Probablity Homework 3

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Problem 3

First we calculate the number of different areas in the city that is 400.
Now we can use binomial and Poisson random variable.
We can see that the Probablity that a specefic conflagaration in a chosen area is $p = \frac{1}{400}$.
If we name has a conflagaration at month ,success, then X(number of conflagaration in a chosen area) is a binomial variable.
Because for $k = 400$, $p = \frac{1}{k}$ is small and $\frac{n}{k}$ is not too large or small we can use Poisson random variable with $\lambda = np = \frac{n}{k}$ and n is 500. So we have:

$\Rightarrow P(X = i) = \frac{\lambda^i * e^{-\lambda}}{i!}$

For calculate at least one conflagaration each month we should calculate $P(X \neq 0)$.
We know that $P(X \neq 0) = 1 - P(X = 0)$.

$\Rightarrow P(X \neq 0) = 1 - e^{-\frac{n}{k}} = 1 - e^{-\frac{5}{4}}$

Problem 5

A

If we suppose that X is variable that represents seconds both leaves until second i, it means that at second i , exactly one of them shoots or both of them shoot, therefore can calculate the probability mass function of X:

$$P(X = i) = \begin{cases} g(i) & : \ i \geq 0 \\ 0 & : \ elsewhere \end{cases}$$

that $g(i)$ is equal to :

$[(1-p)^{i-1}p*(1-q)^{i}] + [(1-q)^{i-1}q*(1-p)^{i}] + [(1-p)^{i-1}p*(1-q)^{i-1}q]$

Now by formula for calculating expectation value E[X] :

$E[X] = \sum_{i=0}^{\infty} i * P(X = i)$

So we will find $E[X]$:

After calculation and use the formula $\sum_{i=0}^{\infty} i * r^{i} = \frac{r}{(1-r)^2}$ we obtain the final value $E[X] = \frac{1}{p-pq+q}$.

B

The Probablity that both of them dies is that both shoot in same second.If we suppose that X is variable that represents seconds both leaves until second i ,it means that at second i ,both of them will shoot.
So the probability mass function is:

$$P(X = i) = \begin{cases} (1-p)^{i-1}p*(1-q)^{i-1}q & : \ i \geq 0 \\ 0 & : \ elsewhere \end{cases}$$

Because ‘i’ can be from 1 to ∞ we should sum up P(X) for all ‘i’s:

$\Rightarrow P(X) = \sum_1^\infty pq * ((1-p)(1-q))^{i-1}$.

$\Rightarrow P(X) = pq \sum_1^\infty ((1-p)(1-q))^{i-1}$

$\Rightarrow P(X) = \frac{pq}{1-(1-p+pq-q)} = \frac{pq}{p-pq+q}$

Problem 6

What the problem wants us to prove is equivalent to showing that the geometric is the only distribution on the positive integers with the memoryless property.It means that:

If X be a discrete random variable with the set of possible values $\{1, 2, \cdots\}$, if for all positive integers n and m, we have:

(I): $P(X > n + m | X > m) = P(X > n)$

then X is a geometric random variable. That is, there exists a number $0 < p < 1$ such that:

$P(X = n) = p(1-p)^{n-1}$

So we start prove that with induction.
First we use conditional probability formulas and break down the (I):

(I) $\Rightarrow (II) : P(X > n + m) = P(X > n)P(X > m)$
Now we start proving by induction:(Suppose $P(X = 1) = p$)
Base case($n = 2$):
$(II) \Rightarrow P(X > 2) = P(X > 1)P(X > 1)$
And we also know that $P(X > 1) = 1 - P(X = 1) = 1 - p$

$\Rightarrow 1 - P(X = 1) - P(X = 2) = (1-p)^2$

$\Rightarrow P(X = 2) = p(1-p)$

So it is true for the base case.
Now we show that if it is true for n, it is true for (n + 1), too:

Induction step:
From induction hypothesis we know that:

$P(X \leq n) = \sum_{i=1}^{n} P(X = i) = \sum_{i=1}^{n} p(1-p)^{i-1}$

If we calculate the experresion above we reach that:

$\Rightarrow P(X \leq n) = 1 - (1-p)^n$

Now we use relation (II) to find $P(X = n+1)$:

$\Rightarrow P(X > n+1) = P(X > n)P(X > 1)$

We know that $P(X > n)$ is equal to $1 - P(X \leq n)$ and $P(X > n-1)$ is equal to $1 - P(X \leq n-1)$ that we can obtain from induction hypothesis. So the relation above is equal to:

$\Rightarrow 1 - P(X \leq n) - P(X = n+1) = (1-p)^n(1-p)$.
If we use value of $P(X \leq n)$ that found earlier we obtain that:

$\Rightarrow P(X = n+1) = p(1-p)^n$

Therefore we have showed that it is true for induction step, too.
So we have proved the problem.

Problem 7

For solving this problem we will use negative binomial random variable with $p = \frac{1}{3}$ because we have three queue and the desired car is in the first queue and each time we choose one of the queues with same probability.
And $r = 10$ because we want the number of experiments until the 10th success occurs.
So we let X be a negative binomial random variable and represents the number of experiments until the 10th success occurs.
Then base on these information we define our PMF as follows:

$P(X = i) = \mathrm{C}_9^{i-1}(\frac{1}{3})^{10}(\frac{2}{3})^{i-10}$

where C_n^k denotes "the number of ways to choose k items out of n".
Now let's see what happens when we sum up all terms in the equation above.
We get:

$\sum_{i=10}^{\infty} \mathrm{C}_9^{i-1}(\frac{1}{3})^{10}(1-p)^{i-10}$

Therefore the equation above is the Probablity of entering desired car in the gas station.
Now lets calculate Var(X) and E(X):
Base on formula in the book we have:

$Var(X) = \frac{r(1-p)}{p^2}$ and $E(X) = \frac{r}{p}$

So the final value of Var(X) and E(X) are:

$E(X) = \frac{10}{(\frac{1}{3})} = 30$ and $Var(X) = \frac{10(\frac{2}{3})}{(\frac{1}{3})^2} = 7.5$