Introduction to Multiplication via Recursion

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1 Recursive Definition

For non-negative integers a and b:

$$a \times b = \begin{cases} 0 & \text{if } b = 0 \quad \text{(Base Case)} \\ a + (a \times (b-1)) & \text{if } b > 0 \quad \text{(Recursive Case)} \end{cases}$$

2 Expanded Recursion Steps

The recursive case reduces multiplication to repeated addition through these steps:

1. **Initial Problem**: $a \times b$ where $b > 0$
2. **Recursive Decomposition**:

   $$a \times b = a + \underbrace{(a \times (b-1))}_{\text{Simpler subproblem}}$$

   This creates:

   - A simpler subproblem: $a \times (b-1)$
   - A pending operation: $+a$

3. **Iterative Reduction**: Repeat until reaching the base case:

$$
\begin{array}{l}
a \times b \\
\Downarrow \\
a + (a \times (b-1)) \\
\Downarrow \\
a + (a + (a \times (b-2))) \\
\Downarrow \\
\quad \vdots \\
\Downarrow \\
\underbrace{a + a + \cdots + a}_{b \text{ times}} + (a \times 0)
\end{array}
$$

4. **Base Case Resolution**: When $b - n = 0$:

$$\underbrace{a + a + \cdots + a}_{b \text{ times}} + \underbrace{0}_{\text{Base case}}$$

5. **Result Construction**:

$$\underbrace{a + a + \cdots + a}_{b \text{ times}} = a \times b$$

3 Complete Recursion Example

For 3×2:

$$
\begin{aligned}
3 \times 2 &= 3 + (3 \times 1) && \text{(First decomposition)} \\
&= 3 + (3 + (3 \times 0)) && \text{(Second decomposition)} \\
&= 3 + (3 + 0) && \text{(Base case applied)} \\
&= 3 + 3 && \text{(Simplify)} \\
&= 6 && \text{(Final result)}
\end{aligned}
$$

Recursion Pattern

General form for $a \times b$:

$$
\begin{aligned}
a \times b &= a + (a \times (b-1)) \\
&= a + (a + (a \times (b-2))) \\
&\vdots \\
&= \underbrace{a + a + \cdots + a}_{b \text{ times}} + (a \times 0) \\
&= \underbrace{a + a + \cdots + a}_{b \text{ times}} \\
&= a \times b
\end{aligned}
$$