Engineering Mathematics
Problem Set 02

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Fall Semester 1398-99

1 Fourier Series

1. Find the Fourier series for the following function:

$$f(x) = \cos(ax) \quad , \quad a \notin \mathbb{Z} \quad , \quad -\pi < x < \pi \quad , \quad f(x + 2\pi) = f(x)$$

2. Show that:

$$\pi \cot \pi x = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2x}{x^2 - k^2}$$

3. Use the result from the previous part to show that:

$$\sin \pi x = \pi x \prod_{k=1}^{\infty} (1 - \frac{x^2}{k^2})$$

4. Show that:

$$\frac{\pi}{2} = \prod_{k=1}^{\infty} \frac{4k^2}{4k^2 - 1}$$

2 Fourier integral

Find the Fourier integral for the following functions:

1. $f(x) = \begin{cases} x^2 & |x| < a \\ 0 & |x| \geq a \end{cases}$

2. $f(x) = \begin{cases} \frac{\pi}{2} \cos x & |x| < \frac{\pi}{2} \\ \\ 0 & |x| \geq \frac{\pi}{2} \end{cases}$

3 Integral Calculation and Fourier Integral

Prove the following identities:

1.

$$\forall x \geq 0, k \geq 0 : \qquad e^{-kx} = \frac{2k}{\pi} \int_0^{\infty} \frac{\cos \omega x}{\omega^2 + k^2} \mathrm{d}\omega$$

2.

$$\forall x \geq 0 : \qquad (1 + x)e^{-x} = \frac{4}{\pi} \int_0^{\infty} \frac{\cos \omega x}{(\omega^2 + 1)^2} \mathrm{d}\omega$$

4 Integral Equation

Solve the following integral equation for $Y(\omega)$:

$$\int_0^\infty Y(\omega)\sin(\omega x)\mathrm{d}\omega = \begin{cases} \dfrac{\pi}{2} & 0 < x < \pi \\ \\ 0 & x > \pi \end{cases}$$

5 Fourier Integral and Real Integrals

1. Find the Fourier integral representation of the following function:

$$f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

2. Find the values of I_1, I_2, and I_3:

$$I_1 = \int_0^\infty \frac{\sin\omega\cos\omega}{\omega}\mathrm{d}\omega$$

$$I_2 = \int_0^\infty \frac{\sin^2\omega}{\omega^2}\mathrm{d}\omega$$

$$I_3 = \int_0^\infty \frac{\sin^4\omega}{\omega^2}\mathrm{d}\omega$$

6 The Fourier Transform

6.1 Basic Identities

Suppose $f(x)$ and $g(x)$ are two functions with Fourier transforms $F(\omega)$ and $G(\omega)$. Find the Fourier transform of each of the following functions in terms of F and G.

1. $af(x) + bg(x) \qquad a, b \in \mathbb{R}$
2. $f(x - x_0)$
3. $e^{i\omega_0 x} f(x)$
4. $f^*(x)$
5. $f(ax) \qquad a \in \mathbb{R}$
6. $f(x) * g(x) \qquad$ (the convolution of f and g)
7. $f(x)g(x)$
8. $\dfrac{\mathrm{d}}{\mathrm{d}x} f(x)$
9. $xf(x)$
10. $F(x)$
11. $f_e(t) = \textbf{Even}\{f(t)\} \qquad f(t) \in \mathbb{R}$
12. $f_o(t) = \textbf{Odd}\{f(t)\} \qquad f(t) \in \mathbb{R}$

6.2 Some More Identities

Prove the following identities:

1. $f(x)$ real and even $\Longleftrightarrow F(\omega)$ real and even
2. $f(x)$ real and odd $\Longleftrightarrow F(\omega)$ purely imaginary and odd
3. Parseval's identity:
$$\int_{-\infty}^{\infty} |f(x)|^2 \mathrm{d}x = \int_{-\infty}^{\infty} |F(\omega)|^2 \mathrm{d}\omega$$
4. Conjugate symmetry for real functions:
$$f(x) \in \mathbb{R} \Longleftarrow \begin{cases} F(\omega) = F^*(-\omega) \\ \mathbf{Re}\{F(\omega)\} = \mathbf{Re}\{F(-\omega)\} \\ \mathbf{Im}\{F(\omega)\} = -\mathbf{Im}\{F(-\omega)\} \\ |F(\omega)| = |F(-\omega)| \\ \sphericalangle F(\omega) = -\sphericalangle F(-\omega) \end{cases}$$

7 Fourier Transform Calculation

Find the Fourier transform of the following functions:

1. $f(x) = e^{-\alpha x} u(x) \quad , \quad \alpha > 0$
2. $f(x) = e^{-\alpha |x|} \quad , \quad \alpha > 0$

8 Fourier Transform and Real Integral

Use the function $f(x) = e^{-\alpha|x|}$ and the Fourier transform to calculate the following integral:

$$I = \int_{-\infty}^{\infty} \frac{1}{(x^2 + \alpha^2)^2} \mathrm{d}x$$

9 Linear Time-Invariant Systems

The input-output relationship of an LTI system is:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 2y(t) = x(t)$$

where $x(t)$ and $y(t)$ are the input and the output of the system, respectively.

1. Taking Fourier transform from both sides of the input-output relationship, find $H(\omega)$, the *transfer function* of the system:
$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$
2. The transfer function $H(\omega)$ is a complex-valued function. Plot its amplitude $|H(\omega)|$ and its phase $\sphericalangle F(\omega)$ against ω. (You may know these plots as *Bode diagrams.*)
3. If the input to the system is $x(t) = e^{-t} u(t)$, find the output $y(t)$.

10 Fourier Transform and Modulation

1. Let ω_0 be a real constant and $f(x)$ be an arbitrary function. We define $g(x)$ as:

$$g(x) = f(x)\cos(\omega_0 x)$$

   Find the Fourier transform of $g(x)$ in terms of $F(\omega)$, the Fourier transform of $f(x)$.

2. The Fourier transform of a function $h(x)$ is plotted in figure 1. Use the result from the previous part to find $h(x)$.

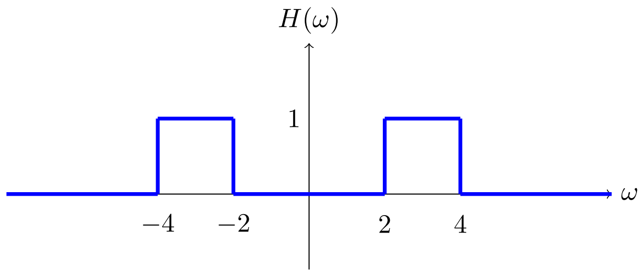

Figure 1

11 Fourier Series and Transform

Let f be an arbitrary function and $f(x) = 0$ for $x \notin [-0.5, 0.5]$. We define g, the *periodization* of f, as:

$$g(x) = \sum_{k=-\infty}^{\infty} f(x-k)$$

It is obvious that g is a periodic function, so it must have a Fourier series representation. On the other hand, f is not periodic, so we can define $F(\omega)$, the Fourier transform of f. Find the complex Fourier series coefficients of g in terms of $F(\omega)$.