Introduction to Mathematical Induction with Examples

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1 What is Mathematical Induction?

Mathematical induction is a powerful proof technique used in mathematics to prove statements that are asserted to be true for all natural numbers. It is especially useful for proving propositions about:

- Summations and series
- Divisibility properties
- Inequalities
- Combinatorial identities

2 The Principle of Mathematical Induction

Theorem 1 (Principle of Mathematical Induction)**.** *To prove that a proposition $P(n)$ is true for all natural numbers $n \geq n_0$, it suffices to:*

1. ***Base Case:*** *Verify $P(n_0)$ is true*
2. ***Inductive Step:*** *Show that if $P(k)$ is true for some arbitrary $k \geq n_0$ (called the induction hypothesis), then $P(k+1)$ must also be true*

3 The Domino Analogy

Mathematical induction works like falling dominos:

- The base case is like knocking over the first domino
- The inductive step ensures each domino will knock over the next one
- Together, these guarantee that all dominos will fall

4 Example: Sum of First n Natural Numbers

Theorem 2. *For all natural numbers $n \geq 1$,*

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Proof. We prove this by mathematical induction.

Base Case ($n = 1$): Left side: 1

Right side: $\frac{1(1+1)}{2} = 1$

Both sides equal. Base case verified.

Inductive Step:

Assume the formula holds for some arbitrary $k \geq 1$ (induction hypothesis):

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$

We must show it holds for $k + 1$:

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \quad \text{(by induction hypothesis)} \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \\ &= \frac{(k+1)((k+1)+1)}{2} \end{aligned}$$

This matches the formula with $n = k + 1$. By induction, the formula holds for all natural numbers $n \geq 1$. □

5 Key Points to Remember

- Always verify both the base case and inductive step
- The induction hypothesis is crucial - you must assume $P(k)$ is true
- Mathematical induction proves statements for *all* natural numbers beyond the base case
- Choose the appropriate base case (n_0) for your proposition

Example 1: Sum of the First n Natural Numbers

Claim: For all $n \in \mathbb{N}$,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Proof: We proceed by mathematical induction.

Base Case: $n = 1$ Left-hand side (LHS): 1 Right-hand side (RHS): $\frac{1(1+1)}{2} = \frac{2}{2} = 1$ LHS = RHS, so the base case holds.

Inductive Step: Assume the formula holds for some $k \in \mathbb{N}$, i.e.,

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

We must show that

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}.$$

Starting from the inductive hypothesis:

$$1+2+\cdots+k+(k+1) = \frac{k(k+1)}{2}+(k+1) = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

Thus, the formula holds for $k+1$.

Conclusion: By induction, the formula holds for all $n \in \mathbb{N}$. □

Example 2: Sum of the First n Odd Numbers

Claim: For all $n \in \mathbb{N}$,

$$1 + 3 + 5 + \cdots + (2n-1) = n^2.$$

Proof:

Base Case: $n = 1$ LHS: 1 RHS: $1^2 = 1$ So the base case holds.

Inductive Step: Assume the formula holds for some $k \in \mathbb{N}$:

$$1 + 3 + \cdots + (2k-1) = k^2.$$

We need to show:

$$1 + 3 + \cdots + (2k-1) + (2k+1) = (k+1)^2.$$

Using the inductive hypothesis:

$$k^2 + (2k+1) = k^2 + 2k + 1 = (k+1)^2.$$

Conclusion: The formula holds for all $n \in \mathbb{N}$ by induction. □

Example 3: Sum of the First n Squares

Claim: For all $n \in \mathbb{N}$,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Proof:
Base Case: $n = 1$ LHS: $1^2 = 1$ RHS: $\frac{1 \cdot 2 \cdot 3}{6} = 1$ So the base case holds.
Inductive Step: Assume the formula holds for $n = k$:

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

We need to prove that:

$$1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}.$$

Start with the inductive hypothesis:

$$\frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}.$$

Factor out $(k+1)$:

$$= \frac{(k+1)\left[k(2k+1) + 6(k+1)\right]}{6} = \frac{(k+1)\left[2k^2 + k + 6k + 6\right]}{6} = \frac{(k+1)(2k^2 + 7k + 6)}{6}.$$

Factor the quadratic:

$$2k^2 + 7k + 6 = (k+2)(2k+3),$$

so the entire expression becomes:

$$\frac{(k+1)(k+2)(2k+3)}{6}.$$

Conclusion: The formula holds for $k+1$, so by induction it is valid for all $n \in \mathbb{N}$. $\square$