1 Question:

Prove that $\sqrt{2}$ is an irrational number.

2 Answer:

Assume, for contradiction, that $\sqrt{2}$ is rational. This means it can be written as a fraction of two integers in lowest terms:

$$\sqrt{2} = \frac{p}{q}, \quad \text{where } p, q \text{ are integers with } \gcd(p, q) = 1. \tag{1}$$

Squaring both sides:

$$2 = \frac{p^2}{q^2}. \tag{2}$$

Multiplying both sides by q^2:

$$2q^2 = p^2. \tag{3}$$

Since p^2 is divisible by 2, it follows that p must also be divisible by 2 (as squares of odd numbers are odd). So we write:

$$p = 2k \quad \text{for some integer } k. \tag{4}$$

Substituting into the equation:

$$2q^2 = (2k)^2 = 4k^2. \tag{5}$$

Dividing both sides by 2:

$$q^2 = 2k^2. \tag{6}$$

This implies that q^2 is also divisible by 2, so q must also be divisible by 2. This contradicts our original assumption that p and q have no common factors. Therefore, $\sqrt{2}$ is irrational.