Engineering Mathematics (25735-2)
Problem Set 01

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1 Fourier series coefficients

Find the Fourier series for the following functions.

1. $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$
2. $f(x) = \begin{cases} \dfrac{1}{2a} & |x| < a \\ \\ 0 & a < |x| < \pi \end{cases} \quad , \quad 0 < a < \pi$
3. $f(x) = \begin{cases} \pi e^{-x} & -\pi < x < 0 \\ \pi e^{x} & 0 \leq x < \pi \end{cases}$
4. $f(x) = \cos(\alpha x) \quad , \quad \alpha \notin \mathbb{Z} \quad , \quad x \in [-\pi, \pi]$
5. $f(x) = \cosh(x) \quad , \quad x \in [-\pi, \pi]$
6. $f(x) = |\cos(x)| \quad , \quad x \in [-\pi, \pi]$
7. $f(x) = \sin^3\left(\frac{n\pi x}{l}\right) \quad , \quad x \in [-l, l]$
8. $f(x) = xe^{ax} \quad , \quad x \in [-\pi, \pi]$

2 Even symmetry

Let f be a function defined on the interval $(0, L)$ for which the following identity holds:

$$f(x) = f(L - x)$$

This means that f is symmetric with respect to the line $x = \dfrac{L}{2}$.

1. Show that even coefficients of the *Fourier sine series* of f are zero.
2. Show that odd coefficients of the *Fourier cosine series* of f are zero.

(*Hint:* To obtain Fourier sine or cosine series, you need different half-range expansions.)

3 Sum of infinite series

Use the Fourier series of the periodic expansion of the function f to calculate the sum S: (You may need to use Parseval's identity)

1.
$$f_1(x) = \begin{cases} \cos x & -\pi < x < 0 \\ 0 & 0 < x < \pi \end{cases} \quad , \qquad S_1 = \sum_{k=1}^{\infty} \frac{4(2k-1)(-1)^k}{\pi\big(4(2k-1)^2 - 1\big)}$$

2.
$$f_2(x) = |x| \quad , \quad x \in [-\pi, \pi] \quad , \qquad S_2 = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4}$$

3.
$$f_3(x) = \begin{cases} -\sin x & -\pi < x < 0 \\ \sin x & 0 < x < \pi \end{cases} \quad , \qquad S_3 = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2(2k+1)^2}$$

4.
$$f_4(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$$

$$S_{4,1} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)} \quad , \qquad S_{4,2} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

$$S_{4,3} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^3} \quad , \qquad S_{4,4} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^6}$$

4 Sum of periodic functions

Consider two periodic functions of periods T_1 and T_2, and their corresponding Fourier series expansions:

$$\begin{aligned} f_1(t) =& a_0^{(1)} + \sum_{n=1}^{\infty} \left(a_n^{(1)} \cos(n\omega_1 t) + b_n^{(1)} \sin(n\omega_1) t\right) \quad , \quad T_1 = \frac{2\pi}{\omega_1} \\ f_2(t) =& a_0^{(2)} + \sum_{n=1}^{\infty} \left(a_n^{(2)} \cos(n\omega_2 t) + b_n^{(2)} \sin(n\omega_2) t\right) \quad , \quad T_2 = \frac{2\pi}{\omega_2} \end{aligned}$$

The function g is defined as sum of f_1 and f_2:

$$g(t) = f_1(t) + f_2(t)$$

1. Is g always periodic? If the answer is yes, prove it. If not, find the condition under which g is periodic. Also, find the fundamental period of g.

2. For this part, assume that $\dfrac{\omega_1}{\omega_2} = \dfrac{k_1}{k_2}$ where k_1 and k_2 are coprime integers (two integers are said to be *coprime* if the only positive integer that divides both of them is 1). If g is periodic with angular frequency ω, find the Fourier series coefficients of g in terms of a_n and b_n.

5 Differential equation with arbitrary input function

Consider the simple RC circuit in figure 1.

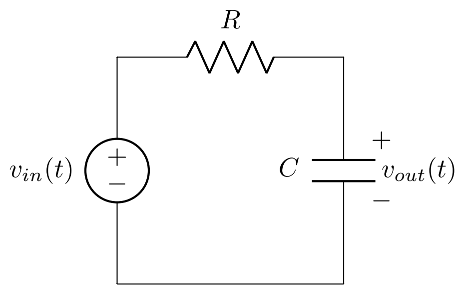

Figure 1

1. Find the differential equation for v_{out} in terms of R, C, and the input v_{in}.
2. Find the steady-state solution for the equation derived in the previous part for each of the following inputs: (You can ignore the transient response.)
   (a) $v_{in}(t) = V_0 \sin \omega t$
   (b) $v_{in}(t) = V_0 \cos \omega t$
   (c) $v_{in}(t) = V_0$
3. Use superposition principle to find the steady-state response to the following input:
$$v_{in}(t) = a_0 + \sum_{n=1}^{\infty} \big(a_n \cos(n\omega t) + b_n \sin(n\omega t)\big)$$
4. Consider the periodic wave in figure 2 as the input to the circuit of figure 1. For simplicity, assume that $R = 1\Omega$ and $C = 1\text{F}$. Find the steady-state response of the circuit.

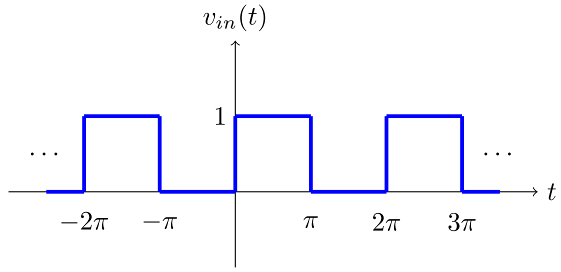

Figure 2

6 Complex Fourier series

You already know the representation of a periodic function as a Fourier series in the following form:
$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\frac{2\pi}{T}x) + \sum_{n=1}^{\infty} b_n \sin(n\frac{2\pi}{T}x)$$
where $a_n, b_n \in \mathbb{R}$.

There exists another form of the Fourier series, which we call *the complex Fourier series*:
$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i(2\pi n/T)x} \qquad , \qquad c_n \in \mathbb{C}$$

1. Considering the uniqueness of Fourier series, find c_n in terms of a_n and b_n.
   (*Hint:* $e^{i\theta} = \cos\theta + i\sin\theta$)

2. Show that:

$$c_n = \frac{1}{T}\int_{<T>} f(x)e^{-i(2\pi n/T)x}\mathrm{d}x$$

   where T is the fundamental period of f and $\int_{<T>}$ is integration over an arbitrary period.

3. Consider an arbitrary periodic function f with complex Fourier series coefficients c_n. Find the complex Fourier series coefficients for each of the following functions in terms of c_n.
   (a) $g(x) = f(x - x_0)$
   (b) $g(x) = f(-x)$
   (c) $g(x) = f(ax) \quad , \quad a > 0$
   (d) $g(x) = \dfrac{\mathrm{d}}{\mathrm{d}x}f(x)$

4. Determine Parseval's identity in terms of c_n.

7 Properties of complex Fourier series

Suppose we are given the following information about a function $f(t)$:

- $f(t) \in \mathbb{R}$
- $f(t)$ is periodic with period $T = 6$ and has complex Fourier coefficients c_k
- $c_n = 0$ for $n = 0$ and $n > 2$
- $f(t) = -f(t-3)$
- $\int_{-3}^{3} |f(t)|^2\mathrm{d}t = 3$
- c_1 is a positive real number

Show that $f(t) = A\cos(Bt + C)$, and determine the values of the constants A, B, and C.

8 Half-wave symmetry

Let f be a periodic function of period 1. One says that f has *half-wave symmetry* if the following identity holds:

$$f(t - \frac{1}{2}) = -f(t)$$

1. Sketch an example of a signal that has half-wave symmetry.

2. If f has half-wave symmetry and its Fourier series is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t}$$

   show that $c_n = 0$ if n is even.

9 Sum of infinite series and the complex Fourier series

1. Compute the complex Fourier series coefficients of periodic expansion of the following function:

$$f(x) = x^2 \qquad , \qquad 0 < x < 2$$

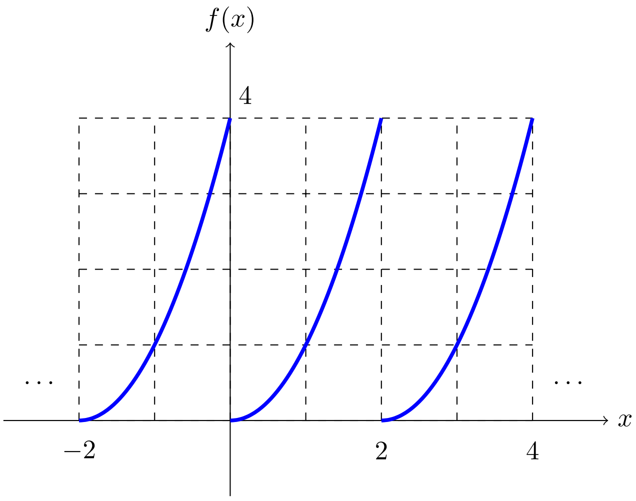

Figure 3

2. Use the results from the previous part to calculate the following sums:

   (a) $S_1 = \sum_{n=1}^{\infty} \frac{1}{n^2}$

   (b) $S_2 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

   (c) $S_3 = \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$

10 Periodic autocorrelation function

Let f be a real, periodic function of period 1. The *autocorrelation* of f with itself is the function

$$R_f(x) = \int_0^1 f(y) f(x+y) \mathrm{d}y$$

1. Show that $R_f(x)$ is also periodic of period $T = 1$.
2. Assume that f has the following Fourier series:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$$

   Show that the Fourier series of R_f is

$$R_f(x) = \sum_{n=-\infty}^{\infty} |c_n|^2 e^{2\pi i n x}$$

11 Superposition of functions and phase difference

The periodic function f with fundamental period $T = \dfrac{2\pi}{\omega}$ has the following Fourier series expansion:

$$f(t) = a_0 + \sum_{n=1}^{\infty} \big(a_n \cos(n\omega t) + b_n \sin(n\omega t)\big)$$

The function g is defined in terms of f:

$$g(t) = f(t - \frac{\theta}{\omega}) + f(t)$$

1. Find P, the average energy of g in one period, in terms of ω, a_n, b_n, and θ.

$$P = \frac{1}{T}\int_0^T \big(g(t)\big)^2 \mathrm{d}t$$

2. Assume that the following condition holds for f:

$$\forall k \in \mathbb{N} : a_{2k} = b_{2k} = 0$$

   Find θ^*, the value of θ which minimizes P. Also find P^*, the minimum of P.

$$\theta^* = \operatorname*{argmin}_{\theta} P(\theta)$$
$$P^* = \min_{\theta} P(\theta)$$

12 Legendre's polynomials

1. Prove that for every $n \in \mathbb{N}$, the elements of the set of functions S_n defined below are linearly independent on the interval $[-1, 1]$.

$$S_n = \big\{f_k(x) = x^k \mid 0 \leq k \leq n, k \in \mathbb{N} \cup \{0\}\big\}$$

2. According to part 1, linear combinations of the functions in S_∞ are a subspace of the space of all functions defined on the interval $[-1, 1]$. In other words, S_∞ is a basis for that subspace. Now we are going to find an orthogonal basis for this subspace. The inner product for this space is defined as follows:

$$\langle f(x), g(x)\rangle = \int_{-1}^{1} f(x)g(x)\mathrm{d}x$$

   Use this inner product and Gram-Schmidt process to find an orthogonal basis for the subspace defined by S_3. (You have to find four mutually orthogonal polynomial functions with orders less than 4.)

3. Now assume that the functions $P_n(x)$ are defined for every $n \in \mathbb{N} \cup \{0\}$ on $[-1, 1]$:

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\big[(x^2 - 1)^n\big]$$

   These functions are called *Legendre's polynomials* and the preceding formula is called *Rodrigues' formula.*

   Prove that $P_n(x)$ is a polynomial of order n. Also find $\langle P_n(x), P_m(x)\rangle$ for two cases $m \neq n$ and $m = n$. You may use the following hint:

$$\int_{-1}^{1} (x^2 - 1)^n \mathrm{d}x = (-1)^n \frac{(n!)^2 2^{2n+1}}{(2n + 1)!}$$

4. Find $P_0(x)$, $P_1(x)$, $P_2(x)$, and $P_3(x)$. Also rewrite your answer in part 2 in terms of these functions.