Engineering Mathematics
Problem Set 03

Department of Electrical Engineering
Sharif University of Technology
Fall Semester 1398-99

1 Solving PDEs

Solve each of the following PDEs.

1.

$$u_{tt} = c^2 u_{xx} - h^2 u \quad , \quad \begin{cases} 0 \leq x \leq 1 \\ t > 0 \\ u(0,t) = 0 \\ u(1,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = 0 \end{cases}$$

2.

$$u_{tt} = 9u_{xx} + 1 - 2x \quad , \quad \begin{cases} 0 \leq x \leq 1 \\ t > 0 \\ u(0,t) = 1 \\ u_x(1,t) = 0 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$$

Hint: Define $u(x,t) = w(x,t) + v(x)$ so as to make a homogeneous PDE for $w(x,t)$.

3.

$$u_{xx} = \frac{1}{c^2} u_t \quad , \quad \begin{cases} 0 < x < L \\ t > 0 \\ u(0,t) = u_x(L,t) = 0 \\ u(x,0) = \frac{T_0}{L} x \end{cases}$$

4.

$$u_{tt} = c^2 u_{xx} + u \quad , \quad \begin{cases} 0 < x < l \\ u(x,0) = f(x) \\ u_t(x,0) = 0 \\ u(0,t) = 0 \\ u(l,t) = 0 \end{cases}$$

5.

$$u_t - t u_x x = 0 \quad , \quad \begin{cases} -\infty < x < \infty \\ t > 0 \\ u(x,0) = \phi(x) \end{cases}$$

Hint: Take Fourier transform from both sides of the equation and find the solution in the form $f(x,t) * \phi(x)$ where you have to find $f(x,t)$.

6. *Cylindrical Coordinates*
   Assume that the solution is independent of z.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\partial^2 V}{\partial z^2}=0$$

7. *Spherical Coordinates*

$$v_t=\nabla^2 v \quad , \quad \begin{cases} 0<r<1 \\ 0<\theta<\frac{\pi}{2} \\ t>0 \\ v(r,0,t)=0 \\ v(r,\frac{\pi}{2},t)=0 \\ v(1,\theta,t)=g(\theta) \\ v(0,\theta,t)\text{ bounded} \\ v(r,\theta,0)=f(r,\theta) \end{cases}$$

8. Find the *steady-state* solution.

$$u_t=4u_{xx} \quad , \quad \begin{cases} 0<x<1 \\ t>0 \\ u(0,t)=1 \\ u(1,t)=0 \\ u(x,0)=\left|x-\frac{1}{2}\right| \end{cases}$$

9.

$$u_t=u_{xx}-2u \quad , \quad \begin{cases} 0\le x<\infty \\ u_x(0,t)=0 \\ u(x,0)=e^{-x} \end{cases}$$

   Hint: Use Fourier cosine transform. Also define $w(x,t)=e^{-2t}u(x,t)$.

10.

$$u_{tt}+a^2u_{xxxx}=0 \quad , \quad \begin{cases} 0\le x\le \pi \\ t\ge 0 \\ u(x,0)=f(x) \\ u_t(x,0)=g(x) \\ u(0,t)=u(\pi,t)=0 \\ u_{xx}(0,t)=u_{xx}(\pi,t)=0 \end{cases}$$

2 The Conducting Cube

Suppose the vertices of a conducting cube are in the following positions:

$$A=(0,0,0) \quad , \quad B=(a,0,0) \quad , \quad C=(0,a,0) \quad , \quad D=(a,a,0)$$

$$E=(0,0,a) \quad , \quad F=(a,0,a) \quad , \quad G=(0,a,a) \quad , \quad H=(a,a,a)$$

Also assume that all the faces of the cube have zero electric potential, except for the upper face ($EFGH$), whose electric potential is V_0. Find the electric potential inside the cube.

3 D'Alembert's Solution of the Wave Equation

Consider the wave equation for a vibrating string of length 2. The propagation speed is denoted by c. Assume that both ends of the string are in fixed positions:

$$u(0,t) = u(2,t) = 0$$

The initial conditions of the string are as below:

$$u(x,0) = 0 \quad , \quad u_t(x,0) = g(x)$$

where $g(x)$ is depicted in figure 1.

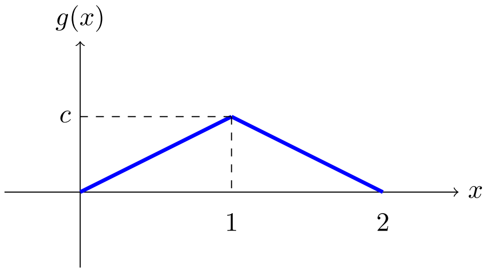

Figure 1

We intend to solve this problem by D'Alambert's method, so we first need to expand $g(x)$.

1. Use the boundary conditions to plot the appropriate expansion of $g(x)$.
2. Find and plot $u(1,t)$.
3. Find t_0, the first time that the speed of the string vanishes.
4. Find and plot $u(x,t_0)$ for t_0 you found in the previous part.

4 The Rectangular Membrane

Consider a rectangular membrane whose four edges have fixed positions ($u = 0$ on the four edges). This membrane starts vibrating with zero initial speed. Find the amplitude of oscillations by solving the following PDE.

$$\frac{1}{c^2}u_{tt} = u_{xx} + u_{yy} \quad , \quad \begin{cases} 0 < x < \pi \\ 0 < y < \pi \\ t > 0 \\ u(x,y,0) = A\sin(x)\sin(2y) \\ u_t(x,y,0) = 0 \end{cases}$$

5 Heat Equation in Unbounded Region

Consider the region:

$$\mathcal{D} = [0,+\infty) \times [0,\pi] = \{(x,y) | x \in [0,+\infty), y \in [0,\pi]\}$$

The temperature inside $\mathcal{D}$ is denoted by $T(x,y)$. We know that $T(x,0) = T(x,\pi) = 0$ and $T(0,y) = f(y)$. Also note that there is no heat source inside $\mathcal{D}$ and $T \to 0$ as $x \to +\infty$.

1. Find $T(x,y)$ for $t \to +\infty$ (the steady-state solution) for arbitrary initial conditions.
2. For this part assume $y = 0$ and $y = \pi$ planes are thermal insulators (instead of having constant temperatures). Find the steady-state solution for this case.