CHAPTER FIVE

Special Types of Linear Programming

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5.1. Transportation problem

- One important application of linear programming is the area of ***physical distribution*** (***transportation***) of goods from several ***supply centers (origins)*** to several ***demand centers (destinations).***

- Transportation problem involves a ***large number of variables*** (transportation/shipping routes) and ***constraints***, it takes a ***long time to solve*** it.

- Therefore, other methods (transportation algorithm) have been developed for this purpose.

- **Objective**: the objective is to determine the amount of commodities which should be transported from several sources to different destinations, at the ***minimum transportation cost and /or time.***
- **Sources or origins** are the places where goods ***originate from*** (like plants, warehouses etc)
- **Destinations** are places where goods are ***to be shipped.***
- It can also be applied to the ***maximization of some total value or utility***, in such a way that the ***profit is maximized***.

5.1.1. General Transportation Problem Model

The transportation algorithm requires the assumptions that:

- ✓All goods are **homogeneous**, so that *any origin is capable of supplying to any destination.*
- ✓Transportation costs are a **linear function** of (or *directly proportional* to) the quantity shipped over any route.
- ✓*Each source has a fixed supply of units*, where this entire supply must be distributed to the destinations.
- ✓Similarly, each destination has a fixed *demand* for units, where this entire demand must be received from the sources.

A transportation problem model, which has *'m' sending locations (origins)* and *'n' receiving locations (destinations),* provides a framework for presenting all relevant data. These are:

- ✓ *Quantity supply* of each origin (**SS_i**)
- ✓ *Quantity demand* of each destination (**DD_j**)
- ✓ Unit *transportation cost* from each origin to each destination (**C_{ij}**)

	Destination					
Source (origins)	To / From	D_1	D_2	...	D_n	Total Supply
	S_1	X_{11} C_{11}	X_{12} C_{12}	...	X_{1n} C_{1n}	SS_1
	S_2	X_{21} C_{21}	X_{22} C_{22}	...	X_{2n} C_{2n}	SS_2
	:	:	:	:	:	:
	S_m	X_{m1} C_{m1}	X_{m2} C_{m2}	...	X_{mn} C_{mn}	SS_m
	Total Demand	dd_1	dd_2	...	dd_m	SS / dd

Where:

- ✓ SS_i - *is total quantity of commodity available at origin I (total supply of origin i).*
- ✓ dd_j - *is total quantity of commodities needed at destinations j (total demand of destination j).*
- ✓ C_{ij} - *measures the costs of transporting one unit of commodity from source i to destination j.*
- ✓ X_{ij} - *is the quantity of commodities transported from i^{th} origin to j^{th} destination.*

The Linear Programming Representation of a Transportation Model

$$Minimize\ Z = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} x_{ij}$$

Subject to:

$$\sum_{j=1}^{n} x_{ij} = SS_i\ ,\ i = 1,2,\ldots,m\ \text{(supply constraints)}$$

$$\sum_{i=1}^{m} xij = DD_j\ ,\ j = 1,2,\ldots,n\ \text{(demand constraint)}$$

$x_{ij} \geq 0$ for all i and j.

Minimize (total transportation cost)

$$Z=C_{11}X_{11}+C_{12}X_{12}+C_{13}X_{13}+\ldots+C_{mn}X_{mn}$$

Subject to:

Capacity constraints (SS constraints)

$$X_{11}+X_{12}+\text{-----}+X_{1n}=SS_1$$
$$X_{21}+X_{22}+\text{-----}+X_{2n}=SS_2$$
$$\begin{matrix} . & . & . & . \\ . & . & . & . \end{matrix}$$
$$X_{m1}+X_{m2}+.\text{----}+X_{mn}=SS_m$$

Requirements constraints (DD constraint)

$$X_{11}+X_{21}+\text{-----}+X_{m1}=SS_1$$
$$X_{21}+X_{22}+\text{-----}+X_{m2}=SS_2$$
$$\begin{matrix} . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{matrix}$$
$$X_{n1}+X_{n2}+.\text{----}+X_{nm}=SS_n$$
$$X_{ij} \geq 0 \text{ for all } i \text{ and } j$$

<u>Before applying the transportation techniques (methods) to solve a specific problem, the problem should satisfy the following conditions.</u>

- *Supplies (SS)and requirements (DD) must be* ***expressed in the same unit****.*
- *This condition means that shipments received at any destination from different sources must be indistinguishable.*
- *In other words, all shipment must be measured in homogenous units.*

- *Total supply must equal to total demand* **$\sum SS = \sum DD$**

- *The problem satisfying this condition is called* ***balanced transportation problem****; otherwise it is known as* ***unbalanced transportation problem.***

- *The condition* $\sum SS = \sum DD$ *is the* ***necessary and sufficient condition*** *for the existence of feasible solution to the transportation problem.*

5.1.2. Methods of Solving Transportation Problem

- **Methods to get the initial solution:** even if there are different methods of such types, the following ***three common methods can be used***.

- ***The North-West Corner Method (NWC)***
- ***Minimum cost method (MCM), and***
- ***The Vogel's Approximation Method (VAM).***

1. North West Corner method

1st. **Balance the problem.** *That is see whether* **ΣDD = ΣSS.**

- ✓ ***If not open a dummy column or dummy row*** as the case may be and balance the problem.

2nd. **Allocate** as much as possible to the selected cell, and adjust the associated amounts of supply and demand by subtracting the allocated amount.

Since : ***The method starts at the northwest-corner cell (route) of the tableau (variable X_{11}).***

3rd. ***Cross out the row or column with zero supply or demand*** to indicate that ***no further assignments*** can be made in that row or column.

- ✓ If both a row and a column net to zero simultaneously, *cross out one only*, and leave a zero supply (demand) in the uncrossed-out row (column).

4th If *exactly one* row or column is left uncrossed out, ***stop.***

- ✓ Otherwise, move to the cell to the right if a column has just been crossed out or below if a row has been crossed out. Go to step 1.

5^{th} Make sure that all the rim conditions are satisfied and ***(m+n-1)*** cells are allocated.

6^{th} Once all the allocations are over, *i.e., both rim requirement (column and row i.e., availability* and requirement constraints) are satisfied, write allocations and ***calculate the cost of transportation***.

Example 1.

Let us consider an example at this juncture to illustrate the application of NWC rule

From \ To	D1	D2	D3	Total supply
S1	X11 5	X12 6	X13 7	70
S2	X21 4	X22 2	X32 5	30
S3	X31 1	X32 5	X33 4	50
Total Demand	65	42	43	150 \ 150

$\Sigma DD = \Sigma SS$. The North West Corner cell X_{11} is chosen for allocation..

- The origin S_1 ***has 70*** items and the destination D_1 ***requires only 65*** items.
- Hence it is enough to *allot 65 items from S_1 to D_1*.

- The origin S_1 which is alive with ***5 more items*** can supply to the destination to the right is alive with *5 more items can supply* to the destination to the right of D_1 namely D_2 whose requirement is 42. So, we supply *5 items to D_2 thereby the origin S_1 is exhausted.*

- D_2 ***requires 37 items*** more. Now consider the origin S_2 that ***has 30 items to spare***. We allot *30 items to the cell (X_{22})* so that the origin S_2 is exhausted.
- Then, move to origin ***S_3 and supply 7 more items*** to the destination D_2. Now the requirement of the destination D_2 is complete and ***S_3 is left with 43 items and the same can be allotted to the destination D_3.***
- Now the origin S_3 is emptied and the requirement at the destination D_3 is also complete. This completes the initial solution to the problem

From \ To	D1	D2	D3	Total supply
S1	65 | 5	5 | 6	7	70
S2	4	30 | 2	5	30
S3	1	7 | 5	43 | 4	50
Total Demand	65	42	43	150 / 150

The total cost of transportation by this method will be:

(65 × 5) + (5 × 6) + (30 × 2) + (7 × 5) + (43× 4) = 325 + 30 + 60 + 35 + 172 = 622.

Example 2.

To / From	X	Y	Z	Dummy	Availability
A	7 [4]	3 [3]	[2]	[0]	10
B	[5]	8 [6]	[1]	[0]	8
C	[6]	1 [4]	4 [3]	[0]	5
D	[3]	[5]	1 [4]	5 [0]	6
Requirement	7	12	5	5	29

The total cost of transportation by this method will be:
(7x4)+(3x3)+(8x6)+(1x4)+(4x3)+(1x4)+(5x0) = 105

Example 3

Source	Destination: N	S	E	W	Supply
A	16 / **100**	13 / **100**	22	17	200
B	14	13 / **40**	19 / **300**	15 / **10**	350
C	9	20	23	10 / **150**	150
Dummy	0	0	0	0 / **90**	90
Demand	100	140	300	250	

Z = 10770

2. Least-Cost Method

Steps in Least-Cost Method:

Step 1: Determine the ***least cost among all the rows*** of the transportation table.

Step 2: Identify the row and ***allocate the maximum feasible quantity*** in the cell corresponding to the least cost in the row.

Then ***eliminate that row*** (column) when an allocation is made.

Step 3: Repeat steps 1 and 2 for the reduced transportation table until all the available quantities are distributed to the required places.

- ✓ If the minimum cost is not unique, the tie can be broken arbitrarily.

Step 4: Make sure that all the rim conditions are satisfied and ***(m+n-1)*** cells are allocated.

To / From	D1	D2	D3	Total supply
S1	5	7	8	70
S2	4	4	6	30
S3	6	7	7	50
Total Demand	65	42	43	150 / 150

- We examine the rows S_1, S_2 and S_3, *4 is the least cost element in the cell (S_2, D_1)* and
- (S_2, D_2) and the tie can be *broken arbitrarily*. Select (S_2, D_1).

- The *origin S_2 can supply 30 items to D_1 and thus origin S_2 is exhausted.*
- *D_1 still requires 35 more units*. Hence, shade the row S_2. Shading S_2, we observe that *5 is the least element in the cell (S_1, D_1)* and examine the supply at S_1 and demand at D_1.

- The destination D_1 requires *35 items and this requirement is satisfied from S_1* so that the column D_1 is shaded next.

- Next, we choose *7 as least element corresponding to the cell (S_1, D_2)*.
- We *supply 35 units from S_1 to D_2.* Now, only one row is left behind.
- Hence, *we allow 7 items from S_3 to D_2 and 43 items S_3 to D_3.*

From \ To	D1	D2	D3	Total supply
S1	35 $\boxed{5}$	35 $\boxed{7}$	$\boxed{8}$	70
S2	30 $\boxed{4}$	$\boxed{4}$	$\boxed{6}$	30
S3	$\boxed{6}$	7 $\boxed{7}$	43 $\boxed{7}$	50
Total Demand	65	42	43	150 \ 150

The cost of the allocation by the least cost method is

(35 x 5) + (35 x 7) + (30 x 4) + (7 x 7) + (43 x 7) = 890

Example 2

From \ To	(A) Albuquerque	(B) Boston	(C) Cleveland	Factory capacity
(D) Des Moines	\$5	\$4	100 \$3	100
(E) Evansville	\$8	\$4	\$3	300
(F) Fort Lauder	\$9	\$7	\$5	300
Warehouse requirement	300	200	200	700

First, \$3 is the lowest cost cell so ship 100 units from Des Moines to Cleveland and cross off the first row as Des Moines is satisfied

Figure C.4

To / From	(A) Albuquerque	(B) Boston	(C) Cleveland	Factory capacity
(D) Des Moines	\$5	\$4	100 \$3	100
(E) Evansville	\$8	\$4	100 \$3	300
(F) Fort Lauderdale	\$9	\$7	\$5	300
Warehouse requirement	300	200	200	700

Second, \$3 is again the lowest cost cell so ship 100 units from Evansville to Cleveland and cross off column C as Cleveland is satisfied

Figure C.4

From \ To	(A) Albuquerque	(B) Boston	(C) Cleveland	Factory capacity
(D) Des Moines	\$5	\$4	\$3 100	100
(E) Evansville	\$8	\$4 200	\$3 100	300
(F) Fort Lauderdale	\$9	\$7	\$5	300
Warehouse requirement	300	200	200	700

Third, \$4 is the lowest cost cell so ship 200 units from Evansville to Boston and cross off column B and row E as Evansville and Boston are satisfied

Figure C.4

From \ To	(A) Albuquerque	(B) Boston	(C) Cleveland	Factory capacity
(D) Des Moines	$5	$4	100 $3	100
(E) Evansville	$8	200 $4	100 $3	300
(F) Fort Lauderdale	300 $9	$7	$5	300
Warehouse requirement	300	200	200	700

Finally, ship 300 units from Albuquerque to Fort Lauder as this is the only remaining cell to complete the allocations

Figure C.4

From \ To	(A) Albuquerque	(B) Boston	(C) Cleveland	Factory capacity
(D) Des Moines	$5	$4	100 $3	100
(E) Evansville	$8	200 $4	100 $3	300
(F) Fort Lauderdale	300 $9	$7	$5	300
Warehouse requirement	300	200	200	700

Total Cost = $3(100) + $3(100) + $4(200) + $9(300)
= $4,100

Figure C.4

3. Vogel Approximation Method

Steps in VAM:

Step 1: For each row (column), determine a penalty measure by *subtracting the smallest unit cost element in the row (column) from the next smallest unit cost element in the same row (column).*

Step 2: Identify the row or column *with the largest penalty*. If ***there is a tie*** (equal penalty) it can be broken by *selecting the cell where maximum allocation can be made.*

Allocate as much as possible to *the variable with the least unit cost in the selected row or column.*

Adjust the supply and demand, and *cross out the satisfied row or column.*

If a row and a column are satisfied simultaneously, only one of the two is crossed out, and the remaining row (column) is assigned zero supply (demand).

Step 3:

(a) *If exactly one row or column with zero supply or demand remains uncrossed out, stop.*

(b) If one row (column) with positive supply (demand) remains uncrossed out, determine the basic variables in the row (column) by the least-cost method. Stop.

(c) If all the uncrossed out rows and columns have (remaining) zero supply and demand, determine the zero basic variables by the least-cost method. Stop.

(d) Otherwise, go to step 1.

Step 4: Make sure that *all the rim conditions are satisfied and cells are allocated.*

To / From	D1	D2	D3	Total supply	Row penalty			
S1	65 $\boxed{5}$	5 $\boxed{7}$	$\boxed{8}$	70	2	2 (ii)	1 (iii)	0
S2	$\boxed{4}$	30 $\boxed{4}$	$\boxed{6}$	30	0	0	0	0
S3	$\boxed{6}$	7 $\boxed{7}$	43 $\boxed{7}$	50	1	1	0	0
Total Demand	65	42	43	150				
Column penalty	1	3 (i)	1					
	1	0	1					
	0	0	1					

The cost of allocation (*i.e.*, the associated objective value) by Vogel's Approximation Method will be: (65× 5) + (5× 7) + (30 × 4) + (7 × 7) + (43× 7) = 325 + 35 +120 + 49 + 301 = 830.

- The *difference between the smallest and next to the smallest element in each row and in each column is calculated.*
- We ***choose the maximum*** from among the differences.
- The ***first* individual allocation will be to the smallest cost of a row or column with the *largest difference.***
- So we select the column D_2 (penalty = 3) for the first individual allocation, and allocate to (S_2, D_2) as much as we can, since this cell has the ***least cost location***.
- Thus *30 units from S_2 are allocated to D_2*. This exhausts the supply from S_2. However, there is *still a demand of 12 units from D_2.*

- The allocations to other cells in that column are 0. The next step is to cross out row S_2 (as it is exhausted).
- The next largest unit difference corresponds to the row S_1. This leads to an *allocation in the corresponding minimum cost location in row S_1, namely cell (S_1, D_1).*
- The maximum possible allocation is only *65 as required by W_1 from S_1* and allocation of 0 to others in the row S_1. Column D_1 is thus crossed out.
- Maximum difference is 1 in row S_3 and in column D_3.
- Select arbitrarily S_3 and allot the least cost cell (S_1, D_2) 5 units. Cross out row S_1 for it is already exhausted.
- Now, we have only one row S_3 and two columns D_2 and D_3 indicating that the entire available amount from S_3 has to be moved to D_2 and D_3 as per their requirements.

Vogel's Method (1): calculate differences

	Destination					
Source	**N**	**S**	**E**	**W**	**Supply**	**diff**
A	16	13	22	17	200	3
B	14	13	19	15	350	1
C	9	20	23	10	150	1
Dummy	0	0	0	0	90	0
Demand	100	140	300	250		
diff	9	13	19	10		

Vogel's Method (2): select x_{DummyE} as basic variable

Source	Destination: N	S	E	W	Supply	diff
A	16	13	22	17	200	3
B	14	13	19	15	350	1
C	9	20	23	10	150	1
Dummy	0	0	0 90	0	90	0
Demand	100	140	300	250		
diff	9	13	19	10		

Vogel's Method (3): update supply, demand and differences

	Destination					
Source	N	S	E	W	Supply	diff
A	16	13	22	17	200	3
B	14	13	19	15	350	1
C	9	20	23	10	150	1
Dummy	0	0	0 90	0	---	---
Demand	100	140	210	250		
diff	5	0	3	5		

Vogel's Method (4): select X_{CN} as basic variable

	Destination					
Source	N	S	E	W	Supply	diff
A	16	13	22	17	200	3
B	14	13	19	15	350	1
C	9 100	20	23	10	150	1
Dummy	0	0	0 90	0	---	---
Demand	100	140	210	250		
diff	5	0	3	5		

Vogel's Method (5): update supply, demand and differences

Source	N	S	E	W	Supply	diff
	Destination					
A	16	13	22	17	200	4
B	14	13	19	15	350	2
C	9 100	20	23	10	50	10
Dummy	0	0	0 90	0	---	---
Demand	---	140	210	250		
diff	---	0	3	5		

Vogel's Method (6): select x_{CW} as basic variable

	Destination					
Source	**N**	**S**	**E**	**W**	**Supply**	**diff**
A	16	13	22	17	200	4
B	14	13	19	15	350	2
C	9 100	20	23	10 50	50	10
Dummy	0	0	0 90	0	---	---
Demand	---	140	210	250		
diff	---	0	3	5		

Vogel's Method (7): update supply, demand and differences

Source	N	S	E	W	Supply	diff
	Destination					
A	16	13	22	17	200	4
B	14	13	19	15	350	2
C	9 100	20	23	10 50	---	---
Dummy	0	0	0 90	0	---	---
Demand	---	140	210	200		
diff	---	0	3	2		

Vogel's Method (8): select x_{AS} as basic variable

Source	N	S	E	W	Supply	diff
	Destination					
A	16	13 (140)	22	17	200	4
B	14	13	19	15	350	2
C	9 (100)	20	23	10 (50)	---	---
Dummy	0	0	0 (90)	0	---	---
Demand	---	140	210	200		
diff	---	0	3	2		

Vogel's Method (9): update supply, demand and differences

Source	N	S	E	W	Supply	diff
A	16	13 140	22	17	60	5
B	14	13	19	15	350	4
C	9 100	20	23	10 50	---	---
Dummy	0	0	0 90	0	---	---
Demand	---	---	210	200		
diff	---	---	3	2		

Vogel's Method (10): select x_{AW} as basic variable

Source	N	S	E	W	Supply	diff
A	16	13 (140)	22	17 (60)	60	5
B	14	13	19	15	350	4
C	9 (100)	20	23	10 (50)	---	---
Dummy	0	0	0 (90)	0	---	---
Demand	---	---	210	200		
diff	---	---	3	2		

(Destination: N, S, E, W)

Vogel's Method (11): update supply, demand and differences

	Destination					
Source	N	S	E	W	Supply	diff
A	16	13 / 140	22	17 / 60	---	---
B	14	13	19	15	350	4
C	9 / 100	20	23	10 / 50	---	---
Dummy	0	0	0 / 90	0	---	---
Demand	---	---	210	140		
diff	---	---				

Vogel's Method (12): select x_{BW} and x_{BE} as basic variables

	Destination					
Source	**N**	**S**	**E**	**W**	**Supply**	**diff**
A	16	13 (140)	22	17 (60)	---	---
B	14	13	19 (210)	15 (140)	---	
C	9 (100)	20	23	10 (50)	---	---
Dummy	0	0	0 (90)	0	---	---
Demand	---	---	---	---		
diff	---	---				

Z = 10330

5.2.Assignment problem

- In many business situations, ***management needs to assign - personnel to jobs***, - ***jobs to machines***, - ***machines to job locations***, or - ***salespersons to territories***.
- Consider the situation of assigning *n jobs to n machines.*
- When a job i (=1,2,....,n) is assigned to machine j (=1,2, ...n) that incurs a ***cost Cij.***
- The ***objective*** is to ***assign the jobs to machines at the least possible total cost.***

- ✓ This situation is a special case of the transportation model and it is known as the ***assignment problem.***

- ✓ ***Here, jobs represent "sources" and machines represent "destinations."***

- ✓ The *supply available at each source is 1 unit* and ***demand at each destination is 1 unit.***

Formulation/construction of the model

	Machine					
		1	2	……..	n	Source
Job	1	C11	C12	……..	C1n	1
	2	C21	C22	……..	C2n	1
	.	.	.		.	.
	.	.	.		.	.
	.	.	.		.	.
	n	Cn1	Cn2	……..	Cnn	1
Destination		1	1	…….	1	

The assignment model can be expressed mathematically as follows:

Xij= *0, if the job j is not assigned to machine i*

1, if the job j is assigned to machine I

Now the problem is ***which work is to be assigned to whom*** so that ***the cost of completion of work will be minimum.***

$$\text{Min} \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij}$$

To minimize z (cost)

where xij =

1; if ith person is assigned jth work

0; if ith person is not assigned the jth work

(Sum of assignments from a source should be exactly equal to 1):

$$\sum_{j=1}^{n} X_{ij} = 1 \qquad \text{For } i=1,2,\ldots,n$$

i.e., ith person will do only one work.

(Sum of assignments to a destination should be equal to the demanded quantity by that destination):

$$\sum_{i=1}^{n} X_{ij} = 1 \qquad \text{For } j=1,2,\ldots,n$$

i.e., jth work will be done only by one person.

(Quantities to be assigned can be either 0 or 1):

$$X_{ij} = 0 \text{ or } 1 \qquad \text{For all i and j.}$$

Example

✓A farm produce different agricultural products and that Products are manufactured on five different assembly lines (1,2,3,4,5).

✓When manufacturing is finished, products are transported from the assembly lines to one of the five different potential customers (A,B,C,D,E).

✓Transporting products from five assembly lines to five inspection areas requires different times (in minutes)

	potential customers				
Assembly Line	A	B	C	D	E
1	10	4	6	10	12
2	11	7	7	9	14
3	13	8	12	14	15
4	14	16	13	17	17
5	19	11	17	20	19

Methods of solving AP
(The Hungarian Method)

- ***In order to find the proper assignment it is essential for us to know the Hungarian method.***

Step I

(A) Row reduction:

- ✓ ***Select the smallest value in each row***.
- ✓ ***Subtract this value from each value*** in ***that row*** in the cost matrix.

(B) Column reduction:

- ✓ After completion of row reduction, ***subtract the minimum entry of each column from all the entries of the respective column.***

Step II

Zero *assignment*:

(A) Starting with first row of the matrix received in first step, ***examine the rows one by one until a row containing exactly one zero is found***.

- Then an experimental ***assignment indicated***

by ‘ ’ is marked to that zero.

- Now ***cross all the zeros in the column in which the assignment is made***.
- **Cross out zero, *if there are other zero in ether column or row.***
- ***This procedure should be adopted for each row assignment.***

(B) When the set of ***rows has been completely examined, an identical procedure is applied successively to columns.***

- Starting with column 1, ***examine all columns until a column containing exactly one zero is found.***

- Then *make an experimental assignment in that position* and

- *Cross other zeros in the row in which the assignment was made*.

- ✓ Continue these successive operations on rows and columns ***until all zero's have either been assigned or corssed-out***.

Now there are ***two possibilities:***

(a) Either all the zeros are assigned or ***crossed out,*** i.e., ***we get the maximal assignment***.

or

(b) At least two zeros are remained by assignment or by crossing out in each row or column.

- ✓ In this situation we ***try to exclude some of the zeros by trial and error method.***
- ✓ This ***completes the second step***.

After this step we can get two situations.

(i) Total assigned zero's = n

- ✓ *The assignment is optimal.*

(ii) Total assigned zero's < n

- ✓ *Use step III and onwards.*

✓ **Since n= # of assignments**

Step III

Draw of minimum lines to cover zero's :

✓In order ***to cover all the zero's at least once you may adopt the following procedure***.

(i) Marks **(√)** to all rows in which the ***assignment has not been done.***

(ii) *See the position of zero in marked (√) row* and then **mark (√) to the corresponding column**.

(iii) ***See the marked (√) column and find the position of assigned zero's*** and then *mark (√) to the corresponding rows which are not marked till now.*

(iv) *Repeat the procedure (ii) and (iii) till the completion of marking.*

(v) ***Draw the lines*** through *unmarked rows and marked columns.*

Note: If the above method **does not work** then make *an arbitrary assignment and then follow step IV.*

Step IV

Select the smallest element from the uncovered elements :

(i) Subtract this smallest element from all those elements which are not covered.

(ii) Add this smallest element to all those elements which are at the intersection of two lines.

Step V

- Thus we *have increased the number of zero's.*
- Now, *modify the matrix with the help of step II and find the required assignment.*

Finally calculate the total minimized cost by summing up numbers from the original table

Example 1.

Four persons A,B,C and D are to be assigned four jobs I, II, III and IV. The cost matrix is given as under, find the proper assignment.

		Man			
		A	B	C	D
Jobs	I	8	10	17	9
	II	3	8	5	6
	III	10	12	11	9
	IV	6	13	9	7

Solution :

In order to find the proper assignment we apply the Hungarian algorithm as follows:

I (A) Row reduction

		Man			
		A	B	C	D
Jobs	I	0	2	9	1
	II	0	5	2	3
	III	1	3	2	0
	IV	0	7	3	1

I (B) Column reduction

		Man			
		A	B	C	D
Jobs	I	0	0	7	1
	II	0	3	0	3
	III	1	1	0	0
	IV	0	5	1	1

II Zero assignment:

		Man			
		A	B	C	D
Jobs	I	$\cancel{0}$	$\boxed{0}$	7	1
	II	$\cancel{0}$	3	$\boxed{0}$	3
	III	1	1	$\cancel{0}$	$\boxed{0}$
	IV	$\boxed{0}$	5	1	1

In this way all the zero's are either crossed out or assigned.

Also total assigned zero's = 4 (i.e., number of rows or columns).

Thus, the assignment is optimal.

From the table we get:

I ➔B;

II ➔ C:

III ➔ D and

IV ➔ A.

Total minimized cost = **10+5+9+6= 30**

Example 2:

There are *five machines* and *five jobs* are to be assigned and the associated cost matrix is as follows. Find the proper assignment.

	machines					
Jobs		I	II	III	IV	V
	A	6	12	3	11	15
	B	4	2	7	1	10
	C	8	11	10	7	11
	D	16	19	12	23	21
	E	9	5	7	6	10

Solution:

- In order to find the proper assignment, we apply the Hungarian method as follows:

I A (Row reduction)

	machines					
Jobs		I	II	III	IV	V
	A	3	9	0	8	12
	B	3	1	6	0	9
	C	1	4	3	0	4
	D	4	7	0	11	9
	E	4	0	2	1	5

IB (Column reduction)

Jobs \ machines	I	II	III	IV	V
A	2	9	0	8	8
B	2	1	6	0	5
C	0	4	3	0	0
D	3	7	0	11	5
E	3	0	2	1	1

II (Zero assignment)

Jobs \ machines	I	II	III	IV	V
A	2	9	$\boxed{0}$	8	8
B	2	1	6	$\boxed{0}$	5
C	$\boxed{0}$	4	3	~~0~~	~~0~~
D	3	7	~~0~~	11	5
E	3	$\boxed{0}$	2	1	1

- From the last table we see that all the zeros are either assigned or crossed out, *but the total number of assignment, i.e.,* **4<5** *(number of jobs to be assigned to machines).*
- Therefore, we have to follow ***step III*** and onwards as follows:

		machines				
		I	II	III √ 2nd	IV	V
Jobs	A	2	9	$\boxed{0}$	8	8 √ 3rd
	B	2	1	6	$\boxed{0}$	5
	C	$\boxed{0}$	4	3	~~0~~	~~0~~
	D	3	7	~~0~~	11	5 √ 1st
	E	3	$\boxed{0}$	2	1	1

Step IV: Here, the smallest element among the uncovered elements is 2.

(i) Subtract 2 from all those elements which are not covered.

(ii) ***Add 2 to those entries which are at the junction(intersection) of two lines.***

Make as it is rows that pass single line .

	machines					
Jobs		**I**	**II**	**III**	**IV**	**V**
	A	0	7	0	6	6
	B	2	1	8	0	5
	C	0	4	5	0	0
	D	1	5	0	9	3
	E	3	0	4	1	1

Step V. using step II again

		machines				
		I	II	III	IV	V
Jobs	A	$\boxed{0}$	7	$\cancel{0}$	6	6
	B	2	1	8	$\boxed{0}$	5
	C	$\cancel{0}$	4	5	$\cancel{0}$	$\boxed{0}$
	D	1	5	$\boxed{0}$	9	3
	E	3	$\boxed{0}$	4	1	1

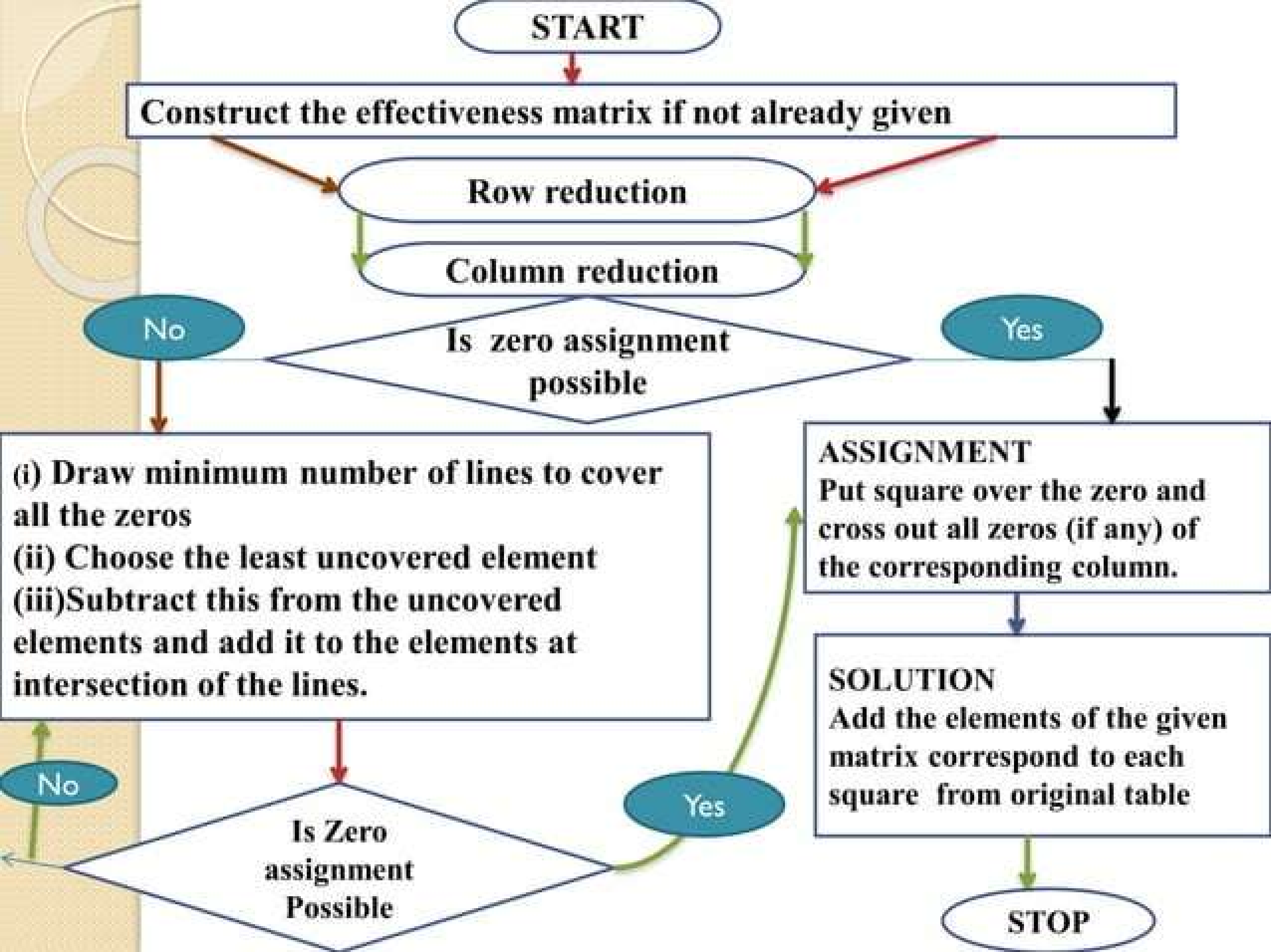
Thus, we have got five assignments as required by the problem.

The assignment is as follows:

A → I, B → IV, C → V, D → III and E → II.

Thus from

the cost matrix the minimum cost = 6+1+11+12+5=35.


START
Construct the effectiveness matrix if not already given
Row reduction
Column reduction
Is zero assignment possible
No
Yes
(i) Draw minimum number of lines to cover all the zeros
(ii) Choose the least uncovered element
(iii)Subtract this from the uncovered elements and add it to the elements at intersection of the lines.
ASSIGNMENT
Put square over the zero and cross out all zeros (if any) of the corresponding column.
SOLUTION
Add the elements of the given matrix correspond to each square from original table
No
Yes
Is Zero assignment Possible
STOP