Time frequency signal analysis (STFT and WT)

Amir M. Mousavi

Department of Computer Engineering
Shahid Rajayi University

Tehran, Iran

Amir.M.Mousavi@gmail.com

***Abstract*—Two different procedures for effecting a frequency analysis of a time-dependent signal locally in time are studied. The first procedure is the short-time or windowed Fourier transform; the second is the wavelet transform, in which high-frequency components are studied with sharper time resolution than low-frequency components.**

This article aims to provide a comprehensive tutorial and survey about review of the Fourier transform and wavelet , Then similarities and the differences between these two methods are discussed. Finally, the MATLAB implementation of scaleogram and spectrum of signal is shown..

Keywords—Digital signal processing , DSP , STFT, EA , Wavelet Transform , Short-time Fourier transform , WT , window , signal analysis , spectrum , scaleogram

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I. Introduction

Time-frequency signal transforms combine traditional Fourier transform signal spectrum information with a time location variable. There results a two-dimensional transformed signal having an independent frequency variable and an independent time variable. Such a signal operation consititutes the first example of a mixed domain signal transform. many applications, and indeed our entire theoretical approach considered signal analysis strategies based upon time, frequency, or scale. Time-domain methods are adequate for tasks such as edge detection, elementary segmentation, correlation-based shape recognition, and some texture analysis problems. But in many situations, the inherent periodicity within signal regions pushes us toward a decomposition of the signal according to its frequency content.

Frequency—or spectral—analysis enters the picture as a tool for discovering a signal's sinusoidal behavior. But this is an inherently global approach. Standard spectral analysis methods, which the Fourier transform in both its analog and discrete guises completely typifies, suffer signal interpretation difficulties when oscillations of interest exist only within a limited signal region. windowing the signal improves local spectral estimates.

Another approach takes a standard signal shape element, shrinks or expands it into a library of local signal forms, and then considers how well different regions of the signal match one or another such local prototypes. This is an analysis based on signal scale. The idea is to mix time-domain methods with either the frequency- or the scale-domain approach. Both combinations provide avenues for structural signal decomposition. The theory is rich and powerful. It has developed rapidly in the last few years. We elect to start with the methods that are most intuitive and, in fact, historically prior: the time-frequency transform techniques..

The Fourier transform is the fundamental tool for frequency-domain signal analysis. It does allow us to solve some problems that confound time-domain techniques. The mapping $X(\omega) = F[x(t)]$ lays out the frequency content of a signal $x(t)$, albeit in complex values, and its large magnitude $|X(\omega)|$ indicates the presence of a strong sinusoidal component of frequency ω radians per second in $x(t)$. We can construct filters and assemble them into filter banks in order to search for spectral components in frequency ranges of interest. All such strategies stem from the convolution theorem, which identifies time-domain convolution—and hence linear, time-invariant processing—with frequency-domain multiplication. The caveat is that standard Fourier techniques depend on a knowledge of the entire time-domain extent of a signal. Even the filter bank highlights ranges of frequencies that existed in the signal for all time: past, present, and future.

But many signals have salient periodic features only over limited time intervals.Although a global analysis is theoretically possible, it may not be practical or efficient. Consider, for example, an orchestra that must play a two-hour symphony, and let us fancy that the composer employs a Fourier transform music style that assigns each instrument just one tone for the entire duration of the performance. The superposition of the various tones, each constantly emitted for two hours by the musicians, does indeed produce the composer's envisioned piece. Of course, the orchestra has but a finite number of musicians, so what is in effect here is really a Fourier series music synthesis. The conductor's job is greatly simplified, perhaps reducing to a few minor pre-concert modifications to the chosen tones. Concert hall owners could well be drawn to encourage such an art form; it would allow them to hire low-paid, unskilled musicians and cut the conductor's hours. The problem of course is that it would be nearly impossible to get the right tonal mix to compose Fourier symphony. A few hertz too far in this or that direction generates not a symphony but a cacophany instead. Localizing the tones works much better. The composer uses a local frequency synthesis, assigning tones to moments in time; the musicians—they must be artists of supreme skill and dedication—read the musical notation and effect the appropriate, time-limited tones; and the conductor orchestrates the entire ensemble, setting the tempo and issuing direction as the performance proceeds. The composition of the signal in terms of time-localized tones is far easier to understand, communicate, replicate, and modify.

In this paper we have proposed to provide a comprehensive tutorial and survey about review of the time frequency signal analysis including Fourier transform and wavelet , etc. which is discussed in sections II to VI Then similarities and the differences between these methods are in section VII . Finally. The results of the MATLAB implementation are presented VIII and the conclusions are discussed in section IX .

II. FOURIER TRANSFORMS

The Fourier Transform X (f) of a continuous-time signal x(t) is given by :

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \ .$$

The continuous function X (f) is the frequency-domain representation of x (t) obtained by summation of an infinite number of complex exponentials. To find X (f) on a digital computer with discrete (sampled) and finite length (time-limited) signals, the Discrete Fourier Transform (DFT) is used. The DFT is defined as :

$$X[k] = \sum_{n=0}^{N-1} x[n] \, e^{-j\frac{2\pi k n}{N}}$$

where 2 [n] is a sequence obtained by sampling the continuous time signal ~ (tev)er y T, seconds for N samples:

$$x[n] = x(nT_s) \quad n = 0, 1, 2, ..., N-1 \ .$$

The DFT produces a sequence of complex values X [k] whose magnitudes are those of discrete frequencies in r [n]. Because the derivation of the DFT strictly requires z [t i] periodic, the representation of a signal by the DFT is best reserved for periodic signals. This and the Nyquist criteria-sampling at least twice as fast as the highest frequency in the signal-are two important caveats to using the DFT.

The limitations of the DFT for non-periodic signals can be illustrated using a signal containing a transient impulse with a ring of 900 Hz superimposed on a 60 HZ fundamental. Such a signal is typical of a capacitor switching transient. The transient signal and the resulting DFT output are shown in Figure 1, (a) and (b), respectively. The presense of significant energy in the sidebands of 900 Hz is a direct result of the non-periocity of the input. This can prevent precise detection of the resonant frequency of the capacitor-compensated system, a potential feature for determining the cause of the transient .

More sophisticated Fourier-based transforms have been developed to reduce the effect of non-periodic signals on the DFT. One such method, the Short-Time Fourier transform (STFT), assumes local periodicity within a continuously translated time window. The section IV discuss the STFT which will be useful in comparing the Wavelet and Fourier methods

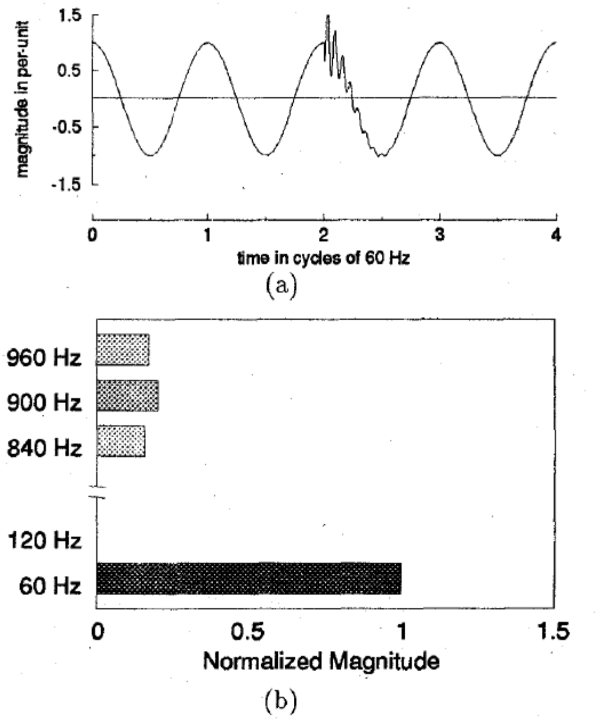


Fig.1. Transient signal with a localized oscillation within the sampling window and its discrete fourier transform

III. GABOR TRANSFORMS

A. Introduction :

The Gabor transform picks a particular time-limiting window—the Gaussian—and generalizes the windowed spectrum computation into a full signal transform. The goal is to capture both the frequency components of a signal and their time locality in the transform equation. Of course, a Gaussian window is not truly finite in extent; its decay is so fast, however, that as a practical computation matter it serves the purpose of localizing signal values. Finite windows are possible with species

- ✓ **Definition (Gabor Transform).** Let *g*(*t*) be some Gaussian of zero mean:

$$g(t) = Ae^{-Bt^2}, \qquad (1)$$

where A, B > 0. If x(t) ∈ L2(R) is an analog signal, then its Gabor transform, written Xg(μ, ω), is the radial Fourier transform of the product x(t)g(t − μ):

$$X_g(\mu, \omega) = \int_{-\infty}^{\infty} x(t) e^{-\frac{(t-\mu)^2}{2\sigma^2}} e^{-j\omega t} dt . \qquad (\ .2)$$

We will occasionally use the "fancy G" notation for the Gabor transform: Xg(μ, ω) = Gg[x(t)](μ, ω). The windowing function g(t) in (10.1) remains fixed for the transform. If its parameters are understood—for instance, it may be the Gaussian of zero mean and standard deviation σ > 0—then we may drop the subscript g for the windowing function.

It is possible to specify a particular normalization for the Gaussian window used in the Gabor transform. For example, we might choose ||g(t)||1 = 1 or ||g(t)||2 = 1, where || · ||p is the norm in the Banach space Lp(R) . Gaussian signals belong to both spaces. Either choice makes some Gabor transform properties look nice but not others. We generally normalize the window with respect to the L1(R) norm, so that our windowing functions are zero-mean Gaussians of standard deviation σ > 0, g0,σ(t):

$$X_g(\mu, \omega) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t) e^{-\frac{(t-\mu)^2}{2\sigma^2}} e^{-j\omega t} dt . \qquad (\quad 3)$$

The exercises explore how these alternative Gabor transform normalizations affect various transform properties

Observe that the Gabor transform, unlike the analog Fourier transform, is a function of two variables. There is a time-domain variable μ, which is the center or mean of the window function, and a frequency-domain variable, ω. Since a time-domain variable—namely the location of the window's center, μ—is a parameter of the transform, the inverse Gabor transform involves a two-dimensional, or iterated integral. Figure .1 shows the Gabor transform scheme.

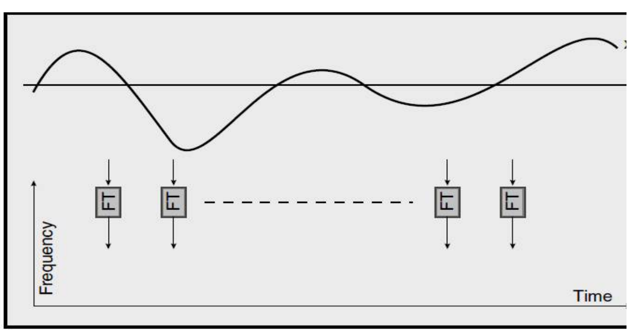


Fig. .1. The Gabor transform finds the spectral content of x(t) within a Gaussian window g(t − μ). The two-dimensional transform function takes parameters μ, the window's center, and ω, the frequency of the exponential exp(jωt).

We interpose a lemma that shows how to compute the Fourier transform of a Gabor transform :

Lemma (Fourier Transform of Gabor Transform). Suppose σ > 0; x(t) ∈ L2(R); g(t) = gμ,σ(t) is the Gaussian window with mean μ and standard deviation σ; and let Xg(μ, ω) ∈ L1(R) be the Gabor transform of x(t). Then, for each ω ∈ R we can Fourier transform the signal Xg(μ, ω), viewing it as a function of μ. So,

$$\int_{-\infty}^{\infty} X_g(\mu, \omega) e^{-j\mu\theta} d\mu = \frac{1}{2\pi} X(\omega+\theta) G(\theta) = \frac{1}{2\pi} X(\omega+\theta) e^{-\frac{\theta^2\sigma^2}{2}} ,$$

where G(θ) is the radial Fourier transform of g(t).

B. Application :

Let us pause the theoretical development for a moment to explore two basic Gabor transform applications: a linear chirp and a pulsed tone. These illustrate the use and behavior of the transform on practical signals. Studying the transform coefficients as dependent on the width of the transform window function will also lead us to important ideas about the relation between the transform's time and frequency resolutions.

✓ Linear Chirp.

This section discusses the Gabor transform for a linear chirp signal. A linear chirp is a sinusoidal function of a squared time variable At2, where A is constant. Thus, as |t| increases, the signal oscillations bunch up. Signal frequency varies with time in a linear fashion, and the Gabor transform will expose this behavior.

Let us consider the analog signal

$$X_a(t) = \begin{cases} \cos(At^2), & t \in [0, L] \\ 0 & \text{otherwise.} \end{cases}$$

The Gabor transform of $x_a(t)$ is

$$\mathcal{G}[x_a(t)](\mu, \omega) = (X_a)_g(\mu, \omega) = \int_0^L x_a(t) g_{\mu, \sigma}(t) e^{-j\omega t} dt .$$

We need to decide upon an appropriate value for the spread of the Gaussian, which is given by its standard deviation σ. Also, (Xa)g is a two-dimensional function, so we seek an image representation of the Gabor transform for a range of values, μ and ω.

$$c(k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{\frac{-2\pi jnk}{N}} ,$$

where 0 δ *k* δ *N* □ 1, are a trapezoidal rule approximation to the Fourier series integral using the intervals [0, *T*/*N*], [*T*/*N*, 2*T*/*N*], ..., [(*N* – 1)*T*/*N*, *T*].

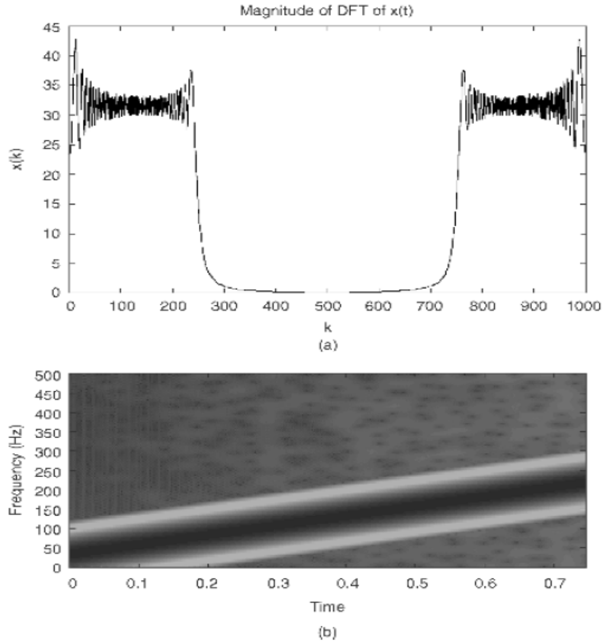


Fig.2. Gabor transform of linear chirp, windowing with a Gaussian of σ = 16. The frequency of xa(t) = cos(At2) rises from 0 to 250 Hz over a 1–s time interval. Its magnitude spectrum X(k) is shown in panel (a). Note the apparent presence of frequencies between 0 and 250 Hz, but that the time of a particular frequency is lost by the discrete Fourier transform (DFT). The Gabor transform reveals the time evolution of frequencies in x(t), as shown in panel (b). Time values are shown along the bottom over the interval [0, 1], which represents samples n from 0 to 255. Image intensities represent Gabor magnitude spectral values |G [xa](μ, ω)|; darker values indicate larger magnitudes

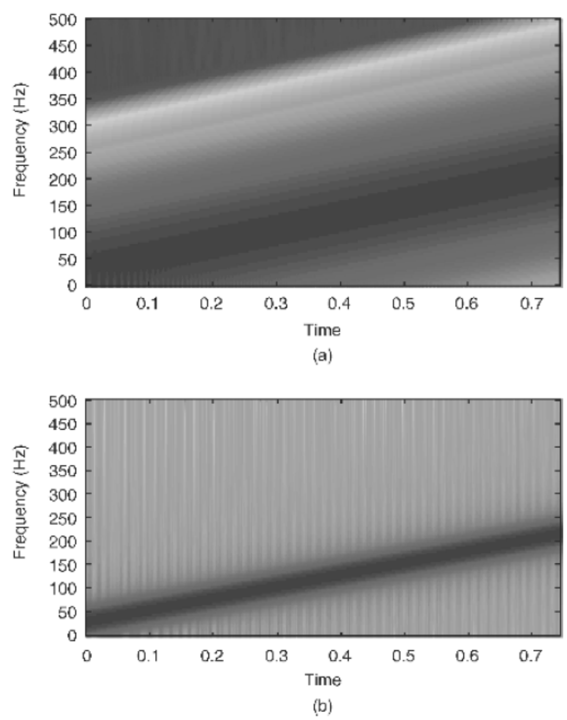


Fig.3. Window width effects in the Gabor transform of a linear chirp. Windowing with a Gaussian of ʃ = 4 is shown in panel (a). Panel (b) shows the case of ʃ = 64.

✓ ***Pulsed Tone.***

Now suppose that we begin with a time-domain pulse:

$$x_a(t) = \exp(-Bt^2)\cos(At).$$

We shall suppose that the pulse frequency is 50 Hz and consider different timedomain durations of xa(t), which are governed by stretching the Gaussian envelope, exp(−Bt2).

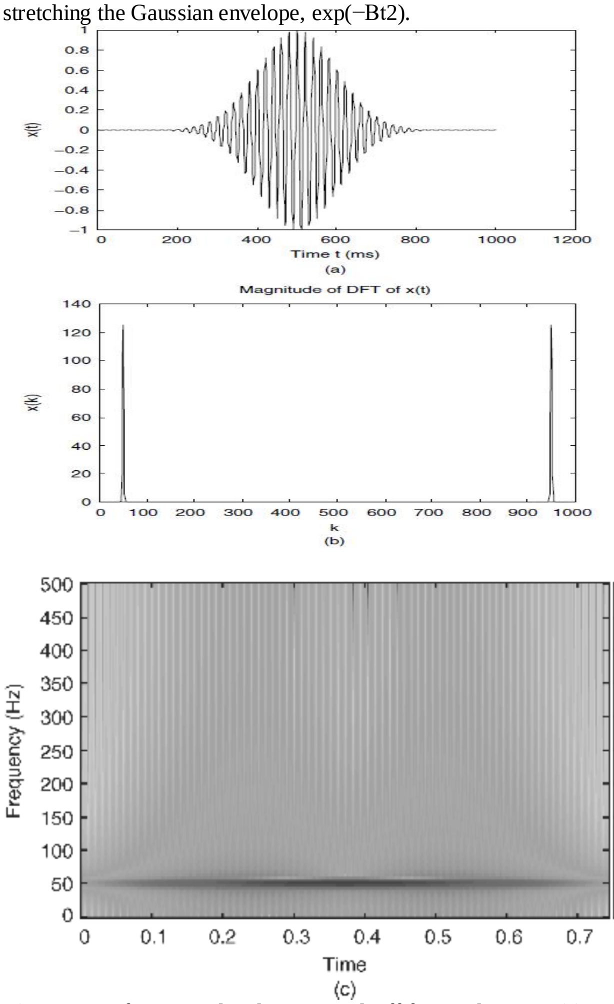


Fig. 4. Time-frequency localization tradeoff for a pulse tone. (a) The 50-Hz tone pulse rising and decaying in a 600-ms interval about *t* = 0.5 *s*. (b) Its Fourier spectrum shows the frequencies present but provides no time information. (c) The Gabor transform.

C. Properties :

Table 10.1 summarizes properties of the Gabor transformation, some of which are left as exercises.

TABLE 1 Gabor Transform Properties[a]

Signal Expression	Gabor Transform or Property
$x(t)$	$X_g(\mu, \omega)$
$ax(t) + by(t)$	$aX_g(\mu, \omega) + bY_g(\mu, \omega)$
$x(t - a)$	$e^{-j\omega a}X_g(\mu - a, \omega)$
$x(t)\exp(j\theta t)$	$X_g(\mu, \omega - \theta)$
$\lVert x\rVert_2 = \sqrt{2\pi}\dfrac{\lVert X_g(\mu, \omega)\rVert_{2, L^2(R^2)}}{\lVert g\rVert_2}$	Plancherel's theorem
$\langle x, y\rangle = \dfrac{1}{2\pi\lVert g\rVert_2^2}\langle X_g, Y_g\rangle$	Parseval's theorem
$x(t) = \dfrac{1}{(2\pi\lVert g\rVert_2^2)}\int_{-\infty}^{\infty} X_g(\mu, \omega)g_{\mu,\sigma}(t)e^{j\omega t}d\omega d\mu$	Inverse, resolution of the identity, synthesis equation

[a]In the table, $x(t)$ is square-integrable, and $g(t)$ is a Gaussian of mean μ and standard deviation σ.

IV. SHORT-TIME FOURIER TRANSFORMS

A. *Short Review of the The Short-Time Fourier Transform* :

The STFT is similar to the Fourier transform except that the input signal X(t) is multiplied by a window function w(t) whose position is translated in time by r :

$$STFT(f, \tau) = \int_{-\infty}^{\infty} x(t)w(t - \tau)e^{-j2\pi ft}dt \ .$$

For digital implementation of the STFT, the Windowed Discrete Fourier 'Transform (WDFT) is used. The WDFT is defined as :

$$WDFT[k, m] = \sum_{n} x[n]\, w[n - m]\, e^{-j\frac{2\pi kn}{N}}$$

where the sequence w [n - m], in its simplest form, is the rectangular window function :

$$w[n] = \begin{cases} 1 & if\ 0 \le n - m \le N - 1 \\ 0 & otherwise \end{cases}$$

For each window wm٠ = w [n - m,], the WDFT produces a sequence of complex values WDFT[k,m,], k = ٠'١' /// N - ١' whose magnitudes are those of the discrete frequencies of the input z [n].

The WDFT of a signal can be represented in a twodimensional grid where the divisions in the horizontal direction represent the time extent of each window w [n - m]; the divisions in the vertical direction represent the frequencies k ; and the shade of each rectangle is proportional to the corresponding magnitude. The DWFT in Figure ٢(b) for the capacitor switching transient, contains the same frequency information as the previous DFT example with an additional dimension of time. The time period of each window in Figure ٢(b) T=١٦/٦٧ ms fixes the frequency resolution Af at ٦٠ IHz (Af = l/T). This however locates the start time of the transient only to within one ٦٠-Hz cycle. Shortening the window period by four as in Figure ٢(c) locates the start of the transient but makes Af four times larger. As a result of the lower frequency resolution of ٢٤٠ Hz, the energy of the ٦٠ Hz fundamental appears in the dc and ٢٤٠ Hz components.

This example illustrates the need for multiple resolution in time and irequency for power signals containing a fundamental frequency superimposed with transienk. More preciscly, fine time resolution for short duration and high frequency signals, and fine frequency resolution for long duration and lower frequency signals are needed. This provides accurate location of the transient component while simultaneously retaining information aboiut the fundamental frequency and its low-order harmonics.

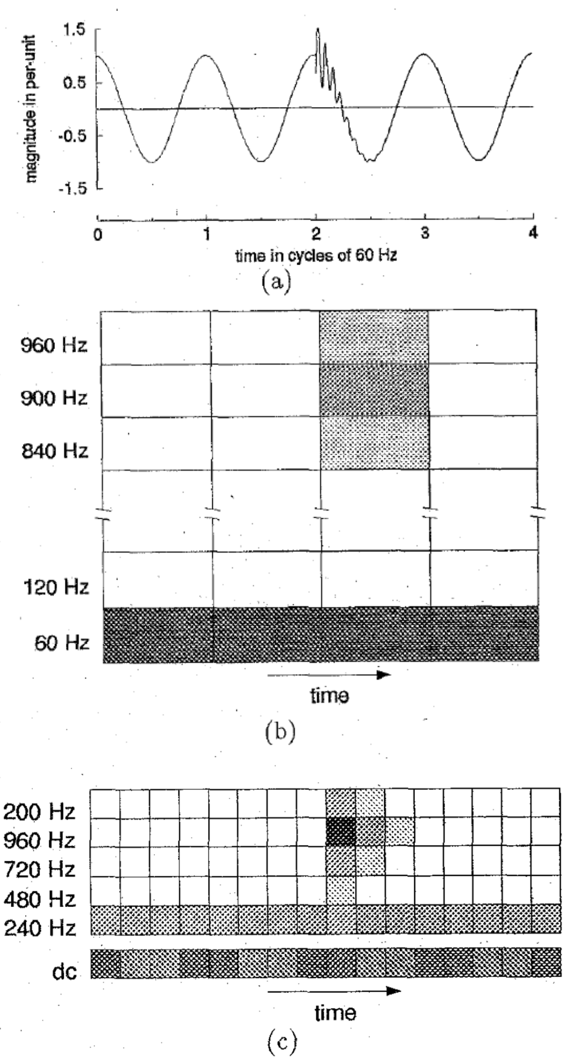


Fig.2. Windowed Discrete Fourier transform output for different window sizes: (a) example input signal, (b) WDFT with one-cycle window size, Af = 60 Hz, (c) WDFT with quarter-cycle window size, Af = 240 Hz.

A short-time Fourier transform (STFT) generalizes the Gabor transform by allowing a general window function. For the supporting mathematics to work, the theory requires constraints on the window functions. These we will elaborate in a moment. Once these theoretical details are taken care of, though, the general transform enjoys many of the same properties as the Gabor transform. One might well ask whether a window shape other than the Gaussian can provide a better time-frequency transform. The answer is affirmative, but qualified. If the window shape matches the shape of signal regions to be analyzed, then an alternative window function offers somewhat better numerical results in signal detection applications. Thus, choosing the window to have roughly the same shape as the signals to be analyzed improves detection performance. These benefits are usually slight, however. We know from the experiments with pulses and chirps at the end of the previous section that there is a tradeoff between time and frequency localization when using the Gabor transform. How does the selection of a transform window affect this behavior? It turns out that there is a hard lower limit on the joint time-frequency resolution of windowed Fourier transforms. Constricting the time-domain window so as to sharpen the time domain resolution results in a proportionately broader, more imprecise frequency-domain localization. This is a fundamental limitation on windowed Fourier methods. Its practical import is that signals with both low and high frequencies or with abrupt transients are difficult to analyze with this transform family. In fact, this limitation—which is a manifestation of the famous Heisenberg Uncertainty Principle—stimulated the search for alternative mixed domain transforms and was an impetus behind the discovery of the wavelet transform (section V). Among all possible window functions, there is one signal in particular that shows the best performance in this regard: the Gaussian. Thus, the Gabor transform is the short-time Fourier transform with the best joint time-frequency resolution. So despite the benefits a special window may have, the Gabor transform prevails in all but certain specialized STFT-based signal analysis applications.

B. Window Functions

This section specifies those functions that may serve as the basis for a windowed transform. We formally define window functions and the resulting general window transform. We also develop some window function properties. This leads to a criterion for measuring joint time-frequency resolution. We prove the uncertainty principle, and the optimality of the Gabor transform follows as a corollary. We should note right away that exponential signals modulated by window functions will play the role of structuring elements for signal analysis purposes. The short-time Fourier transform applies this structuring element at different time locations to obtain a set of time-ordered snapshots of the signal at a given frequency. When we later .discretize the STFT, this idea will become clearer

Definition (Window Function). If x(t) ∈ L٢(R), ||x(t)||٢ ≠ ٠' .and tx(t) ∈ L٢(R), then x(t) is called a window function

So, x(t) is a window function when its squared magnitude, x(t)|٢' has a second order moment. This technical condition | is necessary for many of the properties of the windowed transform. Of course, the familiar functions we have used to .improve signal spectra in Chapter ٩ satisfy this definition

Example (Gaussian). The Gaussian g(t) = Aexp(−Bt٢)' where A ≠ ٠ and B > ٠' is a window function. The Gaussian has moments of all orders, as we can check by integrating by :parts

$$\int_{-\infty}^{\infty} |tg(t)|^2 dt = A\int_{-\infty}^{\infty} t^2 e^{-2Bt^2} dt = \left(\left.\frac{Ate^{-2Bt^2}}{-4B}\right|_{-\infty}^{\infty} + \left(\frac{A}{4B}\right)\int_{-\infty}^{\infty} e^{-2Bt^2} dt \right)$$

$$= \left(\frac{A}{4B}\right)\int_{-\infty}^{\infty} e^{-2Bt^2} dt.$$

The Fourier transform of a Gaussian is still a Gaussian, and therefore G(ω) is a window function in the frequency .domain too

*C. **Transforming with a General Window***

It is not hard to generalize the Gabor transform to work with a general window, now that we have introduced the moment condition that a window function must satisfy. We will define the windowed transform for window functions and make the additional assuption that the Fourier transform of the window is also a window function for some of the properties. Drawing inspiration from the Gabor transform formalizations, we can easily draft a definition for a general .windowed transform

Definition (Short-Time Fourier Transform). Let w(t) be a window function and x(t) ∈ L٢(R). The short-time Fourier transform (STFT) with respect to w(t), written Xw(μ, ω), is :the radial Fourier transform of the product x(t)w(t − μ)

$$X_w(\mu, \omega) = \int_{-\infty}^{\infty} x(t)w(t-\mu)e^{-j\omega t} dt.$$

The STFT is also known as the windowed Fourier transform. There is a “fancy W” notation for the short-time .Fourier transform: Xw(μ, ω) = (Ww)[x(t)](μ, ω)

Remarks. The windowing function w(t) in (١٠/٤١) remains fixed for the transform, as does the Gaussian in a Gabor transform. Indeed, our definition generalizes the Gabor transform: If w(t) is a Gaussian, then the short-time Fourier transform with respect to w(t) of x(t) is precisely the Gabor transform of x(t) using the Gaussian w(t). We do not demand that the Fourier transform (Fw)(ω) = W(ω) must also be a window function; when we turn to study time-frequency localization using the transform, however, we .make this qualification

a. .Standard Windows

We can define an STFT for any of the windowing functions used to improve local spectra estimates . We know that windowing a signal x(t) with a tapered window function reduces the size of Gibbs phenomenon sidelobes. Table_٢ :summarizes possible standard analog windows

i. **rectangle**
ii. **Bartlett (triangle)**
iii. **Hamming**

iv. ,Hanning
v. .Blackman functions

Each of the standard window functions above has a discontinuity in a timedomain derivative of some order. We can develop the STFT using B-splines, however, and achieve smooth time-domain derivatives of arbitrarily high .orders

b. .B-spline Windows

Another window function appropriate for the STFT involves B-splines. Splines are popular in applied mathematics , .computer graphics , and signal processing and analysis

TABLE .2. Short-Time Fourier Transform Window Functions[a]

Name	Definition
Rectangle	$w(t) = \begin{cases} b & \text{if } (\lvert t\rvert \le a) \\ 0 & \text{otherwise.} \end{cases}$
Bartlett (triangle)	$w(t) = \begin{cases} \frac{b}{a}t + b & \text{if } -a \le t \le 0, \\ -\frac{b}{a}t + b & \text{if } 0 \le t \le a, \\ 0 & \text{otherwise.} \end{cases}$
Hanning (von Hann)	$w(t) = \begin{cases} b\cos^2\left(\frac{\pi t}{2a}\right) & \text{if } \lvert t\rvert \le a \\ 0 & \text{otherwise.} \end{cases}$
Hamming	$w(t) = \begin{cases} 0.54b + 0.46b\cos\left(\frac{\pi t}{a}\right) & \text{if } \lvert t\rvert \le a \\ 0 & \text{otherwise.} \end{cases}$
Blackman	$w(t) = \begin{cases} 0.42b + 0.5b\cos\left(\frac{\pi t}{a}\right) + 0.08b\cos\left(\frac{2\pi t}{a}\right) & \text{if } \lvert t\rvert \le a \\ 0 & \text{otherwis} \end{cases}$

[a]Adjust parameter $a > 0$ for a window width appropriate to the signal features of interest. Adjust p meter $b > 0$ in order to normalize the window function.

D.: Properties

Many of the properties of the Gabor transform carry over directly to the short-time Fourier transform. Like the specialized Gabor transform, the STFT obeys basic properties of linearity, time shift, and frequency shift. We state and leave as exercises the STFT Plancherel, Parseval, .and inverse results

Theorem (Short-Time Fourier Transform Parseval's). Suppose x(t), y(t) ∈ L۲(R); w(t) is a window function; and let Xw(μ, ω) and Yw(μ, ω) be the STFTs of x(t) and y(t), :respectively, based on windowing with w(t). Then

$$2\pi\|w\|_2^2\langle x, y\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X_w(\mu, \omega)\overline{Y_w(\mu, \omega)}\, d\omega d\mu = \langle X_w, Y_w\rangle_{L^2(\mathbb{R}^2)} .$$

Theorem (Short-Time Fourier Transform Plancherel's). Suppose σ > ۰; x(t) ∈ L۲(R); w(t) is a window function; and : let Xw(μ, ω) be the STFT of x(t). Then

$$\|x\|_2 = \sqrt{2\pi}\frac{\|X_g(\mu, \omega)\|_{2, L^2(\mathbb{R}^2)}}{\|g\|_2} .$$

Theorem (Inverse Short-Time Fourier Transform). Suppose $x(t)$ □ $L2$(R), $w(t)$ is a window function, and let X_w(⌈, ⌉) be the STFT of $x(t)$. Then for all a □ R, if $x(t)$ is continuous at a, then:

$$x(a) = \frac{1}{(2\pi\|w\|_2^2)}\int_{-\infty}^{\infty} X_w(\mu, \omega)w(a)e^{j\omega a}\, d\omega d\mu .$$

V. TIME-FREQUENCY LOCALIZATION

How precisely we can locate the frequency values within a signal using the shorttime Fourier transform? last Sections showed how the Gaussian window width dramatically affects the transform coefficients. Indeed, an improperly chosen window width—determined by the standard deviation σ—can render the transform information useless for intepreting signal evolution through time. The reason is not too hard to grasp. By narrowing the window, we obtain a more precise time frame in which frequencies of interest occur. But if we calculate the transform from discrete samples, then we cannot shrink σ too far; eventually the number of samples within the window are too few to compute the discrete signal frequencies. This is, of course, the threshold governed by the Nyquist rate. As σ decreases, then, the Gabor transform gains time-domain resolution, but .it loses frequency-domain resolution at the same time

VI. WAVELET TRANSFORMS

A.: Short Review of the wavelet Transform

The Fourier transform is an useful tool to analyze the frequency components of the signal. However, if we take the Fourier transform over the whole time axis, we cannot tell at what instant a particular frequency rises. Short-time Fourier transform (STFT) uses a sliding window to nd spectrogram, which gives the information of both time and frequency. But still another problem exists: The length of window limits the resolution in frequency. Wavelet transform seems to be a solution to the problem above. Wavelet transforms are based on small wavelets with limited duration. The translated-version wavelets locate where we concern. Whereas the scaled-version wavelets allow us to analyze the signal in .dierent scale

The Wavelet Transform (WT) of a continuous signal z(t) is : defined as

$$WT(a,b)=\frac{1}{\sqrt{a}}\int_{-\infty}^{\infty}x(t)g\left(\frac{t-b}{a}\right)dt\ .$$

As with the Short-time Fourier transform (٤)' the signal z (t i s transformed by an analyzing function g (y,) analogous to (w(t - T)e^{-jzTft} in the STFT. The analyzing function g (t) is not limited to the complex exponential. In fact, the only restriction on g (t) is that it must be short and oscillatory; i.e. it must have zero average and decay quickly at both ends.' This restriction ensures that the integral in (٧) is finite and gives the name wavelet or "small wave the transform, with g(t) referred to as the "mother wavelet" and its dilates and translates simply as "wavelets". Figure ٣ gives examples of two mother wavelets showing their oscillatory and potentially non-sinusoidal nature

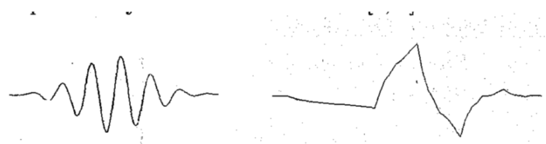


Figure 3: Example mother wavelets.

The second difference is the time-scaling parameter, a, not present in the STFT. The time extent of the wavelet g(٩)is expanded or contracted in time depending on whether a > ١ or a < ١/ A value of a > ١ (a < ١) expands (contracts) g(t) in time and decreases (increases) the frequency of the oscillations in g (y.) H ence, as a is ranged over some interval, usually beginning with unity and increasing, the input is analyzed by an increasingly dilated function that is .becoming less and less focused in time

As with the STFT, the wavelet transform has a digitally implementable counterpart the Discrete Wavelet Tkznsfown : (DWT). The DWT is defined as

$$DWT[m,k]=\frac{1}{\sqrt{a_0^m}}\sum_n x[n]\,g\left[\frac{k-na_0^m}{a_0^m}\right]$$

where g[n] is the mother wavelet, and the scaling and translation parameters a and b of (٧) are functions of an integer parameter m, a = ay and b = nu?. The result is ,geometric scaling, i.e. ١' :' &' ///' and translation by ٠' n, ٢n This scaling gives the DWT logarithmic frequency .... coverage in contrast to the uniform frequency coverage of .the STFT as compared in Figure ٤

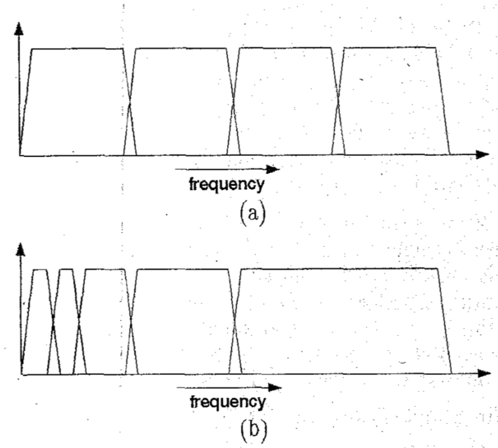


Figure 4: Comparison of (a) the windowed-DFT uniform frequency coverage to (b) the logarithmic coverage of the DWT.

The DWT output can be represented in a twodimensional grid in a similar manner as the STFT but with very different divisions in time and frequcy as shown in Figure ٥(b) for the signal of Figure ٥(a) the rectangles in Figure ٥(b) have equal area or constant timebandwidth product such that they narrow at the lower scales (higher frequencies) and widen at the higher scales (lower frequencies) and are shaded proportionally to the magnitude of the DWT output for the .input signal

In contrast with the WDFT for identical input (shown in Figure ٢)' the DWT isolates the transient component in the top frequency band at precisely the quarter-cycle of its occurrence while the ٦٠ Hz component is represented as a continuous magnitude. This illustrates how the multi-resolution properties of the wavelet transform are well suited to transient signals superimposed on a continuous .fundamental

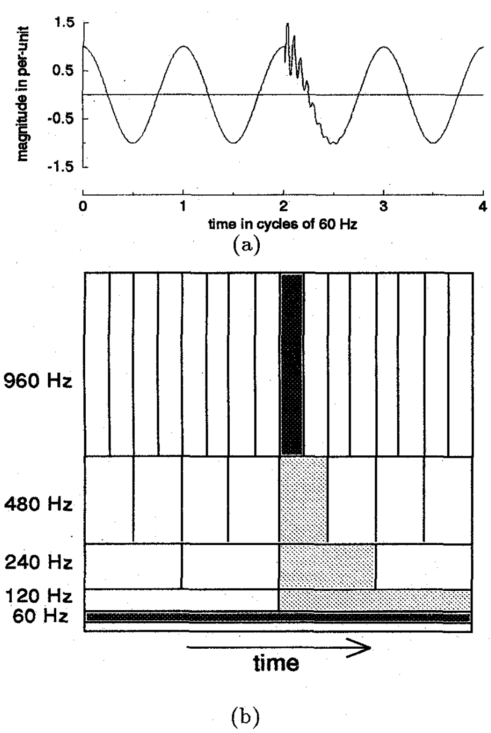


Figure 5: Discrete wavelet transform output: (a) example input signal, (b) DWT on input signal.

B.: History

The frst literature that relates to the wavelet transform is Haar wavelet. It was proposed by the mathematician Alfrd Haar in ۱۹۰۹/ However, the concept of the wavelet did not exist at that time. Until ۱۹۸۱' the concept was proposed by the geophysicist Jean Morlet. Afterward, Morlet and the physicist Alex Grossman invented the term wavelet in ۱۹۸٤/ Before ۱۹۸٥' Haar wavelet was the only orthogonal wavelet people know. A lot of researchers even thought that there was no orthogonal wavelet except Haar wavelet. Fortunately, the mathematician Yves Meyer constructed the second orthogonal wavelet called Meyer wavelet in ۱۹۸٥/ As more and more scholars joined in this feld, the ۱st .international conference was held in France in ۱۹۸۷

In ۱۹۸۸' Stephane Mallat and Meyer proposed the concept of multiresolution. In the same year, Ingrid Daubechies found a systematical method to construct the compact support orthogonal wavelet. In ۱۹۸۹' Mallat proposed the fast wavelet transform. With the appearance of this fast algorithm, the wavelet transform had numerous applications ..in the signal processing feld

C.Discrete Wavelet Transform

:We can approximatea discrete signal in l۲(Z)۱ by

$$f[n] = \frac{1}{\sqrt{M}} \sum_k W_\phi[j_0, k]\phi_{j_0,k}[n] + \frac{1}{\sqrt{M}} \sum_{j=j_0}^{\infty} \sum_k W_\psi[j, k]\psi_{j,k}[n].$$

Here f[n]; -j•;k[n]and j;k[n] are discrete functions dened in M;•]□totally M points. Because the sets f-j•;k[n]gk۲Z ,[۱ and f j;k[n]g(j;k)۲Z۲;jj• are orthogonal to each other. We can simply take the inner product to obtain the wavelet coecients

$$W_\phi[j_0, k] = \frac{1}{\sqrt{M}} \sum_n f[n]\phi_{j_0,k}[n].$$

$$W_\psi[j, k] = \frac{1}{\sqrt{M}} \sum_n f[n]\psi_{j,k}[n] \quad j \geq j_0.$$

- : The Fast Wavelet Transform

Start from the denition, if the form of scaling and wavelet function is known, its coecients is dened in ۳/۲ and ۳/۳/ If we can nd another way to nd the coecients without knowing the scaling and dilation version of scaling and wavelet function. The computation time can be reduced. From ۲/۲۰' we have

$$W_\psi[j, k] = h_\psi[-n] * W_\phi[j + 1, n]\Big|_{n=2k, k\geq 0}.$$

For the commonly used discrete signal, say, a digital image, the original data can be viewed as approximation coeefficients with order J. That is, f[n] = W-[J; n] By ۳/۷ and ۳/۸' next level of approximation and detail can be obtained. This algorithm is "fast" because one can nd the coefficients level by level rather than directly using ۳/۲ and to nd the coefficients. This algorithm was rst proposed ۳/۳ .in]۲[

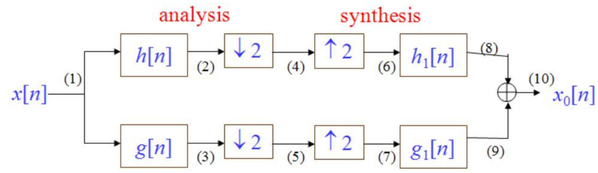


Figure 5: The schematic diagram to realize discrete wavelet transform. Here the filter names are changed.

D.Continous Wavelet Transform

This transform works when we use a continuous wavelet function to find the detailed coefficients of a continuous signal. we have to establish a basis to do such analysis. First, we give the definition of continuous wavelet transform and do some comparison between that and the Fourier transform.
We define a mother wavelet function (t) 2 L2(R), which is limited in time domain. That is, (t) has values in a certain range and zeros elsewhere. Another property of mother wavelet is zero-mean. The other property is that the mother wavelet is normalized. Mathematically, they are :

$$\int_{-\infty}^{\infty} \psi(t)dt = 0$$

$$\|\psi(t)\|^2 = \int_{-\infty}^{\infty} \psi(t)\psi^*(t)dt = 1.$$

As the dilation and translation property states, the mother wavelet can form a basis set denoted by :

$$\left\{\psi_{s,u}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right)\right\}\Bigg|_{u\in\mathbf{R}, s\in\mathbf{R}+}.$$

u is the translating parameter, indicating which region we concern. s is the scaling parameter greater than zero because negative scaling is undefined. The multiresolution property ensures the obtained set f u;s(t)g is orthonormal. Conceptually, the continuous wavelet transform is the coefficient of the basis u;s(t). It is :

$$\begin{aligned} Wf(s,u) &= \ < f(t), \psi_{s,u} > \\ &= \int_{-\infty}^{\infty} f(t)\psi_{s,u}^*(t)dt \\ &= \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{s}}\psi^*\left(\frac{t-u}{s}\right)dt. \end{aligned}$$

Via this transform, one can map an one-dimensional signal f(t) to a two dimensional coefficients W f(s; u). The two variables can perform the time frequency analysis. We can tell locate a particular frequency (parameters) at a certain time instant (parameter u). If the f(t) is a L2(R) function. The inverse wavelet transform is :

$$f(t) = \frac{1}{C_\psi}\int_0^{\infty}\int_{-\infty}^{\infty} Wf(s,u)\frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right)du\frac{ds}{s^2},$$

where C_ψ is defined as

$$C_\psi = \int_0^{\infty}\frac{|\Psi(\omega)|^2}{\omega}d\omega < \infty.$$

$\Psi(\omega)$ is the Fourier transform of the mother wavelet $\psi(t)$. This equation is also called the admissibility condition.

Here we illustrate a famous mother wavelet function, called Mexican hat wavelet with :

$$\psi(t) = \frac{2}{\pi^{1/4}\sqrt{3\sigma}}(\frac{t^2}{\sigma^2} - 1)\exp(-\frac{t^2}{\sigma^2}).$$

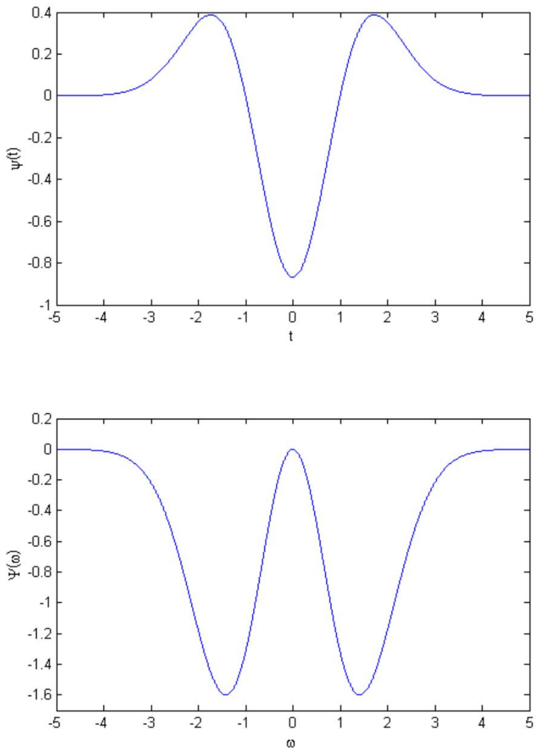


Figure 4: Mexican-hat wavelet for a = 1 and its Fourier transform

VII. Comparison Among the Fourier Transform, Short-time Fourier Transform(STFT) and Wavelet Transform

In this section, we discuss about features among the three .transforms

A. *Forward transform*

- Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\exp(-j\omega t)dt.$$

As we know, Fourier transform convert signal in time domain to frequency domain by integrating over the whole time axis. However, if the signal is not stationary, that is, the frequency composition is a function of time, we cannot tell when a certain frequency rises

- STFT

$$Sf(u,\xi) = \int_{-\infty}^{\infty} f(t)w(t-u)\exp(-j\xi t)dt.$$

The STFT tries to solve the problem in Fourier transform by introducing a sliding window w(t-u). The window is designed to extract a small portion of the signal f(t) and then take Fourier transform. The transformed coefficient has two independent parameters. One is the time parameter E , indicating the instant we concern. The other is the frequency parameter E, just like that in the Fourier transform. However another problem rises. The very low frequency component cannot be detected on the spectrum. It is the reason that we use the window with xed size. Suppose the window size is ١/ If there is a signal with frequency ٠/١ Hz, the extracted data in ١ .second look like at (DC) in the time domain

- Wavelet transform

$$Wf(s,u) = \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{s}}\psi^*(\frac{t-u}{s})dt.$$

Wavelet transform overcomes the previous problem. The wavelet function is designed to strike a balance between time domain (finite length) and frequency domain (finite bandwidth). As we dilate and translate the mother wavelet, we can see very low frequency components at large s while very high frequency . component can be located precisely at small s

B. *Inverse transform*

- Fourier transform

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\exp(j\omega t)dt.$$

- STFT

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Sf(u,\xi)w(t-u)\exp(j\xi t)d\xi du.$$

- Wavelet transform

$$f(t) = \frac{1}{C_\psi}\int_0^{\infty}\int_{-\infty}^{\infty} Wf(s,u)\frac{1}{\sqrt{s}}\psi(\frac{t-u}{s})du\frac{ds}{s^2},$$

$$C_\psi = \int_0^{\infty}\frac{|\Psi(\omega)|^2}{\omega}d\omega < \infty.$$

C. *Basis*

- Fourier transform

Complex exponential function with different :frequencies

$$\exp(j\omega t).$$

- STFT
  Truncated or windowed complex exponential :function

$$w(t-u)\exp(j\xi t).$$

- Wavelet transform

:Scaled and translated version of mother wavelets

$$\frac{1}{\sqrt{s}}\psi(\frac{t-u}{s}).$$

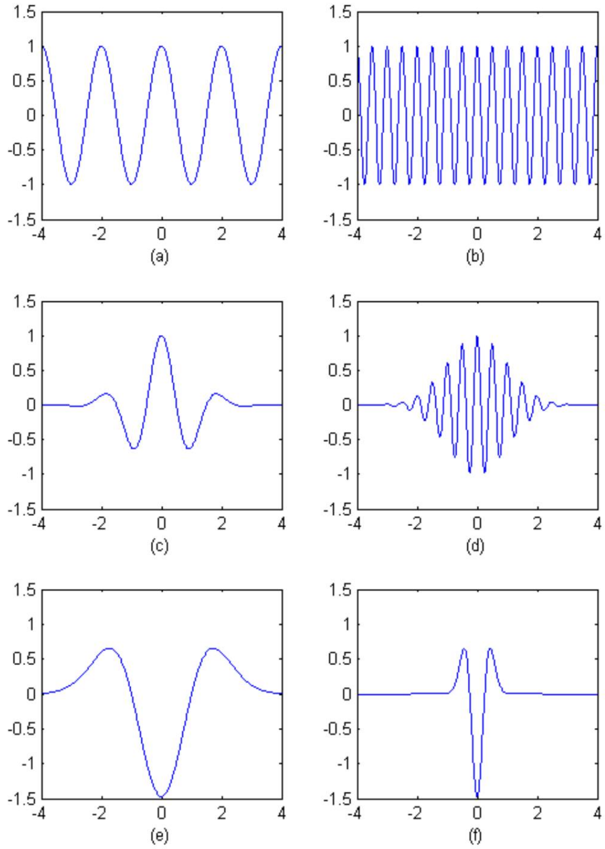


Figure 5 Different basis for the transforms. (a) Real part of the basis for Fourier transform, $\exp(j\pi t)$. (b) Basis for different frequency, $\exp(j4\pi t)$. (c) Basis for STFT, using Gaussian window of $\sigma = 1$. It is $\exp(-t^2/2)\exp(j\pi t)$. (d) Basis for different frequency, $\exp(-t^2/2)\exp(j4\pi t)$. (e) Mexican-hat mother wavelet function and (f) $s = 4$.

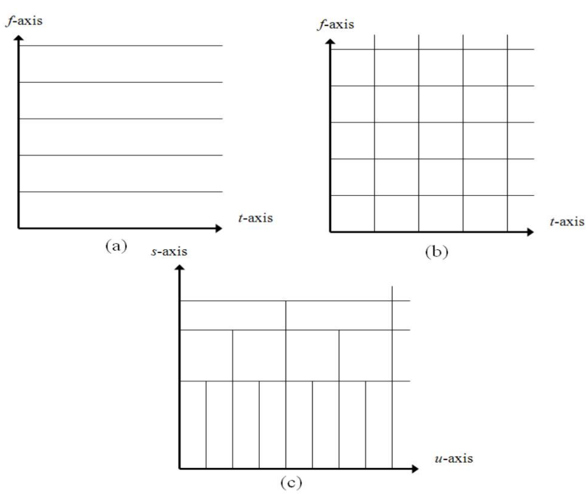
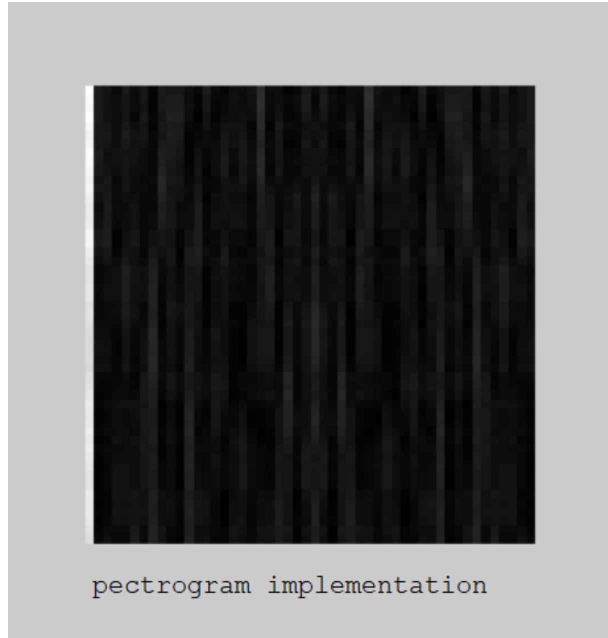


Figure 6 : Different time-frequency tile allocation of the three transforms: (a) Fourier transform, (a) STFT and (a) wavelet transform.

VIII. MATLAB IMPLEMENTATION RESULTS

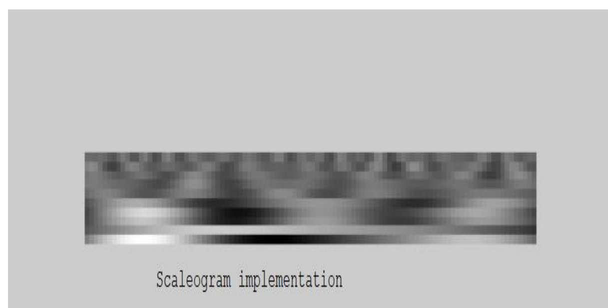
- :Spectrugram (windows size : ٥٠)

We have used the SFTF formula to set a window to roll . the signal and do the sampeling



- Scalogram (sacales of [0.4,0.7,1,1.2,1.8,2.5,3.2,3,8,٤]:(

We have applied the scales to WT formula and then set . the a window to roll the signal and do the sampeling



IX. CONCLUSION

We have described in this paper a (limited) number of aspects of wavelet analysis and time-frequency analysis, with emphasis on some signal processing related problems. Our goal was to give an idea of the broadness of the theory, as well as the huge range of applications and potential .applications

Time-frequency transforms are problematic in certain applications, especially those with transients or local frequency information that defies any a priori demarcation of its frequency- and time-domain boundaries. Quadratic methods have better spectral resolution, but interference terms are sometimes hard to overcome. Finally, the Balian–

Low theorem enforces a fundamental limitation on the joint time-frequency resolution capability of the windowed .Fourier transforms

This situation led to the discovery of another mixed-domain signal analysis tool— the wavelet transform, one of the great discoveries of mathematical analysis in the twentieth century. As we have already indicated, the wavelet transform uses a signal scale variable instead of a frequency variable in its tranform relation. This renders it better able to handle transient signal behavior, without completely giving up frequency selectivity. however we should admit that there are still many problems, connected to wavelet theory .and applications, which are still to be solved

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