Report on OAM

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Chapter 1

Optical Angular Momentum

1.1 Introduction

As we know in physics the electromagnetic fields have their own physical attributes, the carry the energy of

$$u = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) \tag{1.1}$$

And the linear momentum of

$$\mathbf{p} = \varepsilon_0(\mathbf{E} \times \mathbf{B}) \tag{1.2}$$

So it is not outstanding that these fields are also carrying angular momentum. This angular momentum is also known as optical angular momentum, which we are going to have a quick review on in this chapter.

1.2 Angular Momentum

The electromagnetic fields associated with optical beams are transverse, designated by the symbol E_T satisfying $\nabla \cdot \mathbf{E_T} = 0$. The optical angular momentum density vector is defined as the moment of the linear momentum density $\mathbf{p} = \varepsilon_0(\mathbf{E} \times \mathbf{B})$ about a chosen origin r'

$$\mathbf{M} = \varepsilon_0(r - \acute{r}) \times (\mathbf{E} \times \mathbf{B}) \tag{1.3}$$

While the total angular momentum of the field is obtained by integraling over the space so

$$\mathbf{J} = \varepsilon_0 \int \mathbf{d^3 r}(\mathbf{r} - \mathbf{r}') \times (\mathbf{E}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})) \tag{1.4}$$

We now express B in terms of the vector potential A. It can then be shown that the total optical angular momentum splits into two parts

$$\mathbf{J}(\mathbf{r}') = \mathbf{S} + \mathbf{L}(\mathbf{r}') \tag{1.5}$$

where S is identified as the spin angular momentum (SAM)

$$\mathbf{S} = \varepsilon_0 \int d^3\mathbf{r}[\mathbf{E}(\mathbf{r}) \times \mathbf{A}(\mathbf{r})] \tag{1.6}$$

and L(r') is identified as the orbital angular momentum (OAM)

$$\mathbf{L}(\mathbf{r}') = \varepsilon_0 \sum_i \int d^3\mathbf{r} E_i \mathbf{r} - \mathbf{r}' \times \nabla A_i \tag{1.7}$$

Note that L(r'), the OAM, depends on the choice of origin r, while S, the SAM, does not. It is for this reason that SAM is identified as an intrinsic angular momentum.

Chapter 2

Structred Light

2.1 Introduction

Beams with an $\exp(il\phi)$ azimuthal phase function are shown to have orbital angular momentum of $l\hbar$ per photon. It is shown that eleptic beams, although not having this azimuthal function, may also have orbital angular momentum. So in this chapter of the report we will look at the general concept of light beams with orbital angular momentum known through this report as *Structred Light*[1]. We will start our review with paraxial approximation and then

2.2 The paraxial approximation

The change in the transverse beam profile of a reasonbaly well collimated beam occurs only slowl with position z, along the direction of propagation.For an amplitude function $u(a, y, z)$ this approximation is made by ignoring $\partial^2 u/\partial z^2$ compared to $k_z(\partial u/\partial z)$ where k_z is the wave-number, and ignoring $(\partial u/\partial z)$ compared to u, in the scalar wave equation. This gives the paraxial Helmholtz equation :

$$i\frac{\partial u}{\partial z} = -\frac{1}{2k_z}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u. \tag{2.1}$$

2.2.1 Linear Polarized general formulations

As we known Electromagnetic waves are transverse, this means that they have no component in the propagation direction. However the fields of a laser mode like TEM_{nmq}, unlike those in a coaxial metal waveguide of infinte length, are not strictly transverse. They have a small component in the direction of propagation.So because we want to discuss the properties of laser beams that are generated with orbital angular momentum so we should study this not so ideal beams.

A convinent representation of the linearly polarized light is achieved in using the Lorentz guage for vector potential

$$\mathbf{A} = \hat{x}u(x, y, z)e^{-ik_z z} \tag{2.2}$$

The expression $u(x, y, z)$ or $u(\rho, \phi, z)$ is the complex scalar function describing the distribution of the field's amplitude which satisfies the wave equationin the paraxial approximation (eq. (2.1)).
After using the paraxial approximation and the Lorentz guage it yields:

$$\begin{aligned} \mathbf{B} = \mu_0 \mathbf{H} = ik_z\left[u\hat{y} + \frac{1}{k_z}\frac{\partial u}{\partial y}\hat{z}\right]e^{ik_z z} \\ \mathbf{E} = ik_z\left[u\hat{x} + \frac{1}{k_z}\frac{\partial u}{\partial y}\hat{z}\right]e^{ik_z z} \end{aligned} \tag{2.3}$$

[1]Structred light are a more general group of light but here we mean beams with orbital angular momentum.

As noted before there is an z-component for both $\mathbf{E}$ and $\mathbf{B}$ as there is for the TEM modes of a *real laser beam.* We know that the real part of $\varepsilon_0(\mathbf{E} \times \mathbf{B})$ is the time-averaged Poynting vector and the linear momentum density so:

$$\begin{aligned} \varepsilon_0 \langle \mathbf{E} \times \mathbf{B} \rangle = \frac{\varepsilon_0}{2} \left[(\mathbf{E}^* \times \mathbf{B}) + (\mathbf{E} \times \mathbf{B}^*) \right] = \\ i\omega \frac{\varepsilon_0}{2} \left(u \nabla u^* - u^* \nabla u \right) + \omega k_z \varepsilon_0 \left| u \right|^2 \hat{z}. \end{aligned} \tag{2.4}$$

2.2.2 Structred Light

Clearly the result obtained above also applies for cylindrical coordinates, wher u is written as $u(\rho, \phi, z)$. The cycle-averaged momentum density in eq.(2.4) is the linear momentum density in the beam. The angular momentum density along the beam axis j_z, may be found by taking the cross product of the ϕ-component of expression(2.4) with the radius vector ρ.

In case of structred light in form of:

$$u(\rho, \phi, z) = u_0(\rho, \phi, z) e^{il\phi} \tag{2.5}$$

Its easy to see from eq.(2.4) that the ϕ-component of the linear momentum density becomes simply $\varepsilon_0 \langle \mathbf{E} \times \mathbf{B} \rangle = \varepsilon_0 \omega l |u|^2 / \rho$ and the cross production with ρ gives the angular momentum density of magnitude $j_z = \varepsilon_0 \omega l |u|^2$. The energy density in such beams is $w = c\varepsilon_0 \langle \mathbf{E} \times \mathbf{B} \rangle_z = c\varepsilon_0 \omega k |u|^2 = c\varepsilon_0 \omega^2 l |u|^2$. so:

$$\frac{j_z}{w} = \frac{l}{\omega} \tag{2.6}$$

So when these densities are integrated over the x-y plane, the ratio of angular momentum to energy per unit length of beam is obtained. This is:

$$\frac{J_z}{W} = \frac{\iint \rho d\rho d\phi (\mathbf{r} \times \langle \mathbf{E} \times \mathbf{B} \rangle)_z}{c \iint \rho d\rho d\phi \langle \mathbf{E} \times \mathbf{B} \rangle_z} = \frac{l}{\omega} \tag{2.7}$$

We also know that a photon has an energy of hc/λ and also that $\omega = 2\pi c/\lambda$ so then form eq.(2.7) we have:

$$\begin{aligned} \frac{J_z}{hc/\lambda} = \frac{l}{2\pi c/\lambda} \\ J_z = l\hbar \end{aligned} \tag{2.8}$$

Which shows the angular momentum of per photon. The beam is not polarized and the angular momentum is that from orbital angular momentum and not spin.

Now for generalazing this approach, the ligt may also be circularly polarized and have components of electric fields in both the x- and y- directions. That is the vector potential of the field is defined by:

$$\mathbf{A} = (\alpha \hat{x} + \beta \hat{y}) u(x, y, z) exp(ikz) \tag{2.9}$$

Where the amplitude function is in the general form of:

$$u(x, y, z) = u_0(x, y, z) e^{i\Theta} \tag{2.10}$$

Where Θ is the general phase function which for a simple structred light has a form of eq.(2.5). When following the same process we find:

$$\begin{aligned} \mathbf{E} &= i\omega \mathbf{A} + \nabla \left(\frac{c^2}{i\omega} \nabla \cdot \mathbf{A} \right) \\ &= \left[i\omega \alpha u \hat{x} + i\omega \beta u \hat{y} - c \left(\alpha \frac{\partial u}{\partial x} + \beta \frac{\partial u}{\partial y} \right) \right] e^{ik_z z} \end{aligned} \tag{2.11}$$

And after calculating $\varepsilon \langle \mathbf{E} \times \mathbf{B} \rangle$ we have:

$$\begin{aligned}
\varepsilon \langle \mathbf{E} \times \mathbf{B} \rangle &= \frac{\varepsilon}{2}[(\mathbf{E}^* \times \mathbf{B}) + (\mathbf{E} \times \mathbf{B}^*)] \\
&= \frac{\varepsilon}{2} \Bigg[i\omega(u\nabla u^* - u^*\nabla u) + 2\omega k_z \varepsilon |u|^2 \hat{z} \\
&+ i\omega(\alpha\beta^* - \alpha^*\beta) \left(\frac{\partial}{\partial y}\hat{x} - \frac{\partial}{\partial x}\hat{y} \right) |u|^2 \Bigg]
\end{aligned} \tag{2.12}$$

The term $i(\alpha\beta^* - \alpha^*\beta)$ arising from the relative amplitude and phase of the x- and y-components of the electric field is readily identified with the spin in the z-direction, σ. Thus, $\langle \mathbf{E} \times \mathbf{B} \rangle$ has a polarization-independent part and a spin or polarization part.

It is can be shown that

$$\frac{J_z}{W} = \frac{\iint \rho d\rho d\phi (\mathbf{r} \times \langle \mathbf{E} \times \mathbf{B} \rangle)_z}{c \iint \rho d\rho d\phi \langle \mathbf{E} \times \mathbf{B} \rangle_z} = \frac{l + \sigma}{\omega} \tag{2.13}$$

And we see that in the absence of a phase term $e^{il\phi}$ the result would be the usual ratio for spin angular momentum divided by energy. The expression in eq.(2.13) is interesting.When integrated over the beam it is found that the ratio of spin angular momentum to energy depends only on σ, which for left- and right-handed circularly polarized light is ± 1. However, it can be shown that the local value depends on the intensity gradient in the beam. Consequently, in a beam that satisfies the paraxial condition, this means inevitably that the ratio changes from place to place.

The relation(2.13) important because they demonstrate that even for unpolarized light, a beam with an azimuthal phase term $e^{il\phi}$ possesses angular momentum. This may well prove not to be a necessary condition for orbital angular momentum, but within the paraxial approximation it appears to be sufficient.

2.2.3 Lagurre-Gausian (LG) light beams

As we have seen from the equations from this chapter that even unpolarized light, a beam with $e^{il\phi}$ possesses angular momentum. Now its time to see physically realizble light beams.An experimentally realizavle field of this form is that of Lagurre-Gausian moede, whose distribution is an allowed solution of the paraxial equation(2.1). The analytical forms of the amplitude and phase function of the LG set of phase-structred light beams are best displayed in cylindrical coordinates $r = (\rho, \phi, z)$. Each beam propagates along the z-axis with the frequency ω and axial wave-vector $\mathbf{k_z}$.Also each LG mode is characterized by two integers: the winding number l which is an non-zero integer and $p \geq 0$ is the radial number.

The amplitude function is:

$$\begin{aligned}
u_{lp}^{LG} &= \frac{u_{k_z,00} C_{|l|p}}{\omega(z)} \left(\frac{\rho\sqrt{2}}{\omega(z)} \right) \exp\left[-\frac{2\rho^2}{\omega^2(z)} \right] \\
&\times L_p^l \left(\frac{2\rho^2}{\omega^2(z)} \right)
\end{aligned} \tag{2.14}$$

Here $L_p^{|l|}$ are the associated Lagurre polynomials; $u_{k_z 00}$ is the amplitude for a corresponding plane wave of wave-vector k_z; $C_{|l|,p} = \sqrt{\frac{p!}{(|l|+p)!}}$ and $\omega(z)$ is the beam waist at position z such that $\omega^2(z) = \omega_0 \left(\frac{z_R^2 + z^2}{z_R^2} \right)$, where ω_0 is the the width of the beam ($\omega(0)$) and z_R is the Rayleigh range which $z_R = \frac{\pi\omega_0^2}{\lambda} = \frac{1}{2}k_z\omega_0^2$.

Now the phase function is:

$$\Theta_{k_z lp} = k_z z + l\phi + \Theta_{Gouy} + \Theta_{curv} - \omega t \tag{2.15}$$

Where

$$\begin{aligned}
\Theta_{Gouy} &= -(2p + |l| + 1) \tan \frac{z}{z_R} \\
\Theta_{curv} &= \frac{k_z \rho^2 z}{z(z^2 + z_R^2)}
\end{aligned} \tag{2.16}$$

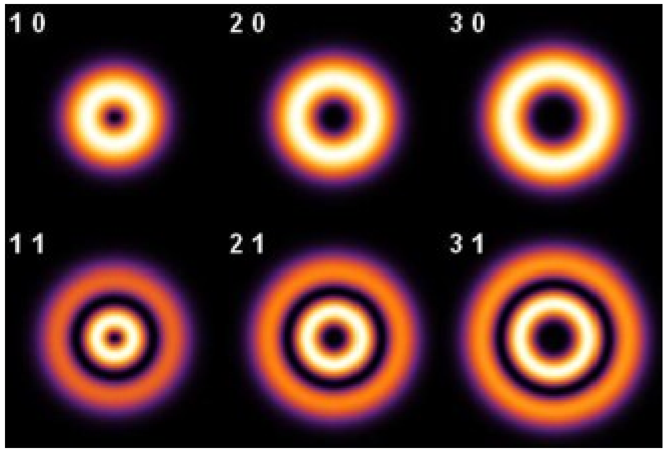

Figure 2.1: The intensity distributions at the focal plane of some Laguerre-Gaussian beams LG_{lp}

For such an amplitude distribution the linear momentum density is readily calculated, and for the purpose of this discussion it suffices simply to consider linearly polarized light.The linear momentum density from eq.(2.4) after ignoring the $\partial u/\partial z$ is :

$$\mathbf{P} = \varepsilon_0 \left\{ \frac{\omega k \rho z}{(z_R^2 + z^2)} \hat{\rho} + \frac{\omega l}{\rho} \hat{\phi} + \omega k \hat{z} \right\} \tag{2.17}$$

while the cross product with $\mathbf{r}$ gives the angular momentum density,

$$\mathbf{j} = \varepsilon_0 \left\{ -\omega l \frac{z}{\rho} \hat{\rho} - \omega k \rho \left(\frac{z_R^2}{z_R^2 + z^2} \right) \hat{\phi} + \omega l \hat{z} \right\} |u|^2 \tag{2.18}$$

We may note that the magnitude of the z-component is again simply $\varepsilon_0 l \omega |u|^2$ and is locally a constant for all z.

Fig.(2.1) displays the intensity distribution of the LG modes LG_{lp} showing only the lowest values of l and p. On the first row are single-ring modes $(l, p) = (1, 0), (2, 0), (3, 0)$ called the doughnut modes and $(l, p) = (1, 1), (2, 1), (3, 1)$ with two rings. In general for an LG_{lp} beam there are p + 1 rings. Note that the doughnut modes are much discussed in optical vortex applications and are often highlighted as the simplest modes displaying the OAM properties of the Laguerre-Gaussian modes. The ring of a doughnut mode has a radius at high intensity given by $\rho_{max} = \omega_0 \sqrt{|l|/2}$.

Small values of z_R or ω_0 indicate strong focusing when the LG beam is generated from an ordinary laser beam using, for example, a spatial light modulator (SLM). Tightly focused laser light can now be generated with spots in the sub-wavelength scale. This should not be confused with a situation where an LG light of a relatively large beam waist once produced can itself be focused further to tight spots of sub-wavelength dimensions

Tightly focused beams, including linearly polarized beams, have been shown to possess a substantial longitudinal electric field component and associated magnetic field components. These additional components are negligibly small for beams with relatively larger waists, and become significant only when the beam waists are sufficiently small. The new field components have important consequences for the properties of the beam itself and its interaction with matter.

Laguerre-Gaussian beams are also characterized by their convergence phase functions, namely the Gouy phase and the curvature phase defined in eq.(2.16). These additional phases functions have significant variations in the vicinity of the focal plane. Beside optical beams the Gouy phase has featured in other focused beams, including acoustic and electron vortex beams. It is clear in the case of LG beams that its magnitude increases with increasing winding number $|l|$ and/or the radial number p and both the Gouy and the curvature phase terms increase in magnitude with tighter focusing corresponding to smaller ω_0 or z_R.

2.3 Non-paraxial Light Beams

The non-paraxial regime is the full theory of optical vortices formally as the solutions of Maxwell's equations without resort to the paraxial approximation.The separation of spin and orbital angular momentum found for paraxial beams is, at fist sight, somewhat surprising. It is necessary to consider whether the result is an artefact of the paraxial approximation and to what extent a well-defined orbital angular momentum is actually possible. The linear momentum and angular momentum densities of an electromagnetic field may be written as

$$\mathbf{p} = \varepsilon_0(\mathbf{E} \times \mathbf{B}); \qquad \mathbf{j} = \varepsilon_0(\mathbf{r}\times(\mathbf{E} \times \mathbf{B})) \tag{2.19}$$

These densities may be integrated over all space to give the total linear momentum and angdar momentum associated with the field. For monochromatic fields we use the complex notation

$$\mathbf{E} = \frac{1}{2}\left(\tilde{\mathbf{E}}e^{-i\omega t} + \tilde{\mathbf{E}}^* e^{+i\omega t}\right); \quad \mathbf{B} = \frac{1}{2}\left(\tilde{\mathbf{B}}e^{-i\omega t} + \tilde{\mathbf{B}}^* e^{+i\omega t}\right) \tag{2.20}$$

With eliminating the magnetic field by using the Maxwell equation $i\omega\tilde{\mathbf{B}} = \nabla\times\tilde{\mathbf{E}}$ We can obtain the linear momentum , angular momentum and energy densities. After that by integration of the respective densities over the x-y plane we can obtain the linear momentum, angular momentum and energy per unit length. The quantities are not in general time-independent and must be cycle-averaged; this gives

$$\begin{aligned}
\mathbf{P} &= \frac{\varepsilon_0}{i\omega}\iint dxdy\,\{\mathbf{E}^* \times (\nabla \times \mathbf{E})\}\,,\\
\mathbf{J} &= \frac{\varepsilon_0}{2i\omega}\iint dxdy\,\{\mathbf{r} \times \mathbf{E}^* \times (\nabla \times \mathbf{E})\}\,,\\
W &= \frac{\varepsilon_0}{2}\iint dxdy\,\{\mathbf{E}^* \cdot \mathbf{E}\}
\end{aligned} \tag{2.21}$$

These expressions do not depend on the paraxial approximation and, within the defined limits, are general results whch follow from Maxwell's equations. From the knowledge gained from the study of paraxial beams, the nonparaxial beams chosen were such that the electric field was of the form

$$\mathbf{E} = (\alpha\hat{x} + \beta\hat{y})E_T(\rho, z)e^{il\phi} + E_z\hat{z} \tag{2.22}$$

where E_T indicates transverse components and E_z longitudinal components. Note that the choice of x- and y-components of the field again allows polarization to be taken into account while the term $e^{il\phi}$ embraces the feature of the field found previously to be sufficient for the presence of orbital angular momentum in the beam. Note, too, that the z-component of the field has been included to ensure it is a transverse electromagnetic field of the kind associated with a laser. When the field is made to satisfy the time-independent wave equation, the Helmholtz equation, as well as $\nabla \cdot \mathbf{E} = 0$, the field is found to be of the form

$$\begin{aligned}
\mathbf{E} &= E_x\hat{x} + E_y\hat{y} + E_z\hat{z}\\
E_x &= \alpha\int_0^k d\kappa E(\kappa)e^{il\phi}\exp\left[i\sqrt{(k^2-\kappa^2)}z\right] J_l(\kappa\rho)\\
E_y &= \frac{\beta}{\alpha}E_x\\
E_z &= i\int_0^k d\kappa\frac{E(\kappa)}{\sqrt{(k^2-\kappa^2)}z}e^{il\phi}\exp\left[i\sqrt{(k^2-\kappa^2)}z\right]\\
&\quad\times\left\{F_{\alpha\beta}(\phi)[J_{l-1}(\kappa\rho) - J_{l+1}(\kappa\rho)] + \frac{i2l}{\kappa\rho}G_{\alpha\beta}(\phi)J_l(\kappa\rho)\right\}
\end{aligned} \tag{2.23}$$

Where

$$F_{\alpha\beta}(\phi) = \alpha\cos\phi + \beta\sin\phi \qquad G_{\alpha\beta}(\phi) = \beta\cos\phi - \alpha\sin\phi \tag{2.24}$$

in the above formulations $\mathbf{k}$ is the wavevector with κ and k_z its transverse and longitudinal components such that $k_z = \sqrt{k^2 - \kappa^2}$. The key point is to exploit the Bessel function set $J_l(\kappa\rho)$ as a complete orthonormal set of functions, which are exact solutions of the Helmholtz equation. In the Barnett–Allen formulation the LG field components are written as integral superpositions of the Bessel set. This leads to the definition of the non-paraxial electric field components as Hankel transforms[2]. So now in the case of LG modes the amplitude function $E(\kappa)$ in eq.(2.23) is

$$E(\kappa) = d_{lp}(\kappa) = \left[\left(\frac{\kappa}{k} \right)^{1+|l|+2p} \exp\left(-\frac{\kappa^2}{2k} z_R \right) \right] \tag{2.25}$$

Note that the form of the longitudinal field component in eq.(2.23) highlights the appearance of the l-dependence term, which changes sign when the sign of l changes. In the paraxial regime the effects of this field component are negligible for LG beams of beam waists ω_0 greater than a wavelength.

In general the calculation is long and complex, but the important result that follows is that

$$\frac{J_z}{W} = \frac{l+\sigma}{\omega} + \frac{\sigma}{\omega} \left[\frac{\int_0^k d\kappa \frac{|E(\kappa)|^2 \kappa}{(k^2-\kappa^2)}}{\int_0^k d\kappa \frac{|E(\kappa)|^2 (2k^2-\kappa^2)}{\kappa(k^2-\kappa^2)}} \right] \tag{2.26}$$

Clearly this is not the simple result of the paraxial beams in eq.(2.13) and it shows that the orbital and spin angular momenta cannot be separated even classically, but that there is an additional correction term. Nevertheless, the result is important in a number of ways. The most important is that the additional term depends only on σ and not on l. It follows, therefore, that the concept of a light beam possessing only orbital angular momentum is a valid one, because for linearly polarized light when $\sigma = 0$, the correction term is zero and the paraxial relation between J , and W is exactly eq.(2.13).

[2] In mathematics, the Hankel transform expresses any given function f(r) as the weighted sum of an infinite number of Bessel functions of the first kind.

Chapter 3

Further Readings

In the further weeks I'm going to concentrate on the following titles:

- Some other points on OAM like: Paraxial beams with small waists, Chirality and helicity, Multiple vortex beams
- Quantization of optical angular momentum
- Quantum features of structured light like: Measurment and entanglement