Selected Problems

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Signals and Systems
Tutorial Session 4

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Overview

Problem 1

Problem 1

Consider the down-sampler system:

$$y[n] = x[Mn]$$

Find $Y(z)$ in terms of $X(z)$.

Problem 1 Solution

$$w_M[n] = \frac{1}{M} \sum_{k=0}^{M-1} e^{j2k\pi n/M}$$

Problem 1 Solution

$$w_M[n] = \frac{1}{M}\sum_{k=0}^{M-1} e^{j2k\pi n/M}$$
$$= \begin{cases} \frac{1}{M}\sum_{k=0}^{M-1} 1 & M \mid n \\ \frac{1}{M}\frac{1-e^{j2k\pi n}}{1-e^{j2k\pi n/M}} & M \nmid n \end{cases}$$

Problem 1 Solution

$$
\begin{aligned}
w_M[n] &= \frac{1}{M}\sum_{k=0}^{M-1} e^{j2k\pi n/M} \\
&= \begin{cases} \frac{1}{M}\sum_{k=0}^{M-1} 1 & M \mid n \\ \frac{1}{M}\frac{1-e^{j2k\pi n}}{1-e^{j2k\pi n/M}} & M \nmid n \end{cases} \\
&= \begin{cases} 1 & M \mid n \\ 0 & M \nmid n \end{cases}
\end{aligned}
$$

Problem 1 Solution

$$Y(z) = \sum_{n=-\infty}^{\infty} x[Mn] z^{-n} = \sum_{n' \in M\mathbb{Z}} x[n'] z^{-n'/M}$$

Problem 1 Solution

$$
\begin{aligned}
Y(z) &= \sum_{n=-\infty}^{\infty} x[Mn] z^{-n} = \sum_{n' \in M\mathbb{Z}} x[n'] z^{-n'/M} \\
&= \sum_{n=-\infty}^{\infty} w_M[n] x[n] z^{-n/M} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{M} \sum_{k=0}^{M-1} e^{j2k\pi n/M} \right) x[n] z^{-n/M}
\end{aligned}
$$

Problem 1 Solution

$$
\begin{aligned}
Y(z) &= \sum_{n=-\infty}^{\infty} x[Mn]z^{-n} = \sum_{n' \in M\mathbb{Z}} x[n']z^{-n'/M} \\
&= \sum_{n=-\infty}^{\infty} w_M[n]x[n]z^{-n/M} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{M} \sum_{k=0}^{M-1} e^{j2k\pi n/M} \right) x[n]z^{-n/M} \\
&= \frac{1}{M} \sum_{k=0}^{M-1} \sum_{n=-\infty}^{\infty} x[n] \left(e^{-j2k\pi/M} z^{1/M} \right)^{-n} = \frac{1}{M} \sum_{k=0}^{M-1} X\left(e^{-j2k\pi/M} z^{1/M} \right)
\end{aligned}
$$

Problem 2

Problem 2 - Part 1

Find $X_1(z)$ and its ROC for the following discrete-time signal:

$$x_1[n] = \left(\frac{1}{2}\right)^{\lfloor n/3 \rfloor}$$

Problem 2 - Part 1 - Solution

$$x_0[n] = \begin{cases} \alpha^{n/3} & 3 \mid n \\ 0 & \text{otherwise} \end{cases}$$

Problem 2 - Part 1 - Solution

$$x_0[n] = \begin{cases} \alpha^{n/3} & 3 \mid n \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow X_0(z) = \frac{1}{1 - \alpha z^{-3}} \quad , \quad |z| > \sqrt[3]{|\alpha|}$$

Problem 2 - Part 1 - Solution

$$x_0[n] = \begin{cases} \alpha^{n/3} & 3 \mid n \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow X_0(z) = \frac{1}{1 - \alpha z^{-3}} \quad , \quad |z| > \sqrt[3]{|\alpha|}$$

$$x_1[n] = x_0[n] + x_0[n-1] + x_0[n-2]$$

Problem 2 - Part 1 - Solution

$$x_0[n] = \begin{cases} \alpha^{n/3} & 3 \mid n \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow X_0(z) = \frac{1}{1 - \alpha z^{-3}} \quad , \quad |z| > \sqrt[3]{|\alpha|}$$

$$x_1[n] = x_0[n] + x_0[n-1] + x_0[n-2]$$

$$\Rightarrow X_1(z) = X_0(z)\left(1 + z^{-1} + z^{-2}\right) = \frac{1 + z^{-1} + z^{-2}}{1 - \alpha z^{-3}}$$

$$|z| > \sqrt[3]{|\alpha|}$$

Problem 2 - Part 2

Find $x_2[n]$ assuming causality:

$$X_2(z) = \frac{1}{1 - (2r\cos\theta)z^{-1} + r^2 z^{-2}}$$

Problem 2 - Part 2 - Solution

$$X_2(z) = \frac{1}{1 - (2r\cos\theta)z^{-1} + r^2 z^{-2}}$$

$$p_1, p_2 = r\cos\theta \pm jr\sin\theta = re^{\pm j\theta}$$

Problem 2 - Part 2 - Solution

$$X_2(z) = \frac{1}{1 - (2r\cos\theta)z^{-1} + r^2 z^{-2}}$$

$$p_1, p_2 = r\cos\theta \pm jr\sin\theta = re^{\pm j\theta}$$

$$X_2(z) = \frac{A}{1 - re^{j\theta}z^{-1}} + \frac{B}{1 - re^{-j\theta}z^{-1}}$$

Problem 2 - Part 2 - Solution

$$X_2(z) = \frac{1}{1 - (2r\cos\theta)z^{-1} + r^2 z^{-2}}$$

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$$X_2(z) = \frac{A}{1 - re^{j\theta} z^{-1}} + \frac{B}{1 - re^{-j\theta} z^{-1}}$$

$$\Rightarrow x_2[n] = A\left(re^{j\theta}\right)^n u[n] + B\left(re^{-j\theta}\right)^n u[n]$$

Problem 2 - Part 2 - Solution

$$X_2(z) = \frac{1}{1 - (2r\cos\theta)z^{-1} + r^2 z^{-2}}$$

$$p_1, p_2 = r\cos\theta \pm jr\sin\theta = re^{\pm j\theta}$$

$$X_2(z) = \frac{A}{1 - re^{j\theta} z^{-1}} + \frac{B}{1 - re^{-j\theta} z^{-1}}$$

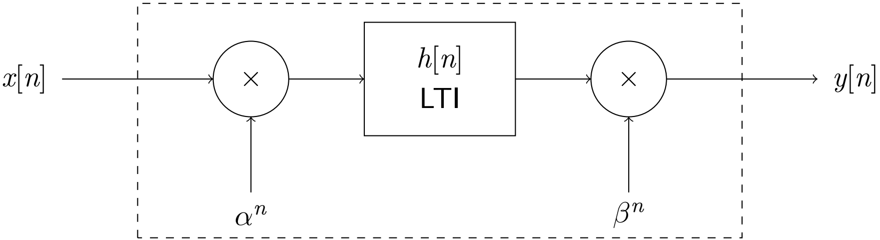
$$\Rightarrow x_2[n] = A\left(re^{j\theta}\right)^n u[n] + B\left(re^{-j\theta}\right)^n u[n]$$

$$\Rightarrow \ldots \Rightarrow x_2[n] = r^n \frac{\sin(n+1)\theta}{\sin\theta} u[n]$$

Problem 3

Problem 3 - Part 1

Consider the following system:



- How should α and β be related so that the whole system is LTI? What would be the impulse response in that case?

Problem 3 - Part 1 - Solution

$$X(z) \longrightarrow X\left(\frac{z}{\alpha}\right) \longrightarrow X\left(\frac{z}{\alpha}\right) H(z) \longrightarrow X\left(\frac{z}{\alpha\beta}\right) H\left(\frac{z}{\beta}\right) = Y(z)$$

Problem 3 - Part 1 - Solution

$$X(z) \longrightarrow X\left(\frac{z}{\alpha}\right) \longrightarrow X\left(\frac{z}{\alpha}\right) H(z) \longrightarrow X\left(\frac{z}{\alpha\beta}\right) H\left(\frac{z}{\beta}\right) = Y(z)$$

$$\alpha\beta = 1 \Rightarrow G(z) = \frac{Y(z)}{X(z)} = H(\alpha z)$$

Problem 3 - Part 1 - Solution

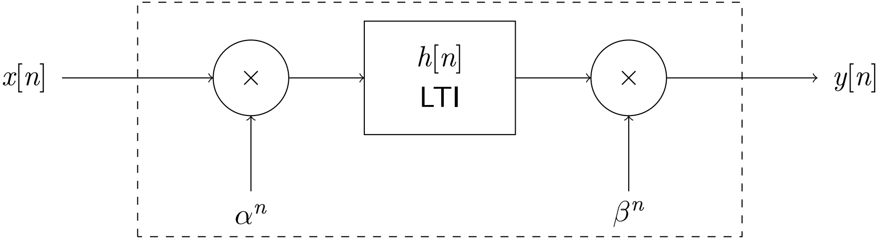
$$X(z) \longrightarrow X\left(\frac{z}{\alpha}\right) \longrightarrow X\left(\frac{z}{\alpha}\right) H(z) \longrightarrow X\left(\frac{z}{\alpha\beta}\right) H\left(\frac{z}{\beta}\right) = Y(z)$$

$$\alpha\beta = 1 \Rightarrow G(z) = \frac{Y(z)}{X(z)} = H(\alpha z)$$

$$g[n] = \alpha^{-n} h[n]$$

Problem 3 - Part 2

Consider the following system:



- Let the ROC of the inner LTI system be $r_1 < |z| < r_2$. What conditions should r_1 and r_2 satisfy in order for the whole system to be stable?

Problem 3 - Part 2 - Solution

$$G(z) = H(\alpha z) \Rightarrow r_1 < |\alpha z| < r_2$$

Problem 3 - Part 2 - Solution

$$
\begin{aligned}
G(z) = H(\alpha z) &\Rightarrow r_1 < |\alpha z| < r_2 \\
&\Rightarrow \frac{r_1}{|\alpha|} < |z| < \frac{r_2}{|\alpha|}
\end{aligned}
$$

Problem 3 - Part 2 - Solution

$$
\begin{aligned}
G(z) = H(\alpha z) &\Rightarrow r_1 < |\alpha z| < r_2 \\
&\Rightarrow \frac{r_1}{|\alpha|} < |z| < \frac{r_2}{|\alpha|} \\
&\Rightarrow \frac{r_1}{|\alpha|} < 1 < \frac{r_2}{|\alpha|}
\end{aligned}
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Problem 3 - Part 2 - Solution

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\begin{aligned}
G(z) = H(\alpha z) &\Rightarrow r_1 < |\alpha z| < r_2 \\
&\Rightarrow \frac{r_1}{|\alpha|} < |z| < \frac{r_2}{|\alpha|} \\
&\Rightarrow \frac{r_1}{|\alpha|} < 1 < \frac{r_2}{|\alpha|} \\
&\Rightarrow r_1 < |\alpha| < r_2
\end{aligned}
$$

Problem 4

Problem 4

Let $H(z)$ be the transfer function of an LTI system:

$$H(z) = \frac{1}{z^3 + 2\sqrt{2}z^{3/2} + \frac{1}{8}}$$

Assuming $|z| = \dfrac{1}{2}$ is inside the ROC, find the output of the system to the following input:

$$x[n] = \left(\frac{1}{2}\right)^n \sin\left(\frac{n\pi}{3}\right)$$

Problem 4 - Solution

Eigenfunctions of Discrete-Time Systems

The output of an LTI system to the input z_0^n is

$$
\begin{aligned}
y[n] &= x[n] * h[n] \\
&= \sum_{k=-\infty}^{\infty} h[k] z_0^{n-k} \\
&= z_0^n \sum_{k=-\infty}^{\infty} h[k] z_0^{-k} \\
&= H(z_0) z_0^n
\end{aligned}
$$

Problem 4 - Solution

$$x[n] = \left(\frac{1}{2}\right)^n \sin\left(\frac{n\pi}{3}\right)$$

Problem 4 - Solution

$$
\begin{aligned}
x[n] &= \left(\frac{1}{2}\right)^n \sin\left(\frac{n\pi}{3}\right) \\
&= \left(\frac{1}{2}\right)^n \frac{1}{2j}\left(e^{j\frac{\pi}{3}n} - e^{-j\frac{\pi}{3}n}\right)
\end{aligned}
$$

Problem 4 - Solution

$$\begin{aligned} x[n] &= \left(\frac{1}{2}\right)^n \sin\left(\frac{n\pi}{3}\right) \\ &= \left(\frac{1}{2}\right)^n \frac{1}{2j}\left(e^{j\frac{\pi}{3}n} - e^{-j\frac{\pi}{3}n}\right) = \frac{1}{2j}\left[\left(\frac{1}{2}e^{j\frac{\pi}{3}}\right)^n - \left(\frac{1}{2}e^{-j\frac{\pi}{3}}\right)^n\right] \end{aligned}$$

Problem 4 - Solution

$$
\begin{aligned}
x[n] &= \left(\frac{1}{2}\right)^n \sin\left(\frac{n\pi}{3}\right) \\
&= \left(\frac{1}{2}\right)^n \frac{1}{2j}\left(e^{j\frac{\pi}{3}n} - e^{-j\frac{\pi}{3}n}\right) = \frac{1}{2j}\left[\left(\frac{1}{2}e^{j\frac{\pi}{3}}\right)^n - \left(\frac{1}{2}e^{-j\frac{\pi}{3}}\right)^n\right] \\
\Rightarrow y[n] &= \frac{1}{2j}\left[H\left(\frac{1}{2}e^{j\frac{\pi}{3}}\right)\left(\frac{1}{2}e^{j\frac{\pi}{3}}\right)^n - H\left(\frac{1}{2}e^{-j\frac{\pi}{3}}\right)\left(\frac{1}{2}e^{-j\frac{\pi}{3}}\right)^n\right]
\end{aligned}
$$

Problem 4 - Solution

$$x[n] = \left(\frac{1}{2}\right)^n \sin\left(\frac{n\pi}{3}\right)$$

$$= \left(\frac{1}{2}\right)^n \frac{1}{2j}\left(e^{j\frac{\pi}{3}n} - e^{-j\frac{\pi}{3}n}\right) = \frac{1}{2j}\left[\left(\frac{1}{2}e^{j\frac{\pi}{3}}\right)^n - \left(\frac{1}{2}e^{-j\frac{\pi}{3}}\right)^n\right]$$

$$\Rightarrow y[n] = \frac{1}{2j}\left[H\left(\frac{1}{2}e^{j\frac{\pi}{3}}\right)\left(\frac{1}{2}e^{j\frac{\pi}{3}}\right)^n - H\left(\frac{1}{2}e^{-j\frac{\pi}{3}}\right)\left(\frac{1}{2}e^{-j\frac{\pi}{3}}\right)^n\right]$$

$$\Rightarrow y[n] = \frac{1}{2j}\left[(-j)\left(\frac{1}{2}e^{j\frac{\pi}{3}}\right)^n - j\left(\frac{1}{2}e^{-j\frac{\pi}{3}}\right)^n\right]$$

Problem 4 - Solution

$$
\begin{aligned}
x[n] &= \left(\frac{1}{2}\right)^n \sin\left(\frac{n\pi}{3}\right) \\
&= \left(\frac{1}{2}\right)^n \frac{1}{2j}\left(e^{j\frac{\pi}{3}n} - e^{-j\frac{\pi}{3}n}\right) = \frac{1}{2j}\left[\left(\frac{1}{2}e^{j\frac{\pi}{3}}\right)^n - \left(\frac{1}{2}e^{-j\frac{\pi}{3}}\right)^n\right] \\
\Rightarrow y[n] &= \frac{1}{2j}\left[H\left(\frac{1}{2}e^{j\frac{\pi}{3}}\right)\left(\frac{1}{2}e^{j\frac{\pi}{3}}\right)^n - H\left(\frac{1}{2}e^{-j\frac{\pi}{3}}\right)\left(\frac{1}{2}e^{-j\frac{\pi}{3}}\right)^n\right] \\
\Rightarrow y[n] &= \frac{1}{2j}\left[(-j)\left(\frac{1}{2}e^{j\frac{\pi}{3}}\right)^n - j\left(\frac{1}{2}e^{-j\frac{\pi}{3}}\right)^n\right] \\
&= -\left(\frac{1}{2}\right)^n \cos\left(\frac{n\pi}{3}\right)
\end{aligned}
$$

Problem 5

Problem 5 - Part 1

Find the Laplace transform of the following signal:

$$x_1(t) = \begin{cases} \dfrac{1}{2^n} & n < t < n + 0.5 \quad , \quad n \in \mathbb{N} \cup \{0\} \\ 0 & \text{otherwise} \end{cases}$$

Problem 5 - Part 1 - Solution

$$x_1(t) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \left(u(t-n) - u\left(t-n-\frac{1}{2}\right)\right)$$

Problem 5 - Part 1 - Solution

$$x_1(t) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \left(u(t-n) - u\left(t-n-\frac{1}{2}\right)\right)$$
$$X_1(s) = \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \left(u(t-n) - u\left(t-n-\frac{1}{2}\right)\right) e^{-st}\mathrm{d}t$$

Problem 5 - Part 1 - Solution

$$x_1(t) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \left(u(t-n) - u\left(t - n - \frac{1}{2}\right)\right)$$

$$X_1(s) = \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \left(u(t-n) - u\left(t - n - \frac{1}{2}\right)\right) e^{-st} \mathrm{d}t$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \int_{-\infty}^{\infty} \left(u(t-n) - u\left(t - n - \frac{1}{2}\right)\right) e^{-st} \mathrm{d}t$$

Problem 5 - Part 1 - Solution

$$x_1(t) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \left(u(t-n) - u\left(t-n-\frac{1}{2}\right)\right)$$

$$X_1(s) = \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \left(u(t-n) - u\left(t-n-\frac{1}{2}\right)\right) e^{-st}\mathrm{d}t$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \int_{-\infty}^{\infty} \left(u(t-n) - u\left(t-n-\frac{1}{2}\right)\right) e^{-st}\mathrm{d}t$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \frac{1}{s} e^{-sn} \left(1 - e^{-s/2}\right)$$

Problem 5 - Part 1 - Solution

$$X_1(s) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \frac{1}{s} e^{-sn} \left(1 - e^{-s/2}\right)$$

Problem 5 - Part 1 - Solution

$$
\begin{aligned}
X_1(s) &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \frac{1}{s} e^{-sn} \left(1 - e^{-s/2}\right) \\
&= \frac{1}{s} \left(1 - e^{-s/2}\right) \sum_{n=0}^{\infty} \left(\frac{1}{2} e^{-s}\right)^n
\end{aligned}
$$

Problem 5 - Part 1 - Solution

$$
\begin{aligned}
X_1(s) &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \frac{1}{s} e^{-sn} \left(1 - e^{-s/2}\right) \\
&= \frac{1}{s} \left(1 - e^{-s/2}\right) \sum_{n=0}^{\infty} \left(\frac{1}{2} e^{-s}\right)^n \\
&= \frac{1}{s} \left(\frac{1 - e^{-s/2}}{1 - \frac{1}{2} e^{-s}}\right)
\end{aligned}
$$

Problem 5 - Part 1 - Solution

$$X_1(s) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \frac{1}{s} e^{-sn} \left(1 - e^{-s/2}\right)$$

$$= \frac{1}{s}\left(1 - e^{-s/2}\right) \sum_{n=0}^{\infty} \left(\frac{1}{2} e^{-s}\right)^n$$

$$= \frac{1}{s}\left(\frac{1 - e^{-s/2}}{1 - \frac{1}{2} e^{-s}}\right)$$

$$\left|\frac{1}{2} e^{-s}\right| < 1 \Rightarrow \left|e^{-2\mathbf{Re}\{s\}}\right| < 2 \Rightarrow \mathbf{Re}\{s\} > -\ln 2$$

Problem 5 - Part 2

Find the Laplace transform of the following signal:

$$x_2(t) = \sin(t)u(\sin(t))u(t)$$

Problem 5 - Part 2 - Solution

$$x_2(t) - x_2(t-\pi) = \sin(t)u(t)$$

Problem 5 - Part 2 - Solution

$$x_2(t) - x_2(t-\pi) = \sin(t)u(t)$$
$$X_2(s) - e^{-\pi s}X_2(s) = \frac{1}{1+s^2}$$

Problem 5 - Part 2 - Solution

$$x_2(t) - x_2(t-\pi) = \sin(t)u(t)$$

$$X_2(s) - e^{-\pi s}X_2(s) = \frac{1}{1+s^2}$$

$$X_2(s) = \frac{1}{\left(1 - e^{-\pi s}\right)\left(1+s^2\right)}$$

Problem 5 - Part 3

Find the Laplace transform of the following signal:

$$x_3(t) = \frac{e^t}{n!} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left(t^n e^{-t}\right) u(t)$$

Problem 5 - Part 3 - Solution

$$
\begin{aligned}
x_3(t) &= \frac{e^t}{n!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(t^n e^{-t}\right) u(t) \\
&= \frac{e^t}{n!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(t^n e^{-t} u(t)\right)
\end{aligned}
$$

Problem 5 - Part 3 - Solution

$$t^n u(t) \leftrightarrow \frac{n!}{s^{n+1}}$$

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$$t^n u(t) \leftrightarrow \frac{n!}{s^{n+1}}$$

$$e^{-t} t^n u(t) \leftrightarrow \frac{n!}{(s+1)^{n+1}}$$

Problem 5 - Part 3 - Solution

$$t^n u(t) \leftrightarrow \frac{n!}{s^{n+1}}$$

$$e^{-t} t^n u(t) \leftrightarrow \frac{n!}{(s+1)^{n+1}}$$

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(e^{-t} t^n u(t)\right) \leftrightarrow \frac{n! s^n}{(s+1)^{n+1}}$$

Problem 5 - Part 3 - Solution

$$
\begin{aligned}
t^n u(t) &\leftrightarrow \frac{n!}{s^{n+1}} \\
e^{-t} t^n u(t) &\leftrightarrow \frac{n!}{(s+1)^{n+1}} \\
\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(e^{-t} t^n u(t)\right) &\leftrightarrow \frac{n! s^n}{(s+1)^{n+1}} \\
\frac{e^t}{n!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(t^n e^{-t}\right) u(t) &\leftrightarrow \frac{(s-1)^n}{s^{n+1}}
\end{aligned}
$$

Problem 6

Problem 6

Consider an LTI stable system with the following transfer function:

$$H(s) = \frac{1}{s+1}$$

Find the system output to the input $x(t)$:

$$x(t) = \cos(2t+1)$$

Problem 6 - Solution

Eigenfunctions of Continuous-Time Systems

The output of an LTI system to the input $e^{s_0 t}$ is

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} h(\tau)\, e^{s_0 (t-\tau)} \mathrm{d}\tau \\ &= e^{s_0 t} \int_{-\infty}^{\infty} h(\tau)\, e^{-s_0 \tau} \mathrm{d}\tau \\ &= H(s_0)\, e^{s_0 t} \end{aligned}$$

Problem 6 - Solution

$$x(t) = \cos(2t+1) = \frac{1}{2}\left(e^{j(2t+1)} + e^{-j(2t+1)}\right)$$

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$$
\begin{aligned}
x(t) = \cos(2t+1) &= \frac{1}{2}\left(e^{j(2t+1)} + e^{-j(2t+1)}\right) \\
&= \frac{1}{2}\left(e^{j} e^{j2t} + e^{-j} e^{-j2t}\right)
\end{aligned}
$$

Problem 6 - Solution

$$
\begin{aligned}
x(t) = \cos(2t+1) &= \frac{1}{2}\left(e^{j(2t+1)} + e^{-j(2t+1)}\right) \\
&= \frac{1}{2}\left(e^{j}e^{j2t} + e^{-j}e^{-j2t}\right) \\
\Rightarrow y(t) &= \frac{1}{2}\left(H(j2)\,e^{j}e^{j2t} + H(-j2)\,e^{-j}e^{-j2t}\right)
\end{aligned}
$$

Problem 6 - Solution

$$
\begin{aligned}
x(t) = \cos(2t+1) &= \frac{1}{2}\left(e^{j(2t+1)} + e^{-j(2t+1)}\right) \\
&= \frac{1}{2}\left(e^{j}e^{j2t} + e^{-j}e^{-j2t}\right) \\
\Rightarrow y(t) &= \frac{1}{2}\left(H(j2)\,e^{j}e^{j2t} + H(-j2)\,e^{-j}e^{-j2t}\right) \\
&= \frac{1}{2}\left(\frac{1-j2}{5}e^{j}e^{j2t} + \frac{1+j2}{5}e^{-j}e^{-j2t}\right)
\end{aligned}
$$

Problem 6 - Solution

$$
\begin{aligned}
x(t) = \cos(2t+1) &= \frac{1}{2}\left(e^{j(2t+1)} + e^{-j(2t+1)}\right) \\
&= \frac{1}{2}\left(e^{j}e^{j2t} + e^{-j}e^{-j2t}\right) \\
\Rightarrow y(t) &= \frac{1}{2}\left(H(j2)\,e^{j}e^{j2t} + H(-j2)\,e^{-j}e^{-j2t}\right) \\
&= \frac{1}{2}\left(\frac{1-j2}{5}e^{j}e^{j2t} + \frac{1+j2}{5}e^{-j}e^{-j2t}\right) \\
&= \frac{1}{5}\left(\cos(2t+1) + 2\sin(2t+1)\right)
\end{aligned}
$$

Problem 7

Problem 7

Let $H(s)$ represent the system function for a causal, stable system. The input to the system consists of the sum of three terms, one of which is an impulse $\delta(t)$ and another a complex exponential of the form $e^{s_0 t}$, where s_0 is a complex constant. The output is

$$y(t) = -6e^{-t}u(t) + \frac{8}{34}e^{4t}\cos(3t) + \frac{36}{34}e^{4t}\sin(3t) + \delta(t)$$

Determine $H(s)$, consistently with this information.

Problem 7 Solution

$$x(t) = \delta(t) + e^{s_0 t} + ?$$

$$y(t) = -6e^{-t}u(t) + \frac{8}{34}e^{4t}\cos(3t) + \frac{36}{34}e^{4t}\sin(3t) + \delta(t)$$

Problem 7 Solution

$$x(t) = \delta(t) + e^{s_0 t} + e^{s_0^* t}$$

$$y(t) = -6e^{-t}u(t) + \frac{8}{34}e^{4t}\cos(3t) + \frac{36}{34}e^{4t}\sin(3t) + \delta(t)$$

Problem 7 Solution

$$x(t) = \delta(t) + e^{s_0 t} + e^{s_0^* t}$$

$$y(t) = -6e^{-t}u(t) + \frac{8}{34}e^{4t}\cos(3t) + \frac{36}{34}e^{4t}\sin(3t) + \delta(t)$$

$$\delta(t) \leftrightarrow -6e^{-t}u(t) + \delta(t) \quad , \quad s_0 =?$$

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$$\Rightarrow H(s) = \frac{-6}{s+1} + 1 = \frac{s-5}{s+1}$$

Problem 7 Solution

$$
\begin{aligned}
e^{s_0 t} + e^{s_0^* t} \longrightarrow & H(s_0)\, e^{s_0 t} + H(s_0^*)\, e^{s_0^* t} \\
= & H(4 + j3)\, e^{(4+j3)\, t} + H(4 - j3)\, e^{(4-j3)\, t} \\
= & \left(\frac{4}{34} + j\frac{18}{34} \right) e^{(4+j3)\, t} + \left(\frac{4}{34} - j\frac{18}{34} \right) e^{(4-j3)\, t} \\
= & \frac{8}{34} e^{4t} \cos(3t) + \frac{36}{34} e^{4t} \sin(3t)
\end{aligned}
$$

Problem 8

Problem 8

Let the Laplace transform of $x(t)$ be

$$X(s) = \frac{e^{-s}}{1 + e^{-2s}}$$

with the ROC $\mathbf{Re}\{s\} > 0$. Calculate the following integral:

$$I = \int_0^2 x(t)\mathrm{d}t$$

Problem 8 Solution

$$X(s) = \frac{e^{-s}}{1 + e^{-2s}}$$

Problem 8 Solution

$$X(s) = \frac{e^{-s}}{1 + e^{-2s}}$$
$$= e^{-s} \sum_{n=0}^{\infty} (-e^{-2s})^n$$

Problem 8 Solution

$$
\begin{aligned}
X(s) &= \frac{e^{-s}}{1+e^{-2s}} \\
&= e^{-s}\sum_{n=0}^{\infty}(-e^{-2s})^n \\
&= e^{-s} - e^{-3s} + e^{-5s} - e^{-7s} + \dots
\end{aligned}
$$

Problem 8 Solution

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X(s) &= \frac{e^{-s}}{1+e^{-2s}} \\
&= e^{-s}\sum_{n=0}^{\infty}(-e^{-2s})^n \\
&= e^{-s} - e^{-3s} + e^{-5s} - e^{-7s} + \dots \\
\Rightarrow x(t) &= \delta(t-1) - \delta(t-3) + \delta(t-5) - \delta(t-7) + \dots
\end{aligned}$$

Problem 8 Solution

$$
\begin{aligned}
X(s) &= \frac{e^{-s}}{1+e^{-2s}} \\
&= e^{-s}\sum_{n=0}^{\infty}(-e^{-2s})^n \\
&= e^{-s} - e^{-3s} + e^{-5s} - e^{-7s} + \dots \\
\Rightarrow x(t) &= \delta(t-1) - \delta(t-3) + \delta(t-5) - \delta(t-7) + \dots \\
\Rightarrow I &= \int_0^2 x(t)\mathrm{d}t = 1
\end{aligned}
$$