Probablity Homework 4

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Problem 1

Let p be the probability that a randomly selected chip has lifetime less than $1.8 * 10^6$.
Now we suppose that X is the normal random variable with mean $1.4 * 10^6$ and standard deviation $3 * 10^5$.
Note that we should use the Lemma 7.1 in the book to standardized X.
For calculating p we should calculate the probability that $P(X < 1.8 * 10^6)$ so we have :

$p = P(X < 1.8 * 10^6) = P(\frac{X - 1.4*10^6}{3*10^5} < \frac{1.8*10^6 - 1.4*10^6}{3*10^5}) = P(\frac{X - 1.4*10^6}{3*10^5} < \frac{4}{3})$.

$\Rightarrow P(Z < \frac{4}{3}) = \Phi(\frac{4}{3}) \simeq 0.9 = p$

Now we calculate the probability of at least 20 out of 200 has this lifetime:

$\Rightarrow \sum_{i=20}^{200} \mathrm{C}_{200}^{i} (0.9)^i (0.1)^{200-i}$

Where C_n^k denotes "the number of ways to choose k items out of n".

Problem 2

A

First we suppose that A is the fire station itself and X is the place we have fire.
Now base on formula in the book for probability density function of X we have:

$$f(t) = \begin{cases} \frac{1}{A} & : \ 0 \leq t \leq A \\ 0 & : \ other \end{cases}$$

So we now calculate $E(|X - a|)$:
Note that we know:

$$|X - a| = \begin{cases} X - a & : \ a \leq X \leq A \\ a - X & : \ 0 \leq X \leq a \end{cases}$$

So we have base on formula for E(X):

$$E(|X - a|) = \int_a^A (X - a) f(t)dt + \int_0^a (a - X) f(t)dt = \frac{a^2 - Aa + \frac{A^2}{2}}{A}$$

Also we know from the calcules that for finding min or max , we should Differentiating the equation and equal it to 0. so we after doing the calculation we will find the min value of$E(|X - a|)$ is equal to $\frac{A}{2}$:

$$\Rightarrow \frac{d}{da} E(|X - a|) = \frac{2a}{A} - 1 + 0 \Rightarrow \frac{2a}{A} = 1 \Rightarrow a = \frac{A}{2}$$

And this is what we wanted.So the station should be at mid of the road.

B

Now we calculate it for infinite length:
As we have in the question, we can suppose that X is exponentially distributed with parameter λ. so for probability density function we have :

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & : \ t \geq 0 \\ 0 & : \ t < 0 \end{cases}$$

Now we do like what we did in part ‘A’:

$E(|X-a|) = \int_a^\infty (X-a)f(t)dt + \int_0^a (a-X)f(t)dt$

Just like part A, after differentiating the equation above and equal it to 0 we get:

$e^{\lambda a} = 2 \Rightarrow a = \frac{\log 2}{\lambda}$.

So the station should be at point $\frac{\log 2}{\lambda}$.

Problem 3

A

We know that the Poisson process is memoryless(because Each occurrence is independently of type (i) with probability (p_i)), and the probability of an occurrence being of type ‘i’ is independent of other occurrences.
We also know that number of occurrences in a time interval follows a Poisson distribution with parameter λ means that:
$np = \lambda$
Because each occurrence is independently of type ‘i’ with probability p_i so if we donate n_i for number of occurrences of ‘i’, then we have:

$n_i = np_i$ and $n = \frac{\lambda}{p}$

So for prove what we wanted all we need to do is to show that: $n_i p = \lambda p_i$

$\Rightarrow n_i = \frac{\lambda}{p} p_i \Rightarrow n_i p = \lambda p_i$

So we have proved that
”The number of occurrences of type ‘i’ can be modeled as a Poisson process with rate λp_i”

B

Let X be the time until 2000 th broken car is entered. We know that $\lambda = np$ for a Poisson process and in this process p is equal to $\frac{1}{10}$ and n is 1300. So we obtain $\lambda = 130$.
Now all we should do is to calculate E(X) and Var(X) for a gamma random variable with parameters $r = 2000$ and $\lambda = 130$.

Base on formulas in the book we obtain that:

$E(X) = \frac{r}{\lambda}$ and $Var(X) = \frac{r}{\lambda^2}$

$E(X) = \frac{2000}{130} \simeq 15.4$ and $Var(X) = \frac{2000}{130*130} \simeq 0.12$.

Problem 6

For distribution function F we know that:

$F(t) = P(x \leq t)$

If we want to prove that $F(X) = Y$ is uniform over (0, 1) we should show this:

$$P(Y \leq t) = \begin{cases} t & : \ 0 \leq t \leq 1 \\ 1 & : \ t > 1 \\ 0 & : \ t < 0 \end{cases}$$

It is easy to show that $P(Y \leq t)$ is equal to 1 for $t > 1$ and is equal to 0 for $t < 0$ because:

X is a function from 'sample space' to $\Re$ and F is a function from $\Re$ to $[0, 1]$.
So F(X) is a function from 'sample space' to $\Re$.

Now we try to prove last and final part:
Base on F 's property from the book we know that inverse of F can't be empty and it also can have more than one element.
So we difine α as follows:

$\Rightarrow a = \{x|F(x) > t\}$

So we obtain that $F(\alpha) = t$. Also because we know that F is nondecreasing so we have:
$x \leq \alpha \Rightarrow F(x) \leq F(\alpha)$ and $x \geq \alpha \Rightarrow F(x) \geq F(\alpha)$.

$\Rightarrow P(Y \leq t) = P(X \leq \alpha)$

And equation above is defenition of distribution function, therefore we get:

$\Rightarrow P(Y \leq t) = F(\alpha) = t$

So we proved that $P(Y \leq t)$ is equal to 't' for $0 \leq t \leq 1$ therefore our prove is completed now.