Engineering Mathematics
Problem Set 06

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1 Residues

1. For each of the following functions, do the following:
   - Find all its singularities in $\mathbb{C}$.
   - For each singularity, determine whether it is a pole, a removable singularity, or an essential singularity.
   - Compute the residue of the function at each singularity.

   (a) $f(z) = \dfrac{1}{\cos^2 z}$

   (b) $f(z) = (1 - z^3)\exp\left(\dfrac{1}{z}\right)$

   (c) $f(z) = \dfrac{\sin z}{z^{2020}}$

   (d) $f(z) = \dfrac{e^z}{1 - z^2}$

   (e) $f(z) = \dfrac{1 - \cos z}{z^2}$

   (f) $f(z) = \tan z$

2. Use residues to compute the following integrals.

   (a) $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{(x-1)(x-2)}\mathrm{d}x$

   (b) $\displaystyle\int_{0}^{2\pi} \sin^4\theta \mathrm{d}\theta$

   (c) $\displaystyle\int_{0}^{2\pi} \frac{1}{a^2 + \sin^2\theta}\mathrm{d}\theta \quad , \quad a > 0$

   (d) $\displaystyle\int_{-\infty}^{\infty} \frac{\sqrt{x}}{1 + x^2}\mathrm{d}x$

   (e) $\displaystyle\int_{0}^{\infty} \frac{\cos ax}{\cosh x}\mathrm{d}x \quad , \quad a > 0$

   (f) $\displaystyle\int_{0}^{\pi} \frac{1}{2 - \cos\theta}\mathrm{d}\theta$

   (g) $\displaystyle\int_{0}^{\pi} \frac{1}{a + b\cos\theta}\mathrm{d}\theta \quad , \quad a, b \in \mathbb{R} \quad , \quad a^2 > b^2$

   (h) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 2x + 2}\mathrm{d}x$

3. Let

$$f(z) = (z - a_1)(z - a_2)\dots(z - a_n)$$

be a complex polynomial with $n \geq 2$ distinct roots $a_1, a_2, \dots, a_n$.

(a) Prove that

$$\int_{|z|=R} \frac{\mathrm{d}z}{f(z)} = 2\pi i \sum_{k=1}^{n} \frac{1}{\prod_{j \neq k}(a_k - a_j)}$$

for $R > |a_k| (k = 1, 2, \dots, n)$

(b) Prove that

$$\sum_{k=1}^{n} \frac{1}{\prod_{j \neq k}(a_k - a_j)} = 0$$

for all distinct complex numbers $a_1, a_2, \dots, a_n$.

2 Mapping

4. Consider the following four types of mappings:

$$w = z + b \quad , \quad w = az \quad , \quad w = \frac{1}{z} \quad , \quad w = \frac{az+b}{cz+d}$$

   Show that each transformation maps circles onto either circles or straight lines.

5. Find a mapping $w = f(z)$ having the following conditions:

   - f maps the circle $|z - i| = 1$ onto itself.
   - $f(2i) = 0$
   - $f(1 + i) = -1 + i$

6. Find the image of the following region under the mapping $w = z^2$.

$$D = \{z = x + iy| - 1 \leq x \leq 2\,,\ -2 \leq y \leq 1\}$$

7. Find all the mappings of the form

$$w = f(z) = \frac{az+b}{cz+d}$$

   having no fixed points. (A *fixed point* of a mapping f is a point z_0 with the property that $f(z_0) = z_0$)

8. (*Brown-Churchill book, page 318, problem 5*) Find the image of the region $x > 1$, $y > 0$ under the transformation $w = 1/z$.

9. (*Brown-Churchill book, page 318, problem 11*) Let the circle $|z| = 1$ have a positive, or counterclockwise orientation. Determine the orientation of its image under the transformation $w = 1/z$.

10. (*Brown-Churchill book, page 318, problem 12*) Show that when a circle is transformed into a circle under the transformation $w = 1/z$, the center of the original circle is *never* mapped onto the center of the image circle.

11. (*Brown-Churchill book, page 362, problem 2*) What angle of rotation is produced by the transformation $w = 1/z$ at the point

    (a) $z_0 = 1$

    (b) $z_0 = i$

12. (*Brown-Churchill book, page 362, problem 3*) Show that under the transformation $w = 1/z$, the images of the lines $y = x - 1$ and $y = 0$ are the circle $u^2 + v^2 - u - v = 0$ and the line $v = 0$, respectively. Sketch all the four curves, determine corresponding directions along them, and verify the conformality of the mapping at the point $z_0 = 1$.

3 The z-Transform

13. (*Signals and Systems by Alan. V. Oppenheim, 2nd edition, problems 10.21 and 10.22*) Determine the z-transform for each of the following sequences. Sketch the pole-zero plot and indicate the region of convergence.

    (a) $\delta[n+5]$

    (b) $\delta[n-5]$

    (c) $(-1)^n u[n]$

    (d) $\left(\frac{1}{2}\right)^{n+1} u[n+3]$

    (e) $\left(-\frac{1}{3}\right)^n u[-n-2]$

    (f) $\left(\frac{1}{4}\right)^n u[3-n]$

    (g) $2^n u[-n] + \left(\frac{1}{4}\right)^n u[n-1]$

    (h) $(\frac{1}{3})^{n-2} u[n-2]$

    (i) $(\frac{1}{2})^n \{u[n+4] - u[n-5]\}$

    (j) $n(\frac{1}{2})^{|n|}$

    (k) $|n|(\frac{1}{2})^{|n|}$

    (l) $4^n \cos[\frac{2\pi}{6}n + \frac{\pi}{4}]u[-n-1]$

14. (*Signals and Systems by Alan. V. Oppenheim, 2nd edition, problem 10.23*) Find the inverse z-transform for each of the following z-transforms.

    (a) $X(z) = \dfrac{1 - z^{-1}}{1 - \frac{1}{4}z^{-2}} \quad , \quad |z| > \dfrac{1}{2}$

    (b) $X(z) = \dfrac{1 - z^{-1}}{1 - \frac{1}{4}z^{-2}} \quad , \quad |z| < \dfrac{1}{2}$

    (c) $X(z) = \dfrac{z^{-1} - \frac{1}{2}}{1 - \frac{1}{2}z^{-1}} \quad , \quad |z| > \dfrac{1}{2}$

    (d) $X(z) = \dfrac{z^{-1} - \frac{1}{2}}{1 - \frac{1}{2}z^{-1}} \quad , \quad |z| < \dfrac{1}{2}$

    (e) $X(z) = \dfrac{z^{-1} - \frac{1}{2}}{(1 - \frac{1}{2}z^{-1})^2} \quad , \quad |z| > \dfrac{1}{2}$

    (f) $X(z) = \dfrac{z^{-1} - \frac{1}{2}}{(1 - \frac{1}{2}z^{-1})^2} \quad , \quad |z| < \dfrac{1}{2}$

15. Let a discrete LTI system have the following input-output relationship:

$$y[n] - 3y[n-1] + 2y[n-2] = 5x[n] - 7x[n-1]$$

    (a) Take the z-transform from both sides of the above equation and find the system transfer function $H(z) = \dfrac{Y(z)}{X(z)}$.

    (b) Find $h[n]$, the impulse response of the system (i.e. the response of the system to the input $x[n] = \delta[n]$). Is the answer to this question unique?