Measure Theory

? How to measure subsets of real line? What about an abstract set?

- X a set & $P(X)$ the power set. $\mathcal{A} \subseteq P(X)$ is a σ-algebra if:
  a) $\emptyset, X \in \mathcal{A}$
  b) $A \in \mathcal{A} \Rightarrow A^c := X \setminus A \in \mathcal{A}$
  c) $A_i \in \mathcal{A}$, $i \in \mathbb{N} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

  $A \in \mathcal{A}$ is called a $\mathcal{A}$-measurable set.

* $\mathcal{A}_i$ σ-algebra on X, $i \in I \Rightarrow \bigcap_{i \in I} \mathcal{A}_i$ is also a σ-algebra.

- For $\mathcal{M} \subseteq P(X)$, there is a smallest σ-algebra that contains $\mathcal{M}$:

$$\sigma(\mathcal{M}) := \bigcap_{\substack{\mathcal{M} \subseteq \mathcal{A} \\ \mathcal{A}\ \sigma\text{-algebra}}} \mathcal{A} \rightsquigarrow \text{σ-algebra generated by } \mathcal{M}$$

○ Ex. $X = \{a, b, c, d\}$, $\mathcal{M} = \{\{a\}, \{b\}\}$

$\sigma(\mathcal{M}) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}, \{a, c, d\}, \{c, d\}\}$

- a topological space is an ordered pair $(X, \underset{\text{open sets}}{\tau})$, where X is a set & $\tau \subseteq P(X)$ where:
  1) $\emptyset, X \in \tau$
  2) τ is closed under (finite or infinite) unions.
  3) τ is closed under finite intersections.

■ (X,τ) a topological space.

$\mathcal{B}(X)$ is Borel σ-algebra on X: $\mathcal{B}(X) = \sigma(\tau)$

the σ-algebra generated by the open sets

■ $(X, \mathcal{A})$ is called a measurable space.

a σ-algebra over X.

A map $\mu: \mathcal{A} \longrightarrow [0,\infty) \cup \{\infty\}$ is a measure if:

1) $\mu(\emptyset) = 0$

σ-additive 2) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ if $A_i \cap A_j = \emptyset$ for $i \neq j$

$(X, \mathcal{A}, \mu)$ is a measure space.

Ex. X, $\mathcal{A} = \mathcal{P}(X)$

a) counting measure $\rightarrow$ $\mu(A) := \begin{cases} \#A & A \text{ has finitely many elements} \\ \infty & \text{else} \end{cases}$

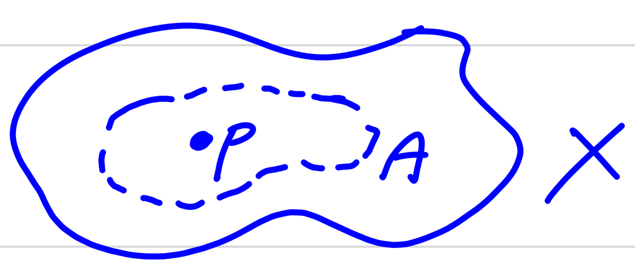
* calculations in $[0,\infty) \cup \{\infty\}$: $x + \infty = \infty \quad \forall x \in [0,\infty]$

$x \cdot \infty = \infty \quad \forall x \in (0,\infty]$

$0 \cdot \infty = 0$ (in most cases in measure theory!)

b) Dirac measure for $p \in X$:

$\delta_p(A) = \begin{cases} 1 & p \in A \\ 0 & p \notin A \end{cases}$



c) We search a measure on $X = \mathbb{R}^n$:

(1) $\mu([0,1]^n) = 1$

(2) $\mu(x+A) = \mu(A) \quad \forall x \in \mathbb{R}^n$

translation invariance

? How to define this?

* Measure problem: search measure μ on $P(\mathbb{R})$ with:

(1) $\mu([a,b]) = b-a$

(2) $\mu(x+A) = \mu(A)$

$\Rightarrow \mu$ does not exist on the whole power set

claim: μ a measure on $P(\mathbb{R})$ with $\mu((0,1]) < \infty$ & translation invariance

$$\Rightarrow \mu = 0$$

proof: Definitions: * $I := (0,1]$ with equivalence relation on I:

$$x \sim y \iff x - y \in \mathbb{Q}$$

$$\text{i.e. } [x] := \{ x + r \mid r \in \mathbb{Q},\ x + r \in I \}$$

* Define A as a set containing exactly one element from each class $[x]$:

$A \subseteq I$: a1) $\forall [x]\ \exists a \in A : a \in [x]$

a2) $\forall a, b \in A : a, b \in [x] \Rightarrow a = b$

We need axiom of choice to make sure A exists.

* $A_n := r_n + A$, where $\{r_n\}_{n \in \mathbb{N}}$ enumeration of $\mathbb{Q} \cap (-1,1]$

Claim: $A_n \cap A_m = \emptyset$ if $n \neq m$.

proof $x \in A_n \cap A_m \Rightarrow \begin{cases} x = r_n + a_n, & a_n \in A \\ x = r_m + a_m, & a_m \in A \end{cases} \Rightarrow r_n + a_n = r_m + a_m$

$\Rightarrow r_n + a_n = r_m + a_m \Rightarrow a_n - a_m = r_m - r_n \in \mathbb{Q} \Rightarrow a_n \sim a_m$

$\Rightarrow a_m, a_n \in [a_m] \overset{(a2)}{\Rightarrow} a_m = a_n \Rightarrow r_n = r_m \Rightarrow m = n$

* Claim: $(0,1] \subseteq \bigcup_{n\in\mathbb{N}} A_n \subseteq (-1,2]$

proof: 1) $\forall x \in (0,1]$: $\exists\, t\in A\cap(0,1]$: $x-t = r \in (-1,1]\cap\mathbb{Q} \Rightarrow x\in A_r$

2) $A_n \subseteq (-1,2]$: obvious!

Now assume: μ measure on $\mathcal{P}(\mathbb{R})$ with $\mu((0,1])<\infty$ & translation invariance:

- $\mu(r_n + A) = \mu(A) \quad \forall n\in\mathbb{N}$
- $\mu((0,1]) \leq \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \mu((-1,2])$ *
- $\mu((0,1]) = C < \infty$

$\mu((-1,2]) = \mu\left((-1,0] \cup (0,1] \cup (1,2]\right) = 3C$

$\overset{*}{\Rightarrow} C \leq \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq 3C$

$\Rightarrow C \leq \sum_{n=1}^{\infty} \mu(A) \leq 3C \Rightarrow \mu(A)=0 \Rightarrow C=0$

$\Rightarrow \mu(\mathbb{R}) = \mu\left(\bigcup_{m\in\mathbb{Z}} (m, m+1]\right) = 0$ ■ the measure problem is not solvable

$\Downarrow$

we will not measure all sets in $\mathcal{P}(\mathbb{R})$; but some of them, called measurable sets.

■ $(\Omega_1, \mathcal{A}_1), (\Omega_2, \mathcal{A}_2)$ measurable spaces:

$f: \Omega_1 \to \Omega_2$ is a measurable map (w.r.t. $\mathcal{A}_1, \mathcal{A}_2$) if

$$f^{-1}(A_2) \in \mathcal{A}_1 \quad \forall A_2 \in \mathcal{A}_2$$

(we want to get a measurable set on x axis if we have a measurable set on y axis!)

Ex. (1) $(\Omega, \mathcal{A})$, $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

Charactristic (indicator) function:

$$\chi_A : \Omega \to \mathbb{R}, \quad \chi_A(\omega) = \begin{cases} 1 & \omega\in A \\ 0 & \omega\notin A \end{cases} \quad \forall A\in\mathcal{A}$$

χ_A is a measurable map:

$$\chi_A^{-1}(\emptyset)=\emptyset^{\in\mathcal{A}},\quad \chi_A^{-1}(\mathbb{R})=\Omega^{\in\mathcal{A}},\quad \chi_A^{-1}(\{1\})=A^{\in\mathcal{A}},\quad \chi_A^{-1}(\{0\})=A^c\in\mathcal{A}$$

(2) $(\Omega_1,\mathcal{A}_1)$, $(\Omega_2,\mathcal{A}_2)$, $(\Omega_3,\mathcal{A}_3)$ measurable spaces.

$$\begin{tikzcd} \Omega_1 \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & \Omega_2 \arrow[r, "g"] & \Omega_3 \end{tikzcd}$$

$$\boxed{f,g \text{ measurable} \Rightarrow g\circ f \text{ measurable}}$$

$$(g\circ f)^{-1}(A_3) = \underbrace{f^{-1}\big(\underbrace{g^{-1}(A_3)}_{\in\mathcal{A}_2}\big)}_{\in\mathcal{A}_1} \Rightarrow g\circ f \text{ measurable}$$

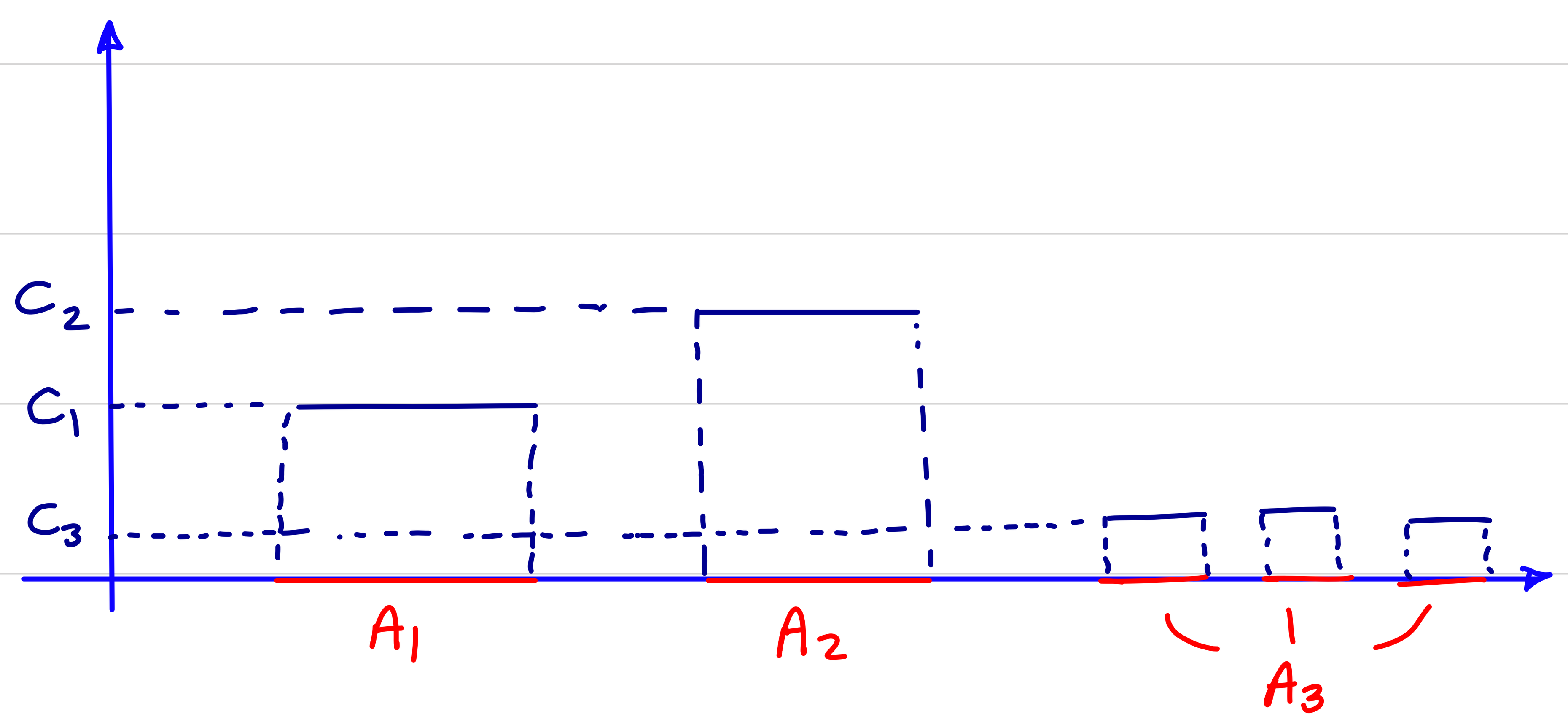
$*$ $(\Omega,\mathcal{A})$, $(\mathbb{R},\mathcal{B}(\mathbb{R}))$

$f,g:\Omega\to\mathbb{R}$ measurable $\Rightarrow$ $f+g$, $f-g$, $f\circ g$, $|f|$ measurable

■ Simple / Step function: $f:\Omega\to\mathbb{R}$ is a step function if $A_1,\dots,A_n\in\mathcal{A}$ and $c_1,\dots,c_n\in\mathbb{R}$ exist such that:

$$f(x)=\sum_{i=1}^{n} c_i\,\chi_{A_i}(x)$$

Ex



■ $S^+ := \{ f : X \to \mathbb{R} \mid f$ simple function $, f \geq 0 \}$

- measurable
- only finitely many values

$f \in S^+ \rightsquigarrow$ choose a representation $f(x) = \sum_{i=1}^{n} c_i \chi_{A_i}(x)$, $c_i \geq 0$

The (Lebesgue) integral of f w.r.t. μ:

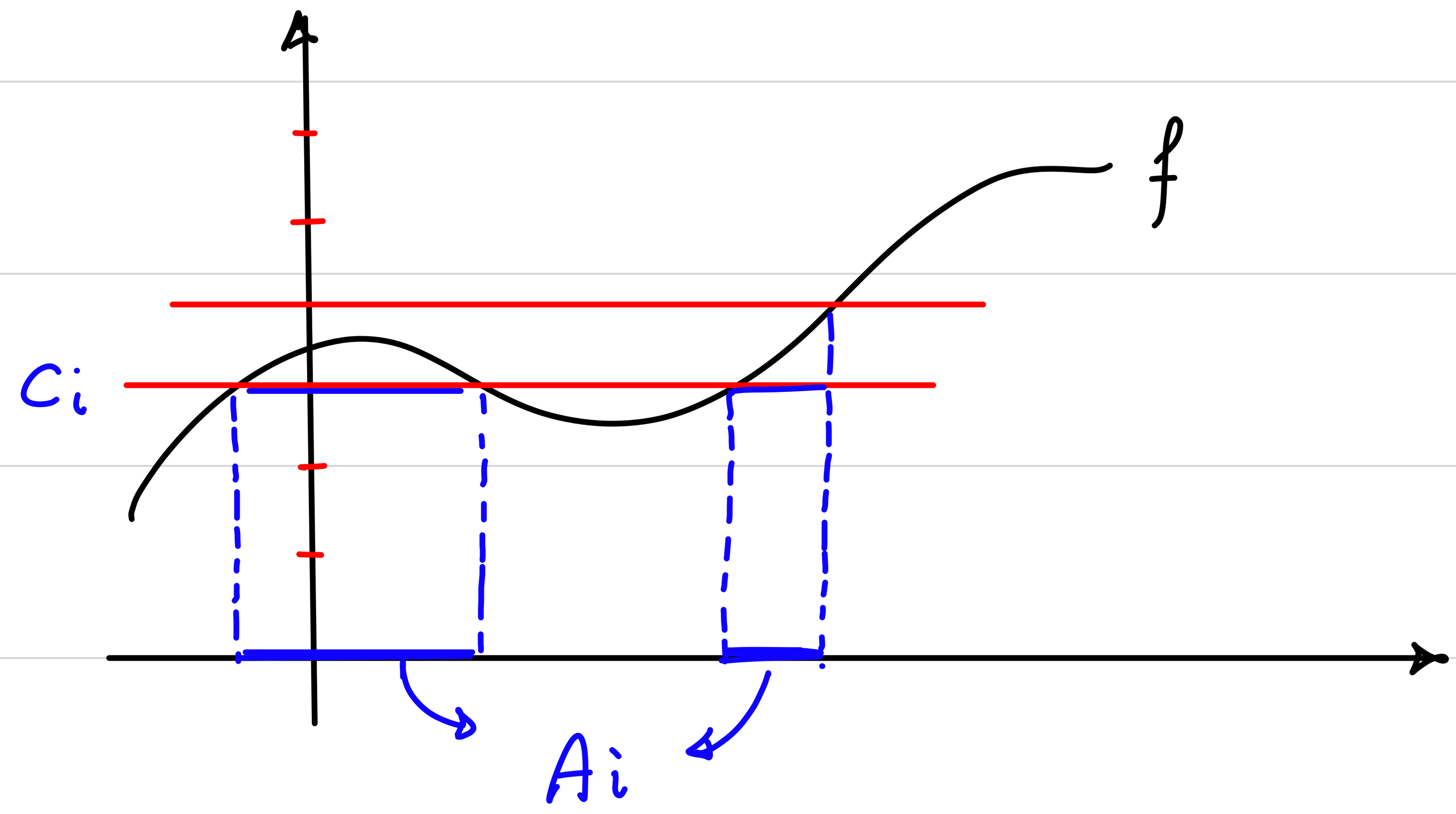
$$\boxed{\int_X f(x)\,d\mu(x) = \int_X f\,d\mu = I(f) = \sum_{i=1}^{n} c_i\, \mu(A_i) \in [0,\infty]}$$

(well-defined; independent of choices of A_i & c_i)

properties: (a) $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g) \quad \forall \alpha, \beta \geq 0$

(b) $f \leq g \Rightarrow I(f) \leq I(g)$ (monotonicity)

💡 generalization to non-simple functions:



$h = \sum c_i \chi_{A_i} \leq f$

■ $f : X \to [0,\infty)$ measurable:

$$\int_X f\,d\mu := \sup \{ I(h) \mid h \in S^+, h \leq f \} \in [0,\infty]$$

Lebesgue integral of f w.r.t. μ

f is called μ-integrable if $\int_X f\,d\mu < \infty$.

* properties :

(a) $f = g$ $\underbrace{\mu\text{-almost everywhere}}_{\mu(\{x\in X \mid f(x)\neq g(x)\})=0}$ (μ- a.e) $\Rightarrow \int_X f\, d\mu = \int_X g\, d\mu$

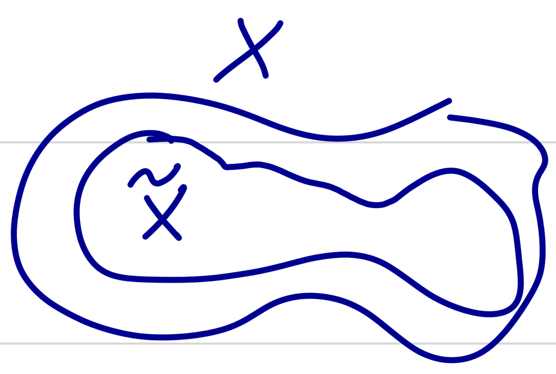
(b) $f \leq g$ μ-a.e. $\Rightarrow \int_X f\, d\mu \leq \int_X g\, d\mu$

(c) $f = 0$ μ-a.e. $\overset{f\geq 0}{\Leftarrow}\!\Rightarrow \int_X f\, d\mu = 0$

proof of b : $h: X \to [0,\infty)$ a simple function :

$$h(x) = \sum_{i=1}^{n} c_i\, \chi_{A_i}(x) = \sum_{t\in h(X)} t\cdot \chi_{\{x\in X \mid h(x)=t\}}$$

Let $X = \tilde{X} \cup \tilde{X}^c$ with $\mu(\tilde{X}^c) = 0$



then define: $\tilde{h}(x) = \begin{cases} h(x), & x\in\tilde{X} \\ a, & x\in\tilde{X}^c \end{cases}$

$$\Rightarrow \tilde{h}(x) = \sum_{t\in h(x)} t\cdot \chi_{\{x\in\tilde{X} \mid h(x)=t\}} + a\cdot \chi_{\tilde{X}^c}$$

$$\Rightarrow I(\tilde{h}) = \sum_{t\in h} t\cdot\mu(\{x\in\tilde{X} \mid h(x)=t\}) + a\,\underbrace{\mu(\tilde{X}^c)}_{0}$$

$$= \sum_{t\in h(X)} t\left(\mu(\{x\in\tilde{X} \mid h(x)=t\}) + \underbrace{\mu(\{x\in\tilde{X}^c \mid h(x)=t\})}_{0}\right)$$

$$= \sum_{t\in h(X)} t\cdot\mu(\{x\in X \mid h(x)=t\})$$

$$= \sum_{t\in h(X)} t\cdot\mu(\{x\in X \mid h(x)=t\}) = I(h) \quad (*)$$

$f \leq g \quad \mu\text{-a-e} \rightsquigarrow \tilde{X} := \{ x \in X \mid f(x) \leq g(x) \}, \quad \mu(\tilde{X}^c) = 0$

$$\int_X f \, d\mu = \sup \{ I(h) \mid h \in S^+, h \leq f \}$$
$$\overset{(*)}{=} \sup \{ I(\tilde{h}) \mid \tilde{h} \in S, \tilde{h} \leq \overset{\leq g}{f} \text{ on } \tilde{X} \}$$
$$\leq \sup \{ I(\tilde{h}) \mid \tilde{h} \in S, \tilde{h} \leq g \text{ on } \tilde{X} \}$$
$$= \int_X g \, d\mu \quad \blacksquare$$

■ Monotone Convergence Theorem:

$(X, \mathcal{A}, \mu)$ measure space, $f_n : X \to [0,\infty)$, $f : X \to [0,\infty)$ measurable for all $n \in \mathbb{N}$

with: $f_1 \leq f_2 \leq f_3 \leq \cdots \quad \mu$-a.e.

$\lim_{n\to\infty} f_n(x) = f(x) \quad \mu$-a.e. $\quad (x \in X)$

$$\Rightarrow \lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$$

alternative more general form:

$(X, \mathcal{A}, \mu)$ measurable space, $f_n : X \to [0,\infty]$ measurable for all $n \in \mathbb{N}$

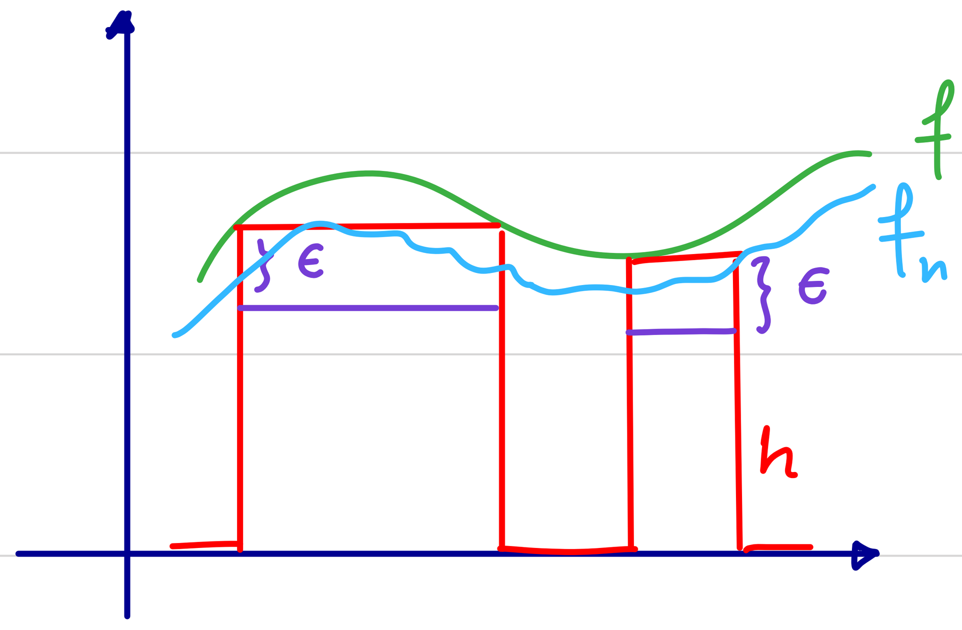
with $f_1 \leq f_2 \leq f_3 \leq \cdots \quad \mu$-a.e.

$$\Rightarrow \lim_{n\to\infty} \int_X f_n \, d\mu = \int_X \overbrace{\lim_{n\to\infty} f_n}^{f} \, d\mu \quad (*)$$

Proof: monotonicity $\rightarrow \int_X f_1\,d\mu \leq \int_X f_2\,d\mu \leq \int_X f_3\,d\mu \leq \cdots \leq \int_X f\,d\mu$

$$\Rightarrow \lim_{n\to\infty} \int_X f_n\,d\mu \leq \int f\,d\mu \qquad (\text{one part of } *)$$

Let h be a simple function $0 \leq h \leq f$ and $\epsilon > 0$.



$X_n := \{x \in X \mid f_n(x) \geq (1-\epsilon)\,h(x)\}$

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \mu\text{-a.e.} \Rightarrow \begin{cases} \tilde{X} := \bigcup_{n=1}^{\infty} X_n \\ \mu(\tilde{X}^c) = 0 \end{cases}$$

$$\int_X f_n\,d\mu \geq \int_{X_n} f_n\,d\mu \geq \int_{X_n} (1-\epsilon)\,h\,d\mu$$

$$\Rightarrow \lim_{n\to\infty} \int_X f_n\,d\mu \geq \lim_{n\to\infty} \int_{X_n} (1-\epsilon)\,h\,d\mu \overset{(?)}{=} \int_{\tilde{X}} (1-\epsilon)\,h\,d\mu$$

$$= \int_X (1-\epsilon)\,h\,d\mu$$

$$\Rightarrow \forall \epsilon > 0: \quad \lim_{n\to\infty} \int_X f_n\,d\mu \geq \int_X (1-\epsilon)\,h\,d\mu$$

$$\Rightarrow \quad \lim_{n\to\infty} \int_X f_n\,d\mu \geq \int_X h\,d\mu$$

$$\underset{h \leq f}{\overset{h \text{ arbitrary}}{\Rightarrow}} \quad \lim_{n\to\infty} \int_X f_n\,d\mu \geq \int_X f\,d\mu \ \blacksquare \qquad (\text{Second part of } *)$$

Application: $(g_n)_{n\in\mathbb{N}}$, $g_n : X \to [0,\infty]$ measurable for all n.

$\Rightarrow \sum_{n=1}^{\infty} g_n : X \to [0,\infty]$ measurable

$$\int_X \sum_{n=1}^{\infty} g_n\,d\mu = \sum_{n=1}^{\infty} \int_X g_n\,d\mu$$

Fatou's Lemma: $(X, \mathcal{A}, \mu)$ measure space.

$f_n : X \to [0,\infty]$ measurable for all $n \in \mathbb{N}$

$$\Rightarrow \int_X \underbrace{\liminf_{n\to\infty} f_n}\, d\mu \le \liminf_{n\to\infty} \int f_n \, d\mu$$

$\hookrightarrow \liminf_{n\to\infty} f_n : X \to [0,\infty]$

$$g(x) := \left(\liminf_{n\to\infty} f_n\right)(x) := \lim_{n\to\infty} \Big(\underbrace{\inf_{k \ge n} f_k(x)}_{g_n(x)} \Big) \in [0,\infty]$$

* note 1: all functions above are measurable.

* note 2: $g_1 \le g_2 \le g_3 \le \cdots$

Proof: $\int_X \lim_{n\to\infty} g_n \, d\mu \underset{\text{monotone conv. th.}}{=} \lim_{n\to\infty} \int_X g_n \, d\mu = \liminf_{n\to\infty} \int_X g_n \, d\mu$

We know $g_n \le f_n$ for all $n \Rightarrow \int_X g_n \, d\mu \le \int_X f_n \, d\mu$

$$\Rightarrow \int_X \liminf_{n\to\infty} f_n \, d\mu \le \liminf_{n\to\infty} \int_X f_n \, d\mu \quad \blacksquare$$

- $(X, \mathcal{A}, \mu)$ measure space:

$$\mathcal{L}^1(\mu) = \left\{ f : X \to \mathbb{R} \text{ measurable} \;\middle|\; \int_X |f| \, d\mu < \infty \right\}$$

for $f \in \mathcal{L}^1(\mu)$, write $f = f^+ - f^-$ where $f^+, f^- \ge 0$

Define:

$$\int_X f \, d\mu := \int_X f^+ \, d\mu - \int_X f^- \, d\mu$$

■ Lebesgue's Dominated Convergence Theorem:

$f_n : X \to \mathbb{R}$ measurable for all $n \in \mathbb{N}$

$f : X \to \mathbb{R}$ with:

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for } x \in X \quad \mu\text{-a.e.}$$

and $|f_n| \leq g$ with $g \in \mathcal{L}^1(\mu)$ for all $n \in \mathbb{N}$.

integrable majorant

Then: $f_1, f_2, f_3, \ldots \in \mathcal{L}^1(\mu)$, $f \in \mathcal{L}^1(\mu)$,

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$$

<u>Proof</u>: $|f_n| \leq g \overset{\text{monotonicity}}{\Longrightarrow} \int_X |f_n| \, d\mu \leq \int_X g \, d\mu < \infty \Rightarrow f_n \in \mathcal{L}^1(\mu)$

$|f| \leq g$ μ-a.e. $\overset{\text{monot.}}{\Longrightarrow} f \in \mathcal{L}^1(\mu)$

We will show $\int_X |f_n - f| \, d\mu \xrightarrow{n\to\infty} 0$

$|f_n - f| \leq |f_n| + |f| \leq 2g$ (μ-a.e. $\leadsto$ not effective in integrals $\Rightarrow$ we can change g so that $|f| \leq g$ holds everywhere & drop μ-a.e.)

$\Rightarrow h_n := 2g - |f_n - f| \geq 0$

$\Rightarrow h_n : X \to [0, \infty]$ measurable for all $n \in \mathbb{N}$.

$$\overset{\text{Fatou}}{\Rightarrow} \underbrace{\int_X \liminf_{n\to\infty} h_n \, d\mu}_{\int_X 2g \, d\mu} \leq \underbrace{\liminf_{n\to\infty} \int_X h_n \, d\mu}_{\int_X 2g \, d\mu - \limsup_{n\to\infty} \int_X |f_n - f| \, d\mu}$$

$$0 \leq \liminf_{n\to\infty} \int_X |f_n - f| \, d\mu \leq \limsup_{n\to\infty} \int_X |f_n - f| \, d\mu \leq 0$$

$\Rightarrow$ limit exists & $\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0$

we include:

$$0 \leq \left| \int_X f_n \, d\mu - \int_X f \, d\mu \right| = \left| \int_X (f_n - f) \, d\mu \right|$$
$$\leq \int_X |f_n - f| \, d\mu \xrightarrow{n\to\infty} 0$$

$$\Rightarrow \lim_{n\to\infty} \int_X f_n \, d\mu \leq \int_X f \, d\mu$$

Carathèodory's extension theorem:

X set, $\mathcal{A} \subseteq P(X)$ semiring of sets

$\mu : \mathcal{A} \to [0, \infty]$ pre-measure

(a) Then μ has an extension $\tilde{\mu} : \sigma(\mathcal{A}) \to [0,\infty]$

measure

i.e. $\mu(A) = \tilde{\mu}(A) \quad \forall A \in \mathcal{A}$

(b) If there is a sequence (S_j) with $S_j \in \mathcal{A}$, $\boxed{\bigcup_{j=1}^{\infty} S_j = X, \mu(S_j) < \infty}$

then the extension $\tilde{\mu}$ from (a) is unique. ($\tilde{\mu}$ is also σ-finite)

def

Explanations:

* Semirings of sets $\mathcal{A} \subseteq P(X)$

(1) $\emptyset \in \mathcal{A}$ (as for σ-algebras)

(2) $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$

(3) for $A, B \in \mathcal{A}$, there are pairwise disjoint sets $S_1, \dots, S_n \in \mathcal{A}$:

$$\bigcup_{j=1}^{n} S_j = A \setminus B$$

* most important example: $\mathcal{A} = \{ [a,b) \mid a, b \in \mathbb{R}, \ a \leq b \}$

not a σ-algebra because $\mathbb{R} \notin \mathcal{A}$

but $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R})$ (Borel σ-algebra)

(1) $\emptyset \in \mathcal{A}$ ✓

(2) $[a,b) \cap [c,d) = \begin{cases} \emptyset & a[\ )b\ \ c[\ )d \\ [c,b) & a[\ c[\ )b\ )d \\ \vdots & \end{cases}$ ✓

$\Rightarrow$ semiring

(3) $[a,b) \setminus [c,d) = \begin{cases} [a,b) & a[\ )b\ \ c[\ )d \\ [a,c) & a[\ c[\ )b\ )d \\ [a,c) \cup [b,d) & a[\ c[\ )d\ )b \end{cases}$ ✓

* **Pre-measure:** $\mu : \mathcal{A} \to [0,\infty]$ with $\mathcal{A}$ semiring of sets:

(a) $\mu(\emptyset) = 0$

(b) $\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$, for $A_j \in \mathcal{A}$

, $A_i \cap A_j = \emptyset$ for $i \neq j$

, $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$

□ **Application** $\mathcal{A} = \{ [a,b) \mid a, b \in \mathbb{R},\ a \leq b \}$ semiring of sets.

$\mu : \mathcal{A} \to [0,\infty]$, $\mu([a,b]) = b - a$ pre-measure

$\xRightarrow{\text{Carathéodory}}$ There is a unique extension to $\mathcal{B}(\mathbb{R}) \Rightarrow$ $\boxed{\text{The Lebesgue Measure}}$

■ Lebesgue - Stieltjes measures

$F : \mathbb{R} \to \mathbb{R}$ monotonically increasing (non-decreasing).

Define $\mu_F([a,b)) = F(b^-) - F(a^-)$ $(*)$

$\mathcal{A} = \{ [a,b) : a, b \in \mathbb{R},\ a \leq b \}$ semiring of sets.

$\xRightarrow{\text{Carathéodory}}$ There exists exactly one measure $\mu_F : \mathcal{B}(\mathbb{R}) \to [0,\infty]$ with $(*)$.

the Lebesgue-Stieltjes measure for F

∘ Examples

(a) $F(x) = x$, $\mu_F([a,b)) = b-a \rightsquigarrow$ Lebesgue measure

(b) $F(x) = 1$, $\mu_F([a,b)) = 0 \rightsquigarrow$ zero measure

(c) $F(x) = \begin{cases} 0 , & x<0 \\ 1 , & x \geq 0 \end{cases} \to \mu_F([-\epsilon,\epsilon)) = 1 \rightsquigarrow$ Dirac measure δ_0

(d) $F : \mathbb{R} \to \mathbb{R}$ monotonically increasing + continuously differentiable

$F' : \mathbb{R} \to [0,\infty)$ $\qquad \mu_F([a,b)) = F(b) - F(a) = \int_a^b F'(x)\,dx$

$\rightsquigarrow \mu_F : A \mapsto \int_A \underbrace{F'(x)}_{\text{density function}}\,dx$

■ $(X, \mathcal{A}, \lambda)$ measure space.

$X = \mathbb{R}$, $\mathcal{A} = \mathcal{B}(\mathbb{R})$, $\lambda \to$ Lebesgue measure ($\rightsquigarrow$ the unique measure that $\lambda([a,b)) = b-a$)

Another measure $\mu : \mathcal{B}(\mathbb{R}) \to [0,\infty)$

∗ Definitions:

(a) μ is called absolutely continuous (w.r.t. λ) if $\lambda(A) = 0 \Rightarrow \mu(A) = 0$ for all $A \in \mathcal{B}(\mathbb{R}) \rightsquigarrow$ one writes: $\mu \ll \lambda$

∘ Ex: λ & the zero measure are absolutely continuous.

(b) μ is called singular (w.r.t λ) if there exists $N \in \mathcal{B}(\mathbb{R})$ with $\lambda(N) = 0$ and $\mu(N^c) = \mu(\mathbb{R}\setminus N) = 0 \rightsquigarrow$ one writes $\mu \perp \lambda$

∘ Ex: δ_0 Dirac measure $(\delta_0(\{0\}) = 1) \Rightarrow \delta_0 \perp \lambda$ $(N = \{0\})$

Theorem: $\mu : \mathcal{B}(\mathbb{R}) \longrightarrow [0, \infty]$ σ-finite

(a) **Lebesgue's Decomposition Theorem:** There are two measures (uniquely determined):

$$\mu_{ac}\,,\ \mu_s \ :\ \mathcal{B}(\mathbb{R}) \longrightarrow [0,\infty]$$

with $\quad \mu = \mu_{ac} + \mu_s\,, \quad \mu_{ac} \ll \lambda\,, \quad \mu_s \perp \lambda$

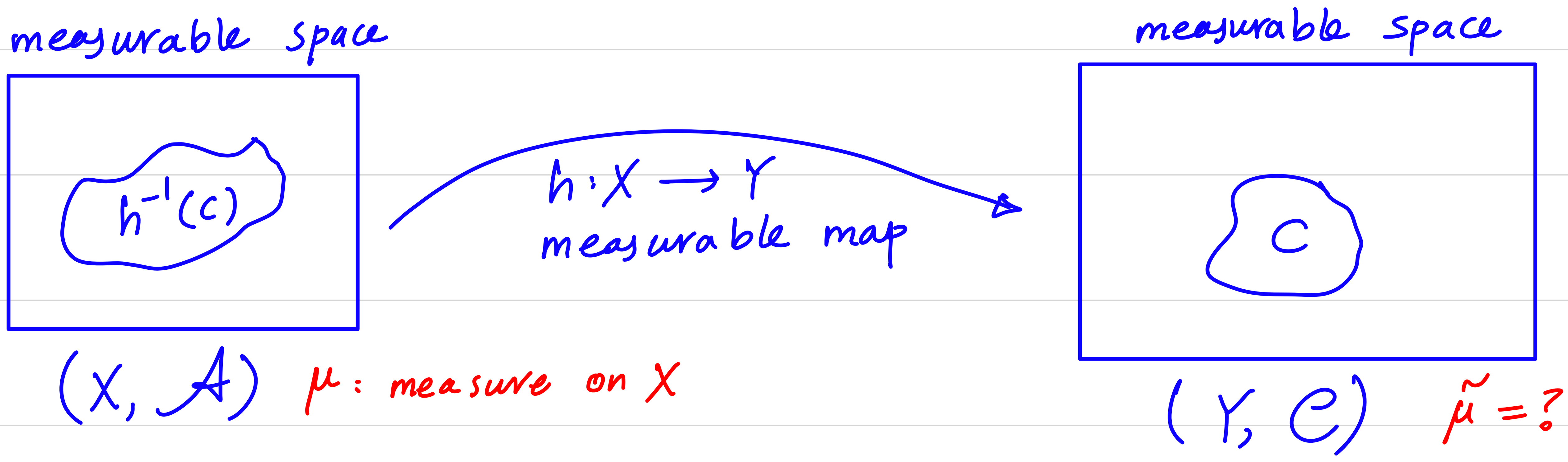
(b) **Radon-Nikodym theorem:** for any given $\mu_{ac} \ll \lambda$, there is a measurable map $h : \mathbb{R} \longrightarrow [0,\infty)$ with

$$\mu_{ac}(A) = \int_A h \, d\lambda$$

density function

for all $A \in \mathcal{B}(\mathbb{R})$.

Image measure & substitution formula:



Define $\tilde{\mu}$ (a measure on $\mathcal{C}$): $\tilde{\mu}(C) = \mu(h^{-1}(C)) \quad \forall C \in \mathcal{C}$

Image Measure
(Push forward measure)

Notations: $h_* \mu$ or $h_\# \mu$ or $\mu \circ h^{-1}$

*** Substitution formula:**

$$g : Y \to \mathbb{R} \implies \int_Y g \, d(h_* \mu) = \int_X g \circ h \, d\mu$$

equivalent notation

$$\int_Y g(y)\, d(\mu \circ h^{-1})(y) = \int_X g(h(x))\, d\mu(x)$$

change of variable $y = h(x)$

Example

$$(\mathbb{R}, \mathcal{B}(\mathbb{R})) \xrightarrow{F} (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

F strictly monotonically increasing + continuously differentiable + surjective

μ_F with $\mu_F(A) = \int_A F'(x)\, dx$

$$(F_* \mu_F)([a,b)) = \mu_F(F^{-1}([b,a))) = \mu_F([F^{-1}(b), F^{-1}(a))) = b-a$$

Lebesgue measure

$\Rightarrow$ Substitution Formula: $\int_Y g \, d(F_* \mu_F) = \int_X g \circ F \, d\mu_F$

simple special case $\leftarrow$ $\int_{\mathbb{R}} g(y)\, dy = \int_{\mathbb{R}} g(F(x))\, dx$

■ Proof of substitution formula.

(1) Let $g = \chi_C$ with $C \subseteq Y$ measurable.

$$\int_Y \chi_C \, d(h_* \mu) = h_* \mu (C) = \mu(h^{-1}(C))$$

$$\int_X \chi_C \circ h \, d\mu = \int_X \underbrace{\chi_C(h(x))}_{\begin{cases} 1 & x \in h^{-1}(C) \\ 0 & x \notin h^{-1}(C) \end{cases}} d\mu(x) = \int_X \chi_{h^{-1}(C)} \, d\mu \rightarrow = \quad \checkmark$$

(2) Let g be a simple function, i.e. $g = \sum_{i=1}^{n} \lambda_i \chi_{C_i}$

$$\begin{aligned} \int_Y g \, d(h_* \mu) &= \int_Y \sum_{i=1}^{n} \lambda_i \chi_{C_i} \, d(h_* \mu) \\ &= \sum_{i=1}^{n} \lambda_i \int_Y \chi_{C_i} \, d(h_* \mu) \\ &\overset{(1)}{=} \sum_{i=1}^{n} \lambda_i \int_X \chi_{C_i}(h(x)) \, d\mu(x) \\ &= \int_X \Big(\sum_{i=1}^{n} \lambda_i \chi_{C_i} \Big)(h(x)) \, d\mu(x) \\ &= \int_X g \circ h \, d\mu \quad \checkmark \end{aligned}$$

(3) Let $g : Y \to [0,\infty)$ measurable

$$\int_Y g \, d(h_* \mu) = \sup \Big\{ \int_Y \tilde{s} \, d(h_* \mu) \;\Big|\; \tilde{s} : Y \to [0,\infty) \text{ simple}, \; \tilde{s} \le g \Big\}$$

$\forall y \in h(X) : \tilde{s}(y) \le g(y)$

$\Leftrightarrow \forall x \in X : \underbrace{\tilde{s}(h(x))}_{\tilde{s} \circ h(x)} \le \underbrace{g(h(x))}_{g \circ h(x)}$

simple function on X

$$= \sup \Big\{ \int_X \tilde{s} \circ h \, d\mu \;\Big|\; \tilde{s} : Y \to [0,\infty) \text{ simple}, \; \tilde{s} \circ h \le g \circ h \Big\}$$

$$\overset{(?)}{=} \sup \Big\{ \int_X s \, d\mu \;\Big|\; s : X \to [0,\infty) \text{ simple}, \; s \le g \circ h \Big\}$$

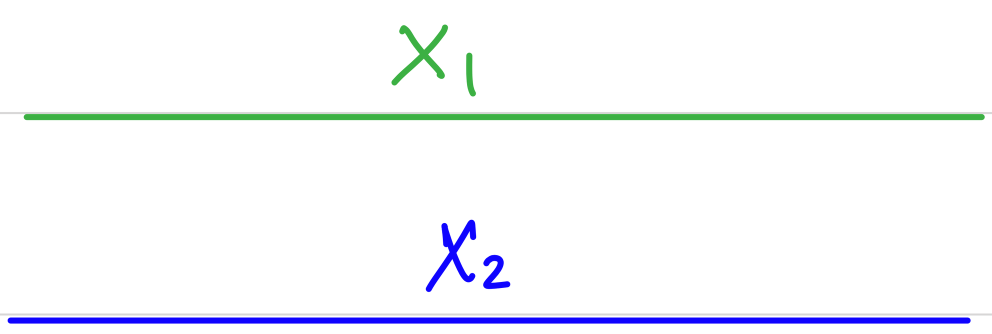
$$= \int_X g \circ h \, d\mu$$

(4) $g : Y \to \mathbb{R} \rightsquigarrow$ decompose g to $g^+ - g^-$ ■

■ Product Measure

$(X_1, \mathcal{A}_1, \mu_1)$ measure space

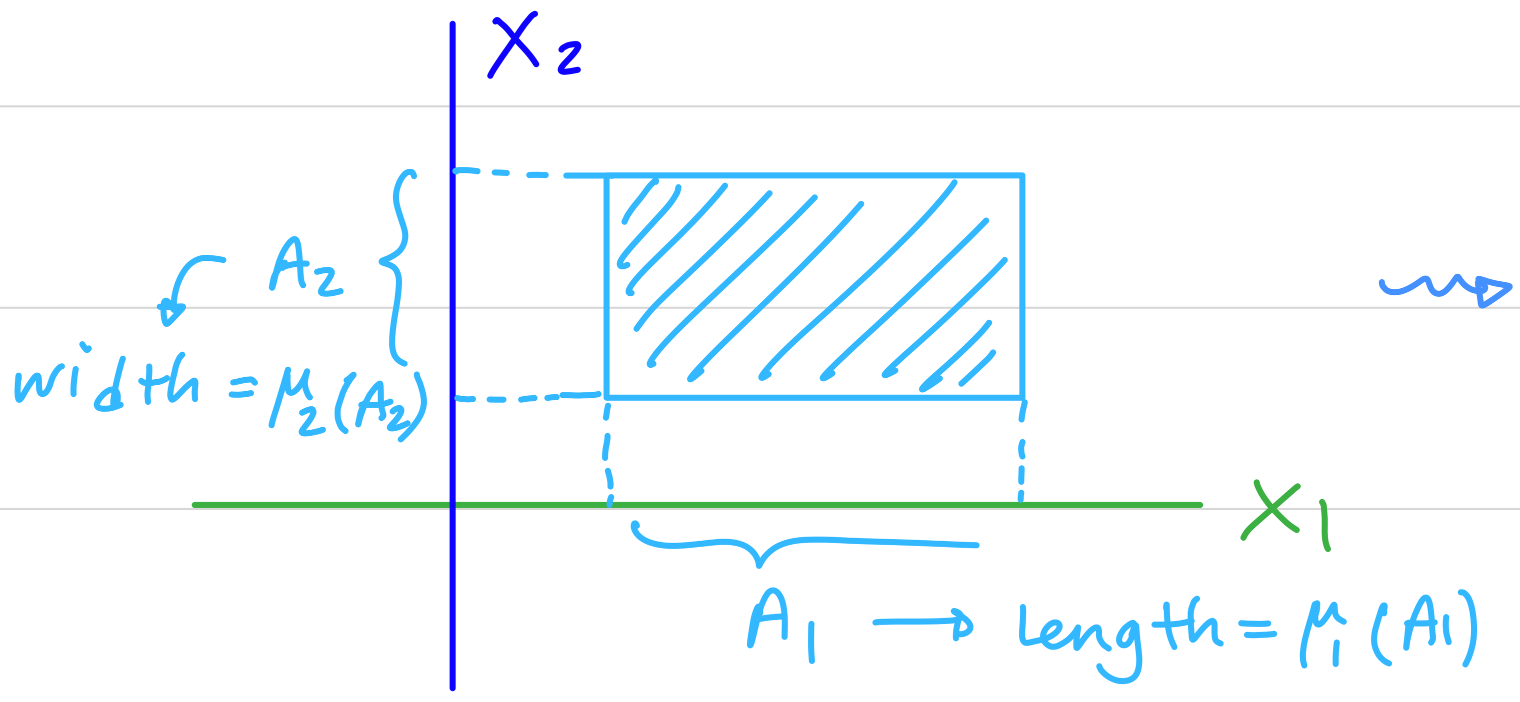
$(X_2, \mathcal{A}_2, \mu_2)$ measure space



$\rightsquigarrow$

$(X_1 \times X_2, \mathcal{A}, \mu)$

product measure



$\rightsquigarrow \mu(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2)$

Product σ-algebra $\rightsquigarrow$ $\mathcal{A}_1 \times \mathcal{A}_2$ is not a σ-algebra (because the union of two rectangles is not necessarily a rectangle).

$\mathcal{A}_1 \times \mathcal{A}_2$ is a semiring.

We define: $\mathcal{A} = \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$

■ **Product Measure**: Define μ as $\mu(A_1 \times A_2) = \mu_1(A_1) \, \mu_2(A_2)$ for all $A_1 \in \mathcal{A}_1$, $A_2 \in \mathcal{A}_2$ and use Carathéodory's extension theorem.

(Product measure is not unique in general)

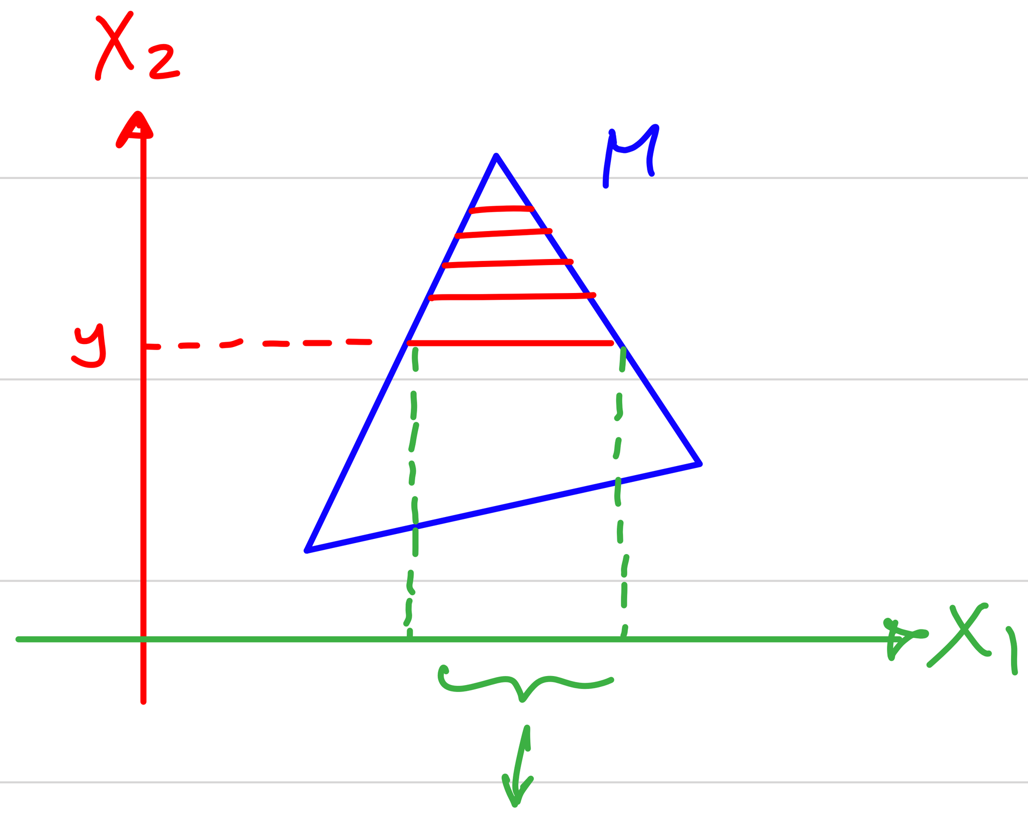
□ **Proposition:** If μ_1, μ_2 are σ-finite, then there is exactly one measure μ with $\mu(A_1 \times A_2) = \mu_1(A_1)\,\mu_2(A_2)$.

It satisfies:

Cavalieri's Principle

$$\boxed{\begin{aligned}\mu(M) &= \int_{X_2} \mu_1(M_y)\, d\mu_2(y) \\ &= \int_{X_1} \mu_2(M_x)\, d\mu_1(x)\end{aligned}}$$

$M_x = \{x_2 \in X_2 \mid (x, x_2) \in M\}$



$M_y := \{x_1 \in X_1 \mid (x_1, y) \in M\}$

Example Calculate the volume of the pyramid with corners:

$(-1,-1,0), (-1,1,0), (1,-1,0), (1,1,0), (0,0,1)$ $\quad (K \subseteq \mathbb{R}^3)$

→ Lebesgue measure in $\mathbb{R}^3$ (product measure construction with Lebesgue measure on $\mathbb{R}$)

$$K = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; 0 \leq z \leq 1,\ |x| \leq 1-z,\ |y| \leq 1-z \right\}$$

μ product measure of μ_1 and μ_2 → Lebesgue measure on $\mathbb{R}^2$ (x-and y-coordinate)

μ_1 → Lebesgue measure in $\mathbb{R}$ (z-coordinate)

$$\mu(K) = \int_{\mathbb{R}} \underbrace{\mu_2(M_{z_0})}_{(2(1-z_0))^2 \text{ for } z_0 \in [0,1]}\, d\mu_1(z_0) \quad , \quad M_{z_0} := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; |x| \leq 1-z_0,\ |y| \leq 1-z_0 \right\}$$

$$= \int_{[0,1]} 4(1-z_0)^2\, d\mu_1(z_0) = \int_0^1 4(1-z_0)^2\, dz_0 = 4\left(\frac{-1}{3}(1-z_0)^3\right)\Big|_0^1 = \frac{4}{3}$$

Fubini's theorem:

Let μ_1, μ_2 be σ-finite, μ be the product measure and

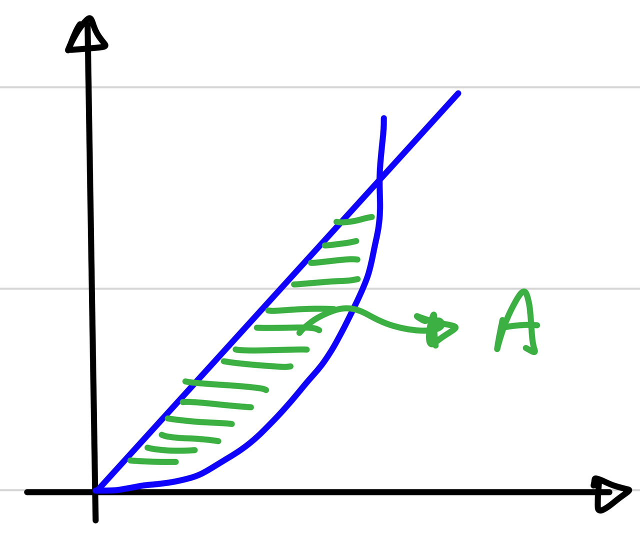
$f: X_1 \times X_2 \to [0,\infty]$ measurable. [or $f \in \mathcal{L}^1(\mu)$]

Then: $$\int_{X_1\times X_2} f\, d\mu = \int_{X_2}\left(\int_{X_1} f(x,y)\, d\mu_1(x)\right) d\mu_2(y) = \int_{X_1}\left(\int_{X_2} f(x,y)\, d\mu_2(y)\right) d\mu_1(x)$$

Example: μ Lebesgue measure for $\mathbb{R}^2$

$f(x,y) = 2xy$ $\qquad$ $A = \{(x,y) \in [0,1]\times[0,1] \mid x \geq y \geq x^2\}$

$\int_A f\, d\mu = ?$



$$\int_A f\, d\mu = \int_{\mathbb{R}^2} f \cdot \chi_A\, d\mu = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(x,y)\, \chi_A(x,y)\, dy\right) dx$$

$$= \int_0^1 \left(\int_{x^2}^{x} 2xy\ dy\right) dx$$

$$= 2\int_0^1 x\left(\int_{x^2}^{x} y\ dy\right) dx = \dots = \frac{1}{12}$$

Outer Measures

Definition: A map $\varphi: P(X) \longrightarrow [0,\infty]$ is called an outer measure if:

(a) $\varphi(\emptyset) = 0$

(b) $A \subseteq B \Rightarrow \varphi(A) \leq \varphi(B)$ (monotonicity)

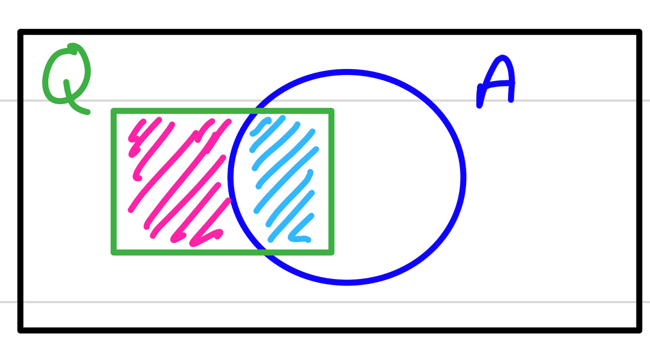
(c) $A_1, A_2, \dots \in P(X) \Rightarrow \varphi\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \varphi(A_n)$ (σ-subadditivity)

Question: $\varphi: P(X) \longrightarrow [0,\infty]$ outer measure $\overset{?}{\rightsquigarrow}$ μ measure

Definition: Let φ be an outer measure. $A \in P(X)$ is called <u>φ-measurable</u>

if for all $Q \in P(X)$ we have:

$$\varphi(Q) \overset{(\geq)}{=} \varphi(Q\cap A) + \varphi(Q \cap A^c)$$



Important Proposition: If $\varphi : P(X) \to [0,\infty]$ is an outer measure, then:

- $\mathcal{A}_\varphi = \{ A \subseteq X \mid A \ \varphi\text{-measurable} \}$ is a σ-algebra.
- $\mu : \mathcal{A}_\varphi \to [0,\infty]$, $\mu(A) = \varphi(A)$, is a measure.

Examples: (1) $\varphi : P(\mathbb{R}) \to [0,\infty]$, $\varphi(A) = \begin{cases} 0 & A = \emptyset \\ 1 & A \neq \emptyset \end{cases}$

$\hookrightarrow$ outer measure but not a measure.

(2) $\varphi : P(\mathbb{N}) \to [0,\infty]$, $\varphi(A) = \begin{cases} |A| & A \text{ finite} \\ \infty & A \text{ not finite} \end{cases}$

(3) $\mathcal{I} = \{ [a,b) \mid a,b \in \mathbb{R},\ a \leq b \}$, $\mu([a,b)) = b-a$ ("length")

Define $\varphi : P(\mathbb{R}) \to [0,\infty]$ by:

$$\varphi(A) = \inf \left\{ \sum_{j=1}^{\infty} \mu(I_j) \,\middle|\, I_j \in \mathcal{I},\ A \subseteq \bigcup_{j=1}^{\infty} I_j \right\}$$

φ is an outer measure: (a) $\varphi(\emptyset) = 0$ ✓

(b) monotonicity ✓

(c) $\varphi\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \varphi(A_n)$

proof of (c): Let $\epsilon > 0$. Choose $\epsilon_n > 0$ with $\sum_{n \in \mathbb{N}} \epsilon_n = \epsilon$

Then there are intervals $I_{j,n}$ with:

$$* \quad \varphi(A_n) \geq \sum_{j=1}^{\infty} \mu(I_{j,n}) - \epsilon_n \quad \text{and} \quad A_n \subseteq \bigcup_{j=1}^{\infty} I_{j,n}$$

Then: $\bigcup_{n \in \mathbb{N}} A_n \subseteq \bigcup_{n \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} I_{j,n} = \bigcup_{j,n} I_{j,n}$

$$\Rightarrow \varphi\left(\bigcup_{n \in \mathbb{N}} A_n\right) \overset{(b)}{\leq} \varphi\left(\bigcup_{j,n} I_{j,n}\right) \leq \sum_{j,n} \mu(I_{j,n}) = \sum_{n \in \mathbb{N}} \left(\sum_{j \in \mathbb{N}} I_{j,n} \right)$$

$$\overset{*}{\leq} \sum_{n \in \mathbb{N}} \left(\varphi(A_n) + \epsilon_n \right) = \sum_{n \in \mathbb{N}} \varphi(A_n) + \epsilon$$

$$\Rightarrow \varphi\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \varphi(A_n) \quad \blacksquare$$