L2: Review of probability and statistics

Probability
- Definition of probability
- Axioms and properties
- Conditional probability
- Bayes theorem

Random variables
- Definition of a random variable
- Cumulative distribution function
- Probability density function
- Statistical characterization of random variables

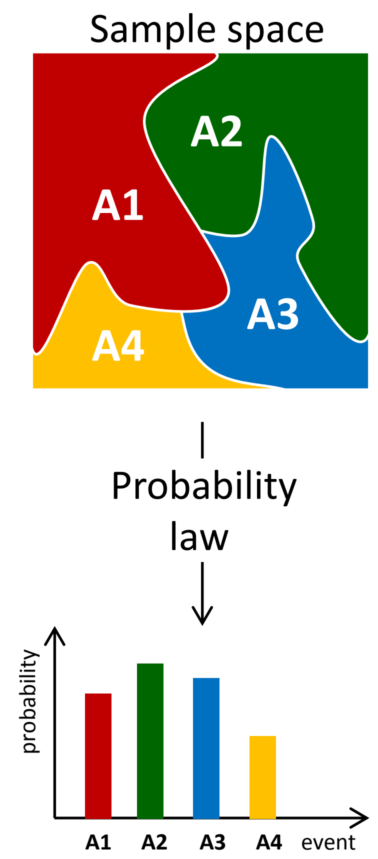
Random vectors
- Mean vector
- Covariance matrix

The Gaussian random variable

Review of probability theory

Definitions (informal)

- Probabilities are numbers assigned to events that indicate "***how likely***" it is that the event will occur when a random experiment is performed
- A probability law for a random experiment is a rule that assigns probabilities to the events in the experiment
- The sample space S of a random experiment is the set of all possible outcomes

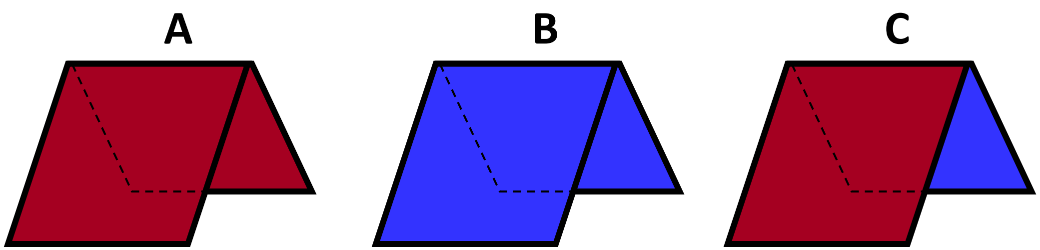


Axioms of probability

- Axiom I: $P[A_i] \geq 0$
- Axiom II: $P[S] = 1$
- Axiom III: $A_i \cap A_j = \emptyset \Rightarrow P[A_i \cup A_j] = P[A_i] + P[A_j]$

Warm-up exercise

– I show you three colored cards
  - One BLUE on both sides
  - One RED on both sides
  - One BLUE on one side, RED on the other



– I shuffle the three cards, then pick one and show you one side only. The side visible to you is RED
  - Obviously, the card has to be either A or C, *right*?

– I am willing to bet $1 that the other side of the card has the same color, and need someone in class to bet another $1 that it is the other color
  - On the average we will end up even, *right*?
  - Let's try it!

More properties of probability

- $P[A^C] = 1 - P[A]$
- $P[A] \leq 1$
- $P[\emptyset] = 0$
- $given\ \{A_1 \dots A_N\}, \{A_i \cap A_j = \emptyset, \forall ij\} \Rightarrow P\left[\bigcup_{k=1}^{N} A_k\right] = \sum_{k=1}^{N} P[A_k]$
- $P[A_1 \cup A_2] = P[A_1] + P[A_2] - P[A_1 \cap A_2]$
- $P\left[\bigcup_{k=1}^{N} A_k\right] =$
  $\sum_{k=1}^{N} P[A_k] - \sum_{j<k}^{N} P\left[A_j \cap A_k\right] + \cdots + (-1)^{N+1} P[A_1 \cap A_2 \dots \cap A_N]$
- $A_1 \subset A_2 \Rightarrow P[A_1] \leq P[A_2]$

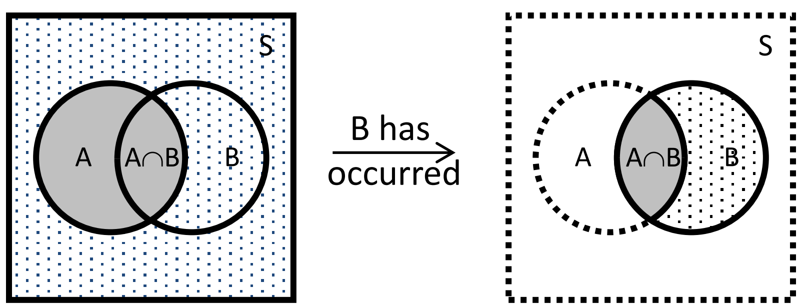
Conditional probability

– If A and B are two events, the probability of event A when we already know that event B has occurred is

$$P[A|B] = \frac{P[A \cap B]}{P[B]} \quad if \;\; P[B] > 0$$

- This conditional probability P[A|B] is read:
  – the "conditional probability of A conditioned on B", or simply
  – the "probability of A given B"

– Interpretation
- The new evidence "*B has occurred*" has the following effects
  - The original sample space S (the square) becomes B (the rightmost circle)
  - The event A becomes A∩B
- P[B] simply re-normalizes the probability of events that occur jointly with B



Theorem of total probability

- Let $B_1, B_2 \dots B_N$ be a partition of S (mutually exclusive that add to S)
- Any event A can be represented as

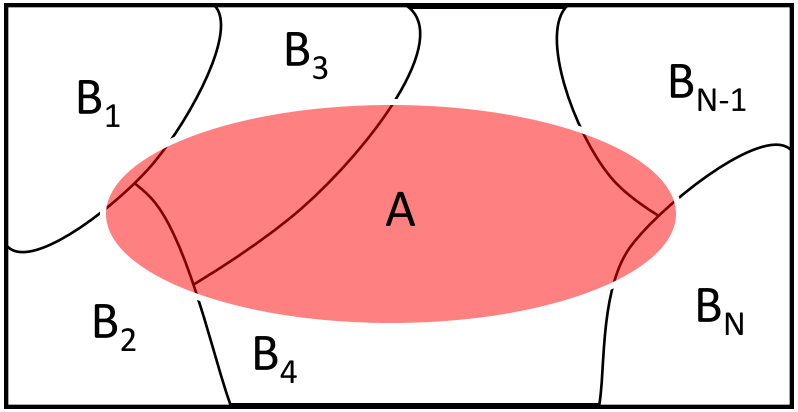
$$A = A \cap S = A \cap (B_1 \cup B_2 \dots B_N) = (A \cap B_1) \cup (A \cap B_2) \dots (A \cap B_N)$$

- Since $B_1, B_2 \dots B_N$ are mutually exclusive, then

$$P[A] = P[A \cap B_1] + P[A \cap B_2] + \cdots + P[A \cap B_N]$$

- and, therefore

$$P[A] = P[A|B_1]P[B_1] + \cdots P[A|B_N]P[B_N] = \sum_{k=1}^{N} P[A|B_k]P[B_k]$$



Bayes theorem

- Assume $\{B_1, B_2 \ldots B_N\}$ is a partition of S
- Suppose that event A occurs
- What is the probability of event B_j?

- Using the definition of conditional probability and the Theorem of total probability we obtain

$$P[B_j|A] = \frac{P[A \cap B_j]}{P[A]} = \frac{P[A|B_j]P[B_j]}{\sum_{k=1}^{N} P[A|B_k]P[B_k]}$$

- This is known as Bayes Theorem or Bayes Rule, and is (one of) the most useful relations in probability and statistics

Bayes theorem and statistical pattern recognition

– When used for pattern classification, BT is generally expressed as

$$P[\omega_j|x] = \frac{p[x|\omega_j]P[\omega_j]}{\sum_{k=1}^{N} p[x|\omega_k]P[\omega_k]} = \frac{p[x|\omega_j]P[\omega_j]}{p[x]}$$

  - where ω_j is the j-th class (e.g., phoneme) and x is the feature/observation vector (e.g., vector of MFCCs)

– A typical decision rule is to choose class ω_j with highest $\mathrm{P}[\omega_j|x]$

  - Intuitively, we choose the class that is more "likely" given observation x

– Each term in the Bayes Theorem has a special name

  - $P[\omega_j]$ <u>prior</u> probability (of class ω_j)
  - $P[\omega_j|x]$ <u>posterior</u> probability (of class ω_j given the observation x)
  - $p[x|\omega_j]$ <u>likelihood</u> (probability of observation x given class ω_j)
  - $p[x]$ normalization constant (does not affect the decision)

Example

- Consider a clinical problem where we need to decide if a patient has a particular medical condition on the basis of an imperfect test
  - Someone with the condition may go undetected (false-negative)
  - Someone free of the condition may yield a positive result (false-positive)
- Nomenclature
  - The true-negative rate $P(NEG|\neg COND)$ of a test is called its SPECIFICITY
  - The true-positive rate $P(POS|COND)$ of a test is called its SENSITIVITY
- Problem
  - Assume a population of 10,000 with a 1% prevalence for the condition
  - Assume that we design a test with 98% specificity and 90% sensitivity
  - Assume you take the test, and the result comes out POSITIVE
  - What is the probability that you have the condition?
- Solution
  - Fill in the joint frequency table next slide, or
  - Apply Bayes rule

	TEST IS POSITIVE	TEST IS NEGATIVE	ROW TOTAL
HAS CONDITION	*True-positive* $P(POS \mid COND)$	*False-negative* $P(NEG \mid COND)$	
FREE OF CONDITION	*False-positive* $P(POS \mid \neg COND)$	*True-negative* $P(NEG \mid \neg COND)$	
COLUMN TOTAL			

	TEST IS POSITIVE	TEST IS NEGATIVE	ROW TOTAL
HAS CONDITION	*True-positive* *P(POS|COND)* **100×0.90**	*False-negative* *P(NEG|COND)* **100×(1-0.90)**	**100**
FREE OF CONDITION	*False-positive* *P(POS|¬COND)* **9,900×(1-0.98)**	*True-negative* *P(NEG|¬COND)* **9,900×0.98**	**9,900**
COLUMN TOTAL	**288**	**9,712**	**10,000**

– Applying Bayes rule

$$P[cond|+] =$$

$$= \frac{P[+|cond]P[cond]}{P[+]} =$$

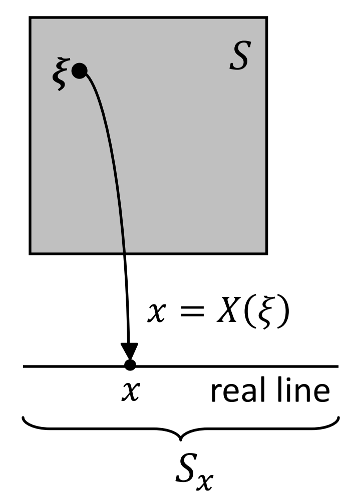
$$= \frac{P[+|cond]P[cond]}{P[+|cond]P[cond] + P[+|\neg cond]P[\neg cond]} =$$

$$= \frac{0.90 \times 0.01}{0.90 \times 0.01 + (1 - 0.98) \times 0.99} =$$

$$= 0.3125$$

Random variables

– When we perform a random experiment we are usually interested in some measurement or numerical attribute of the outcome
    - e.g., weights in a population of subjects, execution times when benchmarking CPUs, shape parameters when performing ATR
– These examples lead to the concept of random variable
    - A random variable X is a function that assigns a real number $X(\xi)$ to each outcome ξ in the sample space of a random experiment
    - $X(\xi)$ maps from all possible outcomes in sample space onto the real line
– The function that assigns values to each outcome is fixed and deterministic, i.e., as in the rule "count the number of heads in three coin tosses"
    - Randomness in X is due to the underlying randomness of the outcome ξ of the experiment
– Random variables can be
    - Discrete, e.g., the resulting number after rolling a dice
    - Continuous, e.g., the weight of a sampled individual

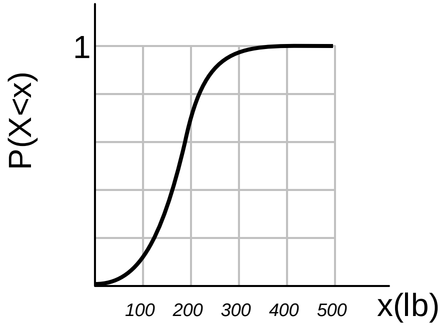


Cumulative distribution function (cdf)

- The cumulative distribution function $F_X(x)$ of a random variable X is defined as the probability of the event $\{X \leq x\}$

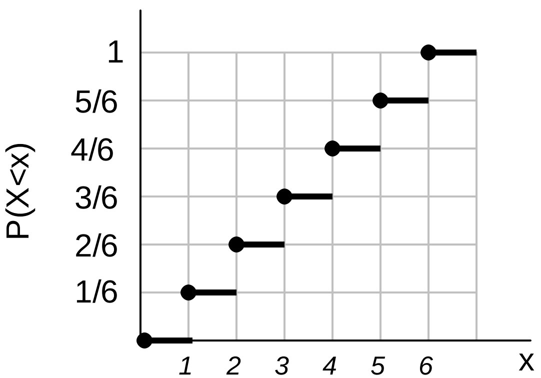
$$F_X(x) = P[X \leq x\,] \quad -\infty < x < \infty$$

- Intuitively, $F_X(b)$ is the long-term proportion of times when $X(\xi) \leq b$



cdf for a person's weight

- Properties of the cdf
  - $0 \leq F_X(x) \leq 1$
  - $\lim_{x\to\infty} F_X(x) = 1$
  - $\lim_{x\to-\infty} F_X(x) = 0$
  - $F_X(a) \leq F_X(b)\ if\ a \leq b$
  - $\mathrm{F_X}(b) = \lim_{h\to 0} F_X(b+h) = F_X(b^+)$



cdf for rolling a dice

Probability density function (pdf)

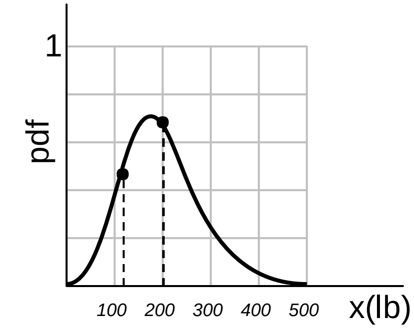
– The probability density function $f_X(x)$ of a continuous random variable X, if it exists, is defined as the derivative of $F_X(x)$

$$f_X(x) = \frac{dF_X(x)}{dx}$$

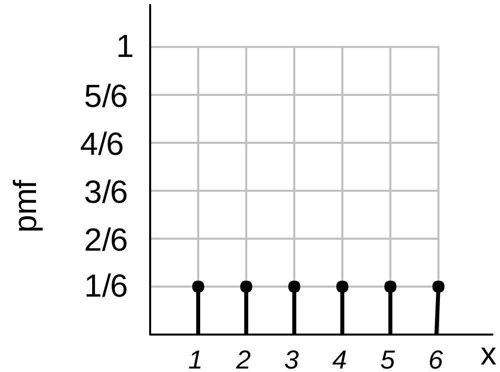
– For discrete random variables, the equivalent to the pdf is the probability mass function

$$f_X(x) = \frac{\Delta F_X(x)}{\Delta x}$$

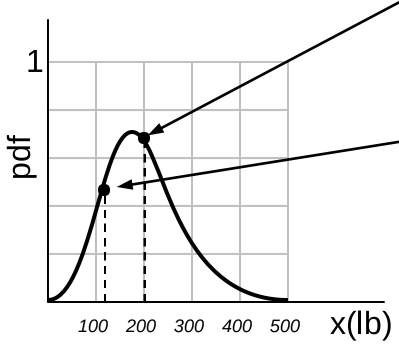
– Properties
  - $f_X(x) > 0$
  - $P[a < x < b] = \int_a^b f_X(x)dx$
  - $F_X(x) = \int_{-\infty}^{x} f_X(x)dx$
  - $1 = \int_{-\infty}^{\infty} f_X(x)dx$
  - $f_X(x|A) = \frac{d}{dx}F_X(x|A) \quad where \quad F_X(x|A) = \frac{P[\{X<x\}\cap A]}{P[A]} \quad if \quad P[A] > 0$



pdf for a person's weight

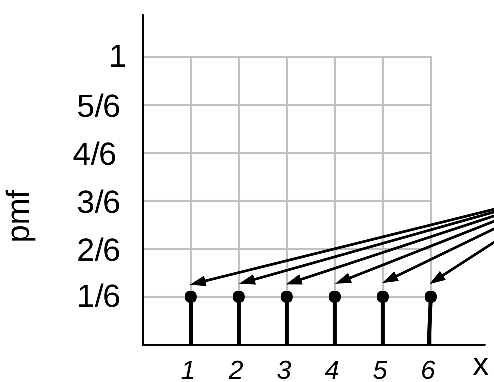


pmf for rolling a (fair) dice



pdf for a person's weight

- **What is the probability of somebody weighting 200 lb?**
  - According to the pdf, this is about 0.62
  - This number seems reasonable, right?
- **Now, what is the probability of somebody weighting 124.876 lb?**
  - According to the pdf, this is about 0.43
  - But, intuitively, we know that the probability should be zero (or very, very small)
- **How do we explain this paradox?**
  - The pdf DOES NOT define a probability, but a probability DENSITY!
  - To obtain the actual probability we must integrate the pdf in an interval
  - So we should have asked the question: what is the probability of somebody weighting 124.876 lb plus or minus 2 lb?



pmf for rolling a (fair) dice

- **The probability mass function is a 'true' probability (reason why we call it a 'mass' as opposed to a 'density')**
  - The pmf is indicating that the probability of any number when rolling a fair dice is the same for all numbers, and equal to 1/6, a very legitimate answer
  - The pmf DOES NOT need to be integrated to obtain the probability (it cannot be integrated in the first place)

Statistical characterization of random variables

- The cdf or the pdf are SUFFICIENT to fully characterize a r.v.
- However, a r.v. can be PARTIALLY characterized with other measures
- Expectation (center of mass of a density)

$$E[X] = \mu = \int_{-\infty}^{\infty} x f_X(x) dx$$

- Variance (spread about the mean)

$$var[X] = \sigma^2 = E[(X - E[X])^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f_X(x) dx$$

- Standard deviation

$$std[X] = \sigma = var[X]^{1/2}$$

- N-th moment

$$E[X^N] = \int_{-\infty}^{\infty} x^N f_X(x) dx$$

Random vectors

- An extension of the concept of a random variable
  - A random vector $\underline{X}$ is a function that assigns a vector of real numbers to each outcome ξ in sample space S
  - We generally denote a random vector by a column vector
- The notions of cdf and pdf are replaced by 'joint cdf' and 'joint pdf'
  - Given random vector $\underline{X} = [x_1, x_2 \dots x_N]^T$ we define the joint cdf as
    $F_{\underline{X}}(\underline{x}) = P_{\underline{X}}[\{X_1 \le x_1\} \cap \{X_2 \le x_2\} \dots \{X_N \le x_N\}]$
  - and the joint pdf as

$$f_{\underline{X}}(\underline{x}) = \frac{\partial^N F_{\underline{X}}(\underline{x})}{\partial x_1 \partial x_2 \dots \partial x_N}$$

- The term <u>marginal pdf</u> is used to represent the pdf of a subset of all the random vector dimensions
  - A marginal pdf is obtained by integrating out variables that are of no interest
  - e.g., for a 2D random vector $\underline{X} = [x_1, x_2]^T$, the marginal pdf of x_1 is

$$f_{X_1}(x_1) = \int_{x_2=-\infty}^{x_2=+\infty} f_{X_1 X_2}(x_1 x_2) dx_2$$

Statistical characterization of random vectors

- A random vector is also fully characterized by its joint cdf or joint pdf
- Alternatively, we can (partially) describe a random vector with measures similar to those defined for scalar random variables
- Mean vector

$$E[X] = \underline{\mu} = \left[E[X_1], E[X_2] \dots E[X_N]\right]^T = [\mu_1, \mu_2, \dots \mu_N]^T$$

- Covariance matrix

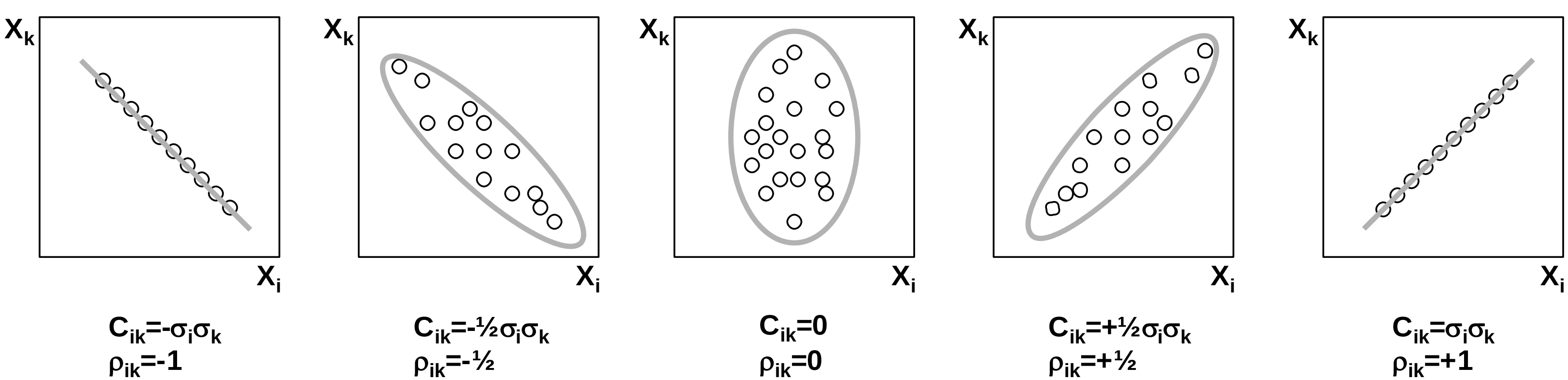
$$cov[X] = \Sigma = E\left[\left(\underline{X} - \underline{\mu}\right)\left(\underline{X} - \underline{\mu}\right)^T\right] =$$

$$= \begin{bmatrix} E[(x_1 - \mu_1)^2] & \dots & E[(x_1 - \mu_1)(x_N - \mu_N)] \\ \vdots & \ddots & \vdots \\ E[(x_1 - \mu_1)(x_N - \mu_N)] & \dots & E[(x_N - \mu_N)^2] \end{bmatrix} =$$

$$= \begin{bmatrix} \sigma_1^2 & \dots & c_{1N} \\ \vdots & \ddots & \vdots \\ c_{1N} & \dots & \sigma_N^2 \end{bmatrix}$$

– The covariance matrix indicates the tendency of each pair of features (dimensions in a random vector) to vary together, i.e., to <u>co-vary</u>*

- The covariance has several important properties
  - If x_i and x_k tend to increase together, then $c_{ik} > 0$
  - If x_i tends to decrease when x_k increases, then $c_{ik} < 0$
  - If x_i and x_k are uncorrelated, then $c_{ik} = 0$
  - $|c_{ik}| \leq \sigma_1 \sigma_k$, where σ_i is the standard deviation of x_i
  - $c_{ii} = \sigma_i^2 = var[x_i]$
- The covariance terms can be expressed as $c_{ii} = \sigma_i^2$ and $c_{ik} = \rho_{ik}\sigma_i\sigma_k$
  - where ρ_{ik} is called the correlation coefficient



A numerical example

Given the following samples from a 3D distribution

– Compute the covariance matrix

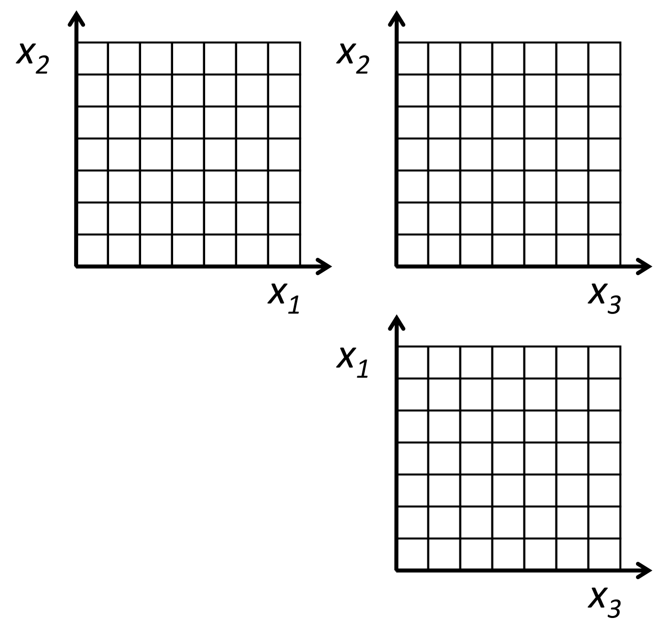
– Generate scatter plots for every pair of vars.

– Can you observe any relationships between the covariance and the scatter plots?

– You may work your solution in the templates below

	Variables (or features)		
Examples	x_1	x_2	$\mathbf{x_3}$
1	2	2	**4**
2	3	4	**6**
3	5	4	**2**
4	**6**	**6**	**4**

Example	x_1	x_2	x_3	$x_1-\mu_1$	$x_2-\mu_2$	$x_3-\mu_3$	$(x_1-\mu_1)^2$	$(x_2-\mu_2)^2$	$(x_3-\mu_3)^2$	$(x_1-\mu_1)(x_2-\mu_2)$	$(x_1-\mu_1)(x_3-\mu_3)$	$(x_2-\mu_2)(x_3-\mu_3)$
1												
2												
3												
4												
Average												



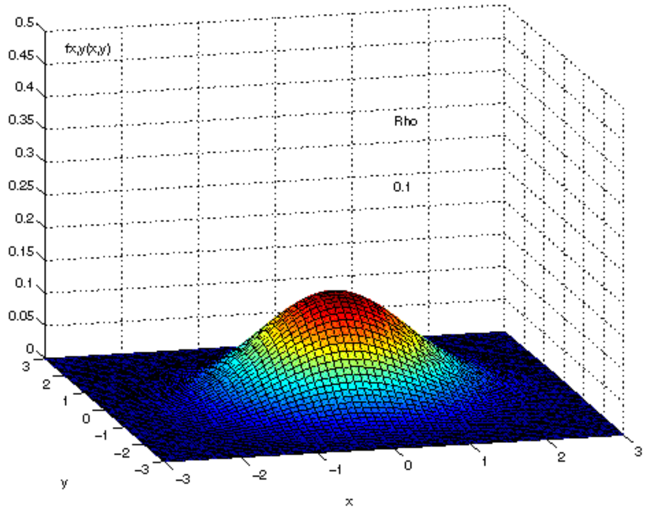
The Normal or Gaussian distribution

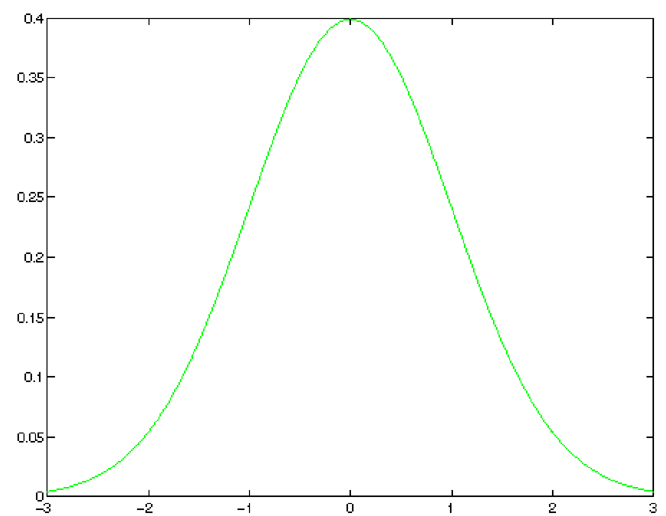
– The multivariate Normal distribution $N(\mu, \Sigma)$ is defined as

$$f_X(x) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)}$$

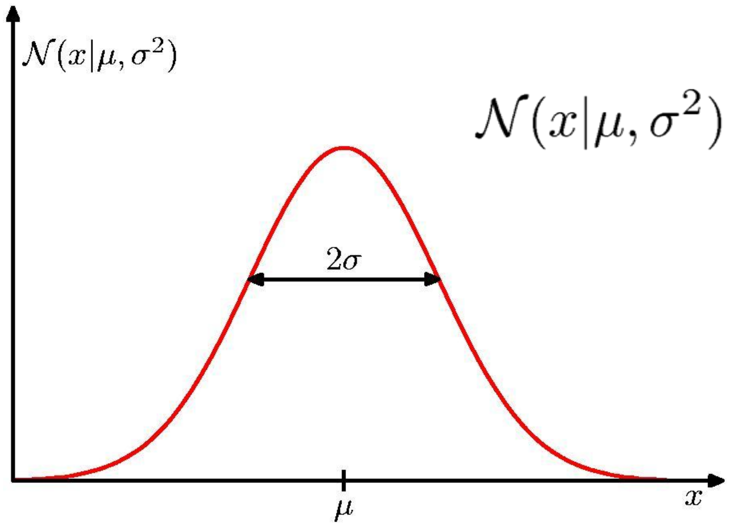
– For a single dimension, this expression is reduced to

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

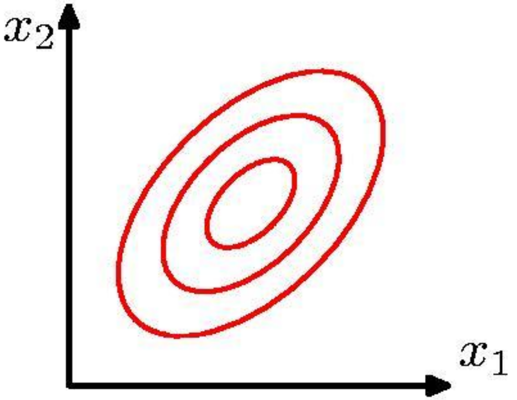




The Gaussian Distribution



$$\mathcal{N}(x|\mu,\sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$



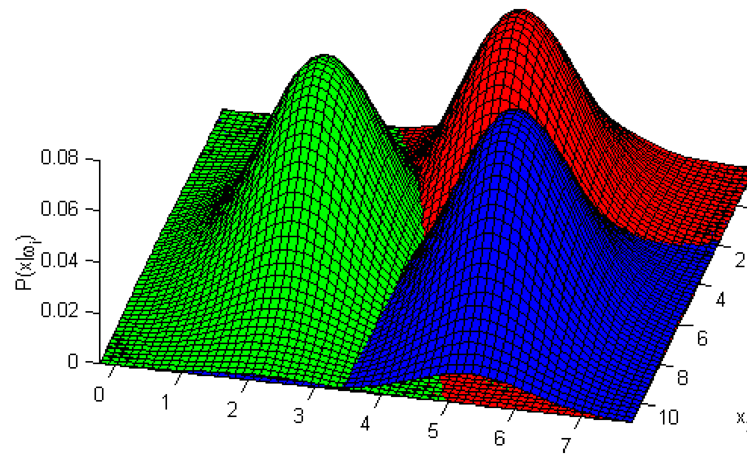
$$\mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}$$

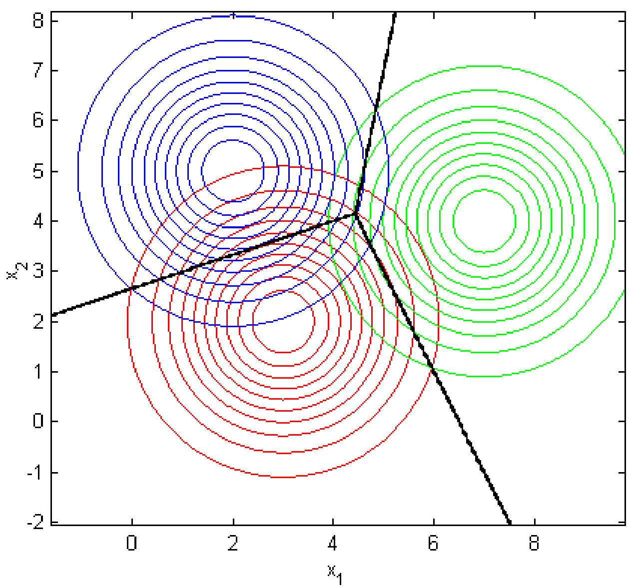
Example

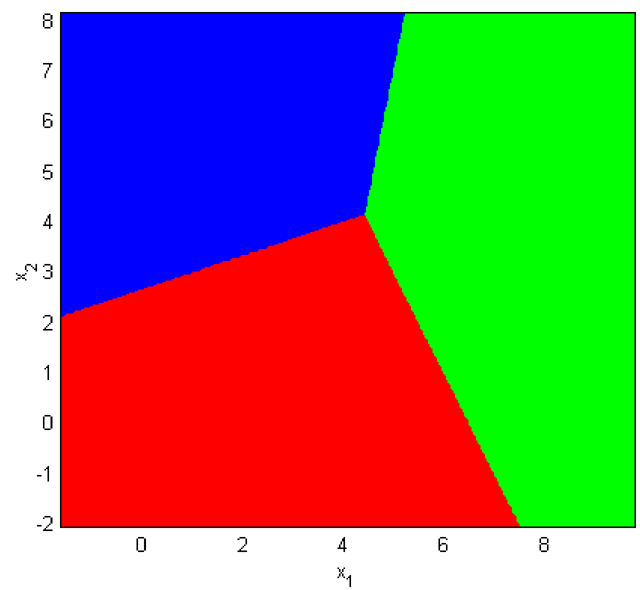
– Three-class 2D problem with equal priors

$$\mu_1 = [3\ 2]^T \qquad \mu_2 = [7\ 4]^T \qquad \mu_3 = [2\ 5]^T$$

$$\Sigma_1 = \begin{bmatrix} 2 & \\ & 2 \end{bmatrix} \qquad \Sigma_1 = \begin{bmatrix} 2 & \\ & 2 \end{bmatrix} \qquad \Sigma_1 = \begin{bmatrix} 2 & \\ & 2 \end{bmatrix}$$





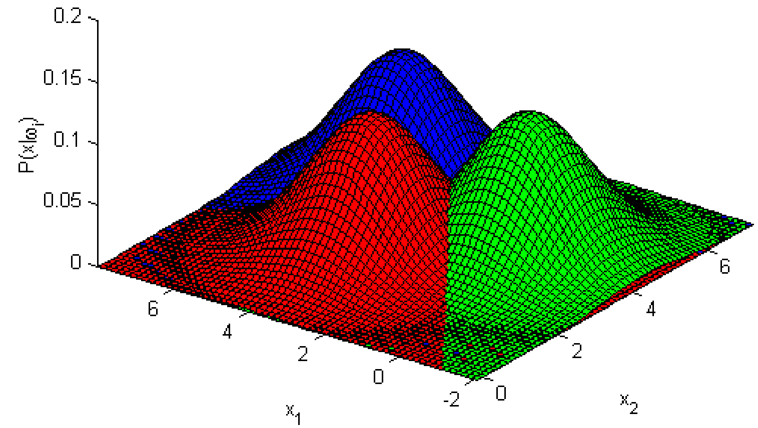


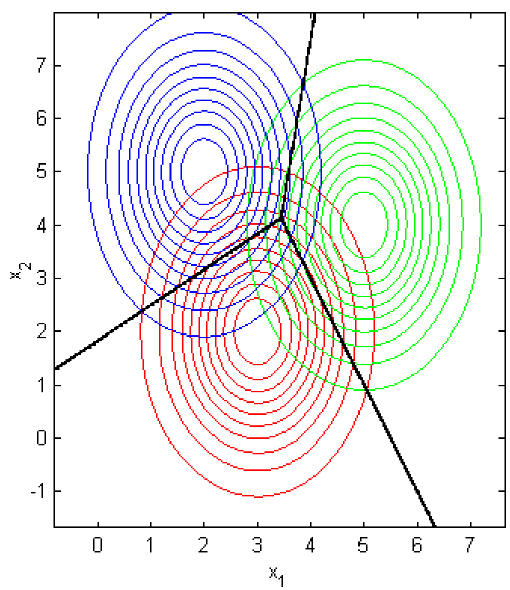
Example

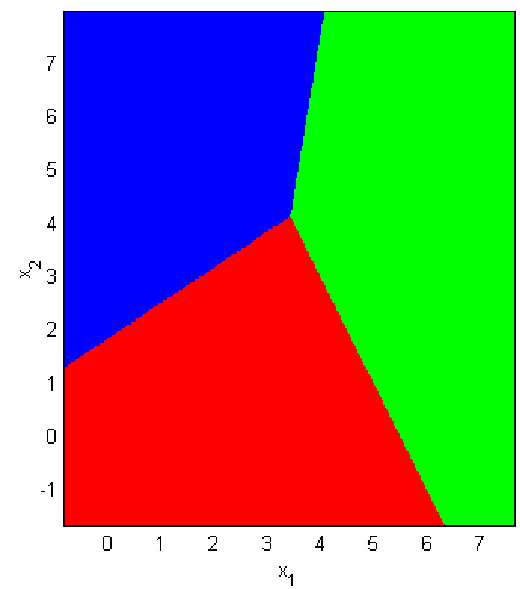
- Three-class 2D problem with equal priors

$$\mu_1 = [3\ 2]^T \qquad \mu_2 = [5\ 4]^T \qquad \mu_3 = [2\ 5]^T$$

$$\Sigma_1 = \begin{bmatrix} 1 & \\ & 2 \end{bmatrix} \qquad \Sigma_2 = \begin{bmatrix} 1 & \\ & 2 \end{bmatrix} \qquad \Sigma_3 = \begin{bmatrix} 1 & \\ & 2 \end{bmatrix}$$

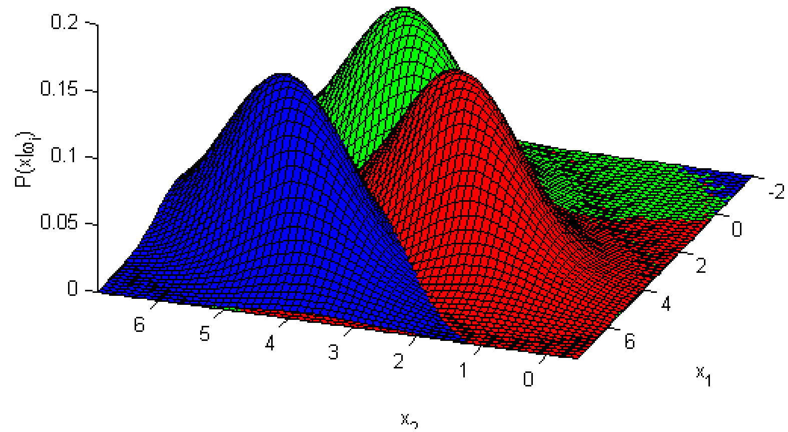






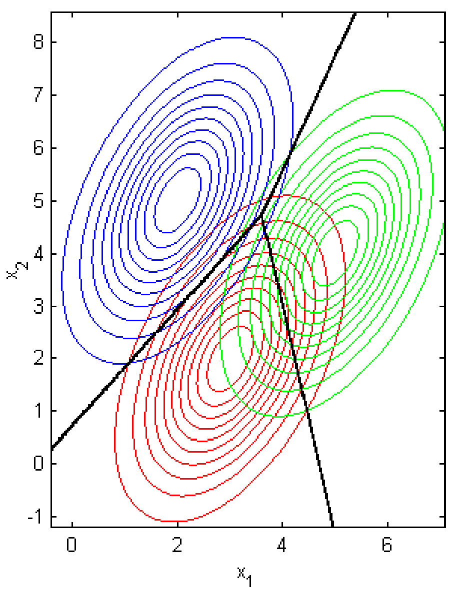
Example

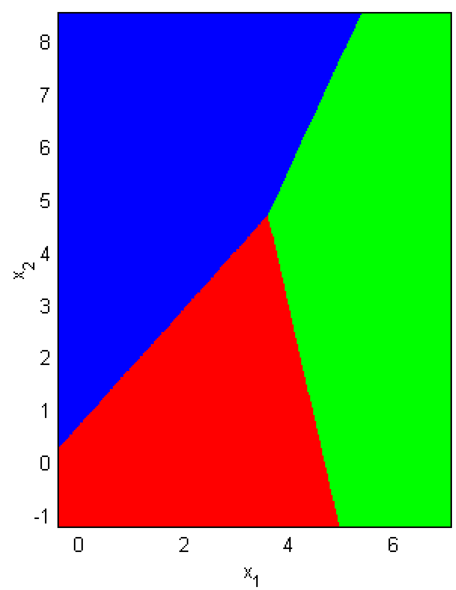
– Three-class 2D problem with equal priors



$$\mu_1 = [3\ 2]^T \qquad \mu_2 = [5\ 4]^T \qquad \mu_3 = [2\ 5]^T$$

$$\Sigma_1 = \begin{bmatrix} 1 & .7 \\ .7 & 2 \end{bmatrix} \qquad \Sigma_2 = \begin{bmatrix} 1 & .7 \\ .7 & 2 \end{bmatrix} \qquad \Sigma_3 = \begin{bmatrix} 1 & .7 \\ .7 & 2 \end{bmatrix}$$



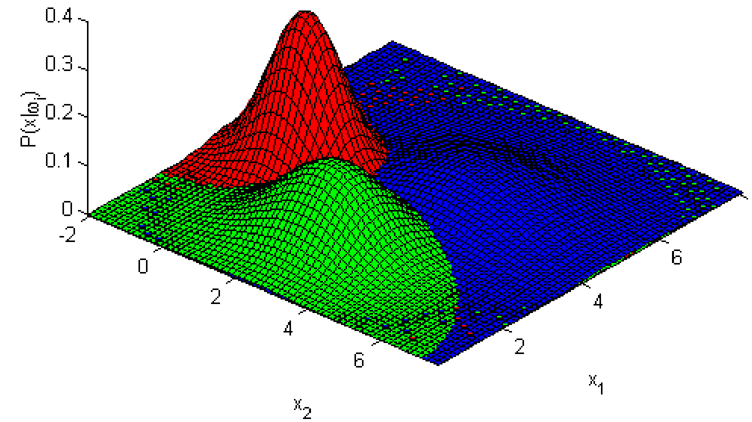


Example

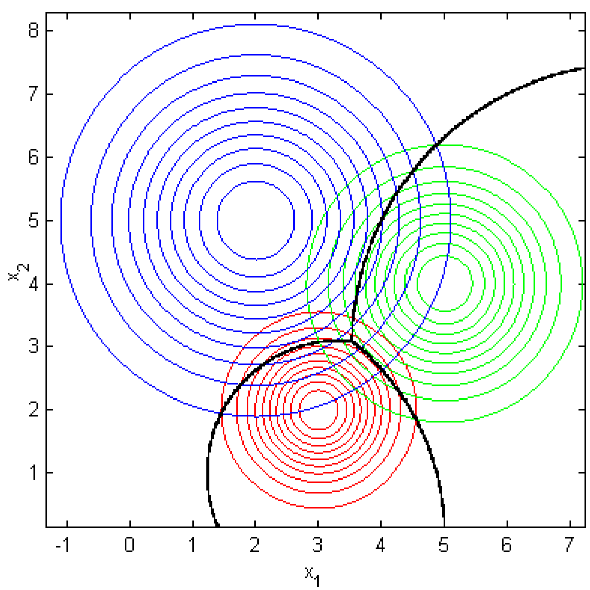
– Three-class 2D problem with equal priors

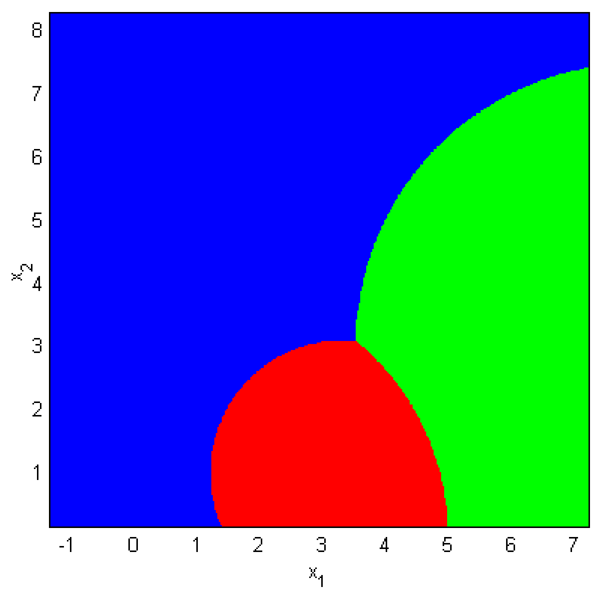
$$\mu_1 = [3\ 2]^T \qquad \mu_2 = [5\ 4]^T \qquad \mu_3 = [2\ 5]^T$$

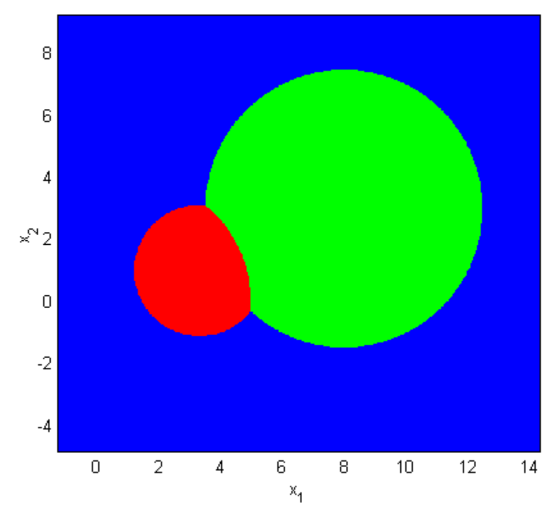
$$\Sigma_1 = \begin{bmatrix} .5 & \\ & .5 \end{bmatrix} \qquad \Sigma_2 = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \qquad \Sigma_3 = \begin{bmatrix} 2 & \\ & 2 \end{bmatrix}$$



Zoom out





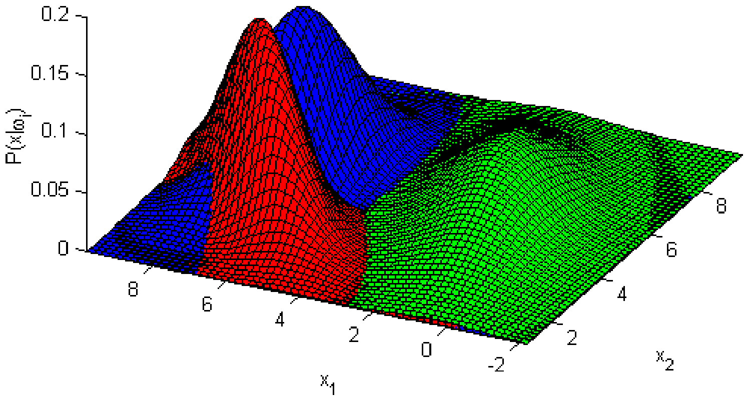


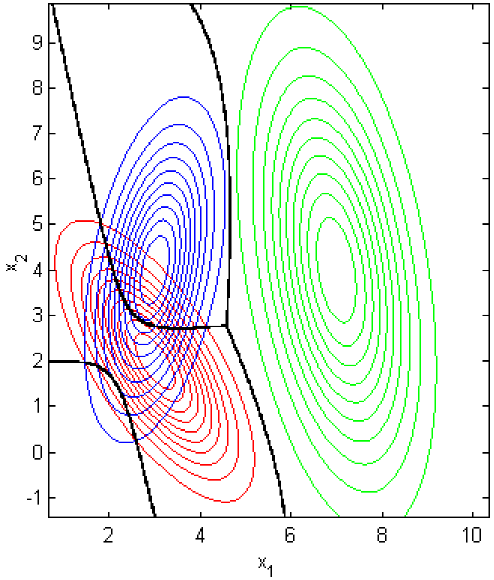
Example

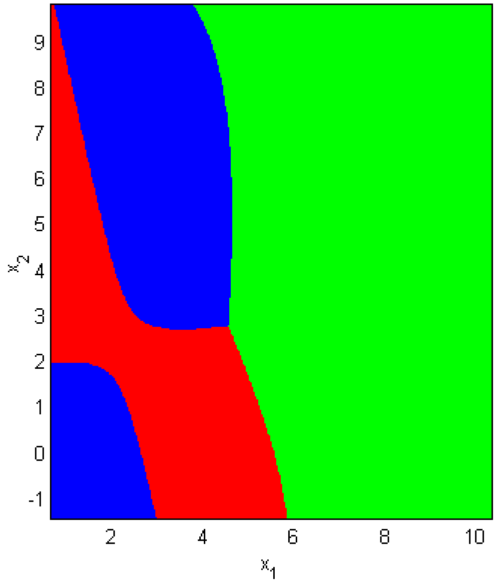
– Three-class 2D problem with equal priors

$$\mu_1 = [3\ 2]^T \qquad \mu_2 = [5\ 4]^T \qquad \mu_3 = [3\ 4]^T$$

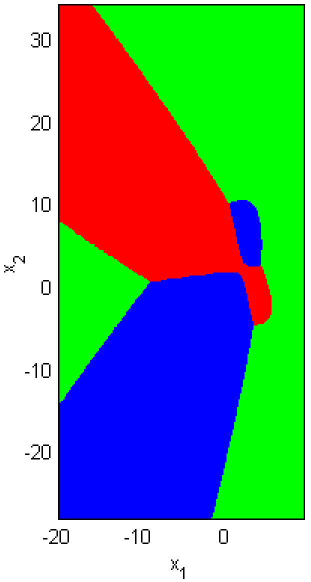
$$\Sigma_1 = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \qquad \Sigma_2 = \begin{bmatrix} 1 & -1 \\ -1 & 7 \end{bmatrix} \qquad \Sigma_3 = \begin{bmatrix} .5 & .5 \\ .5 & 3 \end{bmatrix}$$







Zoom out



– Gaussian distributions are very popular since

- Parameters (μ, Σ) uniquely characterize the normal distribution
- If all variables x_i are uncorrelated $(E[x_i x_k] = E[x_i]E[x_k])$, then
  - Variables are also independent $(P[x_i x_k] = P[x_i]P[x_k])$, and
  - Σ is diagonal, with the individual variances in the main diagonal
- Central Limit Theorem (next slide)
- The marginal and conditional densities are also Gaussian
- Any linear transformation of any N jointly Gaussian rv's results in N rv's that are also Gaussian
  - For $X = [X_1 X_2 \dots X_N]^T$ jointly Gaussian, and $A_{N \times N}$ invertible, then $Y = AX$ is also jointly Gaussian

$$f_Y(y) = \frac{f_X(A^{-1}y)}{|A|}$$

Central Limit Theorem

- Given <u>any</u> distribution with a mean μ and variance σ^2, the sampling distribution of the mean approaches a normal distribution with mean μ and variance σ^2/N as the sample size N increases
  - No matter what the shape of the original distribution is, the sampling distribution of the mean approaches a normal distribution
  - N is the sample size used to compute the mean, not the overall number of samples in the data
- Example: 500 experiments are performed using a uniform distribution
  - $N = 1$
    - One sample is drawn from the distribution and its mean is recorded (500 times)
    - The histogram resembles a uniform distribution, as one would expect
  - $N = 4$
    - Four samples are drawn and the mean of the four samples is recorded (500 times)
    - The histogram starts to look more Gaussian
  - As N grows, the shape of the histograms resembles a Normal distribution more closely