

Solution Manual

Collected Problem Solutions

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Part 1

Domain Theoretic Foundations of Functional Programming

Chapter 1

1-Introduction

Chapter 2

2-PCF and its Operational Semantics

Problem 2.1 (page 14)

Show that the σ with $\Gamma \vdash M : \sigma$ is uniquely determined by Γ and M .

Solution

We prove this by induction on the structure.

- if $M \equiv x$ (variable), then it must be by the variable rule: $\Gamma', x : \sigma \vdash x : \sigma$; thus σ must be unique by the definition of the context Γ . ($\Gamma \equiv x_1 : \sigma_1, \dots, x_n : \sigma_n$, where x_i are pairwise distinct variables).
- if $M \equiv Z$ (zero), then it must be derived by the zero rule: $\Gamma \vdash Z : \mathbb{N}$; thus its type is unique.
- if $M \equiv (\lambda x : \sigma. M)$, then it must be derived by the abstraction rule: $\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash (\lambda x : \sigma. M) : \sigma \rightarrow \tau}$. By IH, M and x have unique types τ and σ , respectively. Thus, the type of the abstraction is uniquely determined as $\sigma \rightarrow \tau$.
- if $M \equiv (M(N))$, then by the application rule, we would have $\frac{\Gamma \vdash M : \sigma \rightarrow \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M(N) : \tau}$. By IH, M and N have unique types $\sigma \rightarrow \tau$ and σ , respectively. Thus, the type of the application $M(N)$ is uniquely determined as τ .
- Same goes for the other cases (*succ*, *pred*, Y_σ , and *ifz*).

Problem 2.2 (Page 16, Lemma 2.1.)

The evaluation relation \Downarrow is deterministic, i.e. whenever $M \Downarrow V$ and $M \Downarrow W$ then $V \equiv W$

Solution

We prove this by induction on the structure of the derivation.

Base cases.

- By the rules of the BigStep semantics for PCF, the lemma for the following base cases is trivial:
 - $M \equiv x$, then $x \Downarrow x$. So V and W can only be x ; thus, $V \equiv W \equiv x$.
 - $M \equiv \lambda x : \sigma.M$, then $\lambda x : \sigma.M \Downarrow \lambda x : \sigma.M$.
 - $M \equiv \underline{0}$, then $\underline{0} \Downarrow \underline{0}$.

Inductive Steps.

- If $M \equiv \text{succ}(M)$, then it must be derived by the rule $\frac{M \Downarrow \underline{n}}{\text{succ}(M) \Downarrow \underline{n+1}}$. Then we would have $V \equiv \underline{n+1}$ and $W \equiv \underline{m+1}$ since the successor rule is the only way to derive $\text{succ}(M)$. By IH, we know that $\underline{n} = \underline{m}$, thus $\underline{n+1} = \underline{m+1}$, and hence $V = W$.
- If $M \equiv M(N)$. The derivation for $M(N)$ must be of the form $\frac{M \Downarrow \lambda x : \sigma.E \quad E[N/x] \Downarrow V}{M(N) \Downarrow V}$. A second derivation for $M(N)$ must use the same rule. i.e., $\frac{M \Downarrow \lambda x : \sigma'.E' \quad E'[N/x] \Downarrow W}{M(N) \Downarrow W}$. But then by IH, we would have $\lambda x : \sigma.E \equiv \lambda x : \sigma'.E'$. So $\sigma \equiv \sigma'$ and $E = E'$. Now, we have $E[N/x] \Downarrow V$ and $E[N/x] \Downarrow W$. By the IH on the sub-derivation for $E[N/x]$, we conclude $V \equiv W$.
- If $M \equiv \text{pred}(M)$, then the rules are $\frac{M \Downarrow \underline{0}}{\text{pred}(M) \Downarrow \underline{0}}$ and $\frac{M \Downarrow \underline{n+1}}{\text{pred}(M) \Downarrow \underline{n}}$. For the derivation $\text{pred}(M) \Downarrow V$, we must have a sub-derivation for $M \Downarrow \underline{x}$ for some numeral \underline{x} . Similarly, for $\text{pred}(M) \Downarrow W$, we must have a sub-derivation for $M \Downarrow \underline{y}$ for some numeral \underline{y} . By the IH on the sub-derivation for M , we can conclude that $\underline{x} \equiv \underline{y} \equiv k$.
 Let's examine k . If $k \equiv \underline{0}$, then both derivations must be $\frac{M \Downarrow \underline{0}}{\text{pred}(M) \Downarrow \underline{0}}$. Thus, $V \equiv W \equiv \underline{0}$. Same argument is valid for the case $k \equiv \underline{n+1}$.
- The other cases (Y_σ , and both cases of $\text{if}z$) can be proved likewise.

Problem 2.3 (Page 16, Theorem 2.2, Subject Reduction)

If $\Gamma \vdash M : \sigma$ and $M \Downarrow V$ then $\Gamma \vdash V : \sigma$.

Solution

We prove this by induction on the structure of the derivation of $M \Downarrow V$.

Base cases.

- If $M \equiv x$, then if $\Gamma \vdash x : \sigma$, then the goal is trivial since $x \Downarrow x$.
 Same goes for the other base cases ($\lambda x : \sigma.M$, $\underline{0}$).

Inductive Steps.

- If $M \equiv \text{succ}(M)$, then the only evaluation rule is $\frac{M \Downarrow n}{\text{succ}(M) \Downarrow n+1}$; so $V \equiv n+1$. We are given $\Gamma \vdash \text{succ}(M) : \sigma$ and by the typing rule for *succ*, we have $\Gamma \vdash M : \text{nat}$.
Now, from $M \Downarrow n$ and $\Gamma \vdash M : \text{nat}$, by IH, $\Gamma \vdash n : \text{nat}$; thus, by the typing rule for *succ*, we have $\Gamma \vdash n+1 : \text{nat}$.
- If $M \equiv M(N)$, the evaluation rule is $\frac{M \Downarrow \lambda x : \sigma.E \quad E[N/x] \Downarrow V}{M(N) \Downarrow V}$. We are given $\Gamma \vdash M(N) : \sigma$ and by the typing rule for application, we have $\Gamma \vdash M : \sigma \rightarrow \tau$ and $\Gamma \vdash N : \sigma$. Now, we have $M \Downarrow \lambda x : \sigma.E$ and $\Gamma \vdash M : \sigma \rightarrow \tau$ and by IH, we have $\Gamma \vdash \lambda x : \sigma.E : \sigma \rightarrow \tau$. The typing rule for abstraction is $\frac{\Gamma, x : \sigma \vdash E : \tau}{\Gamma \vdash \lambda x : \sigma.E : \sigma \rightarrow \tau}$, so we have $\Gamma, x : \sigma \vdash E : \tau$. Using the substitution lemma, we can conclude that $\Gamma \vdash E[N/x] : \tau$. Now, from $E[N/x] \Downarrow V$ and $\Gamma \vdash E[N/x] : \tau$, by IH, we have $\Gamma \vdash V : \tau$.
- Same goes for the other cases (both cases of *pred*, Y_σ , and both cases of *ifz*).

Problem 2.4 (Page 17)

$M \Downarrow V$ iff $M \triangleright^* V$.

Solution

In order to prove this, we prove the following two lemmas:

- (a). if $M \Downarrow V$ then $M \triangleright^* V$
- (b). if $M \triangleright N$ then for all values V , if $N \Downarrow V$, then $M \Downarrow V$.

Then, applying (b) iteratively, it follows that:

- (c). if $M \triangleright^* N$ and $N \Downarrow V$, then $M \Downarrow V$.

and by (a) and (c), we can conclude that $M \Downarrow V$ iff $M \triangleright^* V$.

Proof of (a). We prove this by induction on the structure of the derivation of $M \Downarrow V$.

Base cases.

- All the base cases ($M \equiv x$, $M \equiv \lambda x : \sigma.M$, and $M \equiv 0$) are trivial by the reflexivity of \triangleright^* .

Inductive Steps.

- If $M \equiv \text{succ}(M)$, then the rule is $\frac{M \Downarrow n}{\text{succ}(M) \Downarrow n+1}$; thus, $V \equiv n+1$. By IH on $M \Downarrow n$, we have $M \triangleright^* n$. Now, there are two cases.
 - If $M \equiv n$, then $\text{succ}(M) \equiv \text{succ}(n) \equiv n+1$. So $M \triangleright^* V$.
 - If $M \triangleright M_1 \triangleright \dots \triangleright n$, then by the congruence rule for *succ*, we would have $\text{succ}(M) \triangleright \text{succ}(M_1) \triangleright \dots \triangleright \text{succ}(n) \equiv n+1$. So, $\text{succ}(M) \triangleright^* n+1 \equiv V$.
- If $M \equiv M(N)$, then the rule is $\frac{M \Downarrow \lambda x : \sigma.E \quad E[N/x] \Downarrow V}{M(N) \Downarrow V}$. Then, by IH on $M \Downarrow \lambda x : \sigma.E$, we have $M \triangleright^* \lambda x : \sigma.E$, and on $E[N/x] \Downarrow V$, we have $E[N/x] \triangleright^* V$.

If $M \equiv \lambda x : \sigma.E$, then $M(N) \equiv (\lambda x : \sigma.E)(N)$. By small-step rule $(\lambda x : \sigma.E)(N) \triangleright E[N/x]$. Since $E[N/x] \triangleright^* V$, we have $M(N) \triangleright E[N/x] \triangleright^* V$, so $M(N) \triangleright^* V$.

If $M \triangleright M_1 \triangleright \dots \triangleright \lambda x : \sigma.E$, then by the congruence rule for application $M(N) \triangleright M_1(N) \triangleright \dots \triangleright (\lambda x : \sigma.E)(N)$. Then $(\lambda x : \sigma.E)(N) \triangleright E[N/x]$, and $E[N/x] \triangleright^* V$. Then we would have: $M(N) \triangleright^* (\lambda x : \sigma.E)(N) \triangleright E[N/x] \triangleright^* V$. So $M(N) \triangleright^* V$.

- The other cases are fairly similar to these.

Proof of (b). We prove this by induction on the structure of the derivation of $M \triangleright V$.

Base cases.

- Assume $M \equiv (\lambda x : \sigma.E)(A)$ and $N \equiv E[A/x]$ ($M \triangleright N$). Assume also $N \Downarrow V$, i.e., $E[A/x] \Downarrow V$. We know $\lambda x : \sigma.E \Downarrow \lambda x : \sigma.E$. By the rule: $\frac{\lambda x : \sigma.E \Downarrow \lambda x : \sigma.E \quad E[A/x] \Downarrow V}{(\lambda x : \sigma.E)(A) \Downarrow V}$. So $M \Downarrow V$.
- Assume $M \equiv Y_\sigma(E)$ and $N \equiv E(Y_\sigma(E))$. Assume also $N \Downarrow V$, i.e., $E(Y_\sigma(E)) \Downarrow V$. By the fixpoint rule: $\frac{E(Y_\sigma(E)) \Downarrow V}{Y_\sigma(E) \Downarrow V}$. So $M \Downarrow V$.
- Assume $M \equiv \text{pred}(\underline{0})$ and $N \equiv \underline{0}$. Assume also $N \Downarrow V$, i.e., $\underline{0} \Downarrow V$. This means $V \equiv \underline{0}$. We need to show $\text{pred}(\underline{0}) \Downarrow \underline{0}$. This holds by the rule $\frac{\underline{0} \Downarrow \underline{0}}{\text{pred}(\underline{0}) \Downarrow \underline{0}}$.
- Assume $M \equiv \text{pred}(\underline{k+1})$ and $N \equiv \underline{k}$. Assume also $N \Downarrow V$, i.e., $\underline{k} \Downarrow V$. This means $V \equiv \underline{k}$. We need to show $\text{pred}(\underline{k+1}) \Downarrow \underline{k}$. This holds by the rule $\frac{\underline{k+1} \Downarrow \underline{k+1} \quad \underline{k} \Downarrow V}{\text{pred}(\underline{k+1}) \Downarrow \underline{k}}$.
- Assume $M \equiv \text{ifz}(\underline{0}, E_1, E_2)$ and $N \equiv E_1$. Assume also $N \Downarrow V$, i.e., $E_1 \Downarrow V$. We need to show $\text{ifz}(\underline{0}, E_1, E_2) \Downarrow V$. This holds by the rule $\frac{\underline{0} \Downarrow \underline{0} \quad E_1 \Downarrow V}{\text{ifz}(\underline{0}, E_1, E_2) \Downarrow V}$.
- Assume $M \equiv \text{ifz}(\underline{k+1}, E_1, E_2)$ and $N \equiv E_2$. Assume also $N \Downarrow V$, i.e., $E_2 \Downarrow V$. We need to show $\text{ifz}(\underline{k+1}, E_1, E_2) \Downarrow V$. This holds by the rule $\frac{\underline{k+1} \Downarrow \underline{k+1} \quad E_2 \Downarrow V}{\text{ifz}(\underline{k+1}, E_1, E_2) \Downarrow V}$.

Inductive Steps.

- $\frac{M_1 \triangleright M_2}{\text{succ}(M_1) \triangleright \text{succ}(M_2)}$: $M = \text{succ}(M_1)$, $N = \text{succ}(M_2)$, where $M_1 \triangleright M_2$. Assume $N \Downarrow V$, i.e., $\text{succ}(M_2) \Downarrow V$. This implies $V \equiv \underline{k+1}$ and $M_2 \Downarrow \underline{k}$ for some k . By IH on the sub-derivation $M_1 \triangleright M_2$: since $M_2 \Downarrow \underline{k}$, it follows that $M_1 \Downarrow \underline{k}$. Then, by the rule $\frac{M_1 \Downarrow \underline{k}}{\text{succ}(M_1) \Downarrow \underline{k+1}}$. So $M \equiv \text{succ}(M_1) \Downarrow \underline{k+1}$. Since $V \equiv \underline{k+1}$, we have $M \Downarrow V$.
- $\frac{M_1 \triangleright M_2}{M_1(A) \triangleright M_2(A)}$: $M = M_1(A)$, $N = M_2(A)$. Assume $M_2(A) \Downarrow V$. This means $M_2 \Downarrow \lambda x.E$ and $E[A/x] \Downarrow V$. By IH on $M_1 \triangleright M_2$, since $M_2 \Downarrow \lambda x.E$, then $M_1 \Downarrow \lambda x.E$. Thus, by the rule, $M_1(A) \Downarrow V$.
- $\frac{M_1 \triangleright M_2}{\text{pred}(M_1) \triangleright \text{pred}(M_2)}$: $M = \text{pred}(M_1)$, $N = \text{pred}(M_2)$. Assume $\text{pred}(M_2) \Downarrow V$. This means $M_2 \Downarrow \underline{k}$ and V is $\underline{0}$ (if $k = 0$) or $\underline{k-1}$ (if $k > 0$). By IH on $M_1 \triangleright M_2$, since $M_2 \Downarrow \underline{k}$, then $M_1 \Downarrow \underline{k}$. Thus, by the rule for pred , $\text{pred}(M_1) \Downarrow V$.
- $\frac{M_1 \triangleright M_2}{\text{ifz}(M_1, N_1, N_2) \triangleright \text{ifz}(M_2, N_1, N_2)}$: $M = \text{ifz}(M_1, N_1, N_2)$, $N = \text{ifz}(M_2, N_1, N_2)$. Assume $\text{ifz}(M_2, N_1, N_2) \Downarrow V$. This means $M_2 \Downarrow \underline{k}$ and either $N_1 \Downarrow V$ (if $k = 0$) or $N_2 \Downarrow V$ (if $k > 0$). By IH on $M_1 \triangleright M_2$, since $M_2 \Downarrow \underline{k}$, then $M_1 \Downarrow \underline{k}$. Thus, by the rule for ifz , $\text{ifz}(M_1, N_1, N_2) \Downarrow V$.

Problem 2.5 (Page 19)

Show that the applicative relation \sqsubseteq_{σ} is a preorder on Prg_{σ} , i.e. that \sqsubseteq_{σ} is reflexive and transitive.

Solution

-Reflexivity. We need to show that for any closed PCF term M of type σ , $M \sqsubseteq_{\sigma} M$.

Base Case. For $M \in \text{Prg}_{\text{nat}}$, $M \sqsubseteq_{\text{nat}} M$ means that $\forall n \in \mathbb{N}, M \Downarrow n \Rightarrow M \Downarrow n$. This is trivially true.

Inductive Case. For $M \in \text{Prg}_{\sigma \rightarrow \tau}$, $M \sqsubseteq_{\sigma \rightarrow \tau} M$ means that $\forall P \in \text{Prg}_{\sigma}, M(P) \sqsubseteq_{\tau} M(P)$, which holds by IH.

-Transitivity. We need to show that for any closed PCF terms M, N, K of type σ , if $M \sqsubseteq_{\sigma} N$ and $N \sqsubseteq_{\sigma} K$, then $M \sqsubseteq_{\sigma} K$.

Base Case. For $M, N, K \in \text{Prg}_{\text{nat}}$, assume $M \sqsubseteq_{\text{nat}} N$ and $N \sqsubseteq_{\text{nat}} K$. Then, by definition, we have the followings:

- $\forall n \in \mathbb{N}, M \Downarrow n \Rightarrow N \Downarrow n$
- $\forall n \in \mathbb{N}, N \Downarrow n \Rightarrow K \Downarrow n$

Thus, if $M \Downarrow n$ then $K \Downarrow n$, which means $M \sqsubseteq_{\text{nat}} K$.

Inductive Case. For $M, N, K \in \text{Prg}_{\sigma \rightarrow \tau}$, assume $M \sqsubseteq_{\sigma \rightarrow \tau} N$ and $N \sqsubseteq_{\sigma \rightarrow \tau} K$. Then, by definition, we have the followings:

- $\forall P \in \text{Prg}_{\sigma}, M(P) \sqsubseteq_{\tau} N(P)$
- $\forall P \in \text{Prg}_{\sigma}, N(P) \sqsubseteq_{\tau} K(P)$

Thus, we would have $\forall P \in \text{Prg}_{\sigma}, M(P) \sqsubseteq_{\tau} K(P)$.

Chapter 3

3-The Scott Model of PCF

Problem 3.1 (Page 26)

Show that (Scott) continuous functions between predomains are always monotonic.

Solution

Let $f : (A, \sqsubseteq_A) \rightarrow (B, \sqsubseteq_B)$ be a Scott continuous function between predomains. For any $x, y \in A$ with $x \leq y$, we have that $X = \{x, y\}$ is a directed subset of A . The supremum of the set X is obviously y . Thus $\sqcup X = y$, and since f is continuous, $f(\sqcup X) = f(y) = \sqcup f(\{x, y\})$. Hence, $f(x) \leq f(y)$.

Problem 3.2 (Page 26, Theorem 3.3)

Let $(A_i | i \in I)$ be a family of predomains. Then their product $\prod_{i \in I} A_i$ is a predomain under the componentwise ordering, and the projections $\pi_i : \prod_{i \in I} A_i \rightarrow A_i$ are Scott continuous. If, moreover, all A_i are domains then so is their product $\prod_{i \in I} A_i$.

Solution

Let $D = \prod_{i \in I} A_i = \{f : I \rightarrow \bigcup_{i \in I} A_i \mid \forall i \in I, f(i) \in A_i\}$. We need to show that (D, \sqsubseteq_D) is a poset, and every directed subset of D has a least upper bound. Note that the order \sqsubseteq_D is defined as follows:

$$(d_i)_{i \in I} \sqsubseteq_D (d'_i)_{i \in I} \quad \text{iff} \quad \forall i \in I, d(i) \sqsubseteq_{A_i} d'(i)$$

We now show that (D, \sqsubseteq_D) forms a poset.

- **Reflexivity:** For any $(d_i)_{i \in I} \in D$, $d_i \sqsubseteq_{A_i} d_i, \forall i \in I$ since each A_i is a poset. Thus $(d_i)_{i \in I} \sqsubseteq_D (d_i)_{i \in I}$.
- **Transitivity:** Assume $(d_i)_{i \in I} \sqsubseteq_D (d'_i)_{i \in I}$ and $(d'_i)_{i \in I} \sqsubseteq_D (d''_i)_{i \in I}$. And by each A_i being

transitive, it follows immediately that $d_i \sqsubseteq_{A_i} d_i''$ for all $i \in I$. Therefore, $(d_i)_{i \in I} \sqsubseteq_D (d_i'')_{i \in I}$.

- **Antisymmetry:** Similar to the previous case, it follows immediately from the fact that each A_i is antisymmetric.

Now, suppose that $X \subseteq D = \prod_{i \in I} A_i$ is a directed subset. define $X_i = \{\pi_i(x) | x \in X\}$, that is, the projection of X to A_i . X_i is directed since X is directed. Moreover, X_i has a least upper bound $\bigsqcup X_i \in A_i$. Define $z \in D$ with $z_i = \bigsqcup X_i$ for each $i \in I$. By construction, it is obvious that z is the least upper bound of X in D . Thus, D is a predomain.

Problem 3.3 (Page 27)

Prove that the evaluation map $ev : [A_1 \rightarrow A_2] \times A_1 \rightarrow A_2$ with $ev(f, a) = f(a)$ is continuous in each argument.

Solution

For the first argument, fix $a \in A_1$ and let $F \subseteq [A_1 \rightarrow A_2]$ be a directed set of continuous functions. By Theorem 3.5, we have $\bigsqcup F(a) = g(a) = \bigsqcup_{f \in F} f(a)$. Thus, $ev(\bigsqcup F, a) = g(a) = \bigsqcup_{f \in F} f(a) = \bigsqcup \{ev(f, a) | f \in F\}$.

Now, for the second argument, fix $f \in [A_1 \rightarrow A_2]$ and let $X \subseteq A_1$ be a directed set. Because f is continuous, we have $f(\bigsqcup X) = \bigsqcup \{f(x) | x \in X\}$. Thus,

$$ev(f, \bigsqcup X) = f(\bigsqcup X) = \bigsqcup \{f(x) | x \in X\} = \bigsqcup \{ev(f, x) | x \in X\}.$$

Hence, ev is continuous in both arguments and by Lemma 3.4, it is jointly continuous.

Problem 3.4 (Page 30)

Prove that $\Psi : [[D \rightarrow D] \rightarrow D] \times [D \rightarrow D] \rightarrow D : (F, f) \mapsto f(F(f))$ is continuous in each argument.

Solution

For the first argument, fix $f \in [D \rightarrow D]$ and let $\mathcal{F} \subseteq [[D \rightarrow D] \rightarrow D]$ be a directed set. We know that $(\bigsqcup \mathcal{F})(f) = \bigsqcup \{F(f) | F \in \mathcal{F}\}$. Thus,

$$\underbrace{f((\bigsqcup \mathcal{F})(f))}_{\Psi(\bigsqcup \mathcal{F}, f)} = f(\bigsqcup \{F(f) | F \in \mathcal{F}\}) = \bigsqcup \{f(F(f)) | F \in \mathcal{F}\}$$

For the second argument, fix $F \in [[D \rightarrow D] \rightarrow D]$ and let $X \subseteq [D \rightarrow D]$ be a directed set. We

know that $F(\sqcup X) = \sqcup \{F(f) \mid f \in X\}$. Thus,

$$f(F(\sqcup X)) = f(\sqcup \{F(f) \mid f \in X\}) = \sqcup \{f(F(f)) \mid f \in X\}$$

Hence, Ψ is continuous in both arguments and by Lemma 3.4, it is jointly continuous.

Problem 3.5 (Page 33)

(β -equality). If, $\Gamma, x : \sigma \vdash M : \tau$ and $\Gamma \vdash N : \sigma$ then

$$\llbracket \Gamma \vdash (\lambda x : \sigma. M)(N) \rrbracket = \llbracket \Gamma \vdash M[N/x] \rrbracket$$

Solution

$\llbracket \Gamma \vdash M[N/x] \rrbracket(\vec{d}) = \llbracket \Gamma, x : \sigma \vdash M \rrbracket(\vec{d}, \llbracket \Gamma \vdash N \rrbracket(\vec{d}))$ by Lemma 3.15.

Now, for the other side,

$$\begin{aligned} \llbracket \Gamma \vdash (\lambda x : \sigma. M)(N) \rrbracket(\vec{d}) &= ev(\llbracket \Gamma \vdash \lambda x : \sigma. M \rrbracket(\vec{d}), \llbracket \Gamma \vdash N \rrbracket(\vec{d})) \\ &= \llbracket \Gamma \vdash \lambda x : \sigma. M \rrbracket(\vec{d})(\llbracket \Gamma \vdash N \rrbracket(\vec{d})) = \llbracket \Gamma, x : \sigma \vdash M \rrbracket(\vec{d}, \llbracket \Gamma \vdash N \rrbracket(\vec{d})) \end{aligned}$$

Problem 3.6 (Page 34)

(η -equality). If, $\Gamma \vdash M : \sigma \rightarrow \tau$ then

$$\llbracket \Gamma \vdash \lambda x : \sigma. M(x) \rrbracket = \llbracket \Gamma \vdash M \rrbracket$$

for $x \notin \text{Var}(\Gamma)$.

Solution

$$\begin{aligned} \forall \vec{d} \in \llbracket \Gamma \rrbracket, d' \in D_\sigma, \llbracket \Gamma \vdash \lambda x : \sigma. M(x) \rrbracket(\vec{d})(d') &= \llbracket \Gamma, x : \sigma \vdash M(x) \rrbracket(\vec{d}, d') \\ &= ev(\llbracket \Gamma, x : \sigma \vdash M \rrbracket(\vec{d}, d'), \underbrace{\llbracket \Gamma, x : \sigma \vdash x \rrbracket(\vec{d}, d')}_{\pi_x(\vec{d}, d')=d'}) =^* \llbracket \Gamma \vdash M \rrbracket(\vec{d})(d') \end{aligned}$$

For the last equation (*), because $x \notin \text{Var}(\Gamma)$ and $\Gamma \vdash M$, we have

$$\llbracket \Gamma, x : \sigma \vdash M \rrbracket(\vec{d}, d') = \llbracket \Gamma \vdash M \rrbracket(\vec{d})$$

.

Chapter 4

4-Computational Adequacy

Chapter 5

5-Milner's Context Lemma

Problem 5.1 (Page 44)

Prove that \leq_σ is closed under suprema of directed sets. That is, if $X \subseteq D_\sigma \times D_\sigma$ is directed and $X \subseteq \leq_\sigma$, meaning that for every $(x, y) \in X, x \leq_\sigma y$, then $\bigsqcup X \in \leq_\sigma$.

Solution

Let $X = \{(x_k, y_k) | k \in K\}$ be a directed subset of $D_\sigma \times D_\sigma$ such that for all $k \in K, (x_k, y_k) \in \leq_\sigma$. This means for each $k \in K$, we have $\forall P \in \text{Prg}_\sigma, y_k R_\sigma P \implies x_k R_\sigma P(\star)$.

Let $x = \bigsqcup_{k \in K} x_k$ and $y = \bigsqcup_{k \in K} y_k$ (Note that $\bigsqcup X = (x, y)$). We need to show $x \leq_\sigma y$. That is, $\forall P \in \text{Prg}_\sigma, y R_\sigma P \implies x R_\sigma P$.

Let $P \in \text{Prg}_\sigma$ be an arbitrary closed PCF term and that $y R_\sigma P$. Now, for each $k \in K$, we have $y_k \sqsubseteq y$ and by Lemma 4.2(1), we get $y_k R_\sigma P$ for all $k \in K$. Using (\star) , we have $x_k R_\sigma P$ for each $k \in K$.

By Lemma 4.2(2), $R_\sigma P$ is closed under directed suprema and $\{x_k | k \in K\}$ is a directed subset of D_σ whose elements are all in $R_\sigma P$. So, their suprema x must also be in $R_\sigma P$, meaning that $x R_\sigma P$.

Chapter 6

6-The Full Abstraction Problem

Chapter 7

7-Logical Relations

Problem 7.1 (Page 52)

(Theorem 7.2 When M is Variable) Let R be a logical relation of arity W on the Scott model of PCF. Then for λ -terms $x_1 : \sigma_1, \dots, x_n : \sigma_n \vdash x_k : \sigma_k$ for some k and $d_j \in R_{\sigma_j}$ for $j = 1, \dots, n$ it holds that

$$\underline{\lambda}i \in W. \llbracket x_1 : \sigma_1, \dots, x_n : \sigma_n \vdash x_k \rrbracket (\vec{d}(i)) \in R_{\sigma_k}$$

Solution

By premise, $d_k \in R_{\sigma_k}$. By definition, the goal reduces to $d_k \in R_{\sigma_k}$!

Part 2

Type Theory and Formal Proof

Chapter 8

1-Untyped Lambda Calculus

Problem 8.1 (1.1)

Apply Notation 1.3.10 on the following λ -terms. SO, remove parentheses and combine *lambda*-abstractions:

1. $(\lambda x.(((xz)y)(xx)))$
2. $((\lambda x.(\lambda y.(\lambda z.(z((xy)z)))))(\lambda u.u))$

Solution

1. $(\lambda x.(((xz)y)(xx))) = \lambda x.((xz)y)(xx) = \lambda x.(xz)y(xx) = \lambda x.(xz)y(xx) = \lambda x.xzy(xx)$
2. Final answer: $\lambda xyz.z((xy)z)(\lambda u.u)$

Problem 8.2 (1.2)