Engineering Mathematics
Problem Set 05

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1 Integration

1. Find the contour integral $\int_\gamma \bar{z}\mathrm{d}z$ for:
   (a) γ is the triangle ABC oriented counterclockwise, where $A = 0$, $B = 1+i$, and $C = -2$.
   (b) γ is the circle $|z - i| = 2$ oriented counterclockwise.
2. Evaluate the following integral:
$$I = \int_C \left(z + \frac{1}{z}\right)^2 \mathrm{d}z$$
   where C is the following curve:
$$C: \quad z = x + iy \quad , \quad y + x = 1 \quad , \quad 0 \leq x, y \leq 1$$
3. Evaluated the following integral, where the unit circle is traversed counterclockwise:
$$I = \frac{1}{2\pi i} \oint_{|z|=1} \frac{(z+2)^2}{z^2(2z-1)}\mathrm{d}z$$
4. Let C_R be the circle $|z| = R$ oriented counterclockwise $(R > 1)$. Show that
$$\left| \oint_{C_R} \frac{\mathrm{Log}(z^2)}{z^2}\mathrm{d}z \right| < 4\pi \left(\frac{\pi + \ln R}{R} \right)$$
   and then
$$\lim_{R\to\infty} \oint_{C_R} \frac{\mathrm{Log}(z^2)}{z^2}\mathrm{d}z = 0$$
5. Without evaluating the integral, show that
$$\left| \int_C \frac{1}{\bar{z}^2 + \bar{z} + 1}\mathrm{d}z \right| \leq \frac{9\pi}{16}$$
   where C is the arc of the circle $|z| = 3$ from $z = 3$ to $z = 3i$ lying in the first quadrant.
6. Evaluate
$$I = \oint_C \frac{\sin z}{(z+1)^7}\mathrm{d}z$$
   where C is the circle of radius 5, center 0, positively oriented.
7. Find the value of $\oint_C g(z)\mathrm{d}z$, where C is the circle $|z - i| = 2$ oriented counterclockwise, when
   (a) $g(z) = \dfrac{1}{z^2 + 4}$
   (b) $g(z) = \dfrac{1}{z(z^2 + 4)}$

8. Compute the integrals of the following functions along the curvers C_1 and C_2, both oriented counterclockwise:

$$C_1 = \{|z| = 1\} \quad , \quad C_2 = \{|z-2| = 1\}$$

   (a) $\dfrac{1}{2z - z^2}$

   (b) $\dfrac{\sinh z}{(2z - z^2)^2}$

9. Show that if f is analytic inside and on a simple closed curve C and z_0 is not on C, then

$$(n-1)! \oint_C \frac{f^{(m)}(z)}{(z-z_0)^n} \mathrm{d}z = (m+n-1)! \oint_C \frac{f(z)}{(z-z_0)^{m+n}} \mathrm{d}z$$

   for all positive integers m and n.

10. Let C be the circle $|z| = 1$ oriented counterclockwise.

   (a) Compute

$$I_1 = \oint_C \frac{1}{z^2 - 8z + 1} \mathrm{d}z$$

   (b) Use (or not use) the previous part to compute

$$I_2 = \int_0^\pi \frac{1}{4 - \cos\theta} \mathrm{d}\theta$$

2 Sequences and Series

11. Determine whether each of the following sequences are convergent.

   (a) $x_n = \dfrac{(-1)^n n^2}{n^2 + 1}$

   (b) $x_n = \dfrac{(-1)^n e^n}{e^{n^2} + n}$

   (c) $x_n = \dfrac{1}{n} + \dfrac{1}{n+1} + \cdots + \dfrac{1}{2n}$

12. Let $\{x_n\}_{n=1}^\infty$ be a real sequence, where $x_1 > 0$, $x_2 > 0$, and the following recurrence relation holds for $n \geq 3$:

$$x_n = \frac{x_{n-1} + x_{n-2}}{2}$$

   Show that this sequence is convergent, and find its limit:

$$\lim_{n\to\infty} x_n$$

13. Show that:

   (a) If $\displaystyle\sum_{n=0}^\infty a_n$ is convergent, then $\displaystyle\sum_{n=0}^\infty \frac{a_n}{n}$ is also convergent.

   (b) If $\displaystyle\sum_{n=0}^\infty a_n$ is divergent, then $\displaystyle\sum_{n=0}^\infty n a_n$ is also divergent.

   (c) If $\displaystyle\sum_{n=0}^\infty |a_n|$ and $\displaystyle\sum_{n=0}^\infty |b_n|$ converge, then $\displaystyle\sum_{n=0}^\infty a_n b_n$ is convergent.

14. Determine whether each of the following series are convergent.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

   (b) $\displaystyle\sum_{n=0}^{\infty} (-1)^n n^{(1-n)/n}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ (for different values of p)

15. Find the *region of convergence* for each of the following series.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n(x-1)^n}{2^n}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!(x+2)^n}{2^n}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \left(n^{1/n} - 1\right) x^n$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^n z}{n^2+1}$

16. Find the Taylor series of the following functions and their radii of convergence:

   (a) $z \sinh(z^2)$ at $z = 0$.

   (b) e^z at $z = 2$.

   (c) $\displaystyle\frac{z^2+z}{(1-z)^2}$ at $z = -1$.

   (d) $\cos^2 z$ at $z = \pi$.

17. Let f be a function analytic at 0 and $g(z) = f(z^2)$. Show that $g^{(2n-1)}(0) = 0$ for all positive integers n.

18. Find a power-series expansion of the function $f(z) = \dfrac{1}{1-z}$ in each of the following regions:

   (a) $|z| < 1$ (centered at $z = 0$)

   (b) $|z| > 1$ (centered at $z = 0$)

   (c) $|z+1| < 2$ (centered at $z = -1$)

   (d) $|z+1| > 2$ (centered at $z = -1$)

19. Find a power-series expansion of the function $f(z) = \dfrac{1}{z(1-z)}$ in each of the following regions:

   (a) $|z-1| > 1$

   (b) $|z+1| < 1$

(c) $1 < |z+1| < 2$

(d) $|z+1| > 2$

20. Find a Laurent series expansion for each of the following functions, centered at $z = 0$:

(a) $f(z) = \dfrac{\sin z}{z}$

(b) $f(z) = \dfrac{1 - \cos z}{z^5}$

(c) $f(z) = \sin \dfrac{1}{z}$

21. Find the Laurent series of the function $f(z) = \dfrac{z+4}{z^2(z^2+3z+2)}$ in

(a) $0 < |z| < 1$

(b) $1 < |z| < 2$

(c) $|z| > 2$

(d) $0 < |z+1| < 1$

22. Prove that the coefficients c_n in the expansion

$$\frac{1}{1-z-z^2} = \sum_{n=0}^{\infty} c_n z^n$$

satisfy the recurrence relation $c_0 = c_1 = 1$, $c_n = c_{n-1} + c_{n-2}$ for $n \geq 2$. What is the radius of convergence of the series?