Project Report

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Contents

What is a Qubit

Just as a classical bit has two state – either 0 or 1 – a qubit also

has two state. Two possible states for a qubit are the states $|0\rangle$ and $|1\rangle$.

The difference between bits and qubits is that a qubit can be in a state other than $|0\rangle$ or $|1\rangle$. It is also possible to form linear combinations of states, often called superposition:

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle.$$

Put another way, the state of a qubit is a vector in a two-dimensional complex vector space. The special states $|0\rangle$ and $|1\rangle$ are known as computational basis states. We can examine a bit to determine whether it is in the state 0 or 1.

Thus, in general a qubit's state is a unit vector in a two-dimensional complex vector space.

Bloch Sphere

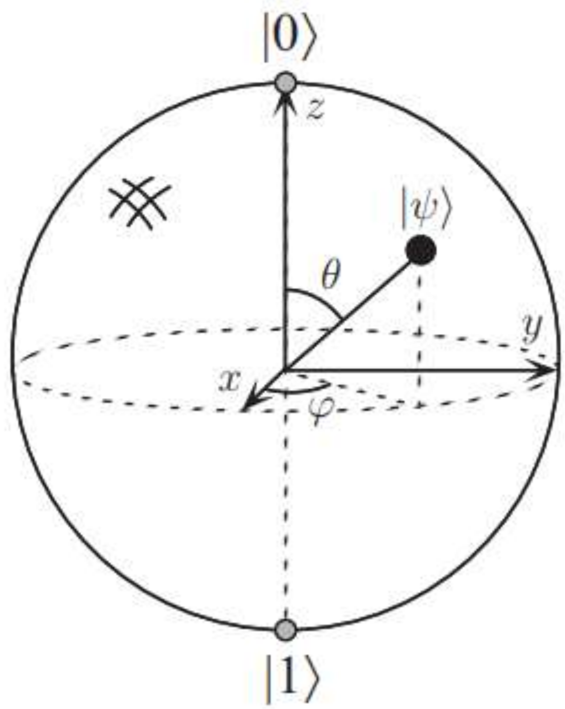
Because |α| 2 + |β| 2 = 1 , we can re-write the state as :

$$|\psi\rangle = e^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right)$$

where θ, φ and γ are real numbers. As We know we can ignore the phase factor because it's not important. We can write:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle.$$

The numbers θ and φ define a point on the unit three-dimensional sphere. This sphere is often called the Bloch sphere; it provides a useful



means of visualizing the state of a single qubit, and often serves as an excellent testbed for ideas about quantum computation and quantum information.

Paradoxically, there are an infinite number of points on the unit sphere, so that in principle one could store an entire text of Shakespeare in the infinite binary expansion of θ. However, this conclusion turns out to be misleading, because of the behavior of a qubit when observed. Recall that measurement of a qubit will give only either 0 or 1. It turns out that only if infinitely many identically prepared qubits were measured would one be able to determine α and β for a qubit in the state.

Optical examples as a Qubit

1- Polarization: We can encode a single qubit into a single photon in the polarization basis using the horizontal and vertical polarization degrees of freedom.
2- Dual-rail (Spatial modes): One can employ ‘dual rail’ encoding, whereby a single photon is placed into a superposition across two spatial modes.
3- Photon Number: We can encode a single qubit into the Fock or the number states of the photons.
4- Time bin: In time-bin encoding we define our basis of modes as distinct, non-overlapping time bins.

Polarization

In order to take into account linear superposition, it is natural to introduce a two-dimensional Hilbert Space for the mathematical description of polarization. Any polarization state can be put into correspondence with a vector in this space. We can, for example, choose as orthogonal basis vectors of H the vectors $|x\rangle$ and $|y\rangle$ corresponding to linear polarizations along Ox and Oy. Any Polarization can be written as:

$$|\Phi\rangle = \lambda|x\rangle + \mu|y\rangle.$$

So like that a state linearly polarized along θwill be denoted |θ⟩ with :

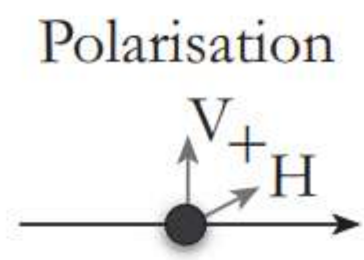
$$|\theta\rangle = \cos\theta\,|x\rangle + \sin\theta|y\rangle$$

The probability amplitude for a photon polarized along to pass through an analyzer oriented along α is given by ⟨α|θ⟩ which is :

$$a(\theta \to \alpha) = \cos(\theta - \alpha) = \langle\alpha|\theta\rangle.$$

And with that given the probability is:

$$\mathrm{p}(\theta \to \alpha) = \cos^2(\theta - \alpha) = |\langle\alpha|\theta\rangle|^2.$$



Photon Number

The quantization of Maxwell's equations thus implies the existence of elementary excitations of the electromagnetic field with quantized energy $\hbar\omega_\lambda$. These excitations, or particles, are the photons. With that said the Hamiltonian of the quantized free electromagnetic field is an infinite collection of uncoupled harmonic oscillators. Each of which is described by a Hamiltonian $\mathbf{H}_\lambda = \hbar\omega_\lambda(\mathbf{n}_\lambda + 1/2)$. Which the $\mathbf{n}_\lambda$ is the number operator which eigenvectors will be the same set as the energy eigenvectors and are known as the Fock states or the number states.

the eigenvalues of this operators will determine the energy level of the system or in other world the numbers of photon in the system.

In other word a single-mode system in the Fock state $|n\rangle$ contains exactly n excitations of energy $\hbar\omega$.

If we can produce single photons, then we can use the existence of a single photon as a qubit. Of course, the photon-number degree of freedom need not be limited to 0 or 1 photons. By fully exploiting the photon-number degree of freedom, we can encode a qudit of arbitrary dimension into a single optical mode.

$$|\psi\rangle_{\text{qudit}} \equiv \sum_{n=0}^{\infty} \alpha_n |n\rangle.$$

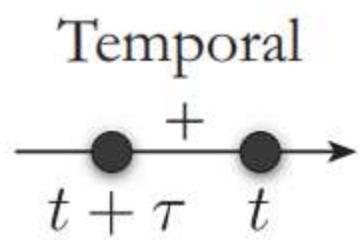
This may give the impression that a single optical mode has infinite information capacity. Needless to say, this sounds too good to be true, and it is. Loss decoherers photon-number-encoded states exponentially with photon number because for large photon number the probability of a number state retaining its photon number exponentially asymptotes to zero.

Dual-rail

Let us consider two cavities, whose total energy is $\hbar\omega_\lambda$, and take the two states of a qubit as being whether the photon is in one cavity ($|01\rangle$) or the other ($|10\rangle$). The physical state of a superposition would thus be written as $c0|01\rangle + c1|10\rangle$; we shall call this the dual-rail representation.

Time bin

In time-bin encoding we define our basis of modes (whether qubits or higher dimensional qudits) as distinct, no overlapping time bins, which are localized wave packets in the temporal degree of freedom, each separated from the next by some fixed interval τ. Time-bin encoding arises naturally in architectures where the photon source driving the system is operating at a high repetition rate, R, in which case τ = 1/R.



Quantum Computation

Changes occurring to a quantum state can be described using the language of quantum computation. a quantum computer is built from a quantum circuit containing wires and elementary quantum gates to carry around and manipulate the quantum information.

Single qubit gates

Classical computer circuits consist of wires and logic gates. Consider, for example, classical single bit logic gates. The only non-trivial member of this class is the gate, whose operation is defined by its truth table, in which $0 \to 1$ and $1 \to 0$, that is, the 0 and 1 states are interchanged.

Can an analogous quantum gate for qubits be defined? Imagine that we had some process which took the state $|0\rangle$ to the state $|1\rangle$, and vice versa. However, specifying the action of the gate on the states $|0\rangle$ and $|1\rangle$ does not tell us what happens to superposition of the states $|0\rangle$ and $|1\rangle$, without further knowledge about the properties of quantum gates. In fact, the quantum gate acts linearly. For example the X gate, it takes the state $\alpha|0\rangle + \beta|1\rangle$ to the corresponding state in which the role of $|0\rangle$ and $|1\rangle$ have been interchanged, $\alpha|1\rangle + \beta|0\rangle$.

There is a convenient way of representing the quantum gate in matrix form, which follows directly from the linearity of quantum gates. Suppose we define a matrix X to represent the quantum gate as follows:

$$X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

If the quantum state $\alpha|0\rangle + \beta|1\rangle$ is written in a vector notation as:

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

with the top entry corresponding to the amplitude for $|0\rangle$ and the bottom entry the amplitude for $|1\rangle$, then the corresponding output from the quantum gate is:

$$X \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$$

So quantum gates on a single qubit can be described by two by two matrices. Are there any constraints on what matrices may be used as quantum gates? It turns out that there are. For the constraints on quantum gates we can refer to the Postulates of quantum mechanics:

"The evolution of a closed quantum system is described by a unitary transformation. That is, the state $|\psi\rangle$ of the system at time t1 is related to the state $|\psi'\rangle$ of the system at time t2 by a unitary operator U which depends only on the times t1 and t2"

$$|\psi'\rangle = U|\psi\rangle$$

This Postulate is true for the quantum gates (even if the system is not close) and that comes from the fact that the sum of the probabilities should remain constant and 1 in every evolution.

Amazingly, this unitarily constraint is the only constraint on quantum gates. Any unitary matrix specifies a valid quantum gate. The interesting implication is that in contrast to the classical case, where only one non-trivial single bit exists (The NOT gate), there are many non-trivial single qubit gates. Two important ones which we shall use later are the Z gate:

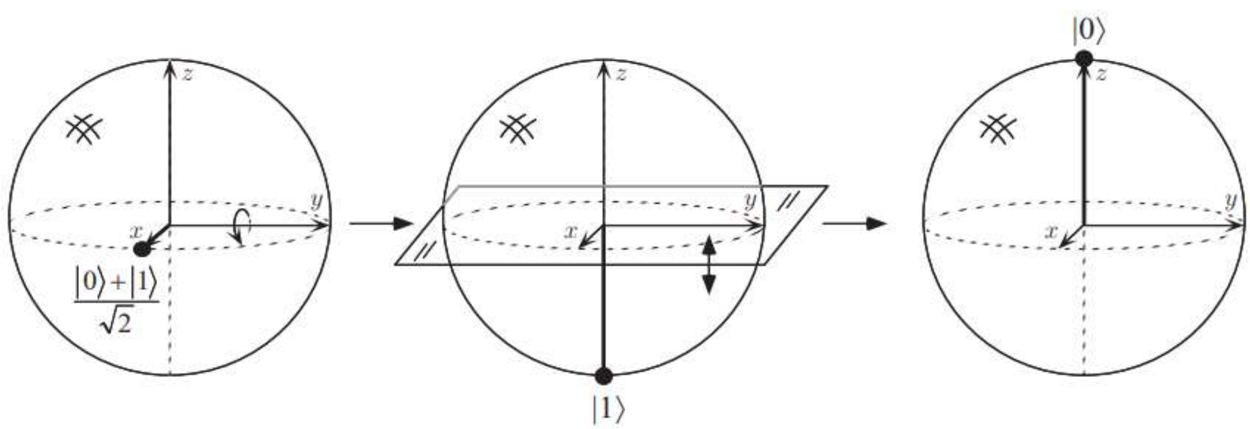
$$Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

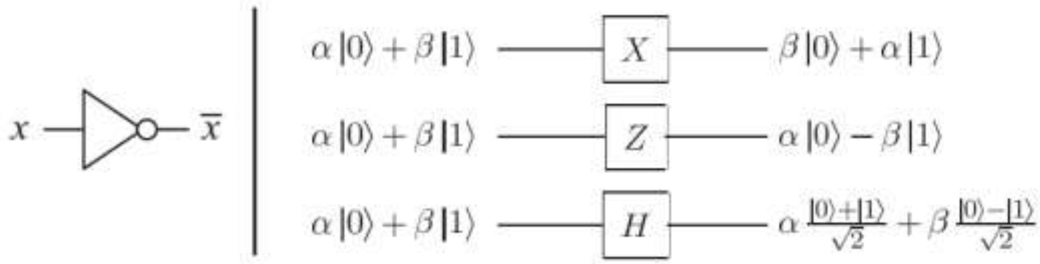
which leaves |0⟩ unchanged, and flips the sign of |1⟩ to give −|1⟩, and the Hadamard gate,

$$H \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

it turns a |0⟩ into (|0⟩ + |1⟩)/ √ 2 (first column of H), 'halfway' between |0⟩ and |1⟩, and turns |1⟩ into (|0⟩ −|1⟩)/ √ 2 (second column of H), which is also 'halfway' between |0⟩ and |1⟩.

Because of the fact that the sum of the probabilities is constant every single bit gate is just rotations in the Bloch sphere for example the Hadamard operation is just a rotation of the sphere about the yˆ axis by 90◦, followed by a rotation about the xˆ axis by 180◦.

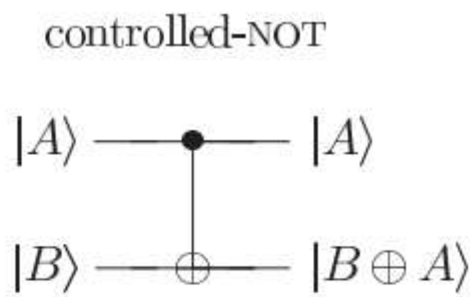




Multiple qubit gates

Now let us generalize from one to multiple qubits. An important theoretical result is that any function on bits can be computed from the composition of NAND gates alone, which is thus known as a universal gate.

The prototypical multi-qubit quantum logic gate is the controlled-NOT or CNOT gate. This gate has two input qubits, known as the control qubit and the target qubit, respectively. The circuit representation for the is shown below.



the top line represents the control qubit, while the bottom line represents the target qubit. The action of the gate may be described as follows. If the control qubit is set to 0, then the target qubit is left alone. If the control qubit is set to 1, then the target qubit is flipped.

$$|00\rangle \rightarrow |00\rangle;\ |01\rangle \rightarrow |01\rangle;\ |10\rangle \rightarrow |11\rangle;\ |11\rangle \rightarrow |10\rangle.$$

Another way of describing the CNOT is as a generalization of the classical XOR gate, since the action of the gate may be summarized as $|A, B\rangle \rightarrow |A, B \oplus A\rangle$, where $\oplus$ is addition modulo two, which is exactly what the XOR gate does. another way of describing the action of the is to give a matrix representation:

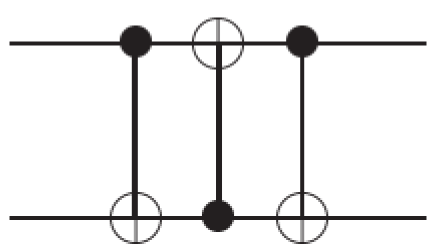
$$U_{CN} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

can easily verify that the first column of UCN describes the transformation that occurs to $|00\rangle$, and similarly for the other computational basis states, $|01\rangle$, $|10\rangle$, and $|11\rangle$.

Of course, there are many interesting quantum gates other than the controlled-NOT. However, in a sense the controlled- and single qubit gates are the prototypes for all other gates because of the following remarkable universality result: Any multiple qubit logic gate may be composed from CNOT and single qubit gates.

Quantum circuits

A simple quantum circuit containing three quantum gates is shown below.

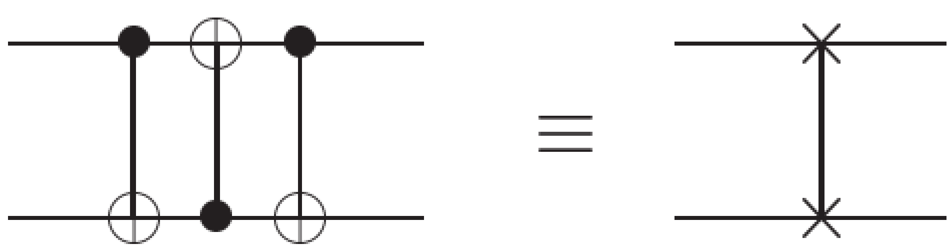


The circuit is to be read from left-to-right. Each line in the circuit represents a wire in the quantum circuit. This wire does not necessarily correspond to a physical wire; it may correspond instead to the passage of time, or perhaps to a physical particle such as a photon – a particle of light – moving from one location to another through space. It is conventional to assume that the state input to the circuit is a computational basis state, usually the state consisting of all $|0\rangle$ s.

The circuit in the Figure accomplishes a simple but useful task – it swaps the states of the two qubits. To see that this circuit accomplishes the swap operation, note that the sequence of gates has the following sequence of effects on a computational basis state $|a, b\rangle$,

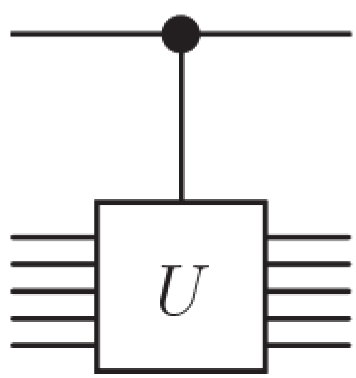
$$\begin{aligned}|a,b\rangle &\longrightarrow |a, a\oplus b\rangle \\ &\longrightarrow |a\oplus(a\oplus b), a\oplus b\rangle = |b, a\oplus b\rangle \\ &\longrightarrow |b, (a\oplus b)\oplus b\rangle = |b, a\rangle\ ,\end{aligned}$$

It is common to show the circuit like below too:

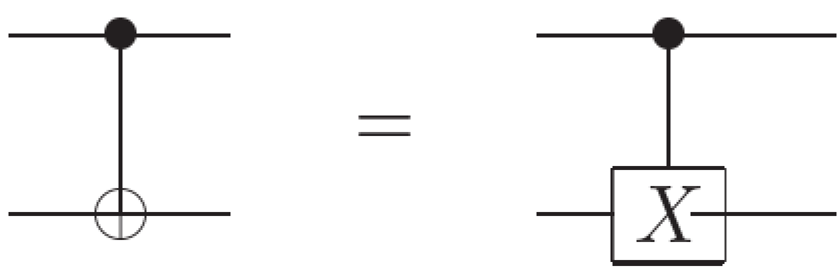


There are a few features allowed in classical circuits that are not usually present in quantum circuits. First of all, we don't allow 'loops', that is, feedback from one part of the quantum circuit to another; we say the circuit is acyclic. Second, classical circuits allow wires to be 'joined' together, an operation known as FANIN, with the resulting single wire containing the bitwise of the inputs. Obviously this operation is not reversible and therefore not unitary, so we don't allow FANIN in our quantum circuits. Third, the inverse operation, FANOUT, whereby several copies of a bit are produced. FANOUT is also not allowed in quantum circuits. In fact, it turns out that quantum mechanics forbids the copying of a qubit, making the operation impossible!

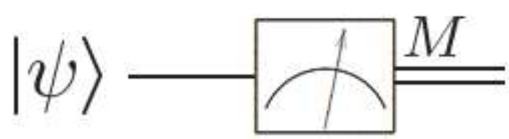
Suppose U is any unitary matrix acting on some number n of qubits, so U can be regarded as a quantum gate on those qubits. Then we can define a Controlled-U gate which is a natural extension of the controlled- gate. Such a gate has a single control qubit, indicated by the line with the black dot, and n target qubits, indicated by the boxed U. If the control qubit is set to 0 then nothing happens to the target qubits. If the control qubit is set to 1 then the gate U is applied to the target qubits.



We could even write the Controlled-NOT gate in the form a Controlled-U gate with U=X.



Another important operation is measurement, which we represent by a ‘meter’ symbol as shown below. This operation converts a single qubit state $|\psi\rangle = \alpha|0\rangle+\beta|1\rangle$ into a probabilistic classical bit M (distinguished from a qubit by drawing it as a double-line wire), which is 0 with probability $|\alpha|^2$, or 1 with probability $|\beta|^2$.



Manipulating qubits

In this part I have looked upon the question how can we build these quantum gates or more specifically how can we change the states of the qubit. For an example I have researched the manipulation of spin-1/2 qubits and The Rabi oscillations.

Interaction of a spin 1/2 with a magnetic field

An elementary calculation of classical physics shows that the magnetic moment $\boldsymbol{\mu}$ of a rotating charged system is proportional to its angular momentum **J**, $\boldsymbol{\mu} = (\gamma/\hbar)\mathbf{J}$, where γ is called the gyromagnetic ratio. The proton spin is in fact an intrinsic angular momentum, rather as though the proton were spinning on its axis like a top. The proton spin is associated with the operator $\hbar\boldsymbol{\sigma}$ /2. The magnetic moment, also a vector, is associated with a corresponding operator which must be proportional to the intrinsic angular momentum, because the only vector (actually, axial vector) at our disposal is $\boldsymbol{\sigma}$:

$$\vec{\mu} = \frac{1}{2}\gamma_{\mathrm{p}}\vec{\sigma}, \qquad \gamma_{\mathrm{p}} = 5.59\,\frac{q_{\mathrm{p}}\hbar}{2m_{\mathrm{p}}},$$

Now let a spin 1/2 be placed in a classical magnetic field with a periodic component as in:

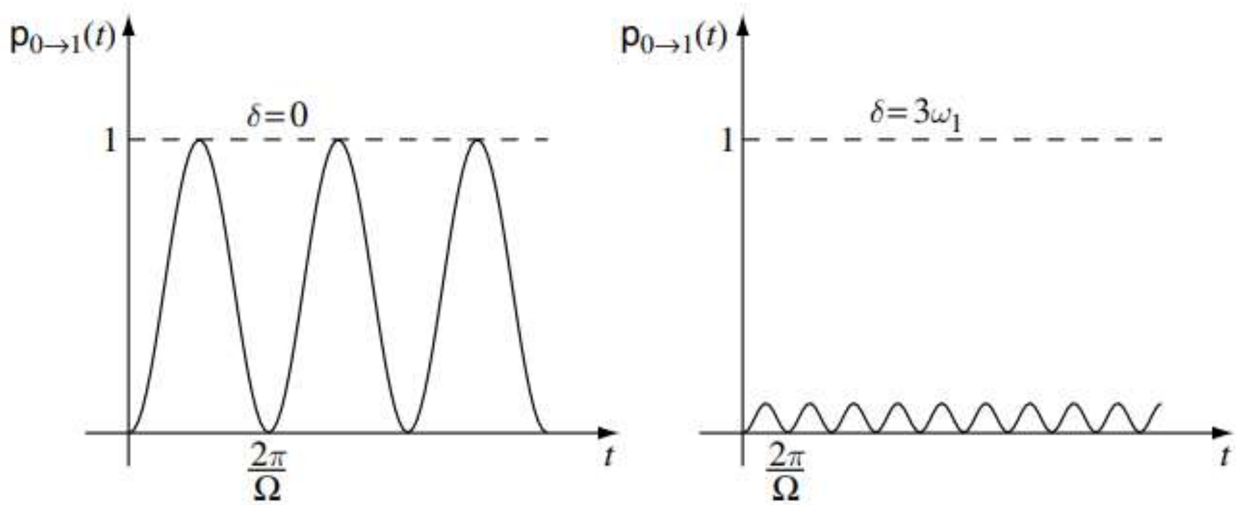
$$\vec{B} = \vec{B}_0\hat{z} + B_1(\hat{x}\cos\omega t - \hat{y}\sin\omega t)$$

The form of **H** (Hamiltonian Operator) then is:

$$\hat{H}(t) = -\frac{\hbar}{2}\begin{pmatrix} \omega_0 & \omega_1 e^{i\omega t} \\ \omega_1 e^{-i\omega t} & -\omega_0 \end{pmatrix}$$

where ω1 is proportional to B1 and can therefore be adjusted at will. The frequency ω1 is called the Rabi frequency. As it can be seen that the energy eigenvalues and eigenstates are variable with time. So after solving the evolution equations with the Hamiltonian that we have we can see that if at time t = 0 the qubit is in the state |0⟩ , at time t it will have a probability p0→1(t) of being found in the state |1⟩ given by:

$$p_{0\to 1}(t) = \left(\frac{\omega_1}{\Omega}\right)^2 \sin^2\frac{\Omega t}{2}, \qquad \Omega = \sqrt{(\omega - \omega_0)^2 + \omega_1^2}$$



The detuning is defined as δ= ω−ω0.

This is the phenomenon of Rabi oscillations. The oscillation between the levels |0⟩ and |1⟩ has maximum amplitude for $\omega = \omega_0$, that is, at resonance:

$$p_{0\to 1}(t) = \sin^2 \frac{\omega_1 t}{2}$$

To go from the state |0⟩ to the state |1⟩ it is sufficient to adjust the time t during which the rotating field acts:

$$\frac{\omega_1 t}{2} = \frac{\pi}{2}, \qquad t = \frac{\pi}{\omega_1}$$

This is called a π pulse. But the cool things happen when we are in a time like $t = \pi/2\omega_1$ then our quantum system states are :

$$|0\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

In summary, Rabi oscillations are the basic process used to manipulate qubits. These oscillations are obtained by exposing qubits to periodic electric or magnetic fields during suitably adjusted time intervals.

Entanglement

Entanglement states are states which are n-particle states that cannot be factored into n single-particle state in any basis.

Let us take n =2 and discuss the meaning of the definition. The Hilbert space H associated with a composite system is the tensor product of the Hilbert spaces H1 and H2, associated with the system's components 1 and 2. So we have:

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

Now we have to choose a basis for the Hilbert space, the most natural basis for the Hilbert space H is constructed from the tensor products of the basis vectors of H1 and H2. So if the basis for the two spaces are:

$$\left\{|0\rangle_1, |1\rangle_1\right\}, \qquad \left\{|0\rangle_2, |1\rangle_2\right\}$$

Then a basis for the Hilbert space H is given by the four vectors:

$$\left\{|0\rangle_1 \otimes |0\rangle_2,\ |0\rangle_1 \otimes |1\rangle_2,\ |1\rangle_1 \otimes |0\rangle_2,\ |1\rangle_1 \otimes |1\rangle_2\right\}$$

But here is where the problem starts because according to the super position principle the most general state in the Hilbert space H can written as follow:

$$|\psi\rangle = \sum_{i,j=0}^{1} c_{ij} |i\rangle_1 \otimes |j\rangle_2$$

Which in general is not a tensor product of two single state unless $c_{00}c_{11}=c_{10}c_{01}$ but this condition has no reason to be valid all the time.

So what are these states? These states are called “Entangled states” which cannot be factored into 2 single-particle state.

Bell states

We have seen a basis for the joint Hilbert space but, alternatively, in such a space we could also choose very different bases which even could be entangled. A maximally entangled basis for two independent particles, two qubits, is

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1\rangle|0\rangle),$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle),$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle),$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle - |1\rangle|1\rangle).$$

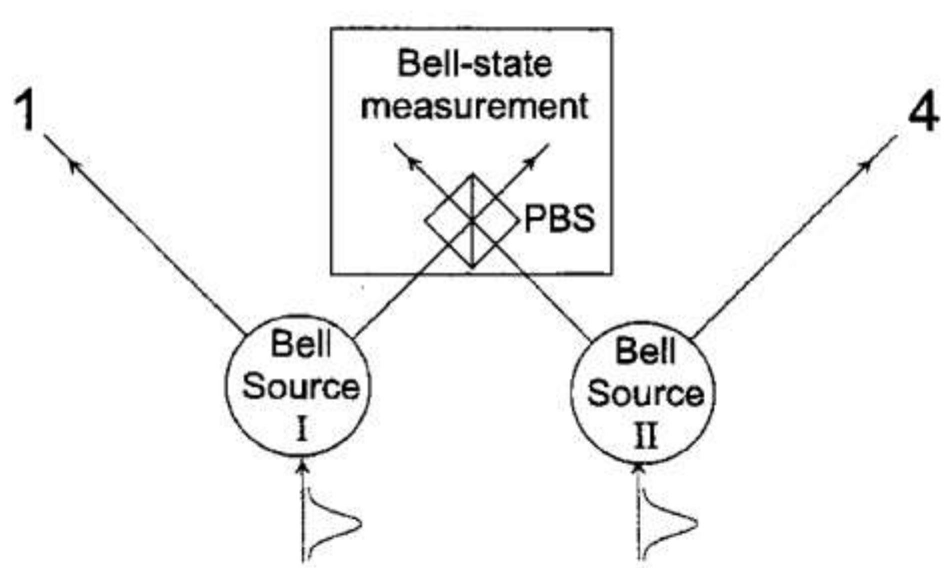
This is the so-called Bell basis. It is important to notice that here we can still encode two bits of information, that is we have four different possibilities, but now this encoding is done in such a way that none of the bits carries any well-defined information on its own. All information is encoded into relational properties of the two qubits.

It thus follows immediately that in order to read out the information one has to have access to both qubits. The corresponding measurement is called a Bell-state measurement. This is to be compared with the classical case where access to one qubit is simply enough to determine the answer to one yes/no question. In contrast, in the case of the maximally entangled basis access to an individual qubit does not provide any information.

Entanglement swap

Entanglement used to be considered as a consequence of the fact that the entangled particles interacted in their past or that they came from a common source.

In the simplest case of entanglement swapping we take two entangled pairs and subject two particles, one from each source, to a Bell-state measurement. Then the other two particles which have never interacted in the past and also did not come from a common source are projected into an entangled state.



But how is it possible? Let us, for simplicity of discussion, just consider the case where we have sources that produce our two qubits in the state $|\psi^-\rangle$. in terms of its information content, the statement is that we know the two qubits are different whatever basis we choose. We thus know simply by the choice of preparation that in each of the two entangled pairs to be used in entanglement swapping the two qubits are completely different.

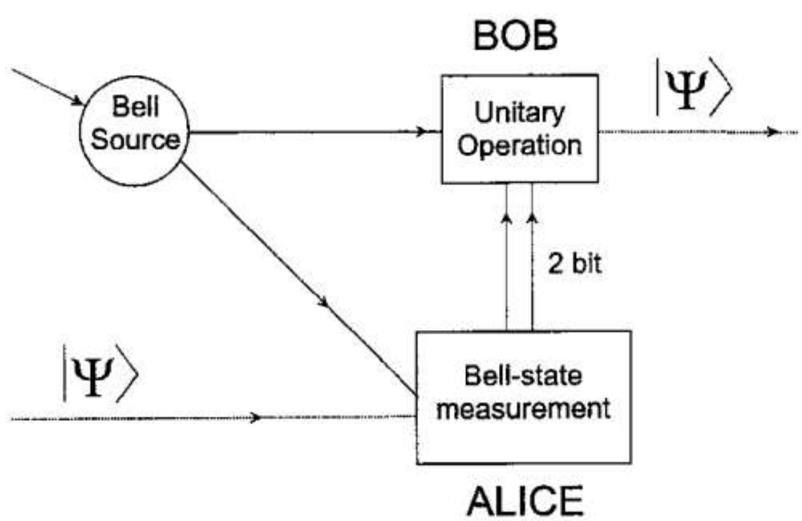
At first we note that, whichever states we would produce at the sources, a fair Bell-state analyzer will return any of the four possible answers with equal probability of 25%. That is, the action of the Bell-state analyzer is such that it projects the two photons onto an entangled state and, since in our case the two qubits are themselves members of maximally entangled pairs and therefore carry no information, this has to happen with equal probability for all four Bell states measured. The Bell-state measurement does not reveal any information about any of the qubits emitted by one of the two sources nor any joint information about each source. Yet, what we gain is joint (or relative) information about the two sources.

Suppose, specifically, that in a certain experimental run we obtain the result $|\psi^-\rangle$ for the Bell-state measurement. We then know that qubits 2 and 3 have been projected by the measurement onto a state which is characterized by the fact that these two qubits are different in whatever basis. Now by the properties of source 1 we know that qubit 1 and qubit 2 are different. By the result of the specific Bell state measurement we know that qubit 2 and 3 are different and, finally, from the property of source 2 we know that qubit 3 and 4 are different. Therefore, since our qubits are defined in a Hilbert-space of dimension 2 only, we conclude that qubits 1 and 4 also have to be different in any basis. Therefore they emerge in the anti-symmetric state $|\psi^-\rangle$. As we can see now that qubits 1 and 4 have been entangled without even being in contact with each other. This technic can be used to create entangle pairs in two distanced locations with the help of a repeater in the middle to shorten the distance.

Teleportation

A most remarkable application of the concept of entanglement is quantum teleportation. Suppose that Alice has an object which Bob, who could be anywhere, might need at a certain time. In classical physics what she can do is perform many precise measurements on the object and send the information to Bob who then can reconstitute the object. Yet, we know that in the end any measurement will run into the limitations imposed by quantum mechanics. no measurement whatsoever performed by Alice can reveal the full quantum state of the object.

To solve this problem there is a way. Alice and Bob have to share in advance an entangled pair of quantum states.



Let us consider for simplicity that the object is simply a two-state system, a qubit. Then Alice and Bob share from the beginning an ancillary entangled pair which for convenience we again consider to be in the state $|\Phi^+\rangle$. Subsequently Alice performs a Bell-state measurement on her qubit and one of the two ancillaries. As discussed above, for the case of entanglement swapping, Alice will obtain each one of the four possible answers with equal probability, that is her original qubit and her qubit from the ancillary pair will be projected onto any one of the four Bell-states each with probability 25%. We note again that this measurement does not reveal any information, neither about the properties of the original qubit nor about the properties of the ancillary pair.

So Alice obtains one of four possible results. She then broadcasts this information, that is two classical bits, such that Bob can receive them. By now Bob is in possession of a specific state as a consequence of Alice's Bell-state measurement. Performing one of four unitary transformations depending on Alice's specific result Bob can transform his particle into the original qubit.

For seeing the process consider The state to be teleported is $|\psi\rangle = \alpha|0\rangle+\beta|1\rangle$, where α and β are unknown amplitudes. And the two Bell states to be in $|\Phi^+\rangle$ then we can write the joint state as :

$$|\psi\rangle\left|\Phi^{+}\right\rangle = \left|\psi^{joint}\right\rangle$$

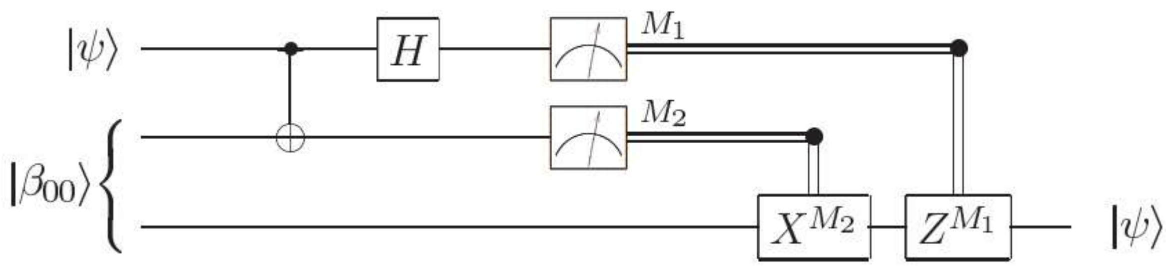
$$= \frac{1}{\sqrt{2}}\left[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|00\rangle + |11\rangle)\right]$$

After we write the above equations in the Bell-state basis for one of the Bell qubits and the state we have then:

$$\left|\psi^{joint}\right\rangle = \frac{1}{2\sqrt{2}}\left[\left|\Phi^{+}\right\rangle (\alpha|0\rangle + \beta|1\rangle) + \left|\Phi^{-}\right\rangle (\alpha|0\rangle - \beta|1\rangle) + \left|\psi^{+}\right\rangle (\alpha|1\rangle + \beta|0\rangle) + \left|\psi^{-}\right\rangle (\alpha|1\rangle + \beta|0\rangle)\right]$$

So we see that for every result the state of the qubit that bob has is identified and can be converted to the original state that we wanted to send trough one qubit gates.

This teleportation could also be done without the Bell-analyzer trough the circuit below:

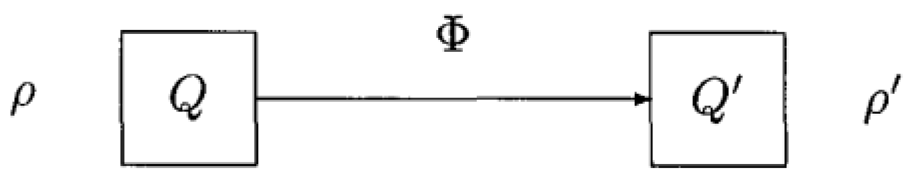


Which uses a normal basis of 00,01,10,11 for measuring but:

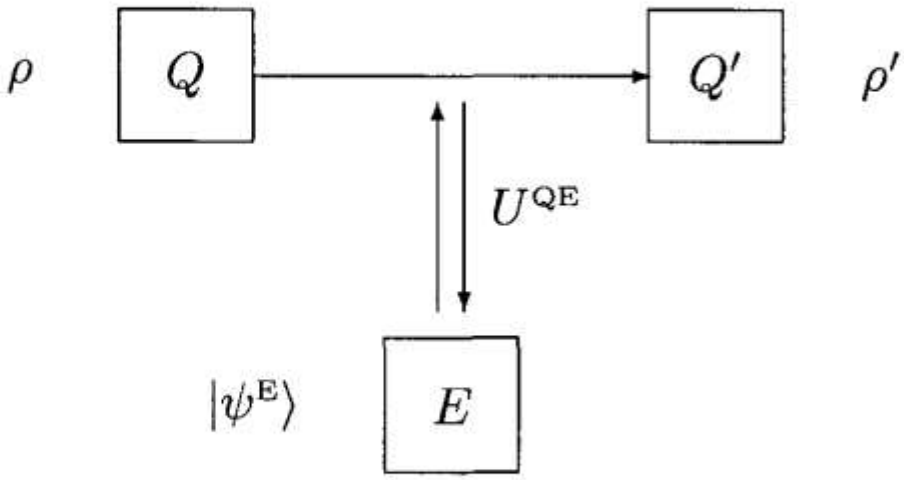
$$(|00\rangle + |11\rangle)/\sqrt{2} \equiv |\beta_{00}\rangle$$

Quantum Channels

Now we will dive in the definition and usage of quantum channels. The subject of quantum information theory has much the same motivation as that of the classical theory: Alice needs to send the preparation of a system—perhaps for the purpose of sending a real message or simply for the sheer pleasure of sending a quantum state—and the only way for it to get to Bob is to travel through territories unknown or at least uncontrollable. Those territories unknown or uncontrollable are known as the quantum channel.



The quantum state p is loaded onto the system Q and ultimately emerges as some state p' on a system Q' (possibly isomorphic to the original system but not necessarily). The unknowns or uncontrollability in the channel's action are represented by way of a mapping Φ: ρ—>ρ'. The question before us is to delineate the structure of the mappings Φ. The technical name for this structure is the set of trace-preserving completely positive linear maps. Perhaps the most interesting thing about the quantum channel at the outset is this: even if Alice and Bob have the maximum knowledge allowed by physical law about the interaction between the carrier and its environment and moreover the precise initial state of the environment (as depicted below), there may still be a necessary degradation in their signal. This is because the environment may become entangled with the carrier system.



Positive linear maps

An operator like is called positive if the expectation value of that operator on every state vector in the Hilbert space H_A is positive:

$$\langle v|X|v\rangle \geq 0 \qquad \forall\ v \in H_A.$$

In this Hilbert space the we show the set of linear operators with $L(H_A)$ so this means if $T \in L(H_A)$ then , $T: H_A \rightarrow H_A$.

The subset of positive linear operators is shown with $L^+(H_A)$ and the subset of $L^+(H_A)$ that has a trace equal to 1 is shown with $D(H_A)$.

$$D(H_A) \subset L^+(H_A) \subset L(H_A)$$

Now consider a map ε That maps ρ ($\rho \in (H_1)$) to ρ'($\rho' \in D(H_2)$) so :

$$\mathcal{E} : D(H_1) \longrightarrow D(H_2).$$

This map has some attributes:

1. It maps a Hermitian matrix to a Hermitian matrix.
2. It maps a positive matrix to a positive matrix.
3. It will preserve the trace.

So we call this Map a trace-preserving positive linear map and also we call ε a "Superoperator" so the conditions of this maps can be wrote as:

$$\begin{aligned} \mathcal{E}(X^\dagger) &= \mathcal{E}(X)^\dagger \\ Tr(\mathcal{E}(X)) &= Tr(X). \end{aligned}$$

And a map is trace-preserving completely positive linear map (CPT map) if:

$$\begin{aligned} &\mathcal{E}(X^\dagger) = (\mathcal{E}(X))^\dagger \\ &tr\,(\mathcal{E}(X)) = tr(X) \\ &\forall\, H_C, \quad \mathcal{E} \otimes id : L(H_A \otimes H_C) \longrightarrow L(H_B \otimes H_C) \geq 0. \end{aligned}$$

Kraus theorem: Every CPT map can be shown as:

$$\mathcal{E}(X) = \sum_m A_m X A_m^\dagger,$$

Where

$$\sum_m A_m^\dagger A_m = I.$$

We call these CPT maps quantum channels.

Now for seeing the effect of a CPT map on a density matrix we can write the Kraus theorem in the form of:

$$\varepsilon(\rho) = \sum_m tr\left(A_m \rho A_m^\dagger\right) \frac{A_m \rho A_m^\dagger}{tr\left(A_m \rho A_m^\dagger\right)} = \sum_m P_m \rho_m$$

Where P_m are the probabilities and is equal to:

$$P_m \equiv tr(A_m \rho A_m^\dagger)$$

And ρ_m are density matrices and are equal to:

$$\rho_m \equiv \frac{A_m \rho A_m^\dagger}{tr\left(A_m \rho A_m^\dagger\right)}$$

The physical meaning of this is that if we use a CPT map on a density operator then this density operator itself will evolve to another from a set of density operators each with a respect to a probability. So we can say that the state ρ is going to have an error of A_m with the probability of P_m but we don't know which so it's a mixture of all of them.

The point about quantum channel is that we don't use this term only in communication or sending a quantum state but we use this for any change and evolution in the state e.g. time evolution or any other change in the state. Mainly in communication we use a channel for the purpose of formulating the change occurring on the state that is sent to another party.

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