Scattering of photon by a charged particle

(why is the sky blue?)

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1 Thomson Scattering

At first, we assume that we have a free charged particle that "gets hit" by a photon. because of the photon's electric field, the charge will do a harmonic oscillation motion. from the Newton's second law, we have:

$$m\frac{dv}{dt} = qE$$

We also know, from the Larmor formula, that the power radiated by an accelerated charged particle is given by:

$$P_{scat} = \frac{\mu q^2(\dot{v})^2}{6\pi c} = \frac{\mu q^2(\frac{qE}{m})^2}{6m^2\pi c} = \frac{\mu q^4 E^2}{6\pi c}$$

On the other hand, the total amount of power radiated by an electromagnetic field in a unit surface is the magnitude of the Poynting vector:

$$|S| = \frac{1}{\mu}|E \times B| = \frac{1}{\mu c}E^2$$

So we can define Thomson scattering cross section as:

$$\sigma_{Th} = \frac{P}{|S|} = \frac{\mu^2 q^4}{6m^2\pi}$$

This cross section will give us the fraction of the power that comes from scattering of the photon. for example, for an electron, this cross section is calculated below:

$$\sigma_{Th} = \frac{P}{|S|} = \frac{\mu^2 e^4}{6m^2\pi} \simeq 10^{-28}m^2$$

If we assume that this cross section is circular, then we can define a radius for it. for electron, this radius will be:

$$r = (\frac{\sigma_{Th}}{\pi})^{\frac{1}{2}} \simeq 2.8 \times 10^{-15}m$$

Now we can find the angular distribution of the radiation by the particle:

$$\frac{dP_{scat}}{d\Omega} = \frac{\mu q^2 (n \times \dot{v})^2}{16\pi c} = \frac{\mu q^4 (n \times E)^2}{16 m^2 \pi c}$$

in which, **n** is the direction of scattering. we also know that

$$(A \times B)^2 = A^2 B^2 - (A \cdot B)^2$$

which leads us to:

$$\frac{dP_{scat}}{d\Omega} = \frac{\mu q^4 [1 - (n \cdot E)^2]}{16 m^2 \pi c}$$

when we average the expression $[1 - (n \cdot E)^2]$ in all of the possibilities, the result will be:

$$1 - (n \cdot E)^2 = E^2 \cdot \frac{1 + cos^2\theta}{2}$$

Therefore:

$$\frac{dP_{scat}}{d\Omega} = \frac{\mu q^4 E^2}{32 m^2 \pi c} \cdot 1 + cos^2\theta$$

Now we must define Differential cross section which gives us the cross section for a scattering which results in the space angle of dΩ :

$$\frac{d\sigma}{d\Omega} = \frac{dP_{scat}/d\Omega}{|S|} = \frac{\mu^2 q^4}{32 m^2 \pi} \cdot 1 + cos^2\theta$$

Then the total cross section can be derived by integration over $d\Omega$:

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega$$

2 Scattering by a bound charge

Now we assume that our charge is bound by a force which causes its motion to become a damping oscillation with a natural frequency of ω_0 and a damping factor of γ:

$$m\ddot{r} + m\gamma\dot{r} + m{\omega_0}^2 r = qE$$

because the electric field **E** is created by the incoming light, it can be written as:

$$E(t) = Re(E_0 e^{i\omega t})$$

so the deviation of the particle from the center of the force is given by:

$$r(t) = \frac{q}{m} Re(\frac{E_0 e^{i\omega t}}{{\omega_0}^2 - \omega^2 - i\gamma\omega})$$

Therefore, the acceleration is given by:

$$\ddot{r}(t) = -\frac{q}{m} Re(\frac{E_0 \omega^2 e^{i\omega t}}{{\omega_0}^2 - \omega^2 - i\gamma\omega})$$

for reminders, the Larmor power formula is

$$P_{scat} = \frac{\mu q^2 \ddot{r}^2}{6\pi c}$$

because it's a complex function, the average of its squared result is:

$$\overline{E(t)^2} = \frac{1}{2} E_0^2$$

So the average Larmor power and Poynting vector's magnitude become:

$$\overline{P_{scat}} = \frac{\mu q^2 \overline{\ddot{r}^2}}{6\pi c} = \frac{\mu q^4}{12 m^2 \pi c} \cdot \frac{E_0^2 \omega^4}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}$$

$$\overline{S} = \frac{1}{\mu c} \overline{E^2} = \frac{1}{2\mu c} E_0^2$$

The cross section can be derived from the fraction of the two formulas above:

$$\sigma = \frac{\overline{P_{scat}}}{\overline{S}} = \frac{\mu^2 q^4}{6 m^2 \pi c} \cdot \frac{\omega^4}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}$$

if $\omega \gg \omega_0$, then the cross section approaches to the Thomson cross section:

$$\sigma \to \frac{\mu^2 q^4}{6 m^2 \pi c} = \sigma_{Th}$$

On the other hand, if $\omega \ll \omega_0$, then:

$$\sigma \to \sigma_{Th} \cdot \frac{\omega^4}{\omega_0^4}$$

Which we call **Rayleigh cross section**. This shows that at the low frequencies, $\sigma \propto \omega^4$. actually, in this domain, the highest frequencies are scattered most strongly. **this frequency behaviour is presumably one of the reasons the sky is blue.**

On another special conditions, we may have the **Resonant scattering**, which happens when $\omega = \omega_0$. at this point:

$$\sigma = \frac{\omega^2}{\gamma^2} \cdot \sigma_{Th}$$

Which for small damping factors $\gamma \ll \omega$, becomes much greater than the Thomson cross section; but the factor can never be equal to zero, because in that case, the particle will radiate due to its acceleration.