In the Name of God

Communication Systems (25751-1)
Problem Set 04

Department of Electrical Engineering
Sharif University of Technology
Fall Semester 1398-99
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Due on ////////////////// at 7:30 a.m.

1 Autocorrelation for Random Processes

Find the autocorrelation function for each of the following processes. Can you define power spectral density function? If so, find the power spectral density.

1. $X(t) = A\cos(2\pi f_0 t + \theta)$
   A is constant, and and θ is a random variable uniformly distributed on $[0, \theta_m]$
2. $X(t) = Y_1 + Y_2$
   where Y_1 and Y_2 are independent, Y_1 is uniform on $[a_1, b_1]$ and Y_2 is uniform on $[a_2, b_2]$.

2 The Random Telegraph Wave

Figure 1 represents a sample function of a *random telegraph wave.* This signal makes independent random shifts between two equally likely values, A and 0. The number of shifts per unit time is governed by a Poisson distribution, with μ being the average shift rate. It means that

$$\mathbb{P}\left[\text{There are } n \text{ shifts in } (t, t+T)\right] = \frac{(\mu T)^n}{n!} e^{-\mu T}$$

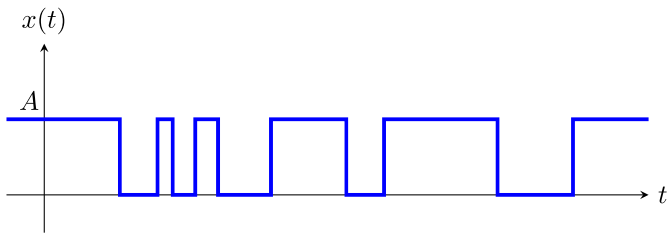

Figure 1: Random Telegraph Wave

1. Show that this random process is WSS.
2. Show that the autocorrelation function of this process is:

$$R_x(\tau) = \frac{A^2}{4}\left(e^{-2\mu|\tau|} + 1\right)$$

3. Find the power-spectral density of this process.

3 Power Spectral Density Estimation

Consider a random signal $x(t)$, its autocorrelation function $R_x(\tau)$ and its power-spectral density $\mathcal{S}_x(f)$. Consider the finite-duration or *truncated* random signal

$$x_T(t) = \begin{cases} x(t) & |t| \leq \frac{T}{2} \\ 0 & |t| > \frac{T}{2} \end{cases}$$

and its Fourier transform, $X_T(f)$

1. Write $X_T(f)$ in terms of $x_T(t)$ and $x(t)$.
2. We want to calculate $\mathbb{E}\left[|X_T(f)|^2\right]$. From the definition of $X_T(f)$, we can write:

$$\mathbb{E}\left[|X_T(f)|^2\right] = \int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} \mathbb{E}[x(t_1)x^*(t_2)]e^{-j2\pi f(t_1-t_2)}dt_1 dt_2$$

   Let $\tau = t_1 - t_2$ and $\mu = t_1$, then calculate the double integral over a appropriate region of the $\tau\mu$ plane and show that:

$$\mathbb{E}\left[|X_T(f)|^2\right] = \int_{-T}^{0} R_x(\tau)e^{-j2\pi f\tau}(T+\tau)d\tau + \int_{0}^{T} R_x(\tau)e^{-j2\pi f\tau}(T-\tau)d\tau$$

3. Show that

$$\mathbb{E}\left[|X_T(f)|^2\right] = T\int_{-T}^{T}\left(1-\frac{|\tau|}{T}\right)R_x(\tau)e^{-j2\pi f\tau}d\tau$$

4. Conclude that

$$\mathcal{S}_x(f) = \lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left[|X_T(f)|^2\right]$$

4 Stochastic Analysis of System Input-Output Relationship

The stationary process $X(t)$ is passed through an LTI system and the output process is denoted by $Y(t)$. Find the output autocorrelation function and the cross-correlation function between the input and the output process in each of the following cases:

1. A system defined by the input-output relation $y(t) = x(t) - x(t-T)$.
2. A system with $h(t) = \frac{1}{t}$
3. A system with $h(t) = e^{-\alpha t}u(t)$ where $\alpha > 0$.
4. A system described by $\frac{d}{dt}Y(t) + Y(t) = \frac{d}{dt}X(t) - X(t)$.
5. A finite time average defined by the input-output relation $y(t) = \frac{1}{2T}\int_{-T}^{T} x(\tau)d\tau$.

5 Stochastic Analysis of System Input-Output Relationship (2)

$X(t)$ is a stationary random process with autocorrelation function $R_x(\tau) = e^{-\alpha|\tau|}$. This process is applied to an LTI system with $h(t) = e^{-\beta t}u(t)$. Find the power-spectral density of the output process $Y(t)$.

6 A Gaussian Process Problem

The goal of this problem is to find the autocorrelation and power spectral density functions for a random process $Y(t)$ defined as:

$$Y(t) = X^2(t)$$

where $X(t)$ is a stationary real normal process with zero mean. Follow the steps to solve this problem.

1. Let X be a normal random variable:

   $$X \sim \mathcal{N}(\mu, \sigma^2)$$

   Find $\Phi(\omega)$, the characteristic function of X. (You may need to review the definition and applications of the characteristic function from your previous probability course.)

2. Consider the following definition: (You must already know it from your probability course)
   Definition: The random variables X_i are *jointly normal* if and only if the sum

   $$a_1 X_1 + \cdots + a_n X_n = \mathbf{a}^\intercal \mathbf{X}$$

   is a normal random variable for any $\mathbf{a}$.
   Suppose $\mathbb{E}[X_i] = 0$ for $1 \le i \le n$. Find the joint characteristic function $\Phi(\omega_1, \ldots, \omega_n)$ for the random vector $\mathbf{X}$ in terms of the covariance matrix C. (Remember the definition of the covariance matrix: $C_{ij} = \mathbb{E}[(X_i - \mu_i)(X_j - \mu_j)]$)
   Hint: Define $W = \omega_1 X_1 + \cdots + \omega_n X_n$ and use the result from the previous part for W. (Is W normal? Why?)

3. The four random variables $X1$, $X2$, $X3$, $X4$ are zero-mean jointly Gaussian random variables with covariance $C_{ij} = \mathbb{E}[X_i X_j]$ and characteristic function $X(\omega_1, \omega_2, \omega_3, \omega_4)$. Show that

   $$\mathbb{E}[X_1 X_2 X_3 X_4] = C_{12}C_{34} + C_{13}C_{24} + C_{14}C_{23}$$

4. a *Gaussian (normal) process* is a stochastic process such that every finite collection of those random variables has a jointly normal distribution. Let $X(t)$ be a stationary real normal process with zero mean. Let a new process $Y(t)$ be defined by

   $$Y(t) = X^2(t)$$

   Determine the autocorrelation function of $Y(t)$ in terms of the autocorrelation function of $X(t)$.

5. Determine the power spectral density function of $Y(t)$ in terms of the autocorrelation function and the power spectral density function of $X(t)$.