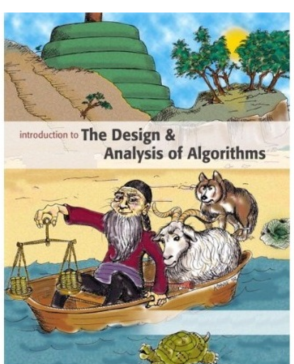
Introduction to

Algorithm Design and Analysis

[5] HeapSort



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In the last class …

- **The *sorting* problem**
  - Assumptions
- **InsertionSort**
  - Design
  - Analysis: inverse
- **QuickSort**
  - Design
  - Analysis

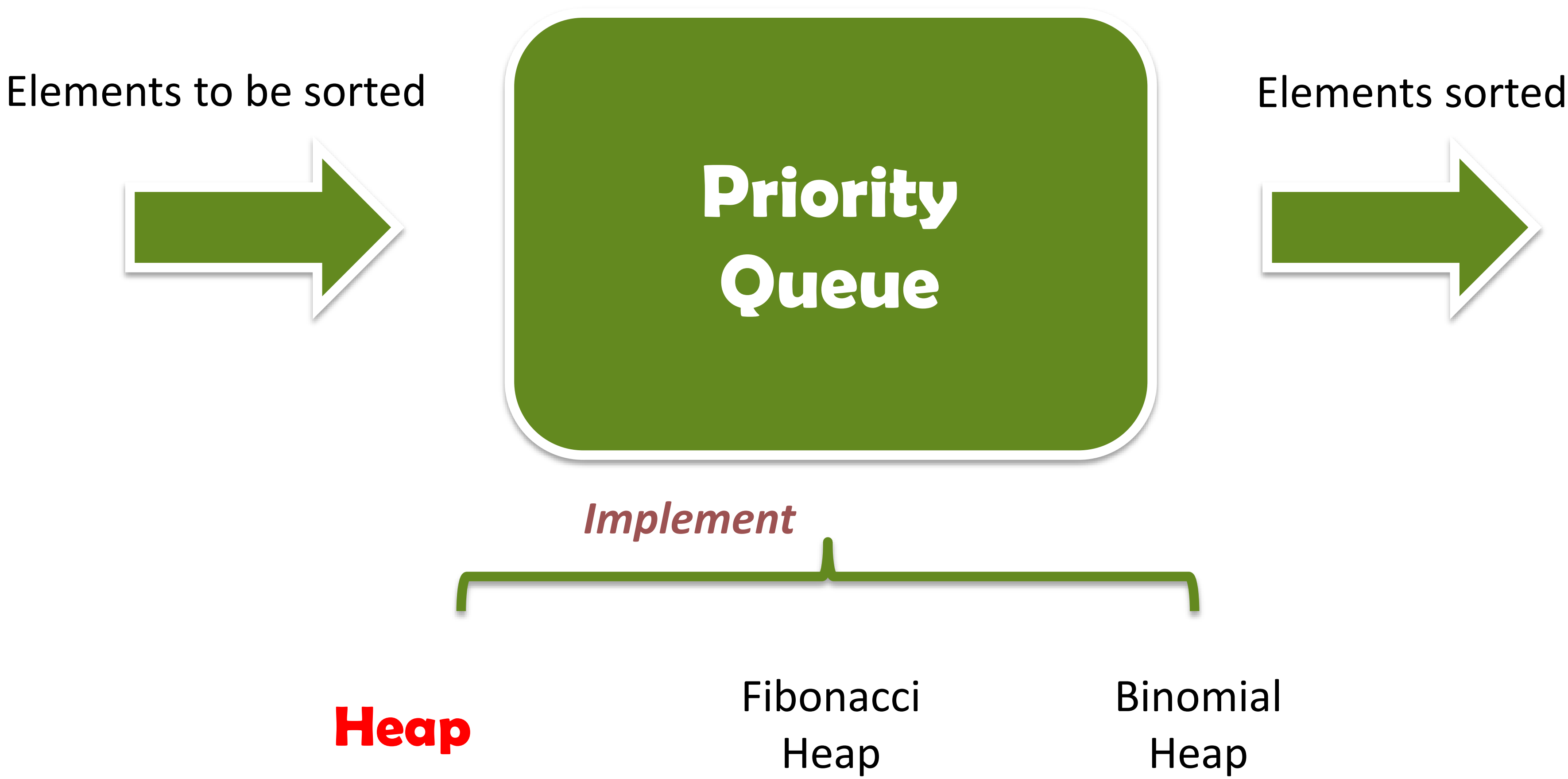
Heapsort

- Heap
- HeapSort

- FixHeap
- ConstructHeap

- Complexity of Heapsort
- Accelerated Heapsort

How HeapSort Works



Elementary Priority Queue ADT

- **"FIFO" in some special sense. The "first" means some kind of "priority", such as value(largest or smallest)**
  - **PriorityQ** create()
    - Precondition: none
    - Postconditions: If pq=create(), then, pq refers to a newly created object and isEmpty(pq)=**true**
  - **boolean** isempty(PriorityQ pq)
    - precondition: none
  - **int** getMax(PriorityQ pq)
    - precondition: isEmpty(pq)=**false**
    - postconditions: **
  - **void** insert(PriorityQ pq, **int** id, **float** w)
    - precondition: none
    - postconditions: isEmpty(pq)=**false**; **
  - **void** delete(PriorityQ pq)
    - precondition: isEmpty(pq)=**false**
    - postconditions: value of isEmpty(pq) updated; **
  - **void** increaseKey(PriorityQ pq, **int** id, **float** newKey)

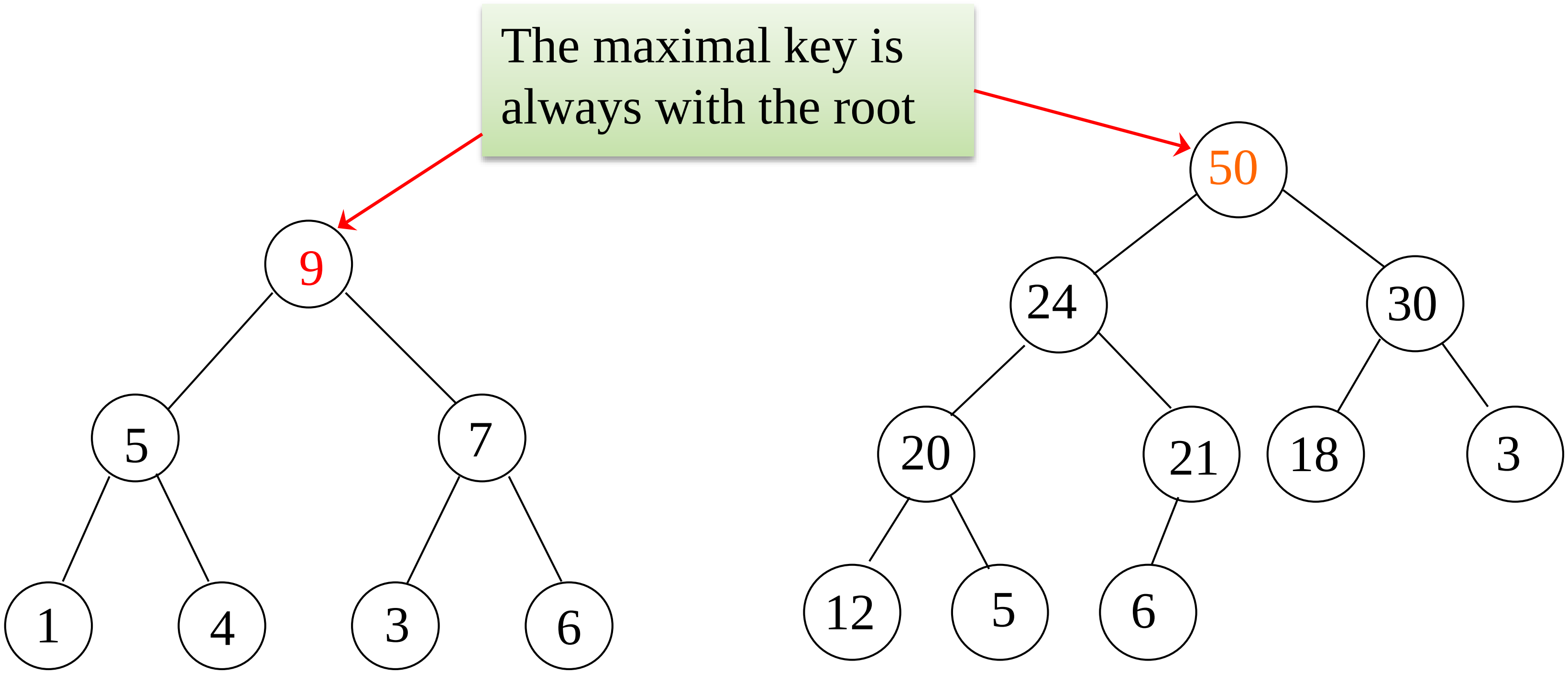
**pq can always be thought as a sequence of pairs (id_i, w_i), in non-decreasing order of w_i

Heap: an Implementation of Priority Queue

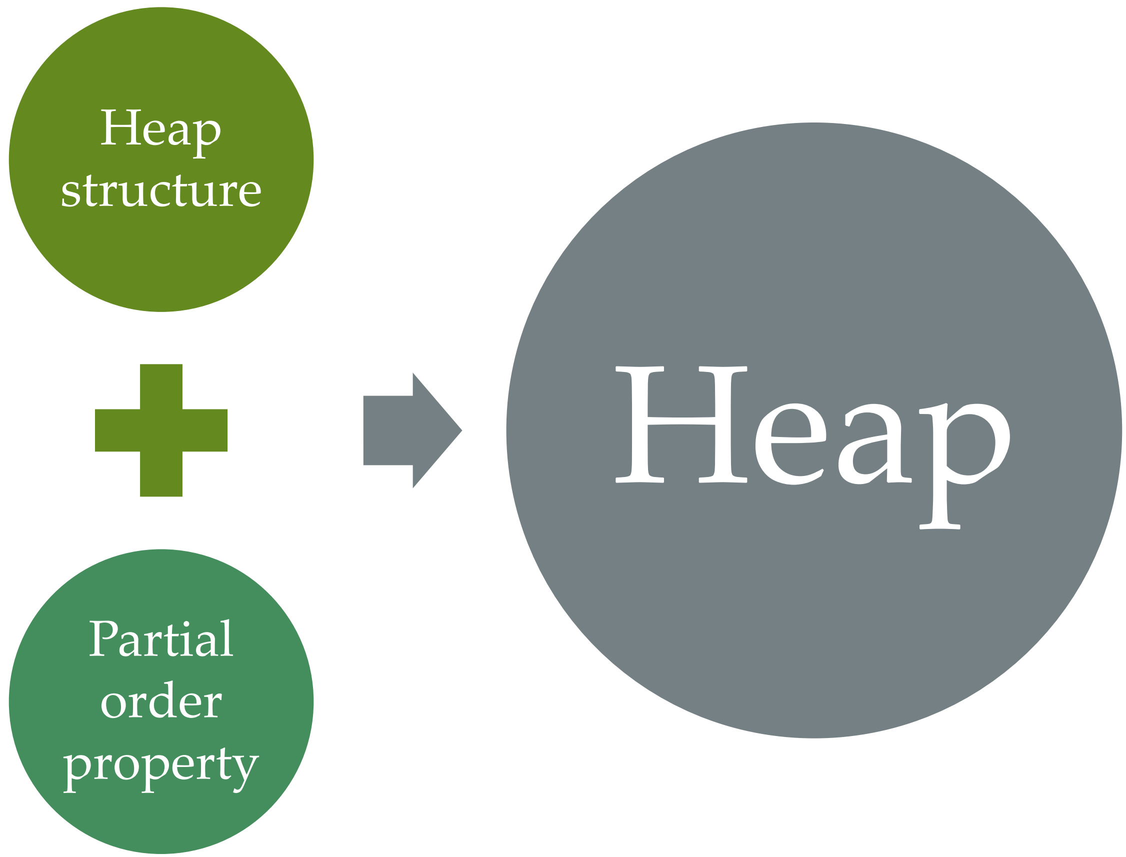
- **A binary tree *T* is a *heap structure* if:**
  - *T* is complete at least through depth *h*-1
  - All leaves are at depth *h* or *h*-1
  - All path to a leaf of depth *h* are to the left of all path to a leaf of depth *h*-1

- **Partial order tree *property***
  - A tree *T* is a (maximizing) partial order tree if and only if the key at any node is greater than or equal to the keys at each of its children (if it has any).

Heap: Examples



Heap: an Implementation of Priority Queue



HeapSort: the Strategy

```
heapSort(E,n)
    Construct H from E, the set of n elements to be sorted;
    for (i=n;i≥1;i--)
        curMax = getMax(H);
        deleteMax(H);
        E[i] = curMax

deleteMax(H)
    Copy the rightmost element on the lowest level of H into K;
    Delete the rightmost element on the lowest level of H;
    fixHeap(H,K)
```

FixHeap

- **Input: A nonempty binary tree H with a "vacant" root and its two subtrees in partial order. An element K to be inserted.**
- **Output: H with K inserted and satisfying the partial order tree property.**
- **Procedure:**

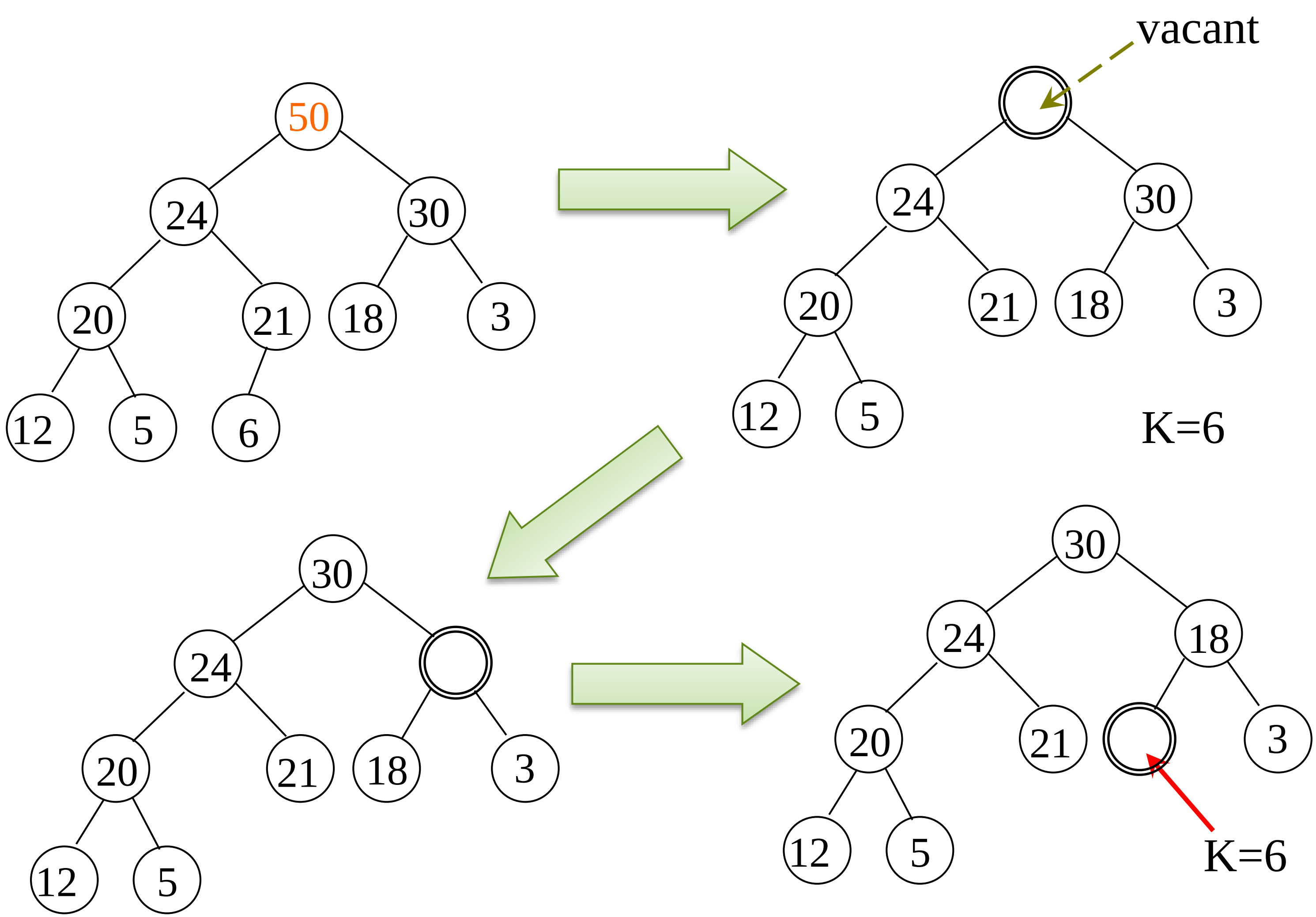
fixHeap(H,K)
    if (H is a leaf) insert K in root(H);
    else
        Set *largerSubHeap*;
        if (K.key≥root(largerSubHeap).key) insert K in root(H)
        else
            insert root(largerSubHeap) in root(H);
            fixHeap(largerSubHeap, K);
    return

One comparison:
largerSubHeap is left- or right-Subtree(H), the one with larger key at its root.
Special case: rightSubtree is empty

Recursion

"Vacant" moving down

FixHeap: an Example



Worst Case Analysis for fixHeap

- *2 comparisons* at most in one activation of the procedure
- The tree *height decreases by one* in the recursive call
- So, *2h comparisons are needed in the worst case*, where *h* is the height of the tree

- **Procesure:**

  fixHeap(H,K)

  if (H is a leaf) insert K in root(H)

  else

  Set *largerSubHeap*;

  if (K.key≥root(largerSubHeap).key) insert K in root(H)

  else

  insert root(largerSubHeap) in root(H);

  fixHeap(largerSubHeap, K);

  return

One comparison:
largerSubHeap is left- or right-Subtree(H), the one with larger key at its root.
Special case: rightSubtree is empty

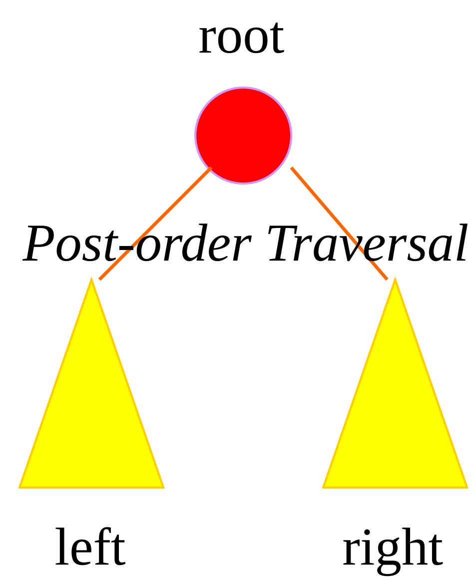
Recursion

"Vacant" moving down

Heap Construction

- **Note**: ***if*** **left subtree and right subtree both satisfy the partial order tree property, then fixHeap(H,root(H))** ***gets the thing done*****.**
- **We begin from a Heap Structure** ***H*****:**

```
void constructHeap(H)
   if (H is not a leaf)
      constructHeap(left subtree of H);
      constructHeap(right subtree of H);
      Element K=root(H);
      fixHeap(H,K)
   return
```



Correctness of *constructHeap*

- **Specification**
  - Input: A heap structure *H*, not necessarily having the partial order tree property.
  - Output: *H* with the same nodes rearranged to satisfy the partial order tree property.

```
void constructHeap(H)
  if (H is not a leaf)
    constructHeap(left subtree of H);
    constructHeap(right subtree of H);
    Element K=root(H);
    fixHeap(H,K)
  return
```

H is a leaf: base case, satisfied trivially.

Preconditions hold respectively?

Postcondition of ***constructHeap*** satisfied?

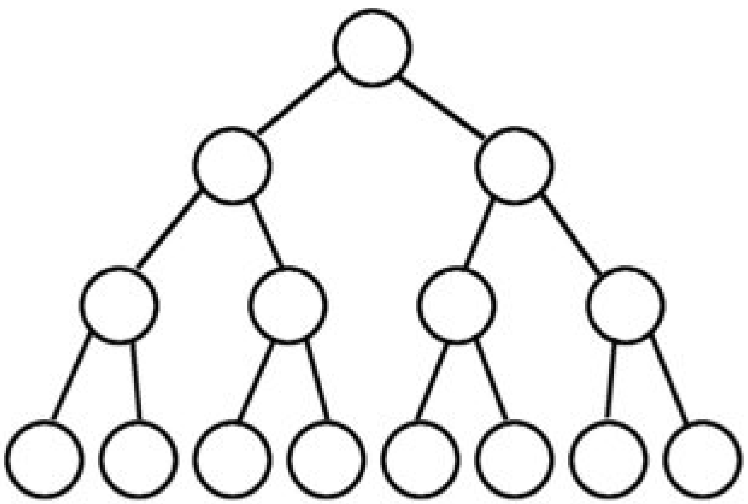
Linear Time Heap Construction!

- **The recursion equation:** **Number of nodes in right subheap**
  W(n)=W(n-r-1)+W(r)+2log(n) **Cost of fixHeap**
- **A special case: *H* is a complete binary tree:**
  - The size $N=2^d-1$,
  (then, for arbitrary n, $N/2<n\le N\le 2n$, so $W(n)\le W(N)\le W(2n)$)
  - Note: $W(N)=2W((N-1)/2)+2\log(N)$
  - The **Master Theorem** applies, with b=c=2, and the critical exponent E=1, $f(N)=2\log(N)$
  - Note: $$\lim_{N\to\infty}\frac{2log(N)}{N^{1-\varepsilon}}=\lim_{N\to\infty}\frac{2\log N}{N^{1-\varepsilon}\log 2}=\lim_{N\to\infty}\frac{2N^{\varepsilon}}{((1-\varepsilon)\log 2)N}$$
  - When $0<\varepsilon<1$, this limit is equal to zero
  - So, $2\log(N)\in O(N^{E-\varepsilon})$, case 1 satisfied, we have $W(n)\in\Theta(n)$

Direct Analysis of Heap construction

- **Heap construction**
  - o From *recursion* to *iteration*
  - o Sum of rowsums

$$\text{Cost} = \sum_{h=0}^{\lfloor logn \rfloor} n\frac{O(h)}{2^{h+1}} = O(n)$$



c = logn fix; h = logn; # = 1

c = 2 fix; h = 2; # = n/8

c = 1 fix; h = 1; # = n/4

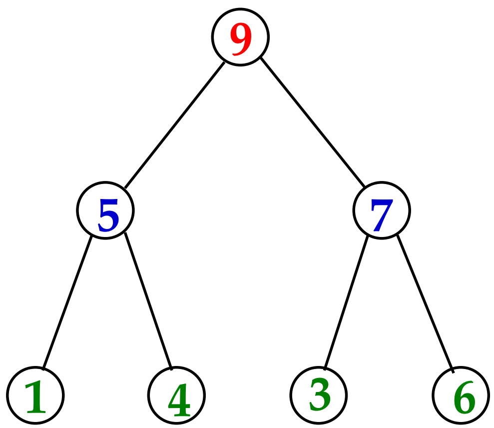
c = 0 fix; h = 0; # = n/2

1 fix = 2 comparisons

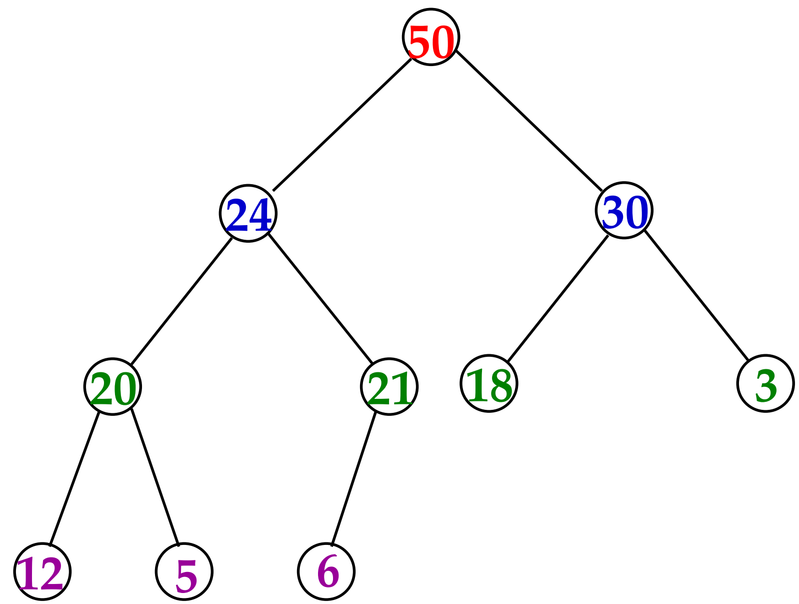
Understanding the Heap

- **Where is the 1st element in the heap?**
  - 2nd ?
  - 3rd?
- **Where is the k^{th} element?**
  - The cost should be f(k)
  - Could you do it in $O(k^2)$?

Implementing Heap Using Array



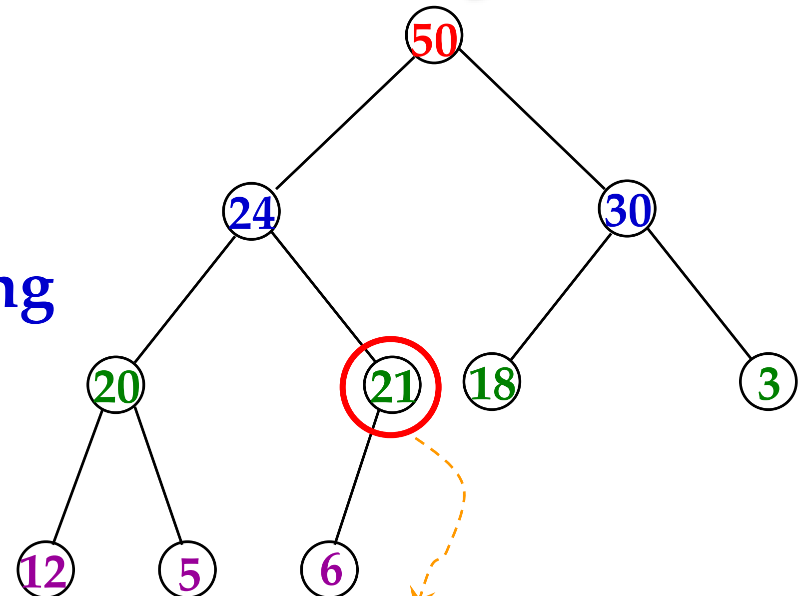
9	5	7	1	4	3	6



50	24	30	20	21	18	3	12	5	6

Looking for the Children Quickly

Starting from 1, not *zero*, then the *j* th level has 2^{j-1} elements. and there are 2^{j-1}-1 elements in the proceeding *j*-1 levels altogether.



50	24	30	20	21	18	3	12	5	6
				<5>					<10>

So, If E[i] is the k^{th} element at level *j*, then $i=(2^{j-1}-1)+k$, and the index of its left child (if existing) is

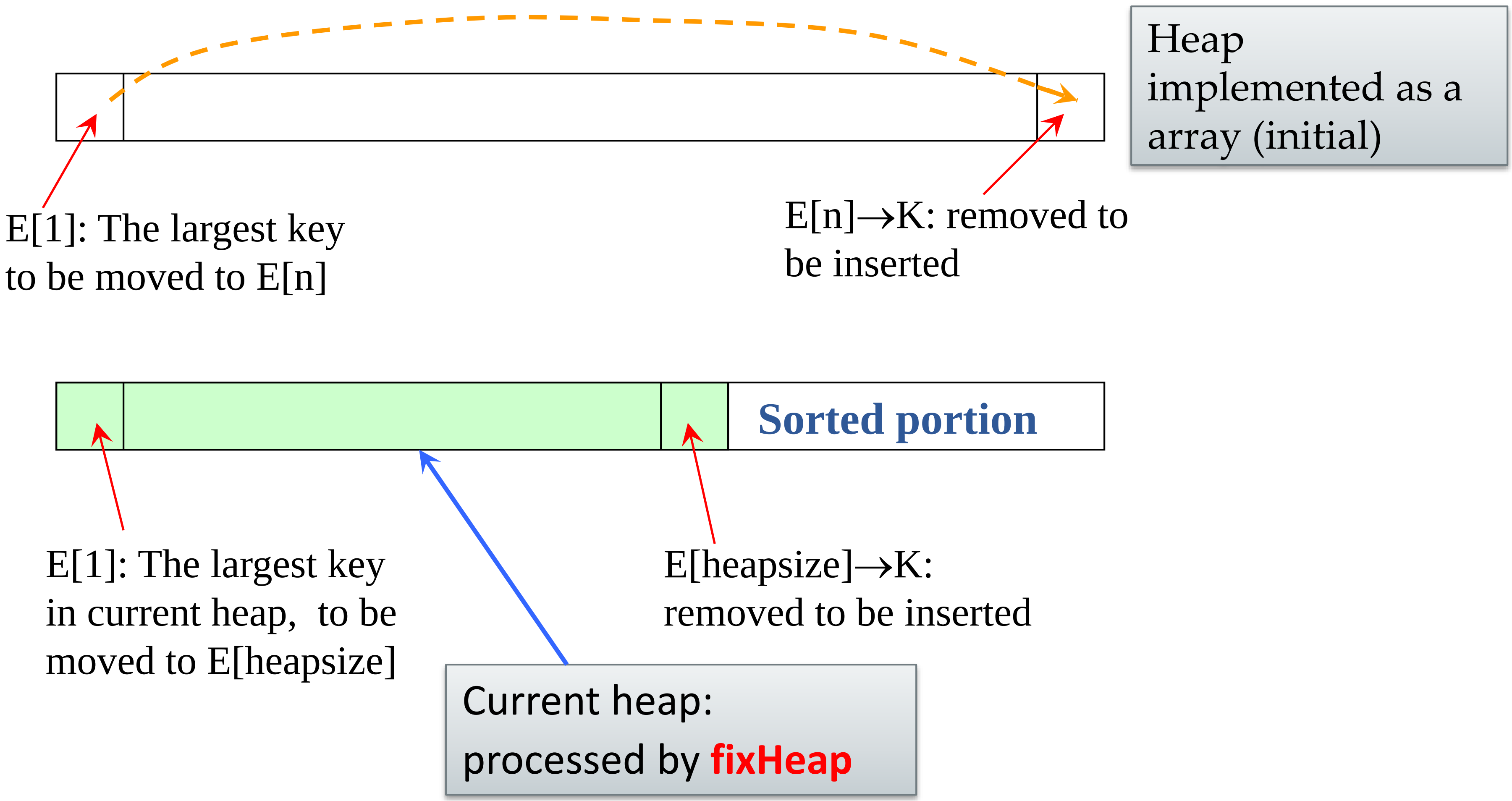
$$i+(2^{j-1}-k)+2(k-1)+1=2i$$

The number of node on the right of E[i] on level j

The number of children of the nodes on level j on the left of E[i]

For E[i]:
Left subheap: E[2i]
right subheap: E[2i+1]

HeapSort: In-Space Implementation



FixHeap: Using Array

- Void fixHeap(Element[ ] E, int heapSize, int root, Element K)
- int left=2*root; right=2*root+1;
- if (left>heapSize) E[root]=K; //*Root is a leaf.*
- else
- int largerSubHeap; //*Right or Left to filter down.*
- if (left==heapSize) largerSubHeap=left; // *No right SubHeap.*
- else if (E[left].key>E[right].key) largerSubHeap=left;
- else largerSubHeap=right;
- if (K.key≥E[largerSubHeap].key) E[root]=K;
- else E[root]=E[largerSubHeap]; //vacant filtering down one level.
- fixHeap(E, heapSize, largerSubHeap, K);
- return

Heapsort: the Algorithm

- Input: E, an unsorted array with n(>0) elements, indexed from 1
- Sorted E, in nondecreasing order
- Procedure:

“array version”

```
void heapSort(Element[] E, int n)
    int heapsize
    constructHeap(E,n,root)
    for (heapsize=n; heapsize≥2; heapsize--;)
        Element curMax=E[1];
        Element K=E[heapsize];
        fixHeap(E,heapsize-1,1,K);
        E[heapsize]=curMax;
    return;
```

Worst Case Analysis of Heapsort

- **We have:** $W(n) = W_{cons}(n) + \sum_{k=1}^{n-1} W_{fix}(k)$
- **It has been shown that:** $W_{cons}(n) \in \Theta(n)$ and $W_{fix}(k) \le 2\log k$
- **Recall that:**

$$2\sum_{k=1}^{n-1} \lceil \log k \rceil \le 2\int_1^n \log e \ln x dx = 2\log e(n \ln n - n) = 2(n \log n - 1.443n)$$

- **So, $W(n) \le 2n\lg n + \Theta(n)$, that is $W(n) \in \Theta(n\log n)$**

Coefficient doubles that of mergeSort approximately

HeapSort: the Right Choice

- For heapSort, $W(n) \in \Theta(n\log n)$
- Of course, $A(n) \in \Theta(n\log n)$
- More good news: HeapSort is an in-space algorithm (using iteration instead of recursion)

- It will be more competitive *if only* the coefficient of the leading term can be decreased to 1

Number of Comparisons in fixHeap

2 comparisons are done in filtering down for one level.

Procedure:

```
fixHeap(H,K)
  if (H is a leaf) insert K in root(H);
  else
    Set largerSubHeap;
    if (K.key≥root(largerSubHeap).key) insert K in root(H)
    else
      insert root(largerSubHeap) in root(H);
      fixHeap(largerSubHeap, K);
 return
```

A One-Comparison-per-Level Fixing

- **Bubble-Up Heap Algorithm:**

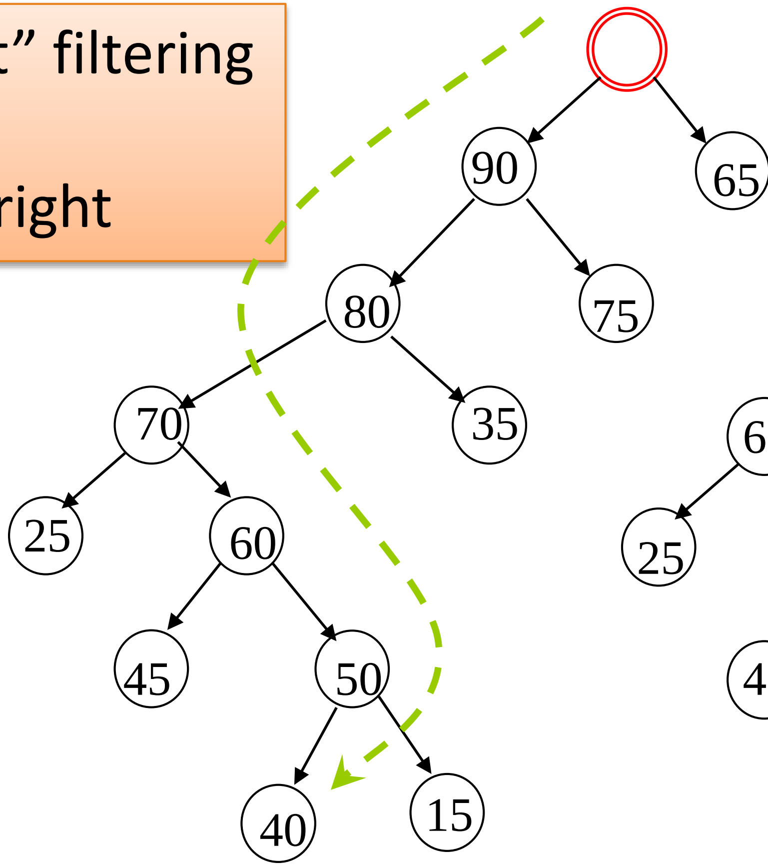
```
void bubbleUpHeap(Element []E, int root, Element K,
    int vacant)
  if (vacant==root) E[vacant]=K;
  else
    int parent=vacant/2;
    if (K.key≤E[parent].key)
      E[vacant]=K
    else
      E[vacant]=E[parent];
      bubbleUpHeap(E,root,K,parent);
```

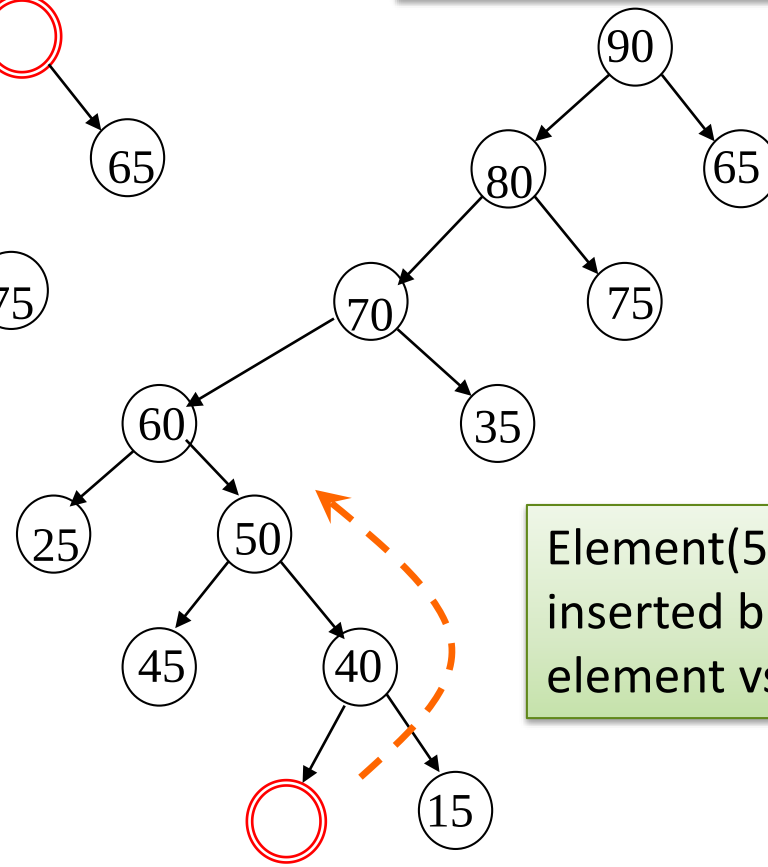
Bubbling up from vacant through to the root, recursively

Risky FixHeap

In fact, the "risk" is no more than "no improvement"

"Vacant" filtering down:
left vs. right

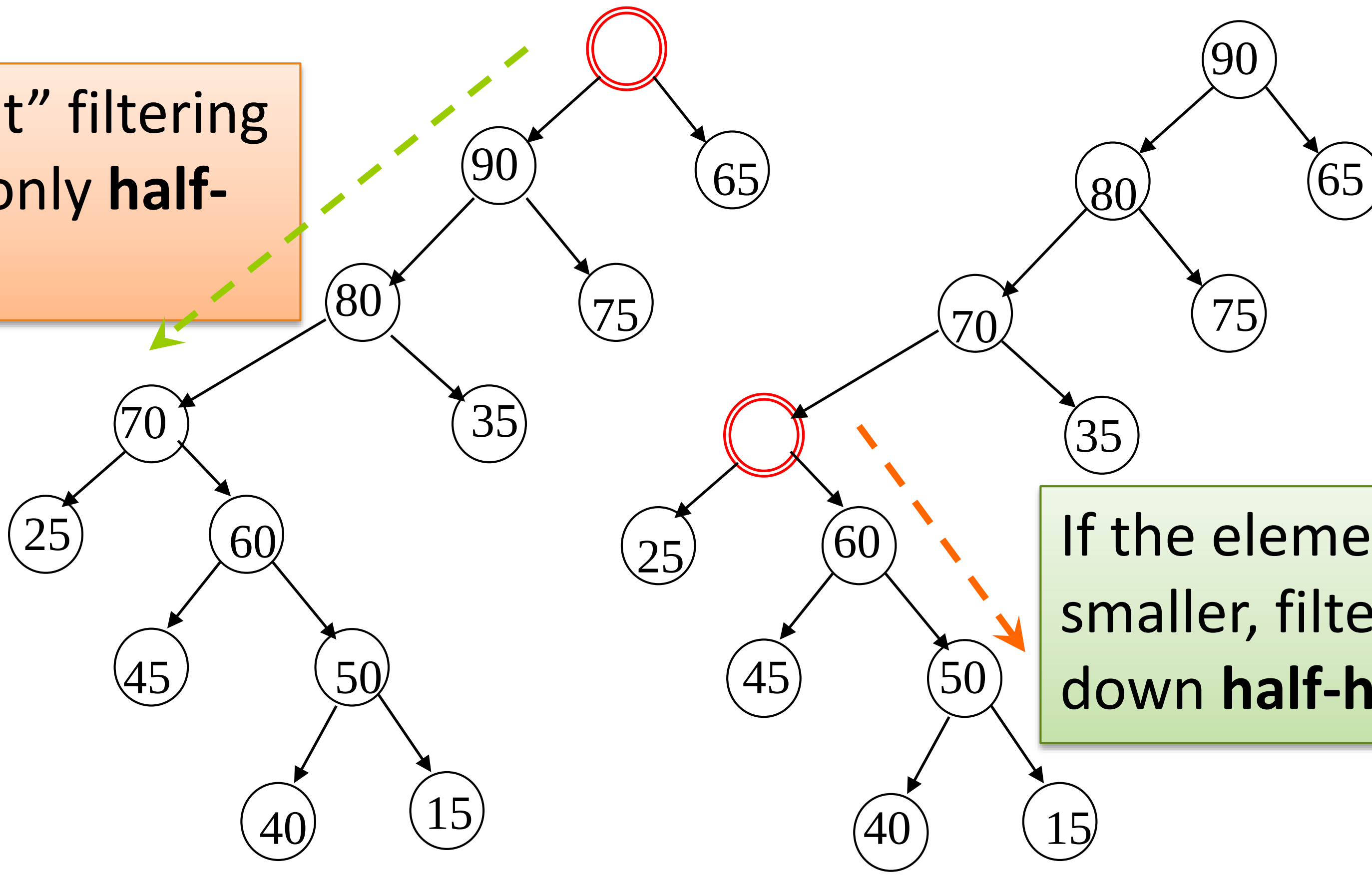




Element(55) to be inserted bubling up: element vs. parent

Improvement by Divide-and-Conquer

“Vacant” filtering down only **half-way**



If the element is smaller, filtering down **half-half-way**

The bubbling up will not beyond last vacStop

Depth Bounded Filtering Down

```
int promote(Element [ ] E, int hStop, int vacant, int h)
    int vacStop;
    if (h≤hStop) vacStop=vacant;
    else if (E[2*vacant].key≤E[2*vacant+1].key)
        E[vacant]=E[2*vacant+1];
        vacStop=promote(E, hStop, 2*vacan+1, h-1);
    else
        E[vacant]=E[2*vacant];
        vacStop=promote(E, hStop, 2*vacant, h-1);
    return vacStop
```

Depth Bound

FixHeap Using Divide-and-Conquer

```
void fixHeapFast(Element [ ] E, Element K, int vacant, int h)
  //h=⌈lg(n+1)/2⌉in uppermost call
  if (h≤1) Process heap of height 0 or 1;
  else
    int hStop=h/2;
    int vacStop=promote(E, hStop, vacant, h);
    int vacParent=vacStop/2;
    if (E[vacParent].key≤K.key)
      E[vacStop]=E[vacParent];
      bubbleUpHeap(E, vacant, K, vacParent);
    else
      fixHeapFast(E, K, vacStop, hStop)
```

Number of Comparisons in Accelerated FixHeap

- **Moving the vacant one level up or down need one comparison exactly in promote or bubbleUpHeap.**
- **In a cycle, *t* calls of promote and 1 call of bubbleUpHeap are executed at most. So, the number of comparisons in promote and bubbleUpHeap calls are:**

$$\sum_{k=1}^{t}\left\lceil \frac{h}{2^k} \right\rceil + \left\lceil \frac{h}{2^t} \right\rceil = h = \log(n+1)$$

- **At most, lg(*h*) checks for reverse direction are executed. So, the number of comparisons in a cycle is at most *h*+log(*h*)**
- **So, for accelerated heapSort: W(*n*)=*n*log*n*+Θ(*n*loglog*n*)**

Recursion Equation of Accelerated heapSort

- The recurrence equation about *h*, which is about log(*n*+1)

$$\begin{cases} T(1) = 2 \\ T(h) = \left\lceil \frac{h}{2} \right\rceil + \max\left(\left\lceil \frac{h}{2} \right\rceil, 1 + T\left(\left\lfloor \frac{h}{2} \right\rfloor \right) \right) \end{cases}$$

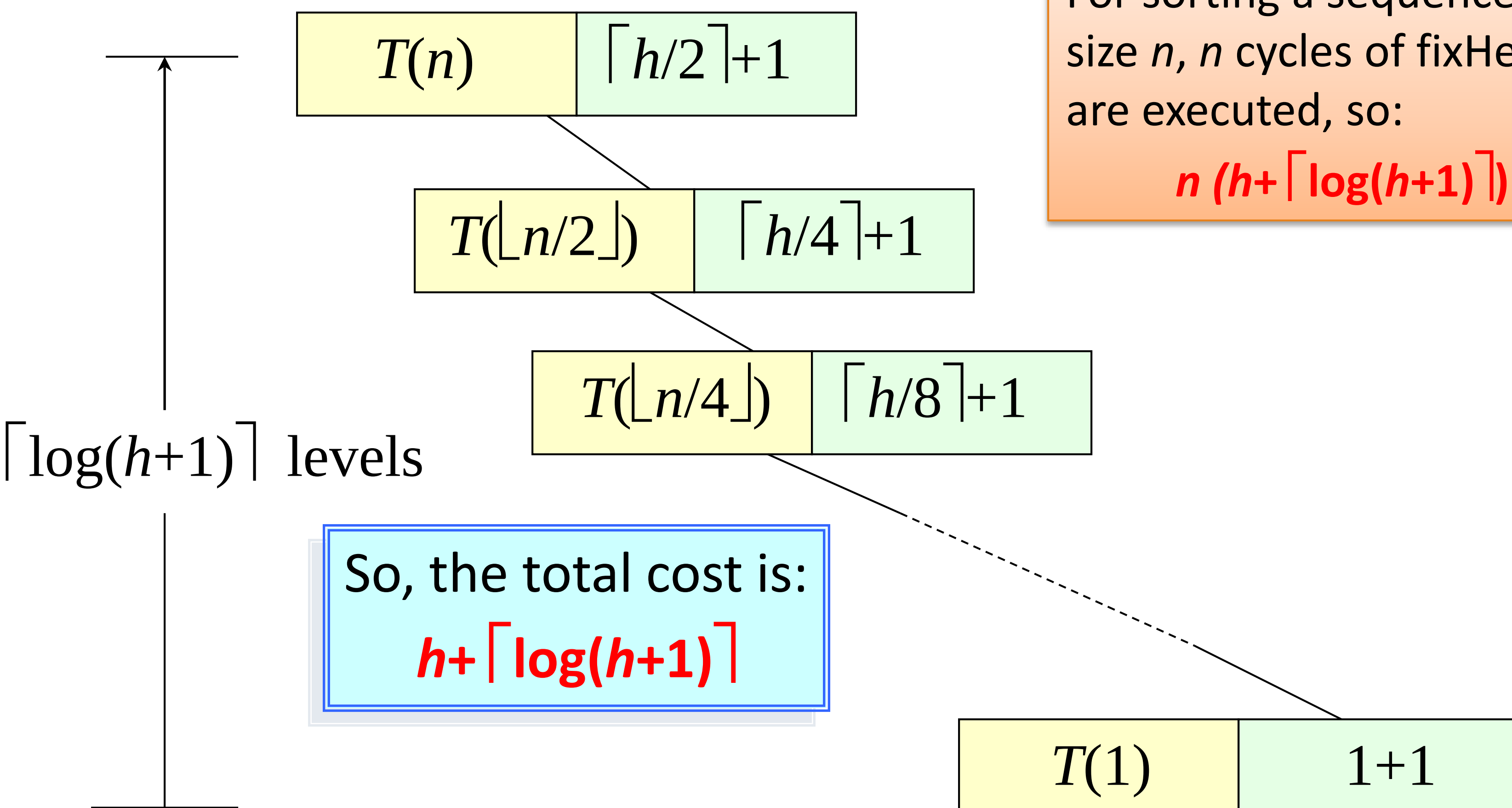
- Assuming T(*h*)≥*h*, then:

$$\begin{cases} T(1) = 2 \\ T(h) = \left\lceil \frac{h}{2} \right\rceil + 1 + T\left(\left\lfloor \frac{h}{2} \right\rfloor \right) \end{cases}$$

Solving the Recurrence Equation by Recursive Tree

For sorting a sequence of size n, n cycles of fixHeap are executed, so:

$$n\,(h+\lceil \log(h+1) \rceil)$$



So, the total cost is:

$$h+\lceil \log(h+1) \rceil$$

Inductive Proof

$$\begin{cases} T(1) = 2 \\ T(h) = \left\lceil \frac{h}{2} \right\rceil + 1 + T\left(\left\lfloor \frac{h}{2} \right\rfloor\right) \end{cases}$$

- **The recurrence equation for fixHeapFast:**
- **Proving the following solution by induction:**

$$T(\mathrm{h}) = \mathrm{h} + \lceil \log(h+1) \rceil$$

  - According to the recurrence equation:

$$T(h+1) = \lceil (h+1)/2 \rceil + 1 + T(\lfloor (h+1)/2 \rfloor)$$

  - Applying the inductive assumption to the last term:

$$T(h+1) = \lceil (h+1)/2 \rceil + 1 + \lfloor (h+1)/2 \rfloor + \lceil \log(\lfloor (h+1)/2 \rfloor + 1) \rceil$$

(It can be proved that for any positive integer:

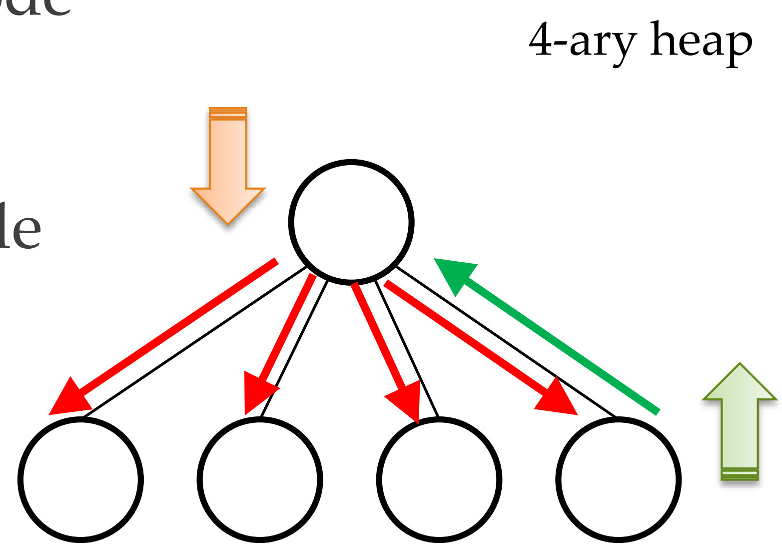
$$\lceil \log(\lfloor (\mathrm{h})/2 \rfloor + 1) \rceil + 1 = \lceil \log(\mathrm{h}+1) \rceil)$$

The sum is h+1

W(n)=$n$log$n$+Θ($n$loglog$n$) for Accelerated HeapSort

Generalization of a Heap

- **d-ary heap**
  - Structure / partial order
- **How to choose "d"?**
  - Top-down: fix the parent node
    - Cost: d comparisons in the worst case
  - Bottom-up: fix the child node
    - Cos: always 1



Not only for Sorting

- Eg1: how to find the k^{th} max element?
  - The cost should be f(k)
- Eg2: how to find the first k elements
  - In sorted order?
- Eg3: how to merge k sorted lists?
- Eg4: how to find the median dynamically?
- …

Thank you!

Q & A

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