Alignment Error and Noise Analysis

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Chapter 1

Alignment Error

1.1 Introduction

In this chapter, we explore the integration of alignment error considerations within the foundational framework of the BB84 protocol, which employs photon polarization for quantum communication. Alignment error refers to the directional disparity between the polarizer axis and the polarization analyzer axis. Within quantum measurement, such misalignment can yield outcomes diverging from the anticipated results.

1.2 Formulation

We need to elucidate the methodology for incorporating misalignment into our scheme. Firstly, we consider the H-V bases (see Fig. 1.1), where the polarizer bases are rotated by an angle θ from the ideal H-V bases. In this scenario, for the two states $|0_S\rangle$ and $|1_S\rangle$, we can express:

$$\begin{aligned} |0_S\rangle &= \cos\theta|0\rangle + \sin\theta|1\rangle \\ |1_S\rangle &= -\sin\theta|0\rangle + \cos\theta|1\rangle \end{aligned} \tag{1.1}$$

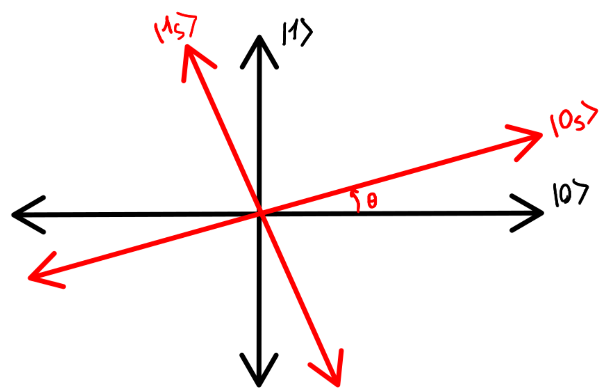

Figure 1.1: The Sender Basis is rotated by an angle of θ

Similarly, we can describe the receiver end, where the polarization analyzer is rotated by an angle α from the ideal H-V bases. Consequently, this yields expressions for the two measured states, $|0_S\rangle$ and $|1_S\rangle$:

$$\begin{aligned} |0_R\rangle &= \cos\alpha|0\rangle + \sin\alpha|1\rangle \\ |1_R\rangle &= -\sin\alpha|0\rangle + \cos\alpha|1\rangle \end{aligned} \tag{1.2}$$

With these considerations, we can proceed to calculate the probabilities associated with transmitting and receiving the correct or incorrect states. The probability amplitudes are as follows:

$$\begin{aligned} \langle 0_R|0_S\rangle &= \cos\theta\cos\alpha + \sin\theta\sin\alpha = \cos(\alpha-\theta) \\ \langle 1_R|1_S\rangle &= \cos\theta\cos\alpha + \sin\theta\sin\alpha = \cos(\theta-\alpha) \\ \langle 1_R|0_S\rangle &= \cos\theta\sin\alpha - \sin\theta\cos\alpha = \sin(\theta-\alpha) \\ \langle 0_R|1_S\rangle &= -\cos\theta\sin\alpha + \sin\theta\cos\alpha = \sin(\alpha-\theta) \end{aligned} \tag{1.3}$$

Now, we extend our analysis to the Diagonal Basis (see Fig. 1.2). Analogous to the H-V Basis, we delineate the sender and receiver states, along with the corresponding probability amplitudes:

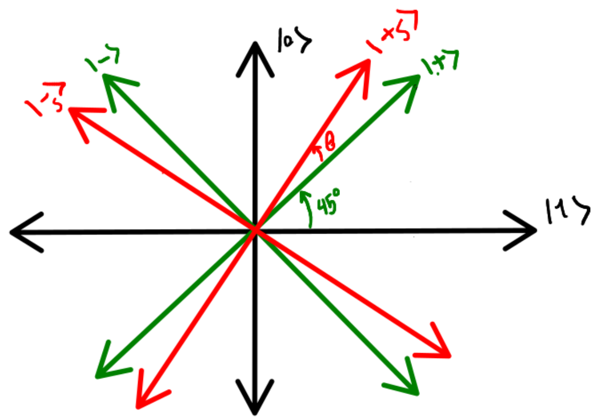

Figure 1.2: The Sender Diagonal Basis is rotated by an angle of θ

$$\begin{aligned} |+_S\rangle &= \cos\theta|+\rangle + \sin\theta|-\rangle \\ |-_S\rangle &= -\sin\theta|+\rangle + \cos\theta|-\rangle \\ |+_R\rangle &= \cos\alpha|+\rangle + \sin\alpha|-\rangle \\ |-_R\rangle &= -\sin\alpha|+\rangle + \cos\alpha|-\rangle \end{aligned} \tag{1.4}$$

$$\begin{aligned} \langle +_R|+_S\rangle &= \cos\theta\cos\alpha + \sin\theta\sin\alpha = \cos(\alpha-\theta) \\ \langle -_R|-_S\rangle &= \cos\theta\cos\alpha + \sin\theta\sin\alpha = \cos(\theta-\alpha) \\ \langle -_R|+_S\rangle &= \cos\theta\sin\alpha - \sin\theta\cos\alpha = \sin(\theta-\alpha) \\ \langle +_R|-_S\rangle &= -\cos\theta\sin\alpha + \sin\theta\cos\alpha = \sin(\alpha-\theta) \end{aligned} \tag{1.5}$$

Chapter 2

Simulation

2.1 Implementation

Now that we have outlined the formulation of the alignment error, our next step is to explore its implementation within the simulation framework devised for the BB84 code. To achieve this, we introduce an additional function as follows:

```
def AlligmentError(cir,bit,basis,theta_radians):
#This is gettingready but with Alligment error
  cir.ry(theta_radians,8)
  if bit == 1:
    cir.x(8)
  if basis == 1:
    cir.h(8)
  return cir
}
```

Listing 2.1: GettingReady function with Alignment error

This function utilizes the 'ry()' gate to introduce the alignment error and modify the initial state based on the specified basis and bit. It specifically targets the 8th qubit, as the other qubits serve as auxiliary ones for error correction.

Another aspect we need to address pertains to the received and measured states. When measuring the incoming qubit, it's advantageous to measure it in the preferred outcome state, i.e., the state aligned with the polarization analyzer. However, rather than modifying the incoming qubit, an alternative approach is to redefine our reference and ideal states to align with the receiver states ($\alpha = 0$). By doing so, since only the relative angle is of significance, we can utilize the existing code designed to measure the ideal states.

Additionally, a modification in the code involves the incorporation of a function named "ExtractKey()." This function serves to extract not only the key but also the error probability, indicating the likelihood of errors in the code (Noise included). Furthermore, it helps to calculates a theoretical probability that demonstrates the likelihood that your key remains unaffected by alignment error.This comparison between the alignment error probability and the error probability can also elucidate the impact of noise within the system.

```
def ExctarctKey(Checked_List):
  length = len(Checked_List)
  result = []
  for d in Checked_List:
      new_d = {}
      for key, value in d.items():
          third_bit = key[2]  # Extract the third bit from the left
          new_d[third_bit] = value
      result.append(new_d)
  output_string = ""
  probability = 1.0
```

```
  for d in result:
      # Determine the most frequent bit
      most_frequent_bit = max(d, key=d.get)
      # Update the output string
      output_string += most_frequent_bit
      # Calculate the probability of the chosen bit
      probability *= d[most_frequent_bit] / sum(d.values())
  return output_string, probability, length
}
```

Listing 2.2: The ExtractKey function

2.2 Results

Now, in this section, we will examine various simulation outcomes. Firstly, let's consider the simple case where there is neither misalignment nor noise:

```
----Information-----
Bit Sequence length is :  64
The angle of mis alignment is (in degree):  0
AliceKey is :  10001000011101100011110100
----- Without Eve -----
your Key is :
10001000011101100011110100
with The length of :26
Probability obtained from theory that your key is un-affected by alignment errors: 1.0
Probabilitythat your key is un-affected by all errors: 1.0
The Connection is Secure with the probability of:
98.6636538989842
----- With Eve -----
your Key is :
10001000111101110101111100
with The length of :26
Probability obtained from theory that your key is un-affected by alignment errors : 1.0
Probabilitythat your key is un-affected by all errors: 1.0
The Connection is not Secure!
```

This demonstrates that in the absence of noise, the probability of the key being correct is 1.0 and affirming the efficacy of the Eve detection functions. Moving forward, let's investigate the impact of noise (without error correction). For this purpose, Although we have a function for generating random angles with a normal distribution, for this specific scenario, we opt to set a fixed angle of 5 degrees for both bit and phase flips.

```
----Information-----
Bit Sequence length is :  64
The angle of mis alignment is (in degree):  0
AliceKey is :  10001000011101100011110100
----- Without Eve -----
your Key is :
10001000011101100011110100
with The length of :26
Probability obtained from theory that your key is un-affected by alignment errors: 1.0
```

```
Probabilitythat your key is un-affected by all errors: 0.6152214829111787
The Connection is Secure with the probability of:
98.6636538989842
----- With Eve -----
your Key is :
00001010010011110011110100
with The length of :26
Probability obtained from theory that your key is un-affected by alignment errors : 1.0
Probabilitythat your key is un-affected by all errors: 0.5635716058698185
The Connection is not Secure!
```

This demonstrates the impact of noise, encompassing both phase and bit flips. It reveals that only in 61.5% of the outcomes, the correct key is obtained. This percentage can vary, either diminishing or amplifying, contingent upon the angle of noise utilized. Now, let's incorporate the error-correction block and observe how it influences the error rate:

```
----Information-----
Bit Sequence length is :  64
The angle of mis alignment is (in degree):  0
AliceKey is :  10001000011101100011110100
----- Without Eve -----
your Key is :
10001000011101100011110100
with The length of :26
Probability obtained from theory that your key is un-affected by alignment errors: 1.0
Probabilitythat your key is un-affected by all errors: 0.9281961507029667
The Connection is Secure with the probability of:
98.6636538989842
----- With Eve -----
your Key is :
10101000011101111011110100
with The length of :26
Probability obtained from theory that your key is un-affected by alignment errors : 1.0
Probabilitythat your key is un-affected by all errors: 0.9281673588570318
The Connection is not Secure!
```

This illustrates the efficiency of our error-correction block. It has significantly reduced the error rate and increased the probability that our key remains unaffected, rising from 61.5% to 92.8%. Now, let's focus solely on misalignment and observe its effects:

```
----Information-----
Bit Sequence length is :  64
The angle of mis alignment is (in degree):  5
AliceKey is :  10010100100100101100111100111010101
----- Without Eve -----
your Key is :
10010100100100101100111100111010101
with The length of :35
Probability obtained from theory that your key is un-affected by alignment errors: 0.7657655441216619
```

```
Probabilitythat your key is un-affected by all errors: 0.7658239951909347
The Connection is Secure with the probability of:
98.6636538989842
----- With Eve -----
your Key is :
00000000100000101100010100111000001
with The length of :35
Probability obtained from theory that your key is un-affected by alignment errors : 0.7657655441216619
Probabilitythat your key is un-affected by all errors: 0.7696390461105446
The Connection is not Secure!
```

In this example, the misalignment angle was set to 5 degrees. We observe that the error probability follows the relation below:

$$P_e = 1 - \cos^{(2Keylength)} \beta \tag{2.1}$$

Now, let's consider both noise and misalignment simultaneously and examine their combined effects:

```
    ----Information-----
Bit Sequence length is :  64
The angle of mis alignment is (in degree):  5
AliceKey is :  10010100100100101100111100111010101
----- Without Eve -----
your Key is :
10010100100100101100111100111010101
with The length of :35
Probability obtained from theory that your key is un-affected by alignment errors: 0.7657655441216619
Probabilitythat your key is un-affected by all errors: 0.46829703774010684
The Connection is Secure with the probability of:
98.6636538989842
----- With Eve -----
your Key is :
10000000000000100000001000100010001
with The length of :35
Probability obtained from theory that your key is un-affected by alignment errors : 0.7657655441216619
Probabilitythat your key is un-affected by all errors: 0.7507460919465523
The Connection is not Secure!
```