The Laplace Transform

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Signals and Systems
Tutorial Session 3

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Overview

Introduction

Introduction

Definition

The Laplace transform of a continuous-time signal $x(t)$ is:

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}\mathrm{d}t \quad , \quad s \in \mathbb{C} \quad , \quad s = \sigma + j\omega$$

Example #1

$$x_1(t) = e^{-at}u(t)$$
$$X_1(s) =$$

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\end{aligned}
$$

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\end{aligned}$$

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Example #2

$$x_2(t) = -e^{-at}u(-t)$$
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x_2(t) &= -e^{-at}u(-t) \\
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Summary - Examples #1 and #2

$$x_1(t) = e^{-at}u(t)$$

$$X_1(s) = \frac{1}{s+a}$$

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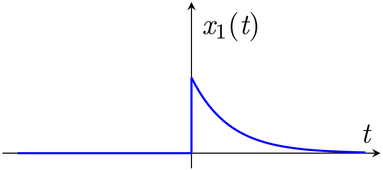

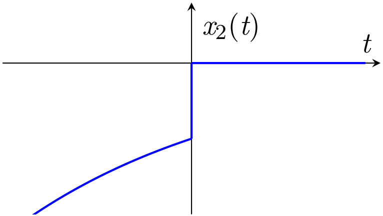

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$$x_1(t) = e^{-at}u(t)$$

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- Note that $X_1(s) = X_2(s)$ while $x_1(t) \neq x_2(t)$.
- The Laplace transform is not unique without specifying where $X(s)$ is defined.

Summary - Examples #1 and #2

$$x_1(t) = e^{-at}u(t)$$
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- Note that $X_1(s) = X_2(s)$ while $x_1(t) \neq x_2(t)$.
- The Laplace transform is not unique without specifying where $X(s)$ is defined.

The Region of Convergence

Definition

The region of the complex plane for which the Laplace transform $X(s)$ is defined is called the **Region Of Convergence (ROC)**.

Remark

The Laplace transform of a continuous-time signal is a pair $(X(s), \mathrm{ROC})$. To answer the question "Find the Laplace transform of the signal $x(t)$", reporting $X(s)$ without specifying the ROC is **an incomplete answer**.

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The Inverse Laplace Transform

The following equation for the inverse Laplace transform is *rarely* used.

Formula

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} \mathrm{d}s$$

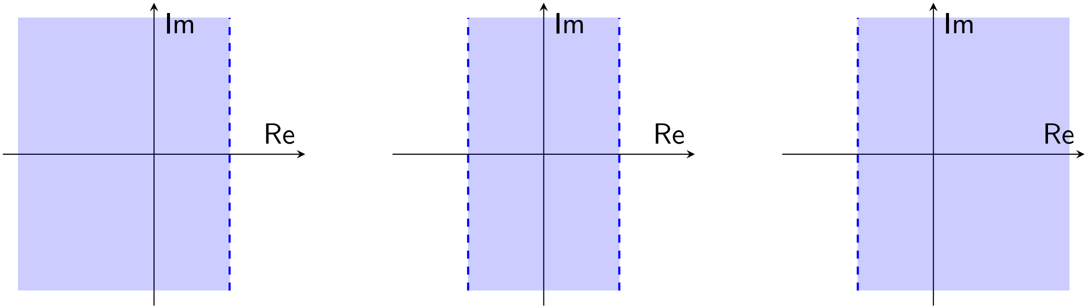
The value of σ can be chosen as any value for which $X(s)$ converges – i.e., any value such that the straight line of integration $\mathbf{Re}(s) = \sigma$ is in the ROC.

Properties of the ROC

Properties of the ROC

Property 1

The ROC of $X(s)$ consists of strips parallel to the $j\omega$-axis in the s-plane.



Properties of the ROC

Property 2

For rational Laplace transforms, the ROC does not contain any poles.

Properties of the ROC

Property 3

If $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire s-plane.

Properties of the ROC

Property 4

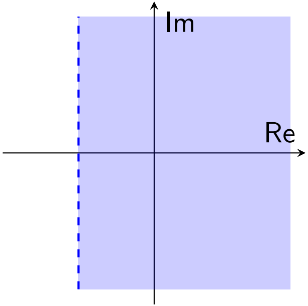
If $x(t)$ is right-sided, and if the line $\mathbf{Re}\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\mathbf{Re}\{s\} > \sigma_0$ will also be in the ROC.

Definition

The signal $x[n]$ or $x(t)$ is called **right-sided** if:

$$\exists n_0 \, , \, \forall n < n_0 \; : \; x[n] = 0$$

$$\exists t_0 \, , \, \forall t < t_0 \; : \; x(t) = 0$$



Properties of the ROC

Property 5

If $x(t)$ is left sided, and if the line $\mathbf{Re}\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\mathbf{Re}\{s\} < \sigma_0$ will also be in the ROC.

Definition

The signal $x[n]$ or $x(t)$ is called **left-sided** if:

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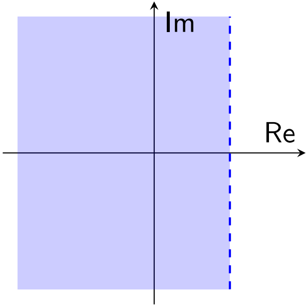
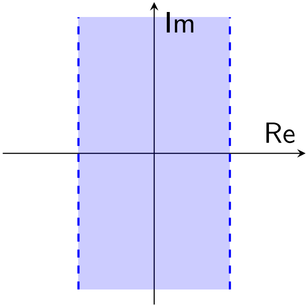


Properties of the ROC

Property 6

If $x(t)$ is two sided, and if the line $\mathbf{Re}\{s\} = \sigma_0$ is in the ROC, then the ROC will consist of a strip in the s-plane that includes the line $\mathbf{Re}\{s\} = \sigma_0$.



Definition

The signal $x[n]$ or $x(t)$ is called **two sided** if it is neither left-sided nor right-sided.

Properties of the ROC

Property 7

If the Laplace transform $X(s)$ of $x(t)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of $X(s)$ are contained in the ROC.

Properties of the ROC

Property 8

If the Laplace transform $X(s)$ of $x(t)$ is rational, then if $x(t)$ is right sided, the ROC is the region in the s-plane to the right of the rightmost pole. If $x(t)$ is left sided, the ROC is the region in the s-plane to the left of the leftmost pole.

Example #3

$$X(s) = \frac{2(s+2)}{s^2+7s+12} \Rightarrow x(t) = ?$$

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$$X(s) = \frac{2(s+2)}{s^2+7s+12} \Rightarrow x(t) =?$$

$$\begin{aligned} X(z) &= \frac{2(s+2)}{(s+3)(s+4)} \\ &= \frac{4}{s+4} - \frac{2}{s+3} \end{aligned}$$

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Do you remember the results from examples #1 and #2?

Example #3

$$X(s) = \frac{2(s+2)}{s^2+7s+12} \Rightarrow x(t) = ?$$

$$X(z) = \frac{2(s+2)}{(s+3)(s+4)} = \frac{4}{s+4} - \frac{2}{s+3}$$

$$\Rightarrow$$

$$x(t) = \begin{cases} 4e^{-4t}u(t) - 2e^{-3t}u(t) \\ 4e^{-4t}u(t) + 2e^{-3t}u(-t) \\ -4e^{-4t}u(-t) + 2e^{-3t}u(-t) \\ -4e^{-4t}u(-t) - 2e^{-3t}u(t) \end{cases}$$

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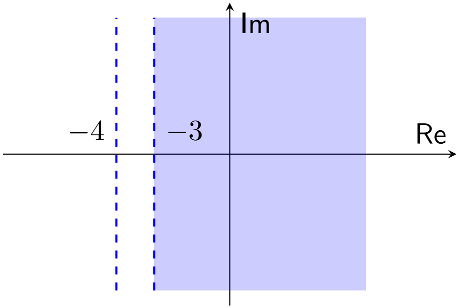

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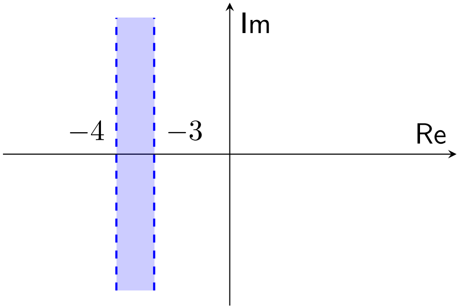

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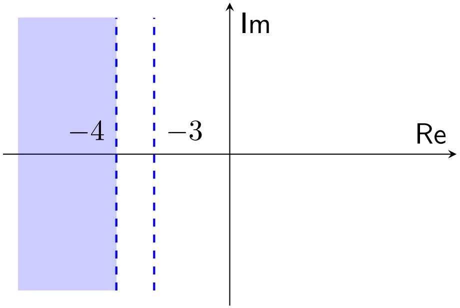
$$X(s) = \frac{2(s+2)}{s^2+7s+12} \Rightarrow x(t) = -4e^{-4t}u(-t) + 2e^{-3t}u(-t)$$



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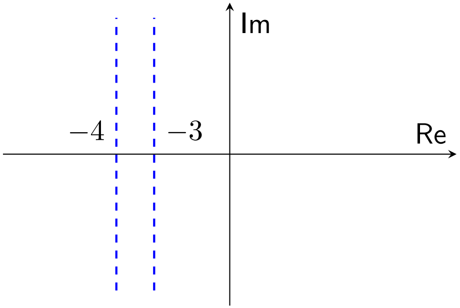


Properties of the Laplace Transform

Linearity

If

$$x_1(t) \xleftrightarrow{\mathcal{L}} X_1(s), \text{ with } \mathrm{ROC} = R_1$$

and

$$x_2(t) \xleftrightarrow{\mathcal{L}} X_2(s), \text{ with } \mathrm{ROC} = R_2$$

then

Property 1

$$ax_1(t) + bx_2(t) \xleftrightarrow{\mathcal{L}} aX_1(s) + bX_2(s)$$

with ROC containing $R_1 \cap R_2$

Time Shifting

If

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \text{ with ROC} = R$$

then

Property 2

$$x(t-t_0) \xleftrightarrow{\mathcal{L}} e^{-st_0}X(s)$$

$$\text{with ROC} = R$$

Shifting in the s-Domain

If

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \text{ with ROC} = R$$

then

Property 3

$$e^{s_0 t} x(t) \xleftrightarrow{\mathcal{L}} X(s - s_0)$$

$$\text{with ROC} = R + \mathbf{Re}\{s_0\}$$

Time Scaling

If

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \text{ with } \mathrm{ROC} = R$$

then

Property 4

$$x(at) \xleftrightarrow{\mathcal{L}} \frac{1}{|a|} X\left(\frac{s}{a}\right)$$

$$\text{with } \mathrm{ROC} = |a| R$$

Conjugation

If

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \text{ with ROC} = R$$

then

Property 5

$$x^*(t) \xleftrightarrow{\mathcal{L}} X^*(s^*)$$

$$\text{with ROC} = R$$

Convolution Property

If

$$x_1(t) \xleftrightarrow{\mathcal{L}} X_1(s), \text{ with } \mathrm{ROC} = R_1$$

and

$$x_2(t) \xleftrightarrow{\mathcal{L}} X_2(s), \text{ with } \mathrm{ROC} = R_2$$

then

Property 6

$$x_1(t) * x_2(t) \xleftrightarrow{\mathcal{L}} X_1(s)X_2(s)$$

with ROC containing $R_1 \cap R_2$

Differentiation in the Time Domain

If

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \text{ with ROC} = R$$

then

Property 7

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} \xleftrightarrow{\mathcal{L}} sX(s)$$

with ROC containing R

Differentiation in the s - Domain

If

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \text{ with ROC} = R$$

then

Property 8

$$-tx(t) \xleftrightarrow{\mathcal{L}} \frac{\mathrm{d}X(s)}{\mathrm{d}s}$$

$$\text{with ROC} = R$$

Integration in the Time Domain

If

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \text{ with ROC} = R$$

then

Property 9

$$\int_{-\infty}^{t} x(\tau)\mathrm{d}\tau \xleftrightarrow{\mathcal{L}} \frac{1}{s}X(s)$$

with ROC containing $R \cap \{\mathbf{Re}\{s\} > 0\}$

The Initial Value Theorem

If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher order singularities at the origin, then

Property 10

$$\lim_{t \to 0^+} x(t) = \lim_{s \to \infty} sX(s)$$

The Final Value Theorem

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \to \infty$, then

Property 10

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

Example #4

Suppose the following facts are given about the signal $x(t)$ with Laplace transform $X(s)$:

- $x(t)$ is real and even.
- $X(s)$ has four poles and no zeros in the finite s-plane.
- $X(s)$ has a pole at $s = \left(\frac{1}{2}\right) e^{j\pi/4}$.
- $\int_{-\infty}^{\infty} x(t)\mathrm{d}t = 4$

Determine $X(s)$ and its ROC.

Example #4

$$p_1 = \left(\frac{1}{2}\right) e^{j\pi/4}$$

$$p_2 = \left(\frac{1}{2}\right) e^{-j\pi/4}$$

$$p_3 = -\left(\frac{1}{2}\right) e^{j\pi/4}$$

$$p_4 = -\left(\frac{1}{2}\right) e^{-j\pi/4}$$

$$X(s) = \frac{K}{(s^2 - \frac{1}{\sqrt{2}}s + \frac{1}{4})(s^2 + \frac{1}{\sqrt{2}}s + \frac{1}{4})}$$

Example #4

$$X(s) = \frac{K}{(s^2 - \frac{1}{\sqrt{2}}s + \frac{1}{4})(s^2 + \frac{1}{\sqrt{2}}s + \frac{1}{4})}$$

$$\int_{-\infty}^{\infty} x(t)\mathrm{d}t = 4$$

$$\Rightarrow X(s) = \frac{\frac{1}{4}}{(s^2 - \frac{1}{\sqrt{2}}s + \frac{1}{4})(s^2 + \frac{1}{\sqrt{2}}s + \frac{1}{4})}$$

$$\text{ROC}: \quad -\frac{1}{2\sqrt{2}} < \mathbf{Re}\{s\} < \frac{1}{2\sqrt{2}}$$

The Laplace Transform and LTI Systems

System Transfer Function

The input-output relationship for an LTI system with impulse response $h(t)$:

$$y(t) = h(t) * x(t)$$

Using the convolution property of the Laplace transform:

$$Y(s) = H(s)X(s)$$

$$H(s) = \frac{Y(s)}{X(s)}$$

Definition

$H(s)$ is called the **transfer function** (or **system function**) of the system.

Note that $H(s)$ is the Laplace transform of the impulse response of the system.

Causality of LTI Systems

Theorem

The ROC associated with the system function for a causal LTI system is a right-half plane.

Note that the converse of this statement is not necessarily true. That is, an ROC to the right of the rightmost pole does not guarantee that a system is causal; rather, it guarantees only that the impulse response is right sided.

$$h(t) = e^{-(t+1)}u(t+1) \Rightarrow H(s) = \frac{e^s}{s+1} \quad , \quad \mathbf{Re}\{s\} > -1$$

Causality of LTI Systems

Theorem

For a system with a rational system function, causality of the system is equivalent to the ROC being the right-half plane to the right of the rightmost pole.

Stability of LTI Systems

Theorem

An LTI system is stable if and only if the ROC of its system function $H(s)$ includes the entire jw-axis [i.e., $\mathbf{Re}\{s\} = 0$].

Example #5

Find the impulse response of the system described by the differential equation

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} - \frac{\mathrm{d}y(t)}{dt} - 2y(t) = x(t)$$

if the system is:

(a) stable.

(b) causal.

(c) non-causal and unstable.

Example #5

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} - \frac{\mathrm{d}y(t)}{dt} - 2y(t) = x(t)$$

$$s^2\, Y(s) - s\, Y(s) - 2\, Y(s) = X(s)$$

Example #5

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} - \frac{\mathrm{d}y(t)}{dt} - 2y(t) = x(t)$$

$$s^2\,Y(s) - s\,Y(s) - 2\,Y(s) = X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 - s - 2}$$

Example #5

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} - \frac{\mathrm{d}y(t)}{dt} - 2y(t) = x(t)$$

$$s^2\, Y(s) - s\, Y(s) - 2\, Y(s) = X(s)$$

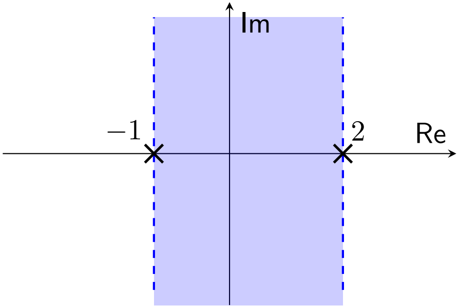
$$\begin{aligned} H(s) = \frac{Y(s)}{X(s)} &= \frac{1}{s^2 - s - 2} \\ &= \frac{1/3}{s-2} - \frac{1/3}{s+1} \end{aligned}$$

Example #5

For a stable system:

$$H(s) = \frac{1/3}{s-2} - \frac{1/3}{s+1}$$

$$h(t) = -\frac{1}{3}e^{2t}u(-t) - \frac{1}{3}e^{-t}u(t)$$



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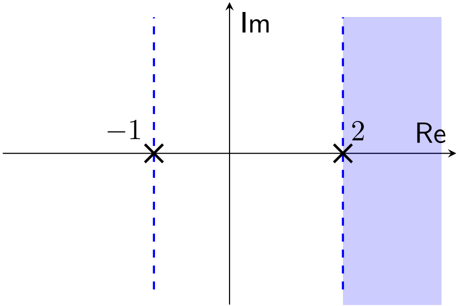

Example #5

For a non-causal and unstable system:

$$H(s) = \frac{1/3}{s-2} - \frac{1/3}{s+1}$$

$$h(t) = -\frac{1}{3}e^{2t}u(-t) + \frac{1}{3}e^{-t}u(-t)$$