

Introduction to Addition via Recursion

1 Recursive Definition

For non-negative integers a and b :

$$a + b = \begin{cases} a & \text{if } b = 0 \quad (\text{Base Case}) \\ (a + (b - 1)) + 1 & \text{if } b > 0 \quad (\text{Recursive Case}) \end{cases}$$

2 Expanded Recursion Steps

The recursive case systematically reduces any addition problem to successive simpler cases through these steps:

1. **Initial Problem:** $a + b$ where $b > 0$
2. **Recursive Decomposition:**

$$a + b = (a + \underbrace{(b - 1)}_{\text{Simpler term}}) + 1$$

This creates:

- A simpler subproblem: $a + (b - 1)$
- A pending operation: $+1$

3. **Iterative Reduction:** Repeat until reaching base case:

$$\begin{aligned} & a + b \\ & \Downarrow \\ & (a + (b - 1)) + 1 \\ & \Downarrow \\ & ((a + (b - 2)) + 1) + 1 \\ & \Downarrow \\ & \vdots \\ & \Downarrow \\ & (\cdots ((a + 0) + 1) + \cdots + 1) \end{aligned}$$

4. **Base Case Resolution:** When $b - n = 0$:

$$\underbrace{(\cdots ((a + 0) + 1) + \cdots + 1)}_{\text{Base case}} \underbrace{}_{b \text{ times}}$$

5. **Result Construction:**

$$a + \underbrace{1 + 1 + \cdots + 1}_{b \text{ times}} = a + b$$

3 Complete Recursion Example

For $3 + 2$:

$$\begin{aligned} 3 + 2 &= (3 + 1) + 1 && \text{(First decomposition)} \\ &= ((3 + 0) + 1) + 1 && \text{(Second decomposition)} \\ &= (3 + 1) + 1 && \text{(Base case applied)} \\ &= 4 + 1 && \text{(First increment)} \\ &= 5 && \text{(Final result)} \end{aligned}$$

Recursion Pattern

General form for $a + b$:

$$\begin{aligned} a + b &= (a + (b - 1)) + 1 \\ &= ((a + (b - 2)) + 1) + 1 \\ &\vdots \\ &= (\cdots ((a + 0) + 1) + \cdots + 1) \\ &= a + \underbrace{1 + 1 + \cdots + 1}_{b \text{ times}} \\ &= a + b \end{aligned}$$