Introduction to Addition via Recursion

1 Recursive Definition

For non-negative integers a and b:

$$a + b = \begin{cases} a & \text{if } b = 0 \quad \text{(Base Case)} \\ (a + (b - 1)) + 1 & \text{if } b > 0 \quad \text{(Recursive Case)} \end{cases}$$

2 Expanded Recursion Steps

The recursive case systematically reduces any addition problem to successive simpler cases through these steps:

1. **Initial Problem**: $a + b$ where $b > 0$
2. **Recursive Decomposition**:

   $$a + b = (a + \underbrace{(b - 1)}_{\text{Simpler term}}) + 1$$

   This creates:

   - A simpler subproblem: $a + (b - 1)$
   - A pending operation: $+1$

3. **Iterative Reduction**: Repeat until reaching base case:

   $$\begin{aligned} & a + b \\ & \Downarrow \\ & (a + (b - 1)) + 1 \\ & \Downarrow \\ & ((a + (b - 2)) + 1) + 1 \\ & \Downarrow \\ & \quad \vdots \\ & \Downarrow \\ & (\cdots((a + 0) + 1) + \cdots + 1) \end{aligned}$$

4. **Base Case Resolution**: When $b - n = 0$:

$$\underbrace{(\cdots((a+0)}_{\text{Base case}}\underbrace{+1)+\cdots+1)}_{b \text{ times}}$$

5. **Result Construction**:

$$a + \underbrace{1+1+\cdots+1}_{b \text{ times}} = a + b$$

3 Example

Complete Recursion Example, for $3 + 2$:

$$\begin{aligned}
3 + 2 &= (3+1)+1 && \text{(First decomposition)} \\
&= ((3+0)+1)+1 && \text{(Second decomposition)} \\
&= (3+1)+1 && \text{(Base case applied)} \\
&= 4+1 && \text{(First increment)} \\
&= 5 && \text{(Final result)}
\end{aligned}$$

Recursion Pattern

General form for $a + b$:

$$\begin{aligned}
a + b &= (a + (b-1)) + 1 \\
&= ((a + (b-2)) + 1) + 1 \\
&\vdots \\
&= (\cdots((a+0)+1)+\cdots+1) \\
&= a + \underbrace{1+1+\cdots+1}_{b \text{ times}} \\
&= a + b
\end{aligned}$$