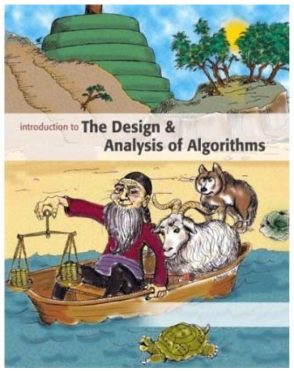
Introduction to

Algorithm Design and Analysis

[20] NP Complete Problems 2



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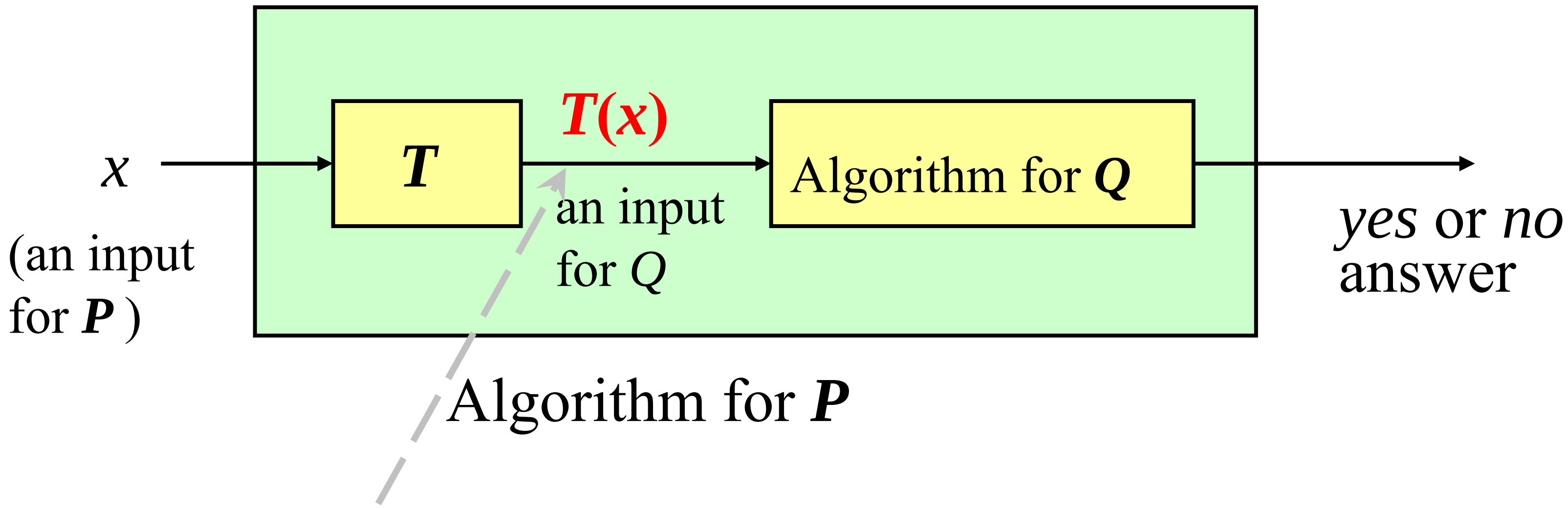
In the Last Class…

- **Decision Problem**
- **The Class *P***
- **The Class *NP***

In This Class

- **Reduction between problems**
- **NP-Complete Problems**
  - No known polynomial time algorithm
  - Computationally related by reduction
- **Other advanced topics**
  - Advanced algorithms
  - Advanced computation models

Reduction



The correct answer for ***P*** on x is *yes* **if and only if** the correct answer for ***Q*** on *T*(*x*) is *yes*.

$\mathcal{NP}$-complete Problems

- **A problem Q is *$\mathcal{NP}$-hard* if every problem P in $\mathcal{NP}$ is reducible to Q, that is $P \leq_P Q$.**

  (which means that Q is at least as hard as any problem in $\mathcal{NP}$)

- **A problem Q is *$\mathcal{NP}$-complete* if it is in $\mathcal{NP}$ and is *$\mathcal{NP}$-hard*.**

  (which means that Q is at most as hard as to be solved by a polynomially bounded nondeterministic algorithm)

Example of an *NP*-hard problem

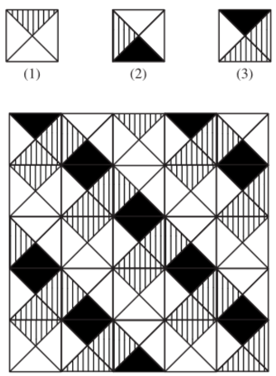
- **Halt problem: Given an arbitrary deterministic algorithm *A* and an input *I*, does *A* with input *I* ever terminate?**
  - A well-known **undecidable** problem, of course not in $\mathcal{NP}$.
  - Satisfiability problem is reducible to it.
    - Construct an algorithm *A* whose input is a propositional formula *X*. If *X* has *n* variables then *A* tries out all 2^n possible truth assignments and verifies if *X* is satisfiable. If it is satisfiable then *A* stops. Otherwise, *A* enters an infinite loop.
    - So, *A* halts on *X* iff. *X* is satisfiable.

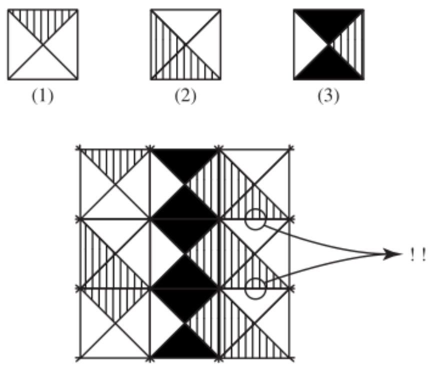
More Undecidable Problems

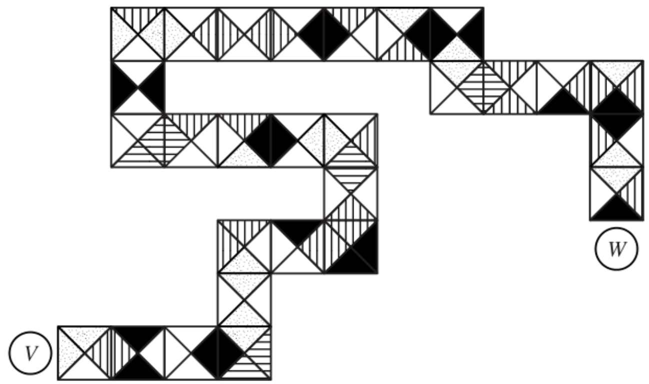
- **Arithmetical SAT**

$$x^3yz + 2y^4z^2 - 7xy^5z = 6$$

- **The *tiling* problem**



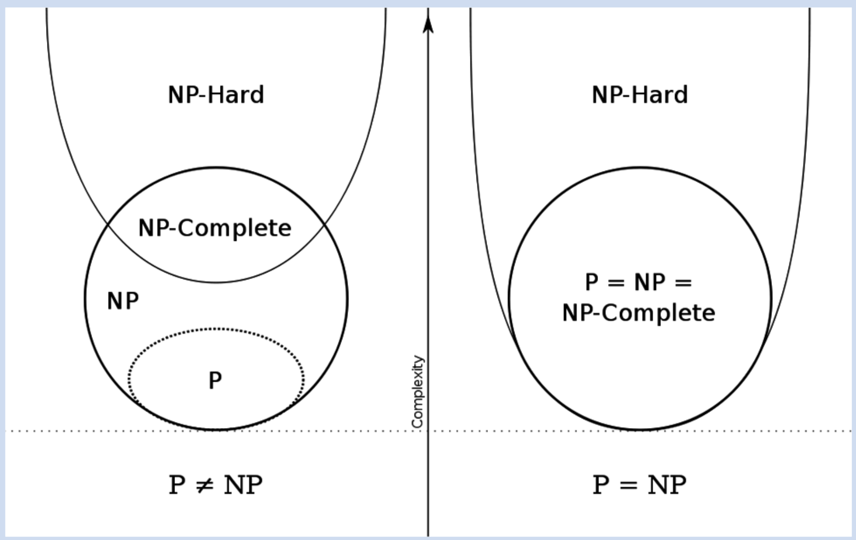


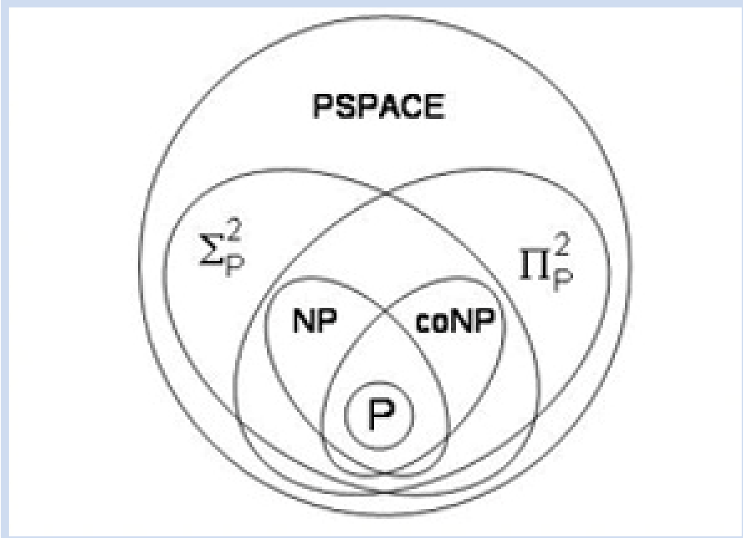


$\mathcal{P}$ and $\mathcal{NP}$ - Revisited

- **Intuition implies that $\mathcal{NP}$ is a much larger set than $\mathcal{P}$.**
  - No one problem in $\mathcal{NP}$ has been proved not in $\mathcal{P}$.
- **If any $\mathcal{NP}$-completed problem is in $\mathcal{P}$, then $\mathcal{NP}=\mathcal{P}$.**
  - Which means that every problems in $\mathcal{NP}$ can be reducible to a problem in $\mathcal{P}$!
  - Much more questionable

$\mathcal{P}$ and $\mathcal{NP}$ - Revisited





Procedure for NP-Completeness

- **Knowledge: *P* is *NPC***
- **Task: to prove that *Q* is *NPC***
- **Approach: to reduce *P* to *Q***
  - For any $R \in \mathbf{NP}$, $R \leq_P P$
  - Show $P \leq_P Q$
  - Then $R \leq_P Q$, by transitivity of reductions
  - Done. Q is NP-complete (given that Q has been proven in NP)

First Known $\mathcal{NPC}$ Problem

- **Cook's theorem:**
  - The SAT problem is NP-complete.
- **Reduction as tool for proving *NP*-completeness**
  - Since *CNF-SAT* is known to be *NP*-hard, then all the problems, to which *CNF-SAT* is reducible, are also *NP*-hard. So, the formidable task of proving *NP*-complete is transformed into relatively easy task of proving of being in $\mathcal{NP}$.

Proof of Cook's Theorem

COOK, S. 1971.

The complexity of theorem-proving procedures.

In

Conference Record of

3rd Annual ACM Symposium on Theory of Computing.

ACM New York, pp. 151–158.

Stephen Arthur Cook: b.1939 in Buffalo, NY. Ph.D of Harvard. Professor of Toronto Univ. 1982 Turing Award winner. The Turing Award lecture: "An Overview of Computational Complexity", CACM, June 1983, pp.400-8

Satisfiability Problem

- **CNF**
  - A literal is a Boolean variable or a negated Boolean variable, as x or $\bar{x}$
  - A clause is several literals connected with $\vee$ s, as $x_1 \vee \overline{x_2}$
  - A CNF formula is several clause connected with $\wedge$ s
- **CNF-SAT problem**
  - Is a given CNF formula satisfiable, i.e. taking the value TRUE on some assignments for all x_i.
- **A special case: 3-SAT**
  - 3-SAT: each clause can contain at most 3 literals

Proving NPC by Reduction

- **The *CNF-SAT* problem is *NP*-complete.**
- **Prove problem *Q* is *NP*-complete, given a problem *P* known to be *NP*-complete**
  - For all $R \in \mathbf{NP}$, $R \leq_P P$;
  - **Show $P \leq_P Q$;**
  - By transitivity of reduction, for all $R \in \mathbf{NP}$, $R \leq_P Q$;
  - So, Q is ***NP***-hard;
  - If Q is in ***NP*** as well, then Q is ***NP***-complete.

Max Clique Problem is NP

```
void nondeteClique(graph G; int n, k)
   set S=ϕ;
   for int i=1 to k do
      int t=genCertif();
      if t∈S then return;
      S=S∪{t };
   for all pairs (i,j) with i,j in S and i≠j do
      if (i,j ) is not an edge of G
         then return;
      Output("yes");
```

Example 1: Max Clique Problem is NPC

In $O(n)$

In $O(k^2)$

So, we have an algorithm for the maximal clique problem with the complexity of $O(n+k^2)=O(n^2)$

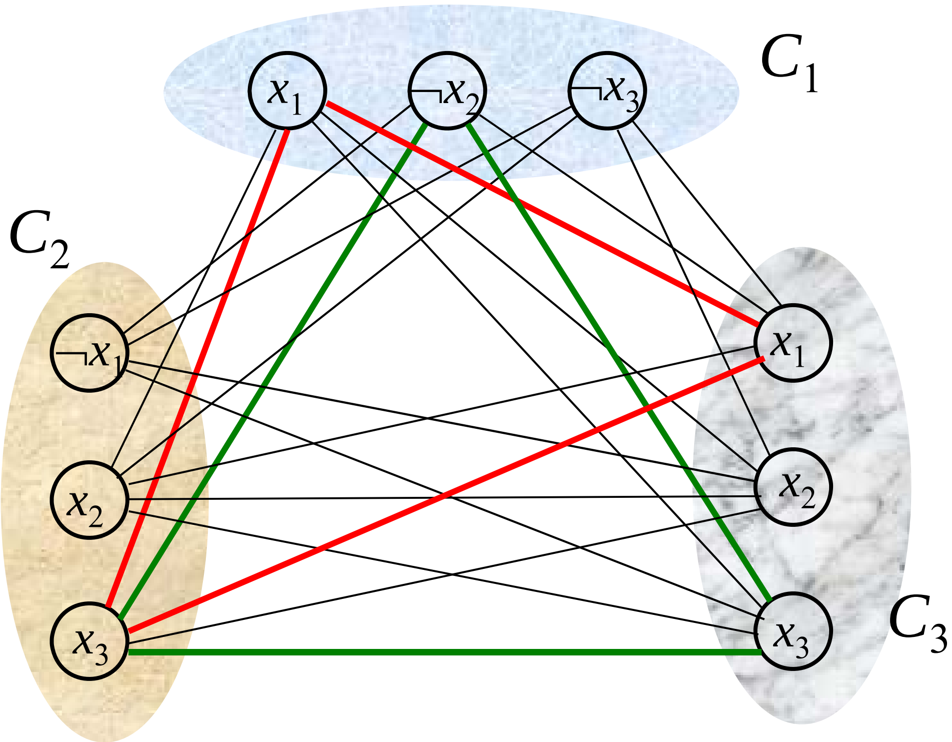
CNF-SAT to Clique

- **Let $\phi=C_1\wedge C_2\wedge\ldots\wedge C_k$ be a formula in CNF-3 with k clauses. For $r=1,2,\ldots,k$, each clause $C_r=(l_1^r\vee l_2^r\vee l_3^r)$, l_i^r is x_i or $\neg x_i$, any of the variables in the formula.**
- **A graph can be constructed as follows. For each C_r, create a triple of vertices v_1^r, v_2^r and v_3^r, and create edges between v_i^r and v_j^s if and only if:**
  - they are in different triples, i.e. $r\neq s$, and
  - they do not correspond to the literals negating each other

(Note: there is no edges within one triple)

3-CNF Graph

$$\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$$



Two of satisfying assignments:

x_1=1/0, x_2=0; x_3=1, or

x_1=1, x_2=1/0, x_3=1

For corresponding clique, pick one "true" literal from each triple

Clique Problem is NP-Complete

- **ϕ, with k clauses, is satisfiable iff. The graph G has a clique of size k.**
- **Proof: $\Rightarrow$**
  - Suppose that ϕ has a satisfying assignment.
  - Then **there is at least one "true" literal in each clause**. Picking such a literal from each clause, their corresponding vertices in G can be proved to be a clique, since any two of them are in different triples and cannot be complements to each other(they are both true).

Known NP-Complete Problems

- **Garey & Johnson:** ***Computer and Intractability: A Guide to the Theory of NP-Completeness*****, Freeman, 1979**
  - About 300 problems, grouped in 12 categories:

1. Graph Theory	2. Network Design	3. Set and Partition
4. Storing and Retrieving	5. Sorting and Scheduling	
6. Mathematical Planning	7. Algebra and Number Theory	
8. Games and Puzzles	9. Logic	
10. Automata and Theory of Languages		
11. Optimization of Programs	12. Miscellaneous	

Advanced Topics

- **Solving hard problems**
  - Approximation algorithms
  - Randomized algorithms
- **Solving more complex problems**
  - Online algorithms
  - External memory models
  - Distributed computation models

Approximation

- **Make modifications on the problem**
  - Restrictions on the input
  - Change the criteria for the output
  - Find new abstractions for a practical situation
- **Find approximate solution**
  - Algorithm
  - Bound of the errors

Bin Packing Problem

- **Suppose we have**
  - An unlimited number of bins each of capacity one, and n objects with sizes s_1, s_2, …, s_n where $0<s_i\leq 1$ (s_i are rational numbers)
- ***Optimization problem***
  - Determine the smallest number of bins into which the objects can be packets (and find an optimal packing) .
- **Bin packing is a NPC problem**

Feasible Solution

- **Set of feasible solutions**
  - For any given input $I=\{s_1,s_2,\ldots,s_n\}$, the feasible solution set, $FS(I)$ is the set of all **valid packing** using any number of bins.
  - In other words, that is the set of all partitions of I into disjoint subsets $T_1,T_2,\ldots,T_p$, for some p, such that the **total of the s_i in any subset is at most 1**.

Optimal Solution

- **In the bin packing problem, the optimization parameter is the number of bins used.**
  - For any given input I and a feasible solution x, ***val***(I,x) is the value of the optimization parameter.
  - For a given input I, the optimum value, ***opt***$(I)=\min\{val(I,x) \mid x \in FS(I)\}$
- **An optimal solution for *I* is a feasible solution which achieves the optimum value.**

Approximate Algorithm

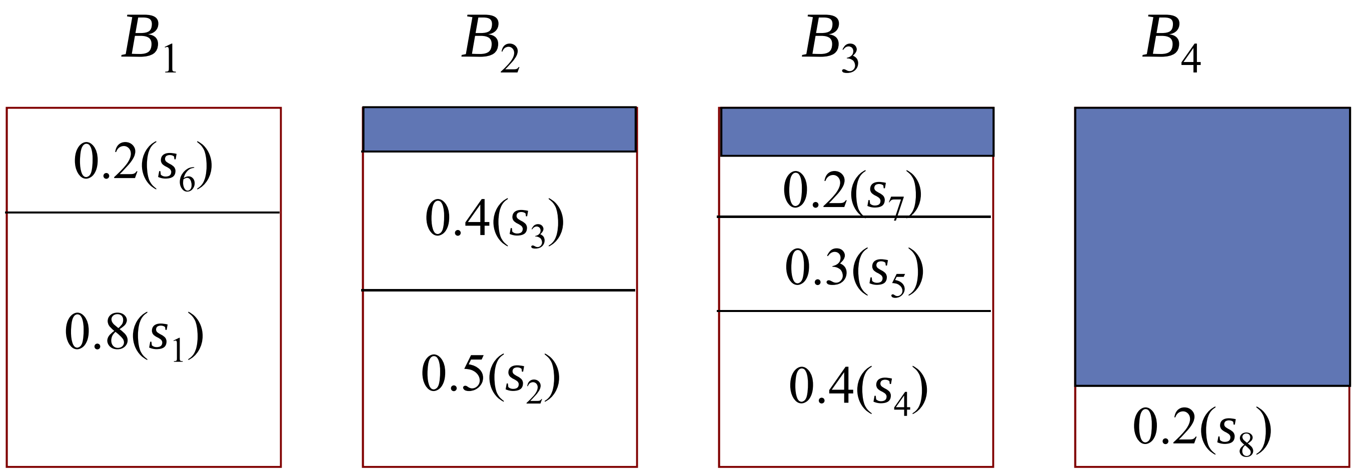
- **An approximation algorithm A for a problem**
  - Polynomial-time algorithm that, when given input I, output an element of FS(I).
- **Quality of an approximation algorithm.**

$$r_A(I) = \frac{val(I, A(I))}{opt(I)} \quad \text{or} \quad r_A(I) = \frac{opt(I)}{val(I, A(I))}$$

  - RA(m) = max $\{r_A(I) \mid I \text{ such that } opt(I)=m\}$
- **Bounded RA(m)**
  - For an approximation algorithm, we hope the value of RA(m) is bounded by small constants.

First Fit Decreasing - FFD

- The strategy: packing the largest as possible
- Example: S=(0.8, 0.5, 0.4, 0.4, 0.3, 0.2, 0.2, 0.2)



This is ***NOT*** an optimal solution!

The Procedure

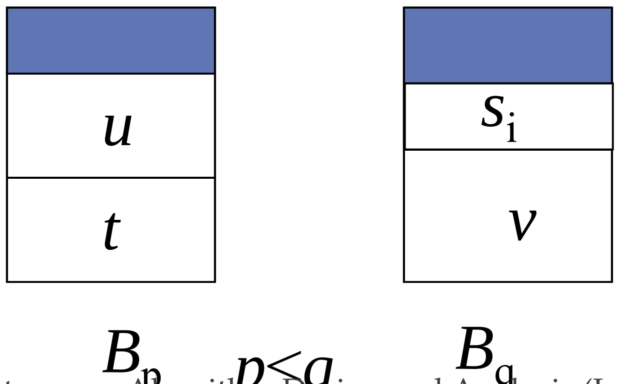
```
binpackFFD(S, n, bin) //bin is filled and output, object i is packed in bin[i]
   float[] used=new float[n+1]; //used[j] is the occupied space in bin j
   int i,j;
   <initialize all used entries to 0.0>
   <sort S into nonincreasing order> // in S after sorted

   for (i=1; i≤n; i++)
      for (j=1; j≤n; j++)
         if (used[j]+S[i]≤1.0)
            bin[i]=j;
            used[j]+=S[i];
            break;
```

Small Objects in Extra Bins

- **Problem formulation**
  - Let $S=\{s_1, s_2, \ldots, s_n\}$ be an input, **in nonincreasing order**
  - Let $opt(S)$ be the minimum number of bins for S.
- **All of the objects placed by FFD in the extra bins have size at most 1/3.**
- **Let i be the index of the first object placed by FFD in bin $opt(S)+1$.**
  - What we have to do for the proof is: $s_i \le 1/3$.

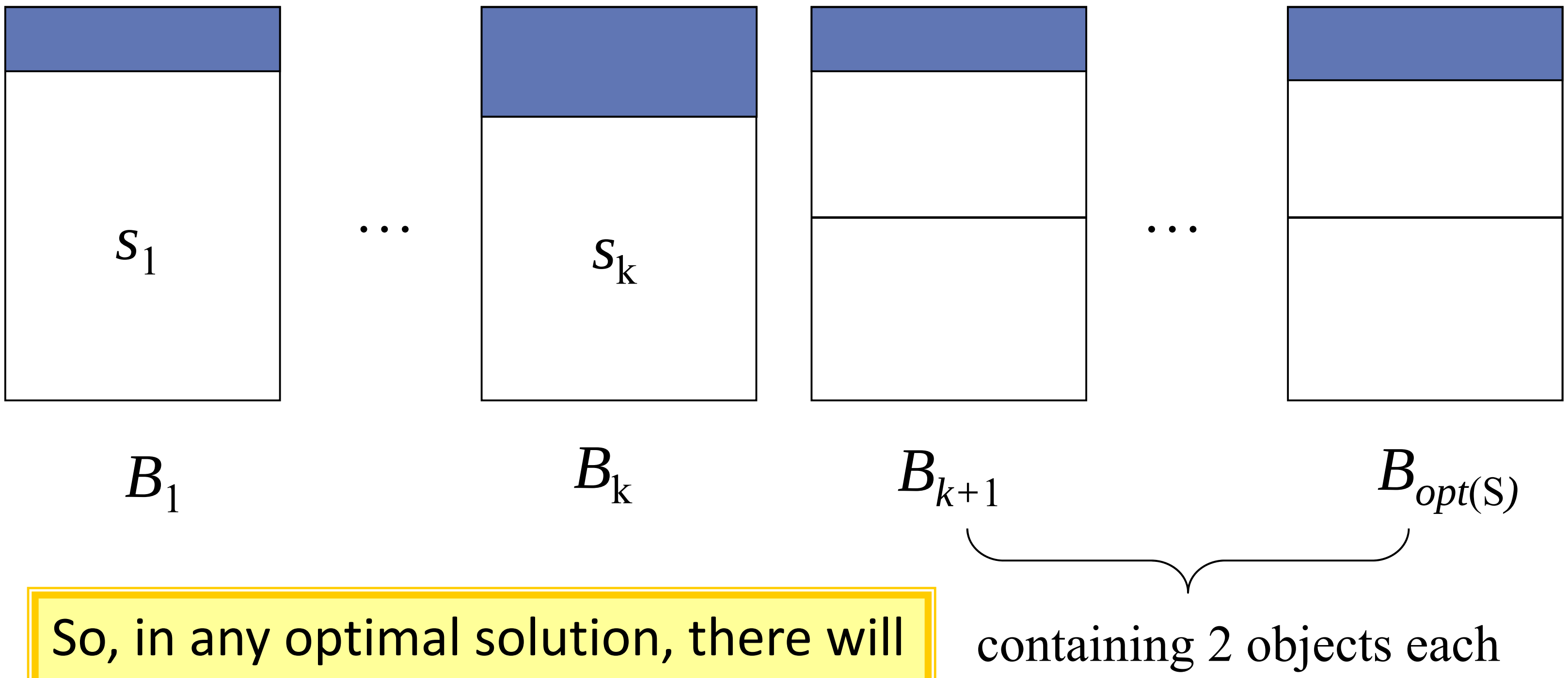
What about a s_i Larger than 1/3?

- **[S is sorted]** **The $s_1, s_2, \ldots, s_{i-1}$ are all larger than 1/3.**
- **So, bin B_j for j=1,…,$opt(S)$ contain at most 2 objects each.**
- **Then, for some $k \geq 0$, the first k bins contain one object each and the remaining $opt(S)$-k bins contain two each.**
  - Proof: no situation (that is, some bin containing 2 objects has a smaller index than some bin containing only one object) as the following is possible



Then: we must have:
$t>v$, $u>s_i$, so $v+s_i<1$, no extra bin is needed!

Considering S_i



So, in any optimal solution, there will be *k* bins that do not contain any of the objects *k*+1,…, *i*.

Contradiction at Last!

- Any optimal solution use only $opt(S)$ bins.
- However, there are k bins that do not contain any of the objects k+1, …, i-1, i. k+1,…, i-1 must occupy $opt(S)$-k bins, with each bin containing 2.
- Since all objects down through to s_i are larger than 1/3, s_i can not fit in any of the $opt(S)$-k bins.
- So, extra bin needed, and contradiction.

Objected in Extra Bins Bounded

- **For any input $S=\{s_1, s_2,\ldots,s_n\}$, the number of objects placed by FFD in extra bins is at most $opt(S)$-1.**

Since all the objects fit in $opt(S)$, $\sum_{i=1}^{n} s_i \le opt(S)$.

Assuming that FFD puts $opt(S)$ objects in extra bins,

and their sizes are : $t_1, t_2, \ldots, t_{opt(S)}$.

Let b_j be the final contents of bin B_j for $1 \le j \le opt(S)$.

Note $b_j + t_j > 1$, otherwise t_j should be put in B_j. So :

$$\sum_{i=1}^{n} s_i \ge \sum_{j=1}^{opt(S)} b_j + \sum_{j=1}^{opt(S)} t_j = \sum_{j=1}^{opt(S)} (b_i + t_i) > opt(S)$$; Contradiction!

A Good Approximation

- **Using FFD, the number of bins used is at most about 1/3 more than optimal value.**

$$R_{FFD}(m) \le \frac{4}{3} + \frac{1}{3m}$$

FFD puts at most *m*-1 objects in extra bins, and the size of the *m*-1 object are at most 1/3 each, so, FFD uses at most $\lceil (m-1)/3 \rceil$ extra bins.

$$r_{FFD}(S) \le \frac{m + \left\lceil \frac{m-1}{3} \right\rceil}{m} \le 1 + \frac{m+1}{3m} \le \frac{4}{3} + \frac{1}{3m}$$

Average Performance is Much Better

- **Empirical Studies on large inputs.**
  - The number of extra bins are estimated by the amount of empty space in the packings produced by the algorithm.
  - It has been shown that for n objects with sizes uniformly distributed between zero and one, the expected amount of empty space in packings by FFD is approximately $0.3\sqrt{n}$.

Randomized Algorithm

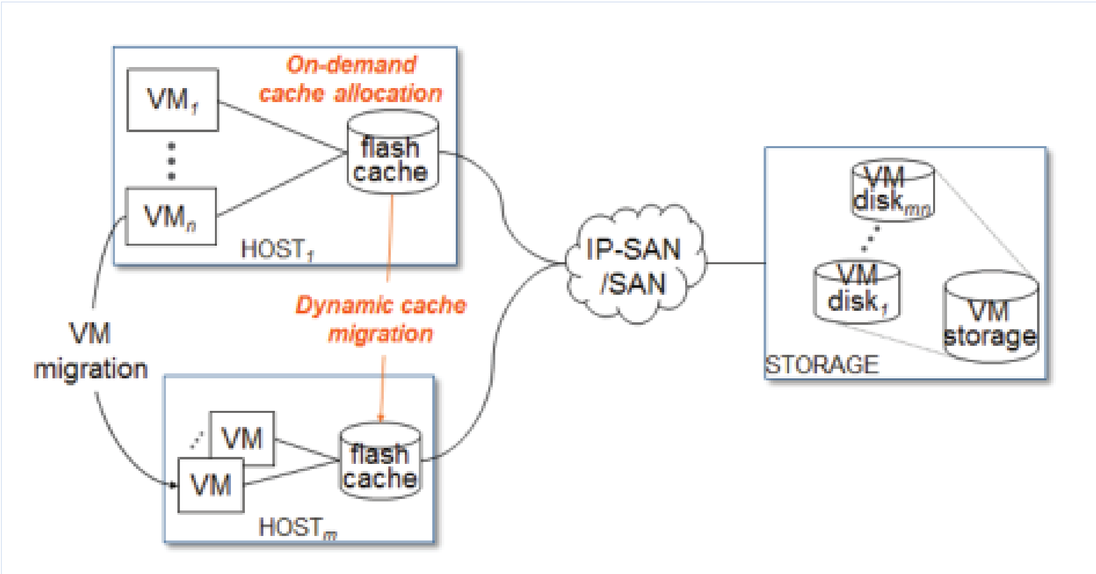
- **Mote Carlo**
  - Always finish in time
  - The answer may be incorrect
- **Las Vegas**
  - Always return the correct answer
  - The running time varies a lot

Online Algorithm

- **The main difference**
  - Offline algorithm: you can obtain all your input in advance
  - Online algorithm: you must cope with unpredictable inputs
- **How to analyze an online algorithm**
  - Competitive analysis: the performance of an online algorithm is compared to that of an optimal offline algorithm

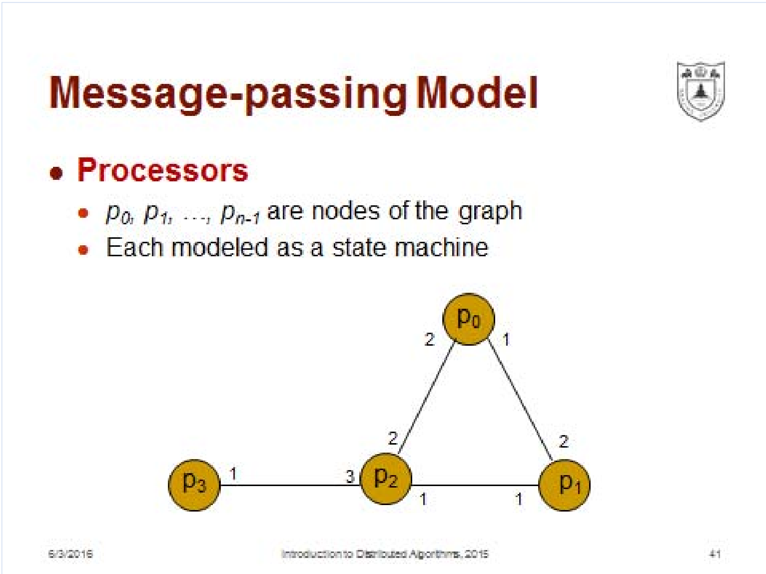
Distributed Data

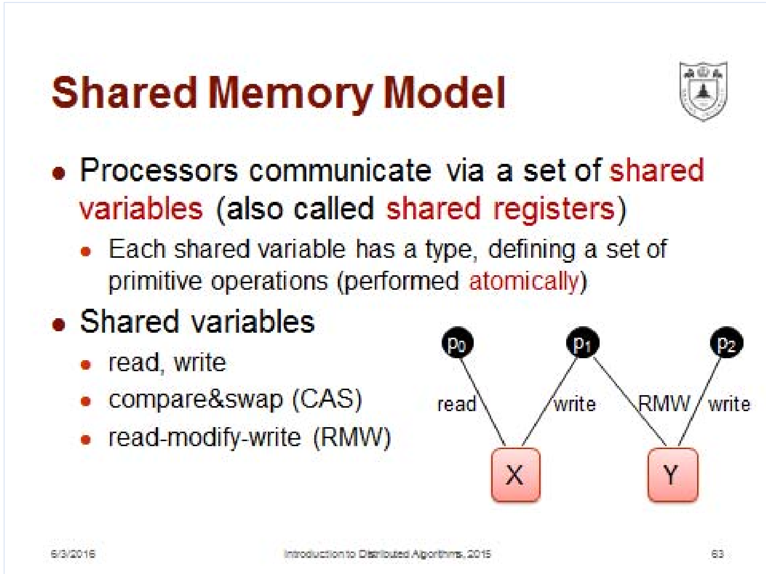
- External memory model



Distributed Computation

- **Model of distributed computation**





Thank you!

Q & A

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