - There are altogether n elements in S, that is, n nodes in F.

# Increasing Sequence of Ranks

- **The ranks of the nodes on a path from a leaf to a root of a tree in *F* form a strictly increasing sequence.**
- **When a *cFind* operation changes the parent of a node, the new parent has higher rank than the old parent of that node.**
    - Note: the new parent was an ancestor of the previous parent.

# A Function Growing Extremely Slowly

- **Function $H$:**

$$\begin{cases} H(0)=1 \\ H(i+1)=2^{H(i)} \end{cases}$$

that is: $H(k)=\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{k\ 2\text{'s}}$

Note:

$H$ grows extremely fast:

$H(4)=2^{16}=65536$

$H(5)=2^{65536}$

- **Function Log-star**

  $\log^*(j)$ is defined as the least $i$ such that:

  $H(i)\geq j$ for $j>0$

- **Log-star grows extremely slowly**

$$\lim_{n\to\infty}\frac{\log^*(n)}{\log^{(p)} n}=0$$

***p* is any fixed nonnegative constant**

**For any $x$: $2^{16}\leq x\leq 2^{65536}-1$, $\log^*(x)=5$ !**

# Grouping Nodes by Ranks

- **Node $v \in s_i$ ($i \geq 0$) iff. log*(1+rank of $v$)=$i$**
  - which means that: if node $v$ is in group $i$, then $r_v \leq H(i)-1$, but not in group with smaller labels
- **So,**
  - Group 0: all nodes with rank 0
  - Group 1: all nodes with rank 1
  - Group 2: all nodes with rank 2 or 3
  - Group 3: all nodes with its rank in [4,15]
  - Group 4: all nodes with its rank in [16, 65535]
  - Group 5: all nodes with its rank in [65536, ???]

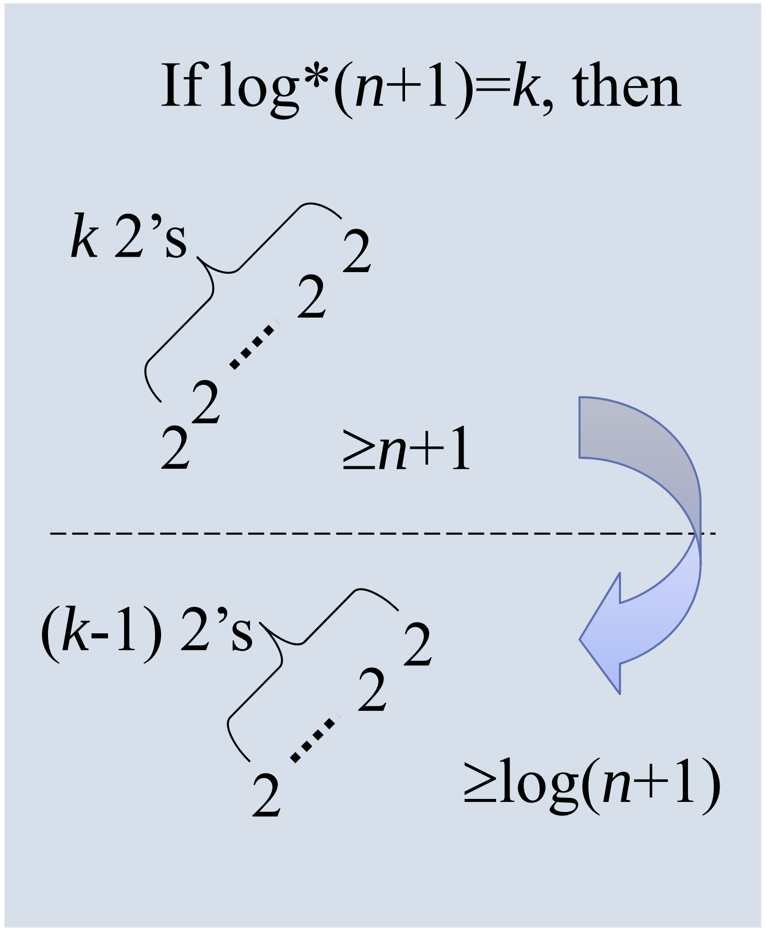
Group 5 exists only when *n* is at least $2^{65536}$. What is that?

# Very Few Groups

- **Node $v \in S_i$ ($i \geq 0$) iff.**

  **log*(1+rank of $v$)=$i$**

- **Upper bound of the number of distinct node groups is log*($n$+1)**
  - The rank of any node in $F$ is at most $\lfloor \log n \rfloor$, so the largest group index is log*(1+$\lfloor \log n \rfloor$)=log*($\lceil \log n+1 \rceil$) = log*($n$+1)-1



If log*($n$+1)=$k$, then
$k$ 2's
$2^{2^{\cdot^{\cdot^{2}}}}$
$\geq n+1$
($k$-1) 2's
$2^{\cdot^{\cdot^{2^{2}}}}$
$\geq \log(n+1)$

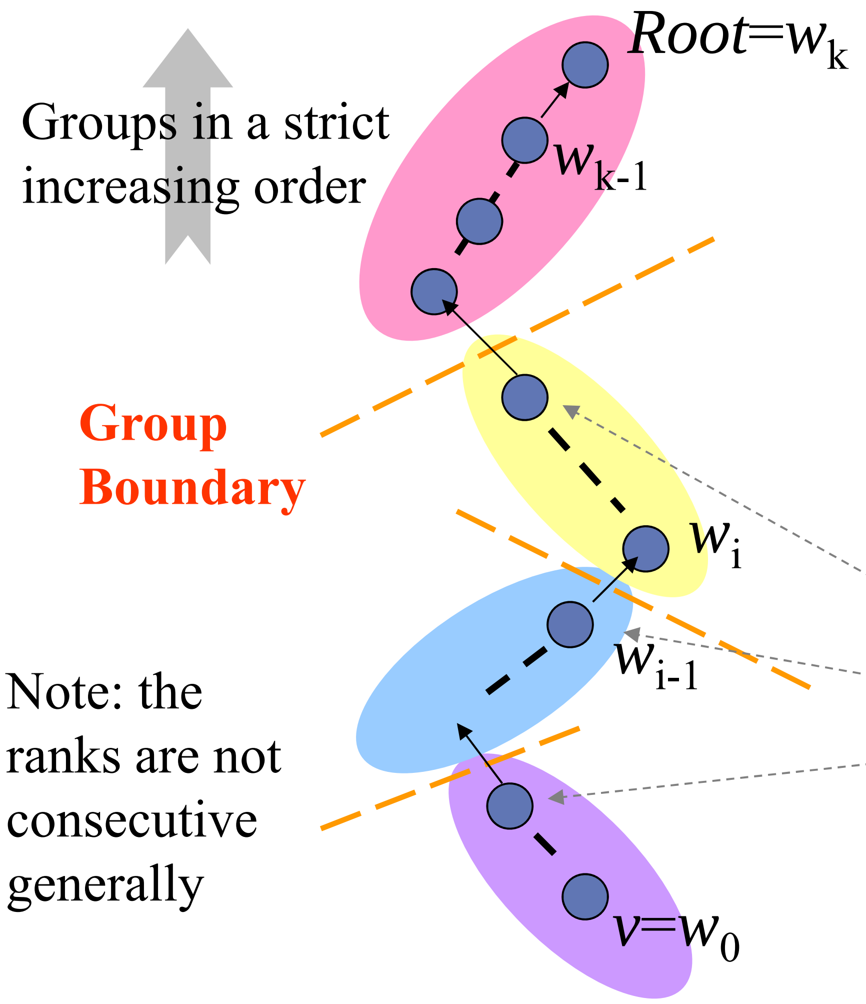
## Amortized Cost of *Union-Find*

- **Amortized Equation Recalled**

$$\boxed{\text{amortized cost} = \text{actual cost} + \text{accounting cost}}$$

- **The operations to be considered:**
  - o *n* makeSets
  - o *m* union & find (with at most *n*-1 unions)

# One Execution of *cFind*($w_0$)



$Root=w_k$
Groups in a strict increasing order
$w_{k-1}$
Group Boundary
$w_i$
$w_{i-1}$
Note: the ranks are not consecutive generally
$v=w_0$

Only when $k$=0,1, there is no parent change

For one *cFind* operation, the actual cost is $2k$ Not $2(k+1)$

Accounting cost is -2 for each pair of ($w_{i-1}$, $w_i$) for the the 2 nodes in ths **same group only**, which we call a **withdrawal**.

# Amortizing Scheme for *wUnion-cFind*

- **makeSet**
  - Accounting cost is $4\log^*(n+1)$
  - So, the amortized cost is $1+4\log^*(n+1)$
- ***wUnion***
  - Accounting cost is 0
  - So the amortized cost is 1
- ***cFind***
  - Accounting cost is describes as in the previous page.
  - Amortized cost $\leq 2k-2(\underbrace{(k-1)-(\log^*(n+1)-1)}_{\text{Number of withdrawal}})=2\log^*(n+1)$ (Compare with the worst case cost of *cFind*, $2\log n$)

## Validation of the Amortizing Scheme

- **We must be assure that the sum of the accounting costs is never negative.**
- **The sum of the negative charges, incurred by *cFind*, does not exceed 4*n*log*(*n*+1)**
    - We prove this by showing that at most 2*n*log*(*n*+1) withdrawals on nodes occur during all the executions of *c*Find.

## Key Idea in the Derivation

- **For any node, the number of withdrawal will be less than the number of different ranks in the group it belongs to**
  - When a *cFind* changes the parent of a node, the new parent is always has higher rank than the old parent.
  - Once a node is assigned a new parent in a higher group, no more negative amortized cost will incurred for it again.
- **The number of different ranks is limited within a group.**

## Derivation

- **Bounding the number of withdrawals**

The number of withdrawals from all $w \in S$ is:

a loose upper bound of ranks in a group

$$\sum_{i=0}^{log^*(n+1)-1} H(i)(\text{number of nodes in group } i)$$

The number of nodes in group $i$ is at most:

$$\sum_{r=H(i-1)}^{H(i)-1} \frac{n}{2^r} \leq \frac{n}{2^{H(i-1)}} \sum_{j=0}^{\infty} \frac{1}{2^j} = \frac{2n}{2^{H(i-1)}} = \frac{2n}{H(i)}$$

So,

$$\sum_{i=0}^{log^*(n+1)-1} H(i) \frac{2n}{H(i)} = 2n \log^*(n+1)$$

## Conclusion

- **The number of link operations done by a *Union-Find* program implemented with *wUnion* and *cFind*, of length $m$ on a set of $n$ elements is in $O((n+m)\log^*(n))$ in the worst case.**
  - Note: since the sum of accounting cost is never negative, the actual cost is always not less than amortized cost. The upper bound of amortized cost is: $(n+m)(1+4\log^*(n+1))$

*Thank you!*

*Q & A*

*Yu Huang*
http://cs.nju.edu.cn/yuhuang