Proof: The Product of Two Odd Integers is Odd

Problem

Prove that the product of any two odd integers is always odd.

Solution 1: Direct Proof Using Definition

Let a and b be two odd integers. By definition, an odd integer can be written as:

$$a = 2m + 1, \quad b = 2n + 1, \quad \text{where } m, n \text{ are integers.} \tag{1}$$

The product of a and b is:

$$a \cdot b = (2m + 1)(2n + 1). \tag{2}$$

Expanding the product:

$$a \cdot b = 4mn + 2m + 2n + 1. \tag{3}$$

Factoring out 2 from the first three terms:

$$a \cdot b = 2(2mn + m + n) + 1. \tag{4}$$

Since $2mn + m + n$ is an integer, $a \cdot b$ is of the form $2k + 1$, which is odd.

Solution 2: Proof by Contradiction

Assume, for contradiction, that the product of two odd integers is even. That is, assume:

$$a \cdot b = 2k, \quad \text{for some integer } k. \tag{5}$$

Since a and b are odd, we write:

$$a = 2m + 1, \quad b = 2n + 1. \tag{6}$$

Multiplying both values:

$$a \cdot b = (2m + 1)(2n + 1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1. \tag{7}$$

Since $2(2mn + m + n) + 1$ is odd, this contradicts our assumption that $a \cdot b$ is even. Therefore, $a \cdot b$ must be odd.

Solution 3: Proof Using Parity

An integer is odd if it has remainder 1 when divided by 2. That is,

$$a \equiv 1 \pmod{2}, \quad b \equiv 1 \pmod{2}. \tag{8}$$

Multiplying both congruences:

$$a \cdot b \equiv 1 \times 1 \equiv 1 \pmod{2}. \tag{9}$$

Since $a \cdot b \equiv 1 \pmod{2}$, it follows that $a \cdot b$ is odd.