The University of California, Los Angeles

Robotics Design Capstone

EE 183DB

Off-center spinning mass controller for Quadcopters

Author:
Lin Li
Angel Jimenez
Wilson Chang
Amirali Omidfar

Professor:
Ankur Mehta

June 11, 2018

Abstract

We aim to design an off-center spinning mass underactuated controller to steer flying objects. A quadcopter with a rotating arm attached to it is used to demonstrate the principle of such controller. By in depth analysis of the system dynamics and results of this project, we wanted to develop a similar model applicable for rocket control systems.

1 Mathematical Model

1.1 Symbols

Here is a list of all symbols used in this paper:

$\boldsymbol{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$	linear position vectors
$\boldsymbol{q} = \begin{bmatrix} q_r \\ q_i \\ q_j \\ q_k \end{bmatrix}$	angular orientation in quaternion
$\boldsymbol{F_T}$	thrust force
$\boldsymbol{F_G}$	gravitational force
$\boldsymbol{F_{AB}}$	reaction force acted from A on B
$\boldsymbol{\tau_{AB}}$	reaction torque acted from A on B
$\boldsymbol{\tau_M}$	torque generated by the motor
$\boldsymbol{\tau_{RF}}$	torque generated by the reaction force
m_A	mass of A
I_A	moment of inertial of A

1.2 Appendix

The Quaternion-derived Rotation matrix is defined as follow,

$$
{}^O_B R = R(\boldsymbol{q_B}) = \begin{bmatrix} q_r^2 + q_i^2 - q_j^2 - q_k^2 & 2q_iq_j - 2q_rq_k & 2q_iq_k + 2q_rq_j \\ 2q_iq_j + 2q_rq_k & q_r^2 - q_i^2 + q_j^2 - q_k^2 & 2q_jq_k - 2q_rq_i \\ 2q_iq_k - 2q_rq_j & 2q_jq_k + 2q_rq_i & q_r^2 - q_i^2 - q_j^2 + q_k^2 \end{bmatrix}
$$

1.3 Quadcopter Body Dynamics

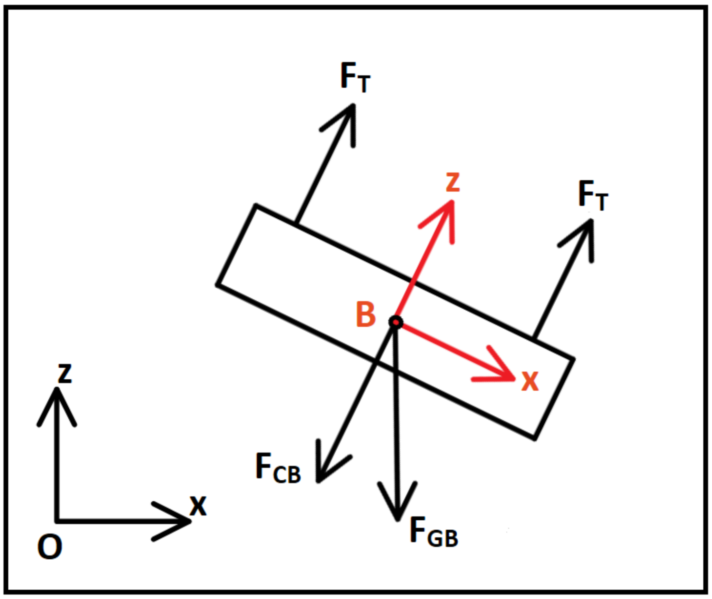

Figure 1: Free-Body diagram of Body

Forces and Torques:

$$
{}^{B}\boldsymbol{F_T} = \begin{bmatrix} 0 \\ 0 \\ F_{TB} \end{bmatrix}
$$

$$
{}^{O}\boldsymbol{F_{GB}} = \begin{bmatrix} 0 \\ 0 \\ -m_b g \end{bmatrix}
$$

$$
{}^{O}\boldsymbol{F_{CB}} = \begin{bmatrix} F_{CBx} \\ F_{CBy} \\ F_{CBz} \end{bmatrix}
$$

$$
{}^{B}\boldsymbol{\tau_{CB}} = \begin{bmatrix} \tau_{CBx} \\ \tau_{CBy} \\ -\tau_M \end{bmatrix}
$$

Net Force and Torque

$$ {}^{O}\boldsymbol{F_{net,B}} = {}^{O}\boldsymbol{F_{GB}} + {}^{O}\boldsymbol{F_{T}} + {}^{O}\boldsymbol{F_{CB}} = m_B \, {}^{O}\boldsymbol{a_B} \tag{1} $$

$$ {}^{O}\boldsymbol{\tau_{net,B}} = R(\boldsymbol{q_B}) \, {}^{B}\boldsymbol{\tau_{CB}} = {}^{O}I_B \, {}^{O}\boldsymbol{\alpha_B} \tag{2} $$

1.4 Controller Dynamics

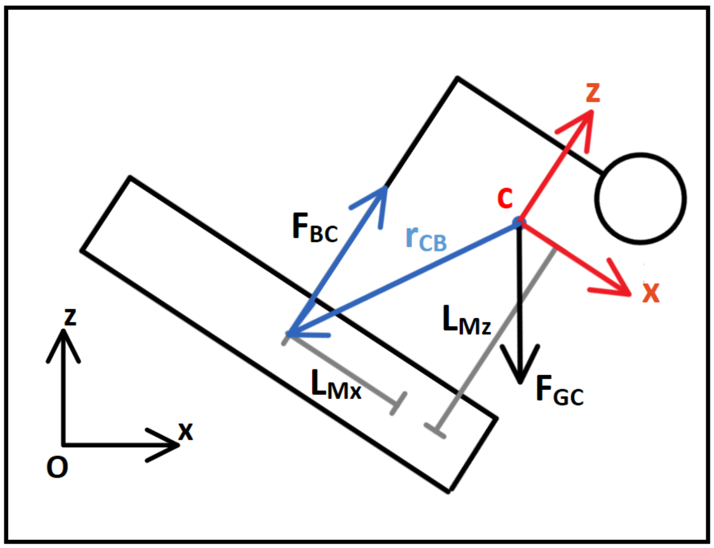

Figure 2: Free-Body diagram of Controller

Forces and Torques:

$$
\begin{aligned}
{}^{O}\boldsymbol{F_{BC}} &= \begin{bmatrix} F_{BCx} \\ F_{BCy} \\ F_{BCz} \end{bmatrix} \\
{}^{O}\boldsymbol{F_{GC}} &= \begin{bmatrix} 0 \\ 0 \\ -m_c g \end{bmatrix} \\
{}^{C}\boldsymbol{\tau_{BC}} &= \begin{bmatrix} \tau_{BCx} \\ \tau_{BCy} \\ \tau_M \end{bmatrix} \\
{}^{O}\boldsymbol{r_{CB}} &= R(\boldsymbol{q_C}) \begin{bmatrix} -L_{Mx} \\ 0 \\ -L_{Mz} \end{bmatrix} \\
{}^{O}\boldsymbol{\tau_{RF}} &= {}^{O}\boldsymbol{r_{CB}} \times {}^{O}\boldsymbol{F_{BC}}
\end{aligned}
$$

Net Force and Net Torque:

$$
{}^{O}\boldsymbol{F_{net,C}} = {}^{O}\boldsymbol{F_{BC}} + {}^{O}\boldsymbol{F_{GC}} = m_C \, {}^{O}\boldsymbol{a_C} \tag{3}
$$

$$
{}^{O}\boldsymbol{\tau_{net,C}} = R(\boldsymbol{q_C}) \, {}^{C}\boldsymbol{\tau_{BC}} + {}^{O}\boldsymbol{\tau_{RF}} = {}^{O}I_c \, {}^{O}\boldsymbol{\alpha_C} \tag{4}
$$

1.5 Constraints and Manipulation

In the derivation below, assume everything is in the inertial frame unless explicitly stated.

The two bodies are contrainted (attached together), there are some relationship between the states and the forces between the body and the controller,

Let $\boldsymbol{p_{sys}} = \boldsymbol{p_B}$ and $\boldsymbol{q_{sys}} = \boldsymbol{q_B}$,

$$
\begin{bmatrix} \boldsymbol{p_C} \\ \boldsymbol{q_C} \end{bmatrix} = \begin{bmatrix} \boldsymbol{p_B} + \boldsymbol{r_{BC}} \\ \boldsymbol{q_\theta q_B} \end{bmatrix} = \begin{bmatrix} \boldsymbol{p_{sys}} + \boldsymbol{r_{BC}} \\ \boldsymbol{q_\theta q_{sys}} \end{bmatrix} \tag{5}
$$

$$
\begin{bmatrix} \dot{\boldsymbol{p}}_{\boldsymbol{C}} \\ \dot{\boldsymbol{q}}_{\boldsymbol{C}} \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{p}}_{\boldsymbol{sys}} + \dot{R}(\boldsymbol{q_{sys}}) \, {}^{B}\boldsymbol{r_{BC}} \\ \boldsymbol{q_\theta} \dot{\boldsymbol{q}}_{\boldsymbol{sys}} + \dot{\boldsymbol{q}}_{\boldsymbol{\theta}} \boldsymbol{q_{sys}} \end{bmatrix} \tag{6}
$$

$$
\begin{bmatrix} \ddot{\boldsymbol{p}}_{\boldsymbol{C}} \\ \ddot{\boldsymbol{q}}_{\boldsymbol{C}} \end{bmatrix} = \begin{bmatrix} \ddot{\boldsymbol{p}}_{\boldsymbol{sys}} + \ddot{R}(\boldsymbol{q_{sys}}) \, {}^{B}\boldsymbol{r_{BC}} \\ \boldsymbol{q_\theta} \ddot{\boldsymbol{q}}_{\boldsymbol{sys}} + 2[\dot{\boldsymbol{q}}_{\boldsymbol{\theta}} \dot{\boldsymbol{q}}_{\boldsymbol{sys}}] + \ddot{\boldsymbol{q}}_{\boldsymbol{\theta}} \boldsymbol{q_{sys}} \end{bmatrix} \tag{7}
$$

Newton's Third Law

$$^{O}\boldsymbol{F_{BC}} = -\,^{O}\boldsymbol{F_{CB}} \tag{8}$$
$$^{O}\boldsymbol{\tau_{BC}} = -\,^{O}\boldsymbol{\tau_{CB}} \tag{9}$$

To limit our degree of freedom in the system, we have set a constraint for our quaternions, namely unit quaternion:

$$q_r^2 + q_i^2 + q_j^2 + q_k^2 = 1 \tag{10}$$
$$q_r\dot{q}_r + q_i\dot{q}_i + q_j\dot{q}_j + q_k\dot{q}_k = 0 \tag{11}$$
$$q_r\ddot{q}_r + q_i\ddot{q}_i + q_j\ddot{q}_j + q_k\ddot{q}_k + \dot{q_r}^2 + \dot{q_i}^2 + \dot{q_j}^2 + \dot{q_k}^2 = 0 \tag{12}$$

Last but not least, in the derivation below we use $\boldsymbol{q_\theta}$ directly for ease of typsetting, however, q_θ is not our state variable but θ, their relationship is defined below,

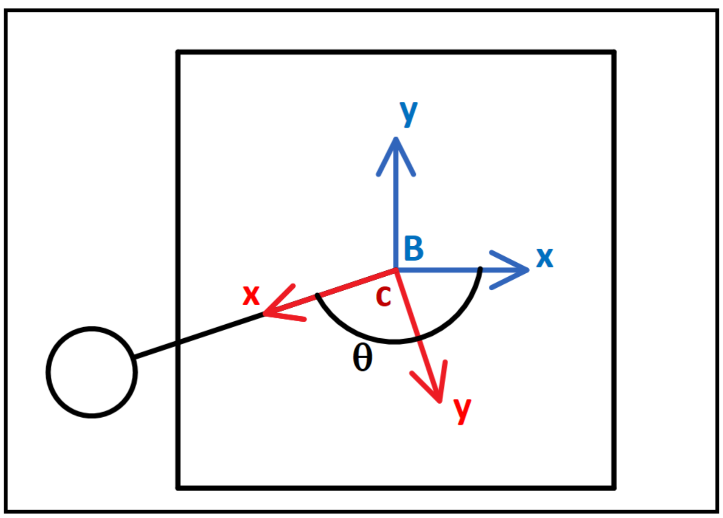

Figure 3: The yaw angle difference between Body and Controller

$$\boldsymbol{q_\theta} = \cos(\frac{\theta}{2}) + \sin(\frac{\theta}{2})R(\boldsymbol{q_{sys}})\,^{B}\hat{\boldsymbol{z_B}}$$
$$\dot{\boldsymbol{q}}_{\boldsymbol{\theta}} = -\frac{1}{2}\sin(\frac{\theta}{2})\dot{\theta} + \frac{1}{2}\cos(\frac{\theta}{2})\dot{\theta}R(\boldsymbol{q_{sys}})\,^{B}\hat{\boldsymbol{z_B}} + \sin(\frac{\theta}{2})R(\dot{\boldsymbol{q_{sys}}})\,^{B}\hat{\boldsymbol{z_B}}$$

where ${}^{B}\hat{\boldsymbol{z_B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

1.5.1 Combining the Force equations

From (1),

$$ {}^{O}\boldsymbol{F_{CB}} = m_B\,{}^{O}\boldsymbol{a_B} - {}^{O}\boldsymbol{F_{GB}} - {}^{O}\boldsymbol{F_T} $$

From (3),

$$ {}^{O}\boldsymbol{F_{BC}} = m_C\,{}^{O}\boldsymbol{a_C} - {}^{O}\boldsymbol{F_{GC}} $$

Using (6),

$$ m_B\,{}^{O}\boldsymbol{a_B} + m_C\,{}^{O}\boldsymbol{a_C} = {}^{O}\boldsymbol{F_{GC}} + {}^{O}\boldsymbol{F_{GB}} + {}^{O}\boldsymbol{F_T} $$

Simplifying the above expression, we get

$$ (m_b + m_c)\ddot{\boldsymbol{p}}_{\boldsymbol{sys}} + m_c\ddot{R}(\boldsymbol{q_{sys}})\,{}^{B}\boldsymbol{r_{BC}} = \boldsymbol{F_{GC}} + \boldsymbol{F_{GB}} + \boldsymbol{F_T} \tag{13} $$

1.5.2 Combining the Torqe equations

From (2),

$$ {}^{O}\boldsymbol{\tau_{CB}} = {}^{O}I_B\,{}^{O}\boldsymbol{\alpha_B} $$

From (4),

$$ {}^{O}\boldsymbol{\tau_{BC}} = {}^{O}I_c\,{}^{O}\boldsymbol{\alpha_C} - {}^{O}\boldsymbol{\tau_{RF}} $$

Using (7),

$$ {}^{O}I_B\,{}^{O}\boldsymbol{\alpha_B} + {}^{O}I_c\,{}^{O}\boldsymbol{\alpha_C} = {}^{O}\boldsymbol{\tau_{RF}} $$

Assuming all the vectors are represented in the inertial O frame, using the quaternion representation for angular acceleration,

$$ I_B 2\left[\ddot{\boldsymbol{q}}_{\boldsymbol{B}}\boldsymbol{q_B^*} - (\dot{\boldsymbol{q}}_{\boldsymbol{B}}\boldsymbol{q_B^*})^2\right] + I_c 2\left[\ddot{\boldsymbol{q}}_{\boldsymbol{C}}\boldsymbol{q_C^*} - (\dot{\boldsymbol{q}}_{\boldsymbol{C}}\boldsymbol{q_C^*})^2\right] = {}^{O}\boldsymbol{\tau_{RF}} \tag{14} $$

Substituting (5)-(7) in the above expression and isolating second derivative on the left, we have

$$ 2I_B[\ddot{\boldsymbol{q}}_{\boldsymbol{sys}}\boldsymbol{q_{sys}^*}] + 2I_C[\boldsymbol{q_\theta}\ddot{\boldsymbol{q}}_{\boldsymbol{sys}}(\boldsymbol{q_\theta q_{sys}})^*] + 2I_C[\ddot{\boldsymbol{q}}_{\boldsymbol{\theta}}\boldsymbol{q_{sys}}](\boldsymbol{q_\theta q_{sys}})^* - \boldsymbol{r_{CB}} \times \boldsymbol{F_{BC}} = \zeta \tag{15} $$

where

$$\zeta = 2I_B(\dot{\boldsymbol{q}}_{\boldsymbol{sys}}\boldsymbol{q}^*_{\boldsymbol{sys}})^2 + 2I_C[(\boldsymbol{q_\theta}\dot{\boldsymbol{q}}_{\boldsymbol{sys}} + \dot{\boldsymbol{q}}_{\boldsymbol{\theta}}\boldsymbol{q}_{\boldsymbol{sys}})(\boldsymbol{q_\theta}\boldsymbol{q}_{\boldsymbol{sys}})^*]^2 - 4I_C(\dot{\boldsymbol{q}}_{\boldsymbol{\theta}}\dot{\boldsymbol{q}}_{\boldsymbol{sys}})(\boldsymbol{q_\theta}\boldsymbol{q}_{\boldsymbol{sys}})^*$$

Note that we put $\boldsymbol{\tau_{RF}}$ on the left hand side, this is because we can express $\boldsymbol{F_{BC}}$ in terms of $\ddot{\boldsymbol{p}}_{\boldsymbol{sys}}$ from (1), a second derivative of positional state

$$\boldsymbol{F_{BC}} = m_B\ddot{\boldsymbol{p}}_{\boldsymbol{sys}} - \boldsymbol{F_{GB}} - \boldsymbol{F_T}$$

1.6 System of equations

From equation (12), (14), and (15), we have the function that relates our state variables together,

$$f(\ddot{\boldsymbol{p}}, \ddot{\boldsymbol{q}}, \ddot{\theta}, \dot{\boldsymbol{p}}, \dot{\boldsymbol{q}}, \dot{\theta}, \boldsymbol{p}, \boldsymbol{q}, \theta) = 0 \tag{16}$$

Assuming we can solve for $\ddot{\boldsymbol{p}}, \ddot{\boldsymbol{q}}, \ddot{\theta}$ given $\dot{\boldsymbol{p}}, \dot{\boldsymbol{q}}, \dot{\theta}, \boldsymbol{p}, \boldsymbol{q}, \theta$, let the state of our system to be

$$\boldsymbol{s_{sys}} = \begin{bmatrix} \dot{\boldsymbol{p}} \\ \dot{\boldsymbol{q}} \\ \dot{\theta} \\ \boldsymbol{p} \\ \boldsymbol{q} \\ \theta \end{bmatrix} \quad \text{so that} \quad \dot{\boldsymbol{s}}_{\boldsymbol{sys}} = \begin{bmatrix} \ddot{\boldsymbol{p}} \\ \ddot{\boldsymbol{q}} \\ \ddot{\theta} \\ \dot{\boldsymbol{p}} \\ \dot{\boldsymbol{q}} \\ \ddot{\theta} \end{bmatrix}$$

We have our state evolution equations as

$$s_{t+1} = s_t + \dot{s}_t\Delta t \tag{17}$$

1.7 Matlab Implementation

Implementing the systems of equations in (16), and solve for $\ddot{\boldsymbol{p}}, \ddot{\boldsymbol{q}}, \ddot{\theta}$ given $\dot{\boldsymbol{p}}, \dot{\boldsymbol{q}}, \dot{\theta}, \boldsymbol{p}, \boldsymbol{q}, \theta$ in Matlab doesn't yield a solution. There must be something wrong with the equations / the implementation.