In the Name of God

Communication Systems (25751-1)
Problem Set 01

Department of Electrical Engineering
Sharif University of Technology
Fall Semester 1398-99
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Due on Mehr 28, 1398 at 7:30 a.m.

1 Fourier Series of Periodic Signals

Determine the Fourier series expression of following signals:

1. $x_1(t) = \sum_{n=-\infty}^{\infty} (-1)^n \Lambda(t-2n)$
2. $x_2(t) = \sum_{n=-\infty}^{\infty} (-1)^n \delta'(t-nT)$
3. $x_3(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \sum_{n=-\infty}^{\infty} e^{-\frac{(t-\mu-nT_s)^2}{2\sigma^2}}$
4. $x_4(t) = |\cos(2\pi f_0 t)|$
5. $x_5(t) = f(t)^* e^{j\frac{2\pi t}{T_0}}$
   (in terms of $f(t)$ (periodic with period T_0) Fourier series coefficients)
6. $x_6(t) = y_1(t)y_2(t)$
   (where y_1 and y_2 are signals of period T, whose Fourier series coefficients are a_n and b_n. Find the answer in terms of a_n and b_n.)

2 Parseval's Theorem

Let $x(t)$ and $y(t)$ be two energy-type signals, and let $X(f)$ and $Y(f)$ denote their Fourier transforms, respectively. Show that:

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y(f)^* df$$

3 Averaging Operator

Let $f(t)$ be a periodic signal of period T and define the *averaging operator* depending on a parameter $h > 0$ by

$$\mathcal{A}_h f(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t) \mathrm{d}t$$

Thus $\mathcal{A}_h f(x)$ is a new signal.

1. Show that $\mathcal{A}_h f(x)$ is periodic of period T as a function of x, i.e.,

$$\mathcal{A}_h f(x+T) = \mathcal{A}_h f(x)$$

2. Find the Fourier series of $\mathcal{A}_h f(x)$ in terms of the Fourier series of $f(t)$.

4 Poisson Sum Formula

1. By computing the Fourier series coefficients for the periodic signal $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$, shows that:

$$\sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} e^{jn\frac{2\pi t}{T_s}}$$

2. Using the result of part (1), prove that for any signal $x(t)$ and any T_s ,the following identity holds:

$$\sum_{n=-\infty}^{\infty} x(t - nT_s) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_s}\right) e^{jn\frac{2\pi t}{T_s}}$$

3. Conclude the following relation known as *Poisson's sum formula.*

$$\sum_{n=-\infty}^{\infty} x(nT_s) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_s}\right)$$

5 Types of Signals

Classify the following signals into energy-type, power-type, and neither energy-type nor power-type signals. For energy-type or power-type signals find the energy or the power contents of the signal.

1. $x_1(t) = e^{-\alpha|t|}\cos(\beta t) \qquad (\alpha > 0)$
2. $x_2(t) = \dfrac{1}{\pi t}$
3. $x_3(t) = \sum_{n=-\infty}^{\infty} (-1)^n \Lambda(t - 2n)$
4. $x_4(t) = Ae^{j(2\pi f_0 t + \theta)}$
5. $x_5(t) = \begin{cases} Kt^{-\frac{1}{4}} & t > 0 \\ 0 & t \leq 0 \end{cases}$

6 Fourier Transform

Determine the Fourier transform of each of the following signals:

1. $x_1(t) = \dfrac{t}{a^2 + t^2}$
2. $x_2(t) = \Lambda(2t + 3) + \Lambda(3t - 2)$
3. $x_3(t) = t^n \mathrm{sinc}(t) \qquad (n > 1)$
4. $x_4(t) = te^{-\alpha|t|}\cos(\beta t) \qquad (\alpha > 0)$

7 Fourier Transform and Real Integrals

Use the known properties of the Fourier transform to obtain the following:

1. $I_1 = \displaystyle\int_0^{+\infty} \frac{1}{(a^2 + x^2)^2} dx$
2. $I_2 = \displaystyle\int_0^{+\infty} e^{-\alpha t}\mathrm{sinc}^2(\beta t) dt \qquad (\alpha > 0)$
3. $I_3 = \displaystyle\int_0^{+\infty} \frac{\sin^4(t)}{t^4} dt$

8 Fourier Transform Properties

(*Hint:* In each part of this problem, you may use the results from previous parts or the Fourier transform of common functions such as $\Pi(t)$, $\Lambda(t)$, etc.)

Consider the functions $g(x)$ and $f(x)$, shown in figure 1.

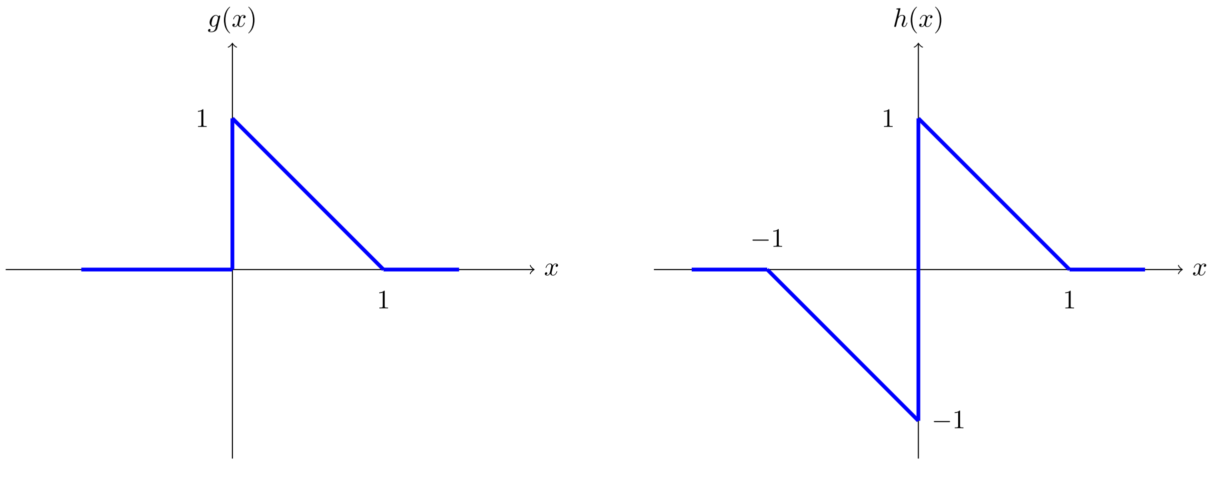

Figure 1

Denote the Fourier transforms by $G(f)$ and $H(f)$, respectively.

1. Consider the imaginary part of $G(f)$:

$$\mathbf{Im}\, G(f) = \frac{\sin(2\pi f) - 2\pi f}{4\pi^2 f^2}$$

How do you explain the singularity of $\mathbf{Im}G(f)$ at $f = 0$ while $g(x)$ is absolutely integrable?

2. What are the two possible values of $\measuredangle\{H(f)\}$, i.e., the phase of $H(f)$? Express your answer in radians.
3. Evaluate $\int_{-\infty}^{\infty} G(f)\cos(\pi f)\mathrm{d}f$.
4. Evaluate $\int_{-\infty}^{\infty} H(f)e^{j4\pi f}\mathrm{d}f$.
5. Without performing any integration, what is the real part of $G(f)$? Explain your reasoning.
6. Without performing any integration, what is $H(f)$? Explain your reasoning.
7. Suppose $h(x)$ is periodized to have period $T = 2$. Without performing any integration, what are the Fourier series coefficents, c_k, of this periodic signal?