Introduction to Signals and Systems

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Signals and Systems
Tutorial Session 1

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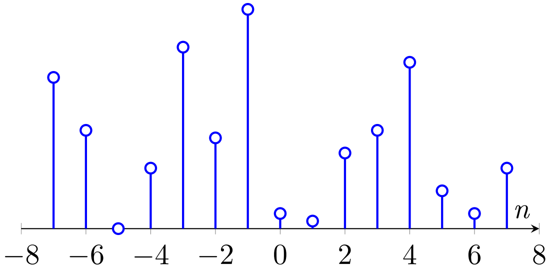
Overview

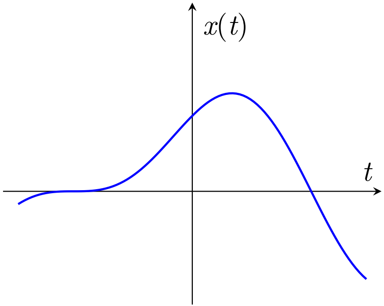
Introduction

Signals

Definition

From a mathematical point of view, a **signal** is simply a function.





Systems

Definition

From a mathematical point of view, a **system** is a function on the space of signals.

If $\mathcal{T}$ is a system, then $\mathcal{T}\{x\} = y$, where x and y are signals.

Each of the two signals x and y can be either discrete or continuous.

Energy of Signals

Definition

The energy of a continuous-time signal $x(t)$ is

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t$$

Definition

The energy of a discrete-time signal $x[n]$ is

$$E_x = \sum_{-\infty}^{\infty} |x[n]|^2$$

Power of Signals

Definition

The power of a continuous-time signal $x(t)$ is

$$P_x = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, \mathrm{d}t$$

Definition

The power of a discrete-time signal $x[n]$ is

$$P_x = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{-N}^{N} |x[n]|^2$$

Power-Type and Energy-Type Signals

Definition

A signal $x(t)$ or $x[n]$ is said to be **energy-type** if

$$E_x < \infty$$

Definition

A signal $x(t)$ or $x[n]$ is said to be **power-type** if

$$0 < P_x < \infty$$

- The power of an energy-type signal is 0.
- The energy of a power-type signal is ∞.
- A signal can be energy-type, power-type, or neither energy-type nor power-type.

System Properties

Memory

Definition

A system is said to be **memoryless** if its output for each value of the independent variable at a given time is dependent only on the input at that same time.

Invertibility

Definition

A system is said to be **invertible** if distinct inputs lead to distinct outputs.

Causality

Definition

A system is **causal** if the output at any time depends only on values of the input at the present time and in the past.

Stability

Definition

A system is said to be **stable** if for every bounded input (i.e., the magnitude of the input does not grow without bound), the output of the system is also bounded.

Time Invariance

Definition

a system is **time invariant** if a time shift in the input signal results in an identical time shift in the output signal.

$$\mathcal{T}\{x[n]\} = y[n] \Rightarrow \mathcal{T}\{x[n-n_0]\} = y[n-n_0]$$

$$\mathcal{T}\{x(t)\} = y(t) \Rightarrow \mathcal{T}\{x(t-t_0)\} = y(t-t_0)$$

Linearity

Definition

A system is said to be **linear** if has the property of superposition, i.e. for every input-output pairs (x_1, y_1) and (x_2, y_2), the output of the system to the signal $ax_1 + bx_2$ is $ay_1 + by_2$.

Linear Time-Invariant Systems – Introduction

The Impulse Response - Convolution

Definition

The response of a system to the impulse input ($\delta[n]$ or $\delta(t)$) is called **the impulse response** of the system.

For a discrete-time system with impulse response $h[n]$, if the input is $x[n]$:

$$x[n] = \cdots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Because of time invariance, the output to $\delta[n-k]$ is $h[n-k]$. Taking linearity into consideration, the input-output relationship is:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \Rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

The Impulse Response

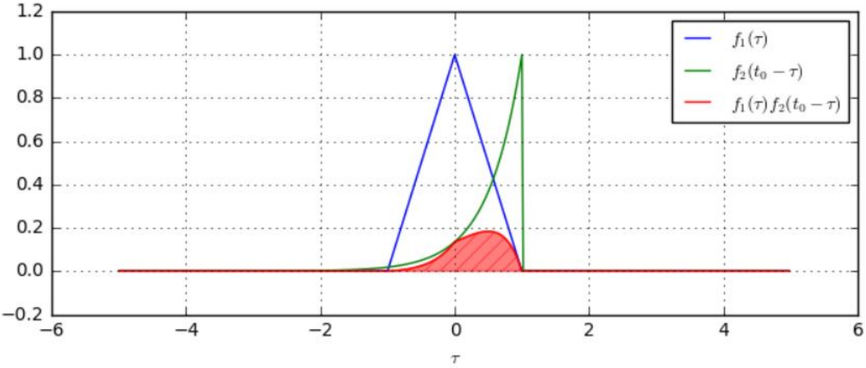
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$
$$= (x * h)[n] \qquad \text{discrete convolution}$$

For continuous-time systems, the results are similar:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau$$
$$= (x * h)(t) \qquad \text{continuous convolution}$$

Convolution - Graphical Intuition

Check this link for animated procedure of continuous-time convolution.



Properties of LTI Systems

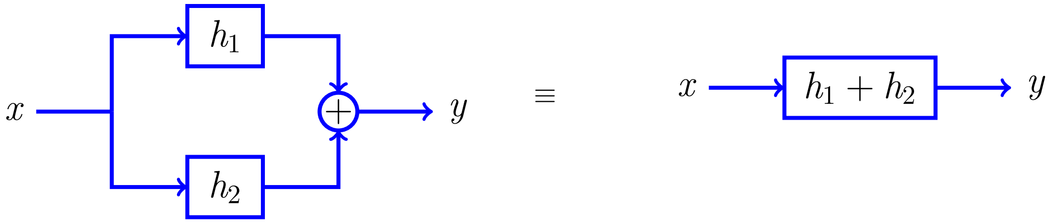
The Commutative Property

$$(x * h)[n] = (h * x)[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$(x * h)(t) = (h * x)(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\mathrm{d}\tau = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau$$

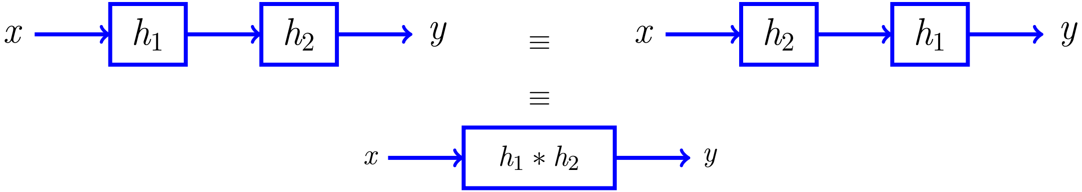
The Distributive Property

$$x * (h_1 + h_2) = x * h_1 + x * h_2$$



The Associative Property

$$x * (h_1 * h_2) = (x * h_1) * h_2$$



Memorylessness

An LTI system is memoryless if and only if $h[n] = 0$ for $n \neq 0$, so the impulse response has the form

$$h[n] = K\delta[n]$$

or

$$h(t) = K\delta(t)$$

As a result (convolution), a memoryless LTI system has the simple input-output relationship

$$y = Kx$$

Invertibility

An LTI system with impulse response h is invertible if there exists a signal h_i such that

$$h * h_i = \delta$$

$$x \longrightarrow \boxed{h} \longrightarrow \boxed{h_i} \longrightarrow y \qquad \equiv \qquad x \longrightarrow \boxed{\begin{matrix}\text{Identity}\\ \text{System}\end{matrix}} \longrightarrow y$$

Causality

An LTI system with impulse response $h[n]$ or $h(t)$ is causal if and only if

$$h[n] = 0 \quad \text{for } n < 0$$

or

$$h(t) = 0 \quad \text{for } t < 0$$

Remark: While causality is a property of systems, it is common terminology to refer to a signal as being causal if it is zero for $n < 0$ or $t < 0$.

Stability

Definition

A discrete-time signal $h[n]$ is said to be **absolutely summable** if

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

Definition

A continuous-time signal $h[n]$ is said to be **absolutely integrable** if

$$\int_{-\infty}^{\infty} |h(\tau)| \mathrm{d}\tau < \infty$$

Stability

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$$\leq \sum_{k=-\infty}^{\infty} |h[k]| \, |x[n-k]|$$

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&\leq B \sum_{k=-\infty}^{\infty} |h[k]| < \infty
\end{aligned}
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If $\sum_{k=-\infty}^{\infty} |h[k]| = \infty$, we show that there exists a bounded input $x[n]$ for which the system output grows unbounded.

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$$x[n] = \begin{cases} 0 & h[-n] = 0 \\ \frac{|h[-n]|}{h[-n]} & h[-n] \neq 0 \end{cases} \quad \Rightarrow \quad |x[n]| \leq 1$$

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$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[-k] = \sum_{k=-\infty}^{\infty} |h[-k]| = \infty$$