Sharif University of Technology
Electrical Engineering Department

Machine Learning and Vision Lab
Pre-Report 1

Amir Hossein Yari
99102507

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Contents

PCA Proof

$$\max_{u \in R^D} u^T C u \qquad \text{subject to } ||u||_2 = 1$$

C is a covariance matrix and all covariance matrix is symmetric and positive semi-definite. Based on Spectral Theorem, all eigenvalues are real and eigenvectors are orthonormal.

$$\Rightarrow C = P \Lambda P^T = \sum_{i=1}^{n} \lambda_i p_i p_i^T$$

Assume the maximum spread direction is u and consider the following definition:

$$b = P^T u \Rightarrow u = Pb$$

$$u^T C u = (Pb)^T (P \Lambda P^T)(Pb) = b^T P^T P \Lambda P^T P b = b^T \Lambda b = \sum_{j=1}^{D} \lambda_j b_j^2 \leq \lambda_1 \sum_{j=1}^{D} b_j^2$$

$$||b||^2 = ||P^T u||^2 = (P^T u)^T (P^T u) = u^T P P^T u = u^T u = ||u||^2 = 1$$

$$u^T C u \leq \lambda_1 \sum_{j=1}^{D} b_j^2 \Rightarrow u^T C u \leq \lambda_1$$

According to above equations, maximum occurs when:

$$u = p_1 \Rightarrow b = P^T p_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow p_1^T C p = b^T \Lambda b = \lambda_1$$

First Principle Component

At first we compute the mean of the data:

$$\bar{x} = \frac{1+2+3}{3} = 2 \quad , \quad \bar{y} = \frac{1+3+2}{3} = 2$$

Now we center the data by subtracting the mean from each data point:

$$\begin{aligned} &\text{Centered point 1:} \quad p_1 = (-1, -1) \\ &\text{Centered point 2:} \quad p_2 = (0, 1) \\ &\text{Centered point 3:} \quad p_3 = (1, 0) \end{aligned}$$

Now should compute the covariance matrix and eigenvalues and eigenvectors of it:

$$C = \frac{1}{N}\sum_{i=1}^{N} p_i p_i^T = \frac{1}{3}[\begin{pmatrix} -1 \\ -1 \end{pmatrix} \begin{pmatrix} -1 & -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}]$$

$$= \frac{1}{3}[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}] = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

$$\begin{vmatrix} \frac{2}{3} - \lambda & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} - \lambda \end{vmatrix} = 0 \Rightarrow 3\lambda^2 - 4\lambda + 1 = 0 \Rightarrow \lambda = 1 \, , \, \frac{1}{3}$$

$$\lambda = 1 \Rightarrow \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix} v_1 = 0 \Rightarrow v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\lambda = \frac{1}{3} \Rightarrow \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} v_2 = 0 \Rightarrow v_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

The first principle component is $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ that is corresponds to $\lambda = 1$.

Reconstruction Error

Data in 1D space is:

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ \frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \end{pmatrix}$$

Reconstruction data is:

$$\begin{pmatrix} \sqrt{2} \\ \frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}^T = \begin{pmatrix} 1 & 1 \\ 2.5 & 2.5 \\ 2.5 & 2.5 \end{pmatrix}$$

Reconstruction error calculate as below:

$$\text{error} = \frac{1}{N} \sum_{i=1}^{N} (x_i - \hat{x}_i)^2 = \frac{1}{3}(0 + (0.5^2 + 0.5^2) + (0.5^2 + 0.5^2)) = \frac{1}{3}$$