Proof: The Square Root of 2 is Irrational

Problem

Prove that $\sqrt{2}$ is an irrational number.

Solution 1: Proof by Contradiction

Assume, for contradiction, that $\sqrt{2}$ is rational. This means it can be written as a fraction of two integers in lowest terms:

$$\sqrt{2} = \frac{p}{q}, \quad \text{where } p, q \text{ are integers with } \gcd(p, q) = 1. \tag{1}$$

Squaring both sides:

$$2 = \frac{p^2}{q^2}. \tag{2}$$

Multiplying both sides by q^2:

$$2q^2 = p^2. \tag{3}$$

Since p^2 is divisible by 2, it follows that p must also be divisible by 2 (as squares of odd numbers are odd). So we write:

$$p = 2k \quad \text{for some integer } k. \tag{4}$$

Substituting into the equation:

$$2q^2 = (2k)^2 = 4k^2. \tag{5}$$

Dividing both sides by 2:

$$q^2 = 2k^2. \tag{6}$$

This implies that q^2 is also divisible by 2, so q must also be divisible by 2. This contradicts our original assumption that p and q have no common factors. Therefore, $\sqrt{2}$ is irrational.

Solution 2: Proof Using Prime Factorization

Suppose $\sqrt{2}$ were rational, meaning $\sqrt{2} = \frac{p}{q}$ for some integers p and q with no common factors. Squaring both sides gives:

$$p^2 = 2q^2. \tag{7}$$

In prime factorization, p^2 has an even number of factors, and $2q^2$ has an odd number of factors of 2. This contradiction shows that p^2 cannot equal $2q^2$, proving that $\sqrt{2}$ is irrational.

Solution 3: Proof Using Infinite Descent

Suppose $\sqrt{2}$ is rational, so $\sqrt{2} = \frac{p}{q}$ in lowest terms. Then:

$$p^2 = 2q^2. \tag{8}$$

Since p^2 is even, p is even, so we write $p = 2k$. Substituting:

$$(2k)^2 = 2q^2 \Rightarrow 4k^2 = 2q^2 \Rightarrow q^2 = 2k^2. \tag{9}$$

Thus, q^2 is even, meaning q is also even. This creates an infinite descent of smaller and smaller even numbers, contradicting the assumption that p and q are in lowest terms. Hence, $\sqrt{2}$ is irrational.