Engineering Mathematics
Problem Set 04

Department of Electrical Engineering
Sharif University of Technology
Fall Semester 1398-99

1 Cartesian Form of a Complex Number

Write each of the following complex numbers in the form $a + ib$:

1. $(1+i)^3$
2. $\dfrac{1+i}{1-i}$
3. $e^{2+i\pi/4}$
4. $\sin\left(\dfrac{\pi}{4} + 2i\right)$
5. $\cosh\left(2 + \dfrac{i\pi}{4}\right)$
6. $\dfrac{(1+i\sqrt{3})^8}{2^7(-1+i\sqrt{3})}$

2 Regions in $\mathbb{C}$

Plot each of the following regions in the complex plane. Are they open, closed, or none? Are they bounded?

1. $|z - 2 + i| \leq 1$
2. $|2z + 3| > 4$
3. $\operatorname{Im} z > 1$
4. $\operatorname{Im} z = 1$
5. $0 \leq \arg z \leq \dfrac{\pi}{4} \quad , \quad z \neq 0$
6. $|z - 4| > |z|$
7. $\left|\dfrac{z - 3i}{z + i}\right| = 1$
8. $|z|^2 + 3 \operatorname{Re} z^2 = 4$
9. $z\bar{z} = 1$

3 Continuity in $\mathbb{C}$

Are the following functions continuous at $z = 0$? (Let $f(0) = 0$ for all these functions.)

1. $\dfrac{\operatorname{Re} z^2}{|z|}$
2. $z^2 \operatorname{Im} \dfrac{1}{z}$
3. $\dfrac{\bar{z}}{z}$

4 Differentiability and Analytic Functions

For each of the following functions, find the largest domain in which they are analytic. Also find the largest domain in which they are differentiable. What is their derivative?

1. $f_1(z) = f_1(x + iy) = x^3 + iy^3$
2. $f_2(z) = e^{\bar{z}}$
3. $f_3(z) = z + |z|$
4. $f_4(z) = \dfrac{1}{1 + z^2}$
5. $f_5(z) = \arg z$
6. $f_6(z) = \dfrac{1 + z}{\sin z}$
7. $f_7(z) = \log(z^2 - z + 2)$
8. $f_8(z) = z \log z$
9. $f_9(z) = \dfrac{\operatorname{Log}(z + 5)}{z^2 + 3z + 2}$

5 Analytic Functions

1. Let $f(z)$ and $\bar{f}(z)$ both be analytic functions on the domain D. Show that f is a constant function on D.
2. Let f be analytic on D and $|f| = c$, where c is a constant. Show that f must be constant on D.
3. Let $f(z) = u(r, \theta) + iv(r, \theta)$ be an analytic function. Show that:
$$r^2 u_{rr}(r, \theta) + r u_r(r, \theta) + u_{\theta\theta}(r, \theta) = 0$$
4. Show that $u(x, y)$ is harmonic and find its harmonic conjugate function:
$$u(x, y) = e^{-2xy} \sin(x^2 - y^2)$$
5. Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function on the domain D. Show that $U(x, y)$ and $V(x, y)$ are analytic in D. Also show that $V(x, y)$ is the harmonic conjugate function of $U(x, y)$:
$$U(x, y) = e^{u(x,y)} \cos(v(x, y))$$
$$V(x, y) = e^{u(x,y)} \sin(v(x, y))$$
6. Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function on the domain D. Show that:
$$\exists a, b, c \in \mathbb{R} : a^2 + b^2 \neq 0, au + bv = c \Rightarrow f \text{ is constant}$$

6 Complex Functions

1. Prove the following identities:
   (a) $|\sin z|^2 = \sin^2 x + \sinh^2 y$
   (b) $|\sinh z|^2 = \sinh^2 x + \sin^2 y$
2. Find the roots of the following equations:
   (a) $\sinh z = i$
   (b) $\cosh z = \dfrac{1}{2}$
   (c) $\sin z = -2$
   (d) $\log(z^2 + 1) = i$

7 Line Integrals

1. Let C be the circle $(x-1)^2 + y^2 = 2$ traversed counterclockwise. Find I:
$$I = \oint_C \frac{\mathrm{d}z}{e^z + e^{-z}}$$
2. Find I, where z changes from $\sqrt{2}$ to $i + 1$ on the circle $x^2 + y^2 = 2$.
$$I = \oint_C (\bar{z} + |z|)\mathrm{d}z$$
3. Make the proper change of variables to calculate the following integral using complex line integrals:
$$I = \int_0^\pi \frac{\sin^2 \theta}{2 + \cos^2 \theta} \mathrm{d}\theta$$
4. consider the following integral:
$$I = \int_{-1}^{1} \frac{z+1}{z^2} \mathrm{d}z$$
   (a) Find I if the integration path is the upper half of the unit circle.
   (b) Find I if the integration path is the lower half of the unit circle.
   (c) Can you find I, using the function $F(z) = \log z - z^{-1}$? What is the result?
   (d) Find J, where C is a circle of radius a, centered at the origin:
$$J = \oint_C \frac{z+1}{z^2} \mathrm{d}z$$
5. Let C be the unit circle. Calculate the following integrals:
   (a)
$$I_1 = \oint_C \frac{z^3 + 1}{z} \mathrm{d}z$$
   (b)
$$I_2 = \oint_{C + \{2\}} \frac{z^2 + 3z + 7}{z - 2} \mathrm{d}z$$
   (c)
$$I_3 = \oint_C \frac{z^3 + 3z^2 + 2z + 7}{z^4} \mathrm{d}z$$

6. Let C be the path depicted in figure 1. Calculate the following integrals:

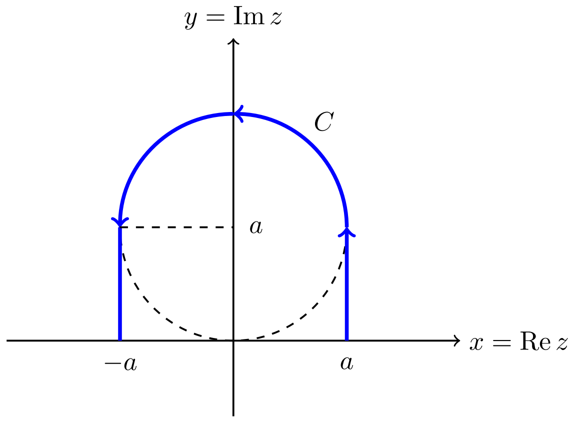

Figure 1

(a)

$$I_1 = \int_C z^2 \mathrm{d}z$$

(b)

$$I_2 = \int_C \sin z \, \mathrm{d}z$$

(c)

$$I_3 = \int_C \frac{1}{z} \mathrm{d}z$$

(d)

$$I_4 = \int_C \frac{1}{z - ia} \mathrm{d}z$$