Cubic Spline

Professor : Dr. Safari

St : AmirAbbas Saberi

University of Tehran

May,24,2023

A cubic spline is a spline constructed of piecewise third-order polynomials which pass through a set of control points. The second derivative of each polynomial is commonly set to zero at the endpoints, since this provides a boundary condition that completes the system of. equations.

1 Cubic Spline and its error

In general, the interpolation error with polynomials is obtained by the following method (the function must be differentiable up to degree n)

$$e_n(x) = f(x) - p(x) = \frac{f^{n+1}(\zeta)}{(n+1)!} \prod (x - x_i) \tag{1}$$

SOwith some simplicity ($x_0 = a$, $x_n = b$, $h = \frac{b-a}{n}$)

$$|f(x) - p_n(x)| \leq \frac{1}{4(n+1)} M(\frac{b-a}{n}) \tag{2}$$

Now the important question is how to reduce the error?

- Frist , Adding sampling points, which along with the possibility of derivability according to the formula we obtained, can cause problems
  for example :
  Assume a function $f = sin(x)$ such that $a = 0$, $b = 1.6875$ and n = 9 then

$$f^{10}(x) = sin(x) \rightarrow |f^{10}(x)| \leq 1$$
$$|sin(x) - p_n(x)| \leq 1.34 * 10^{-9}$$

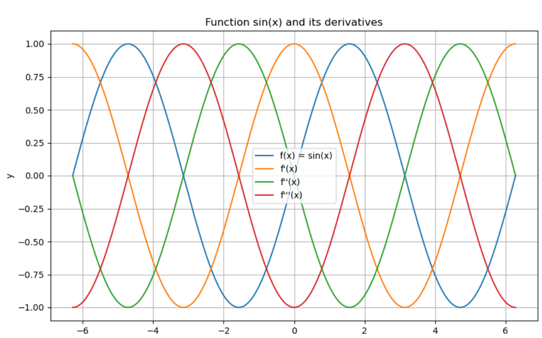

Figure 2: f = sin(x)

Here, adding the number of error points reduces, but let's consider another example: Assume a function $f = \frac{1}{1+x^2}$

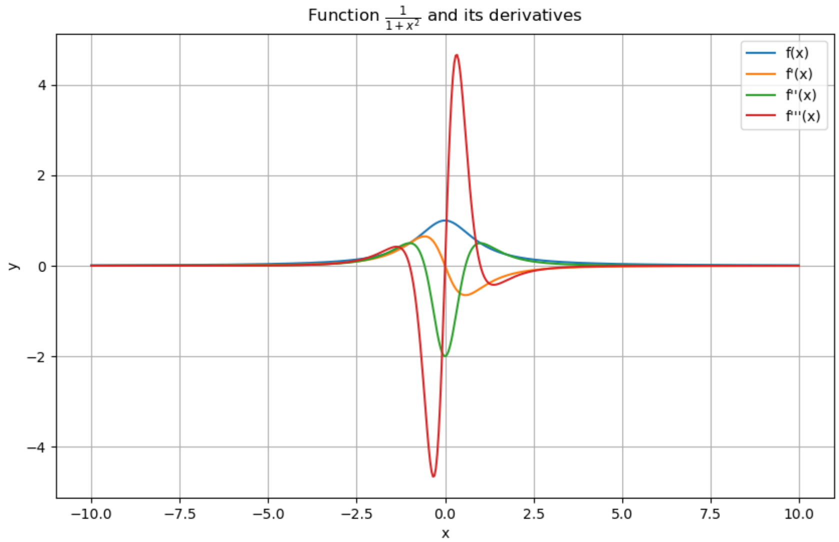

Figure 3: f = $\frac{1}{1+x^2}$ and its derivatives

here is resualt of n = 20 degrees of newtoinian interpolation.

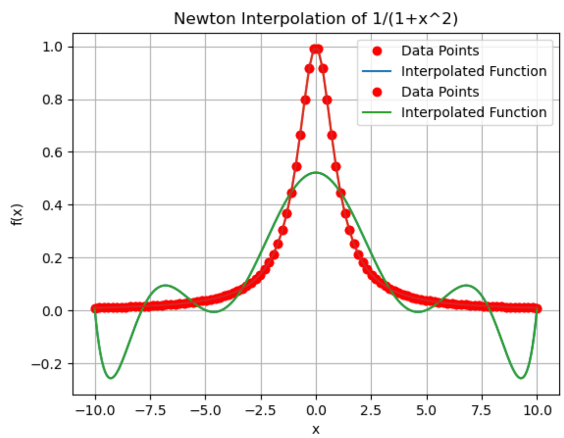

Figure 4: Interpolation of f = $\frac{1}{1+x^2}$ with Newtonian method

$$P = \prod (x_i - x_j) \tag{3}$$

Adding points in this function does not reduce the error, so we have to look for another way.

- The second is to make the intervals smaller and define a separate polynomial for each small interval.

$$S_{m,n} = \begin{cases} p_1(x) = a_0^1 + a_1^1 x + a_2^1 x^2 + ... + a_m^1 x^m \\ p_2(x) = a_0^2 + a_1^2 x + a_2^2 x^2 + ... + a_m^2 x^m \\ . \\ . \\ . \\ p_n(x) = a_0^n + a_1^n x + a_2^n x^2 + ... + a_m^n x^m \end{cases} \tag{4}$$

In this project, we work with a cubic spline that:

$$S_{3,n} = \begin{cases} p_1(x) = a_3^1 x^3 + a_2^1 x^2 + a_1^1 x + a_0^1 \\ p_2(x) = a_3^2 x^3 + a_2^2 x^2 + a_1^2 x + a_0^2 \\ . \\ . \\ . \\ p_n(x) = a_3^n x^3 + a_2^n x^2 + a_1^n x + a_0^n \end{cases} \tag{5}$$

$$4n - 2 = \begin{cases} 1) p_i'(x) = p_{i+1}'(x), [i = 1, ..., n-1] \\ 2) \ddot{p}_i(x) = \ddot{p}_{i+1}(x), [i = 1, ..., n-1] \\ 3) p_i'(x_i) = f_{i-1}), [i = 1, ..., n] \\ 4) p_i'(x) = f_i, [i = 1, ..., n] \end{cases} \tag{6}$$

We have two conditions to obtain low coefficients :

  - Natural

$$\begin{cases} 1) s_{3,n}'(a) = s_{3,n}'(b) = 0 \\ 2) \ddot{s}_{3,n}(a) = \ddot{s}_{3,n}(b) = 0 \end{cases} \tag{7}$$

  - Periodic

$$s_{3,n}^k(a) = s_{3,n}^k(b) \leftarrow k = [0, 1, 2] \tag{8}$$

– Conditional

$$\begin{cases} 1)s'_{3,n}(a) = f'_0 \\ 2)s'_{3,n}(b) = f'_n \end{cases} \tag{9}$$

Theorem: Suppose $\ddot{f}$ is continuous in a and b, and if S is the cubic interpolator function of f in the nodes ti and $0 \leq i < n$ then:

$\int_a^b [\ddot{s}(x)]^2\,dx \leq \int_a^b [\ddot{f}(x)]^2\,dx$

A conclusion can be drawn from the above case :

$$s(x) = \sum_{i=0}^{m} a_i x + \sum_{i=0}^{m} b_i (x - x_i)^{2m+1} \tag{10}$$

and

$$e_n(x) = \frac{(b-4)^4}{24n^4} max|f^{(4)}(x)| \tag{11}$$

The error is reduced by a factor of $\frac{1}{n^4}$

$$s(x) = Ab \tag{12}$$

$$A = \begin{bmatrix} 1 & x_1 & 0 & 0 & 0 & . & . & . & 0 \\ 1 & x_2 & (x_2 - x_1)^3 & 0 & 0 & . & . & . & 0 \\ 1 & x_3 & (x_3 - x_1)^3 & (x_3 - x_2)^3 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 1 & x_n & (x_n - x_1)^3 & (x_n - x_2)^3 & 0 & . & . & . & (x_n - x_{n-1})^3 \\ 0 & 0 & 1 & 1 & . & . & . & 1 & 1 \\ 0 & 0 & x_1 & x_2 & . & . & . & . & x_n \end{bmatrix} \tag{13}$$

$$b = A^{-1}s \tag{14}$$

2 Example of Natural Cubic Spline and Linear Spline

2.1 $\mathbf{f} = \frac{1}{1+x^2}$

Consider the f function as the $\frac{1}{1+x^2}$ function

We consider the sampling step length of the function to be 0.25 and apply a cubic

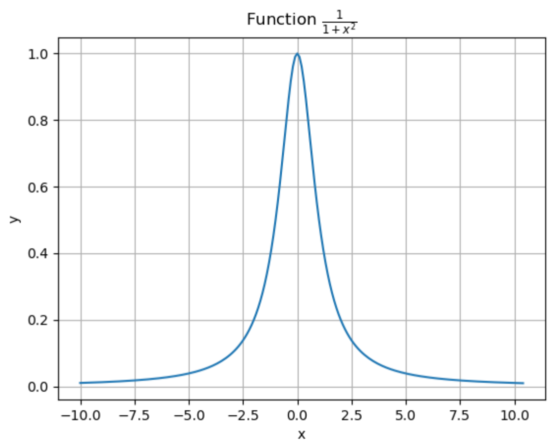

Figure 5: $\mathrm{f} = \frac{1}{1+x^2}$

spline to the resulting sample

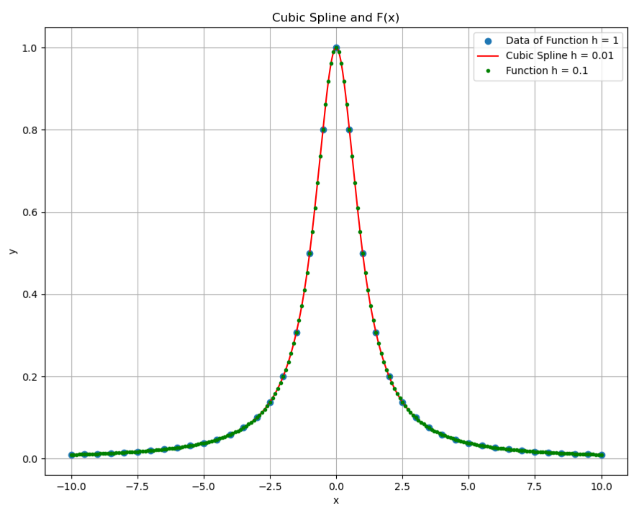

Figure 6: Spline of f , sample (h = 1) with $h_s = 0.01$

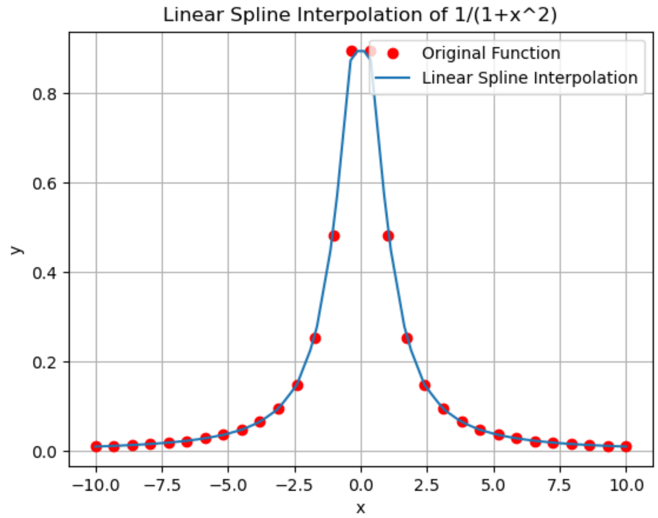

Figure 7: Spline of f , sample (h = 1) with $h_s = 0.01$

error with real functions (step length of the spline to be 0.01) :

Table 1: Table of Spline Error.

e = $f - \hat{s}$	
Cubic	Linear
0.00155	0.0255

2.2 f $= 11(cos(0.5x))^2 + x^4 + 7sin(2x)$

Consider the f function as the $\frac{1}{1+x^2}$ function

We consider the sampling step length of the function to be 0.25 and apply a cubic

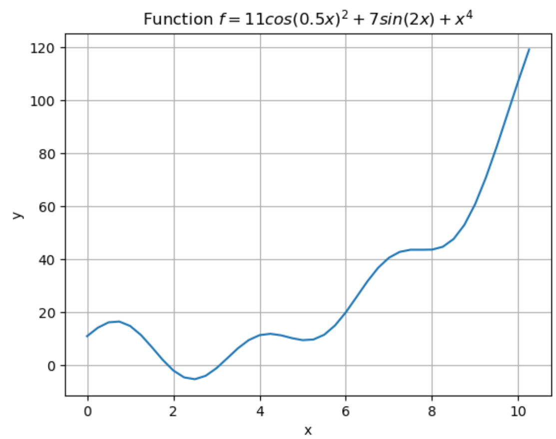

Figure 8: f $= 11(cos(0.5x))^2 + x^4 + 7sin(2x)$

spline to the resulting sample

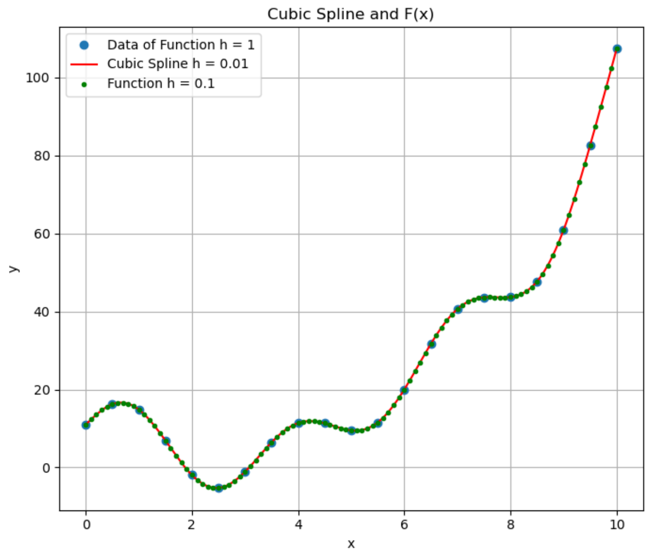

Figure 9: Spline of f , sample (h = 1) with $h_s = 0.01$

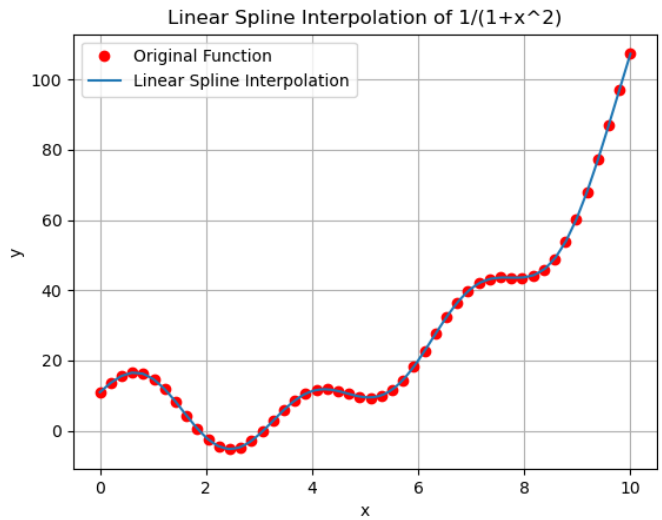

Figure 10: Spline of f , sample (h = 1) with $h_s = 0.01$

error with real functions (step length of the spline to be 0.01) :

Table 2: Table of Spline Error.

$e = f - \hat{s}$	
Cubic	Linear
0.0726	0.1327