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MASTER THESIS

Modelling Meta-Agreement through Deliberation: An Adaptation of the DeGroot Model

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Declaration of Authorship

I, Amir Sahrani, declare that this thesis, entitled ‘Modelling Meta-Agreement through Deliberation: An Adaptation of the DeGroot Model’ and the work presented in it are my own. I confirm that:

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Signed:

Amir Sahrani

Date: July 10, 2025

“The majority, standing in for the people, wills everything and therefore wills nothing”

Joshua Cohen

Abstract

Deliberation is often proposed as a remedy to democratic dysfunction, enabling voters to reach more informed and coherent preferences. In particular, deliberation may promote a shared understanding of the relevant issue dimensions (meta-agreement), which can lead to single-peaked preference profiles and circumvent classic impossibility results in social choice theory. This thesis investigates whether and how deliberation fosters such structured preferences by adapting the DeGroot model of opinion dynamics.

We begin by reviewing theoretical foundations from social choice, focusing on domain restrictions like single-peakedness, and by discussing deliberative democracy and the concept of meta-agreement. We then replicate some of the results of Rad and Roy, published in the *American Political Science Review* in 2021, which models deliberation as preference updating under various distance metrics. While their model can increase proximity to single-peakedness, it does not capture meta-agreement and is vulnerable to strategic manipulation.

To address the lack of meta-agreement, we introduce an extended DeGroot-based model in which agents deliberate not only over their substantive preferences but also over their beliefs about candidate positions on multiple policy dimensions. Trust dynamics are modeled via bias, ego, similarity, and knowledge, and are used to guide how individuals weight others' opinions. Using data from the well-known AMERICA IN ONE ROOM experiment, we calibrate and validate the model, showing it reproduces some empirical patterns of opinion change. Notably, we find that ego-based trust better fits observed data than knowledge-based trust, suggesting that being informed does not necessarily translate to persuasive influence.

We further extend the model to simulate deliberation-driven meta-agreement and show that it can reduce cyclic preferences and increase proximity to single-peakedness. Finally, a sensitivity analysis identifies drivers of opinion change and highlights the interaction effects between trust, bias, and group composition. These results suggest that while deliberation can indeed structure preferences, the dynamics are more complex and depend on voters' understanding of the broader issue landscape.

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I would like to thank Ulle Endriss for helping me walk the tight rope between scientific disciplines, a path I find myself walking all too often. His guidance has been of tremendous help whenever I found myself lost, confused or when I was simply getting ahead of myself.

I thank my friends, and family for keeping me sane. Though stress and tension sometimes ran high; the long talks, cards, and plastic rock have kept my mind from fraying.

Thanks, but I'll take some more fish.

Contents

List of Figures

List of Tables

ABBREVIATIONS

PBS	**P**olicy-**B**ased (ideology) **S**core
PtS-V	**P**roximity **t**o **S**ingle-peakedness (through) **V**oter (deletion)
PtS-C	**P**roximity **t**o **S**ingle-peakedness (through) **C**andidate (deletion)

SYMBOLS

N	The set of all voters
X	The set of all alternatives
$\succ$	A preference relationship
$\mathcal{D}$	A domain of possible profiles
D	A deterministic deliberative procedure
BD	A deliberative procedure with biased voters
$\mathcal{L}(A)$	Set of all possible preference orders over A
R	Set of a preference relations over all candidates
$\boldsymbol{R}$	Set of preferences of all voters
f	A function mapping a strict profile to a candidate
$\triangleleft$	A geometric order over candidates
Ψ	Vector of all policies
ψ	An instance of a policy
S	Vector of support for each policy
Σ	matrix of shape $\|A\| \times \|\Psi\|$, estimating support of policies for each alternative

CHAPTER 1

INTRODUCTION

Claims such as “vaccines are deadly” and “nuclear energy is dangerous” run counter to expert consensus, despite experts overwhelmingly vouching for both their safety and efficacy[1]. Many democracies suffer this kind of misinformation, leading to a general dissatisfaction among the electorate. Misinformation not only pushes voters to more extreme opinions, but skews their views of fellow citizens. Elections thus face a dual challenge: not only must they select broadly appealing candidates, but they must do so in a context where people have drastically differing opinions on the nature of the problems, the possible solutions and the roles of the candidates.

For democracy to function effectively, voters need a shared foundation of understanding — a “shared reality.” This consists of commonly accepted facts and causal relationships allowing meaningful debate about values and priorities. For example, while nuclear energy is considered safe by experts, it comes at high initial cost, and long construction times. Renewable sources such as solar and wind, by contrast, can be scaled up quickly but provide less consistent energy output. When voters share this understanding, they can engage in productive disagreement about whether the time and money investments for nuclear are worth the consistent energy production. However, when some voters believe nuclear to be unsafe, an election seemingly about the trade-off between nuclear and solar becomes a referendum on the perceived safety of nuclear energy.

Traditionally, people’s understanding of the world was shaped by family, friends, and in legacy media. These sources tend to reinforce shared viewpoints, friends and family

[1] While nuclear energy has seen catastrophic failures, such as the Chernobyl Disaster, evidence suggests that, on average, it results in fewer deaths and less environmental harm than fossil fuel-based energy [33]. This nuance does not contradict the broad expert consensus supporting its relative safety.

often consumed similar media and held similar beliefs, while newspapers and broadcasters curated a common public narrative — even if this narrative is not entirely factually accurate. Increasingly, however, algorithmic curation shapes individual worldviews creating a fundamental problem: a fragmented understanding of reality. Because algorithms tailor content to each individual's preferences, people are exposed to unique and sometimes incompatible sets of claims about the world.

This fragmentation creates a problem for collective decision-making. Voters might be supporting the same candidate for fundamentally different, and possibly opposing, reasons. In this work we formalize this notion of a "reason" using the concept of the *issue dimension* introduced by List [25], when people have a common issue dimension, their disagreement over outcomes can be explained through different trade-offs along these dimensions.

From the perspective of social choice, shared issue dimensions can be beneficial. In particular if the problem can be reduced to a singular shared issue dimension, we might get "single-peaked" preferences, a special structure in the preferences of voters. We provide a formal definition in Chapter 3, informally however, single-peaked preferences allow for election mechanisms that encourage voters to report their preferences honestly. We elaborate on what we mean by an election mechanism in Chapter 2, but intuitively, it is a procedure for aggregating individual preferences into a collective choice.

To promote the single-peakedness of preferences, List et al. [26] propose deliberation as a potential strategy, building on List's earlier concept of *meta-agreement* [25], being the idea that voters agree on which issue dimensions matter and where candidates stand on these dimensions. List et al. [26] argue that deliberation can help voters develop more coherent preference structures. Deliberation, then, helps restructure voters' opinions in a more coherent way, particularly on low-salience issues that receive little media coverage.

Given deliberation's potential to generate meta-agreement and more structured preferences, we aim to understand deliberation more rigoriously. With the rise of in-silico experiments in computational social science, we take a computational approach to understanding deliberation. Specifically, we adapt the classic DeGroot model of opinion dynamics [11] to the context of political deliberation, both on voter opinions and perceived candidate positions.

To this end, we begin in Chapter 3 we review work on single-peakedness, deliberation, and experiments, and present a deliberation model by Rad and Roy [32]. In Chapter 4 we formally define some properties of deliberation, and prove negative results regarding "Honesty" during deliberation, showing deliberation is not strategyproof under a variety of circumstances. We also define an adaptation to the DeGroot model, as a mechanistic explanation of deliberation through a computational model. In doing so, we find a limitation in the applicability of this model in the form of a negative computational complexity result. Specifically, we show NP-completeness of mapping voter opinions to trust matrices. In Chapter 5 we explain the experimental setup we use to test our model, the result of which we present in Chapter 6. Finally, we reflect on the results, and broader implications of this thesis in Chapter 7.

CHAPTER 2

PRELIMINARIES

We begin with a short introduction to social choice. We outline the basic voting model, closely following the notation and definitions by Brandt et al. [5], and restate classical results relevant to the following chapters. These provide the theoretical context for the remainder of the thesis.

2.1 The Basic Model

To model elections, we represent voters by the set N consisting of n voters. The possible outcomes of an election, we represent with the set A consisting of $|A|$ possible outcomes, usually called the alternatives. Since our focus is on political elections, we will refer to the outcomes of an election as candidates instead. Each voter represents their preference on candidates through a preference relation $\succ_i$, for example if voter i prefers outcome a to outcome b, we write $a \succ_i b$. When a voter's preference is antisymmetric, complete, and transitive, i.e. it orders all candidates and $a \succ_i b$ and $b \succ_i c$ implies $a \succ_i c$, we call this a linear order, denoted by R_i. We call the set of possible linear orders over the candidates $\mathcal{L}(A)$. For an election, all voters report a linear order. The vector consisting of each voter's preference is called a profile, denoted by $\boldsymbol{R} = (R_1, \dots R_n) \in \mathcal{L}(A)^n$. Finally, a social choice function (SCF) f decides the outcome of the election based on the profile. We discuss the specifics of these functions in Section 2.2.

The last simple definition we will need is the *majority relation* [27]. Given some profile $\boldsymbol{R}$ we can construct a majority relationship as follows: for each pair of candidates x, y, we ask how many voters strictly prefer x to y; if this number of people is greater than $\frac{n}{2}$ we get $x \succ_{\text{maj}} y$. If it is exactly equal to $\frac{n}{2}$ and thus is a tie, we simply write $x \sim \text{maj} y$ (breaking ties arbitrarily), otherwise we write $y \succ_{\text{maj}} x$. We proceed with an example.

Example 1: *Majority relation*

1	2	3
a	b	a
b	c	c
c	a	b

Given the profile on the left, we first start by comparing a to b, both voters 1 and 3 prefer a to b thus the majority prefers a to b. Comparing b to c the majority prefers b to c. Finally, comparing a to c, a is preferred again. Thus, the majority relation is $a \succ_{\text{maj}} b \succ_{\text{maj}} c$.

From this, it is easy to see that the majority relation is in some sense a summary of the voter's preferences. In Section 2.3 we show how a divided population can lead to an inconsistent majority relation.

Using this majority relationship, we can formulate our first notion of when a candidate is winning. We call a candidate x a Condorcet winner, if for each pairwise comparison between x and $y \in A \setminus \{x\}$ we have $x \succ_{\text{maj}} y$. For example, in Example 1 a would be the Condorcet winner. We can relax the requirement of always winning to never losing, i.e. we never have $y \succ_{\text{maj}} x$ but $x \sim \text{maj}y$ is allowed. A candidate that never loses in any pair wise comparison is called a weak Condorcet winner.

2.2 Social Choice Functions

As mentioned, in order to decide the outcome of an election we need a social choice function f, this function should map all possible profiles to an outcome, thus $f : \mathcal{L}(A)^n \to A$. A famous and simple example of a SCF is the plurality rule, which simply elects the candidate voted into first place most often, i.e. "most first place votes wins". This rule presents one of the first challenges for many SCF: handling ties.

For elections, organizers likely will want to ensure the SCF has certain nice properties, such as not favoring a candidate. In social choice, these properties are called axioms, and the procedure of designing a SCF based on desired axioms is called the axiomatic approach. The property just described is the axiom of neutrality, stating that the SCF should be neutral with respect to the candidates. In this work, six main axioms are of importance.

Axiom of Resoluteness. A SCF f is resolute, if for every profile $\boldsymbol{R}$ we have $|f(\boldsymbol{R})| = 1$.

Axiom of Surjectivity. A SCF f is surjective, if for every candidate x, there exists a profile $\boldsymbol{R}$ such that $f(\boldsymbol{R}) = x$.

Axiom of Non-Dictatorship. A SCF f is non-dictatorial, if there does not exist a voter i such that $f(\boldsymbol{R}) = \text{top}(i, \boldsymbol{R})$ for all profiles $\boldsymbol{R}$, where $\text{top}(i, \boldsymbol{R})$ extracts voter i's most preferred candidate from profile $\boldsymbol{R}$.

Axiom of Strategyproofness. A SCF f is strategyproof if, for any voter $i \in N$, i cannot report an untruthful preference $\succ_i'$, such that $\boldsymbol{R}' = (\succ_1, \ldots, \succ_i', \ldots, \succ_n)$ and $f(\boldsymbol{R}') \succ_i f(\boldsymbol{R})$.

Axiom of Anonymity. A SCF f is anonymous if, when the labels of voters are shuffled, the winning candidate stays the same.

Axiom of Neutrality. A SCF f is neutral if, when the labels of the candidates are shuffled, the winner in the shuffled election should correspond to the winner in the original election, under the relabeling.

There are many more axioms one could reasonably argue for, however, these are enough to lead to the main impossibility results this work focuses on.

2.3 Negative Results

Classic social choice theory has many negative results, one such example is the Condorcet cycle. This is a specific profile that results in a cycle in the majority relation, as shown in the following example.

Example 2: *Condorcet cycle*

1	2	3
a	b	c
b	c	a
c	a	b

Voters 1 and 3 prefer a to b resulting in $a \succ_{\text{maj}} b$, next voters 1 and 2 prefer b to c, resulting in $b \succ_{\text{maj}} c$. However, voters 2 and 3 prefer c to a, resulting in $c \succ_{\text{maj}} a$. This yields the cycle $a \succ_{\text{maj}} b \succ_{\text{maj}} c \succ_{\text{maj}} a$.

It can be shown that under weak preferences, the Condorcet cycle can occur anytime there are 3 or more candidates and voters. While Under strict preferences, Condorcet cycles can occur whenever there is an odd number of candidates greater than one, and the number of voters is a multiple of the number of candidates. As we will show later, this profile can be the cause of some impossibility results.

One of the major negative results in social choice is that of the Gibbard-Satterthwaite theorem [21, 35].

Theorem 2.1. [Gibbard-Satterthwaite] There exists no resolute social choice function for elections with $|A| \geq 3$ that is surjective, strategyproof, and non-dictatorial.

Unless we accept a dictatorship, it is impossible to have a voting rule that incentivizes voters to report their preferences truthfully, when we want to pick a singular winner from at least 3 candidates.

Though we do not provide a full proof, the Condorcet cycle offers some intuition for why this result holds. Following Example 2, suppose we have a social choice function (SCF) f that elects candidate a. Voter 1 is very happy with this outcome, but voters 2 and 3 would prefer c instead. Voter 2 could then misreport their preferences by swapping c and b, thereby causing c to become the Condorcet winner. Now, if f is both strategyproof and resolute, it must still elect a despite c being the Condorcet winner. Since f is also surjective, a cannot be the outcome for all preference profiles. Taken together, the only apparent reason a continues to win in this profile is because voter 1 wants it to—suggesting that voter 1 effectively dictates the outcome.

Fortunately, there seem to be ways around these negative results. Mainly through the assumption that there is some structure in the preferences of voters.

2.4 Domain Restrictions

Negative results often are a result of a small set of ill-behaved profiles. If there is reason to conclude these profiles are impossible in the election at hand, there is some hope of constructing SCF's satisfying our axioms. To speak more formally about profiles "not occurring", we introduce Domain restrictions, for this we use the definition by Elkind et al. [14].

Definition 1: *Domain*

Given a set of voters N, candidates A, and conditions C, the domain $\mathcal{D}$ of an election is the set of all profiles $\boldsymbol{R}$ such that all conditions C are satisfied.

This definition is different from usual definitions in social choice in so far as it talks about allowed profiles instead of allowed votes.

As stated earlier, the Condorcet profile is one such ill-behaved profile, as each candidate, holds a majority preference over another candidate. Naturally one might consider if this profile might even come up in practice, though conceivable, it seems generally unlikely for there to exist a perfect split in opinions. Quite naturally one of the first "solutions" one might consider is when the number of voters is not a multiple of the number of candidates, though this offers little practical guidance for real elections, this is the first example of a domain restriction, we define a simple domain that prevents these cycles as follows.

DEFINITION 2: $\mathcal{D}_{\text{No-tie}}$

Let A be the set of candidates and N be the set of voters, of size n such that $n \neq k \cdot |A|$ for any $k \in \mathbb{N}$. We call this domain $\mathcal{D}_{\text{No-tie}}$.

This allows us to state our first proposition.

> **Proposition 2.2.** The plurality rule never returns an $|A|$-way tie between candidates when applied to $\mathcal{D}_{\text{No-tie}}$.

> *Proof.* Assume, for the sake of contradiction, the plurality rule in fact does return an $|A|$-way tie, this means all candidates were ranked first an equal number of times call this k. Necessarily, we need exactly $k \cdot |A|$ voters, but this leads to a contradiction, as this would no longer be inside $\mathcal{D}_{\text{No-tie}}$.

This is a simple result, but it serves as an example on how we can use the properties of the domain to prove things about the election. Gaertner [19] establishes two ways in which a domain can be restricted. Firstly, we can restrict the domain to a number of voters or candidates, which is what we did in $\mathcal{D}_{\text{No-tie}}$. Secondly, the domain can be restricted to have a certain structure, such as being single-peaked.

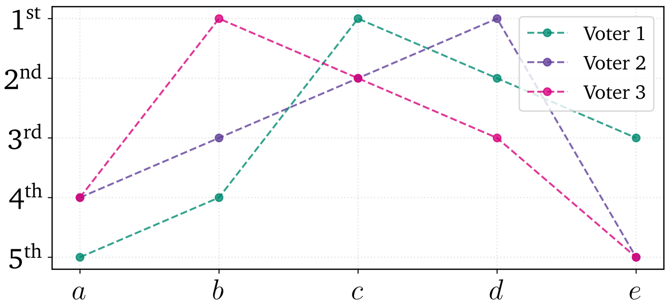
In an election, the candidates might represent a point on an axis, such that a voter prefers a candidate more if they are closer to them on the axis. For example, if the candidates represent the minimum wage, where each cent-value constitutes a candidate. Imagine a voter thinks the minimum wage should be some value x and prefers candidates that are closer to this value x. This results in each voter having a "peak" value, and all other values are ranked in terms of their distance to x. Figure 2.1b shows what this might look like for 3 voters. More generally, we call a profile single-peaked if there exists an axis on which we can place the candidates such that all voters' preferences have a single peak on this axis. Definition 3 makes this notion formal.

DEFINITION 3: *Single-peaked Profiles*

A profile $\boldsymbol{R}$ is single-peaked, if given some ordering $\triangleleft$ over the candidates, it holds that for all voters i, and all $a, b, c \in A$, if $a \triangleleft b \triangleleft c$, then voter i cannot prefer both a and c to b; that is either $a \succ_i b$ or $c \succ_i b$ but not both.

In the following chapters, we explore whether deliberation can serve as a mechanism for increasing single-peakedness in voter preferences

1	2	3
c	d	b
d	c	c
e	b	d
b	a	a
a	e	e



(A) Preference profile

(B) Single-peaked profile visualization

FIGURE 2.1: An election with three voters and five candidates. Each voter has a unique peak, and the profile is single-peaked with respect to a shared axis.

CHAPTER 3

LITERATURE REVIEW

In this chapter we review the theoretical foundations that inform our computational approach to modeling deliberation. We examine three interconnected areas: domain restrictions in social choice theory (particularly single-peaked preferences and hereditary domains), the literature on deliberation and meta-agreement theory, and computational models of deliberative processes. Together, these establish both the theoretical motivation for understanding how deliberation can produce well-structured preference domains and the methodological foundation for our computational modeling approach.

3.1 Condorcet Domain

If our goal is to prevent Condorcet cycles, or in general have transitive majority relations, the best we could hope to do is to apply our domain restriction such that our domain contains all profiles $\boldsymbol{R}$ such that $\boldsymbol{R}$ has a (weak) Condorcet winner. We call this domain $\mathcal{D}_{\text{Condorcet}}$. Under this domain, let $f_{Condorcet}$ be the Condorcet Rule, which picks a Condorcet winner. Then $f_{Condorcet}$ is strategyproof over $\mathcal{D}_{\text{Condorcet}}$ [14].

> *Proof*. (Elkind et al. [14]). Assume, for the sake of a contradiction, we have profiles $\boldsymbol{R} = (\succ_1, \dots, \succ_i, \dots, \succ_n)$ and $\boldsymbol{R}' = (\succ_1, \dots, \succ_{i'}, \dots, \succ_n)$ such that:
>
> $$f_{Condorcet}(\boldsymbol{R}) = a, \quad f_{Condorcet}(\boldsymbol{R}') = b, \quad \text{and } a \neq b$$
>
> Assume that i has $b \succ_i a$, thus strictly prefers b to a. Then under $\boldsymbol{R}$ there is a strict majority $C \subseteq N$ who have $a \succ b$, but $i \notin C$. Thus, in $\boldsymbol{R}'$, C is still a majority preferring a to b, making a the Condorcet winner in $\boldsymbol{R}'$. This is in contradiction to b winning in $\boldsymbol{R}'$.

This result is strengthened by Campbell and Kelly [6, 8], showing that for an odd number of candidates, $f_{\text{Condorcet}}$ is the only voting rule over $\mathcal{D}_{\text{Condorcet}}$ that is strategyproof, surjective and non-dictatorial.

When surjectivity is strengthened to neutrality, and non-dictatorship to anonymity, $f_{\text{Condorcet}}$ is the only strategyproof voting rule over $\mathcal{D}_{\text{Condorcet}}$ for an odd number of voters [7].

Though this result is positive, we might wonder how stable it is. For this we need to define a notion of stability. One natural way to think about it is as follows: suppose one of the candidates or voters drops out, do we keep the nice structure of the domain? If this is true we consider the domain stable and call it *hereditary*.

Definition 4: *Hereditary* (Elkind et al. [14])

A domain $\mathcal{D}$ is called *hereditary* if for every profile $\boldsymbol{R} \in \mathcal{D}$, and every subprofile $\boldsymbol{R}'$ obtained by deleting voters and candidates from $\boldsymbol{R}$, it holds that $\boldsymbol{R}' \in \mathcal{D}$.

$\mathcal{D}_{\text{Condorcet}}$ is not hereditary. This is easy to see through an example:

Example 3: $\mathcal{D}_{\text{Condorcet}}$ *is not hereditary*

1	2	3	4
a	b	c	a
b	c	a	c
c	a	b	b

We can see that in this example, a is the weak Condorcet winner, as it beats b and is tied with c. If we remove voter 4 however, we return to the original Condorcet cycle.

If a domain fails to be hereditary, designing an election with the domain in mind becomes hard. In the case of $\mathcal{D}_{\text{Condorcet}}$ it might be reasonable to make use of a rule such as Black's rule [3], which uses the Condorcet rule only if there is a Condorcet winner and the Borda rule otherwise. This however is not a strategyproof voting rule in general. Instead, we might want to look at hereditary strategyproof domains. We present the single-peaked domain $\mathcal{D}_{\text{SP}}$, which will be the main focus of this thesis. This is the domain of all single-peaked profiles. We first proceed to show that this domain indeed is hereditary.

Proposition 3.1. (Elkind et al. [14]). $\mathcal{D}_{\text{SP}}$ is hereditary.

Proof. (Voter Deletion). If we remove a voter, this does not affect the other voters, so the profile is still single-peaked. ✓

> (Candidate Deletion). Consider any voter i and their single-peaked vote, if we remove some candidate x, to this voter all candidates which they preferred to x stay in the same position, while all other candidates move up one rank, thus preserving the order, and thus single-peakedness. ✓

We have demonstrated that $\mathcal{D}_{\text{SP}}$ possesses the desired properties. However, we currently lack a method to ensure that we operate within $\mathcal{D}_{\text{SP}}$. Deliberation may provide a mechanism to ensure that preference profiles move toward single-peakedness. We will now provide a concise overview of the literature on deliberation.

3.2 The History of Deliberation and Meta-Agreement

We have provided an overview of different domain restrictions and their properties, showing they avoid Condorcet cycles. Bochsler [4] however, argues that Condorcet cycles are empirically rare. The next section is dedicated to explaining how deliberation might explain this is so through examining the historical ideas around deliberation and deliberative democracy, as well as that of meta-agreement.

3.2.1 Deliberation

Deliberation, though intuitively familiar as the process of multiple people talking through a problem with the goal of coming to an agreement, compromise, or solution. Providing a definition that is both clear and consistent with the literature in Political Science, Philosophy and Social Choice is difficult.

Instead of defining deliberation in full generality, we instead focus on deliberation in a political sense. Freeman [18] gives an overview of deliberative democracy. He notes that there is no settled definition of deliberative democracy, however, one account is that of public discussions before voting. Furthermore, he shares the intuitive idea that a deliberative democracy contains open legislative deliberation and a pursuit of the common good. He further proceeds to give a more detailed conception of deliberative democracy, according to which a deliberative democracy is one in which political agents or their representatives:

1. Aim to collect, deliberate and vote
2. Represent their sincere and informed judgments
3. Vote and deliberate on measures beneficial to the common good for the citizens
4. Are seen and see each other as political equals
5. Have Constitutional rights and their social means enable them to participate in public life

6. Are individually free, such that they have their own freely determined conceptions of the good
7. Have diverse and disagreeing conceptions of the good
8. Recognize and accept their duty as democratic citizens, and do not engage in public argument on the basis of their particular moral views incompatible with public reason
9. Agree reason is public, in so much as it is related to and advances common interests of citizens
10. Agree that their common interest lies primarily in freedom, independence and equal status as citizens

These features allow us to be more precise when we talk about a deliberative democracy, and in turn be more careful about what deliberation must entail. Cohen [9] further argues that deliberation is needed for democratic legitimacy. By this he means that without deliberation, a democracy is simply the will of the majority, but since majority rule is unstable, as shown through the Condorcet cycles, it is simply a reflection of the particular institutional constrains at the time, which end up dictating where the cycle breaks. He further goes on to describe the *ideal deliberative procedure* as follows:

1. Ideal deliberation is *free*. Participants regard themselves as only bound by the results of the deliberation, and the preconditions thereof. Participants act in accordance with the decision made through deliberation, and it being agreed on is sufficient reason to do so.
2. Ideal deliberation is *reasoned*. Participants must state their reasons for supporting proposals.
3. In ideal deliberation, parties are *equal*, both formally and substantively. There are no rules that single individuals out, and existing distributions of power do not lend a party the opportunity to contribute to deliberation.
4. Ideal deliberation aims to arrive at rationally defensible *consensus*.

From both Cohen's and Freeman's accounts, there is clear overlap, with Freeman formulating the necessary preconditions for participants to engage in ideal deliberation. Both Cohen and Freeman require freedom in a broad sense. Freedom to have a personal conception of the good, and to acknowledge and act in accordance to a decision that was made through deliberation.

3.2.2 Meta-Agreement

Consensus, sometimes referred to as substantive agreement, then seems like a natural goal for deliberation. Elster [15] argues that this is not only the goal, but through unanimous agreement this process completely replaces voting, thereby circumventing social choice's classic impossibility theorems: "Or rather, there would not be any need for an aggregation mechanism, since a rational discussion would tend to produce unanimous preferences." (p. 112). Though it would be desirable to circumvent these negative results, in practice people, even after deliberation, might not and indeed often do not come to full substantive agreement. List [25] instead proposes another perspective on deliberation based on meta-agreement

Under meta-agreement individuals do not need to agree on their most preferred outcome, instead they only need to agree on the dimensions of the problem. To contrast this with Substantive-agreement, under which individuals do not need to conceive of the problem in the same way, only requiring agreement on the preferred outcome. This means that under substantive agreement, voters can agree outcome $a > b$ for different reasons, while under meta-agreement, if voters disagree on $a > b$ it must be for the same reason.

According to List [25] there are three hypotheses that need to be satisfied for deliberation to induce meta-agreement:

D1 Deliberation leads people to discover a single *issue*-dimension

D2 Deliberation lets people place all possible candidates in this *issue*-dimension

D3 After deliberation, people update their preferences picking a preferred outcome, and ranking all other candidates based on the distance to this outcome in the *issue*-dimension

These are necessary conditions for meta-agreement. From this is it also clear to see that, given that there is exactly one *issue*-dimension, single-peaked profiles are, by definition, a direct consequence. This property of inducing single-peakedness makes meta-agreement particularly desirable, as it enables circumvention of the Gibbard–Satterthwaite theorem [21, 35] through domain restriction to $\mathcal{D}_{\text{SP}}$.

List et al. [26] provide empirical evidence for this theory of deliberation, showing deliberation increases proximity to single-peakedness through voter deletion (PtS-V), which they quantify as $S = \frac{m}{n}$ where $n = |N|$ and m is the largest subset of voters such that their profile is single-peaked. Furthermore, they also introduce the notion of salience, which represents to what extent a topic is salient in the voting population. In order to test whether deliberation increases single-peakedness *through* meta-agreement, they test the following four hypotheses: (H1) deliberation increases PtS-V. (H2')[1] high salience issues show less increase in PtS-V than low salience issues. (H3) Effective deliberation, in the sense that more is learned during deliberation, results in bigger increases of PtS-V. (H4) All things equal, the increase is largest for issues with natural *issue*-dimensions. They find support for all these hypotheses, showing that on low-moderate salience issues PtS-V increases following deliberation.

It is important to note that these claims simply predict what will happen, there is not much explanatory power to these claims. Little is known about the process through which voters would signal the issue dimensions, nor how they decide on which ones to present.

Furthermore, Ottonelli and Porello [30] show single-peakedness from meta-agreement to be a stronger requirement than it may seem at a first glance. Firstly, for (D1) to hold, the *issue*-dimension must hold some semantic meaning, as it is unclear how people can exchange conceptualization of the problem otherwise. Furthermore, the issues must consist of two semantic issues, with only one issue voters simply reach substantive agreement. A further restriction on these two dimensions is that they need to be opposite, with mutually exclusive justifications. If this is not the case, a voter can agree with both justifications, and thereby introduce a new implicit dimension "balance", which then violates the conditions under which single-peaked profiles guarantee the existence of fair, strategyproof voting rules. D2 requires that all voters share the exact same semantic understanding of the dimension, and the outcome associated with each candidate. Finally, D3 requires D1 and D2 to have happened before in order, indeed this is the weakest of the three requirements.

Thus, meta-agreement as a means for single-peaked profiles is still quite restrictive, needing multiple forms of unanimity, and only applying to problems with certain properties. Nonetheless, meta-agreement might still play a crucial part in a deliberative process. In the next section, we will look into a specific computational model of deliberation.

3.3 Models of Deliberation

Rad and Roy [32] model deliberation and its effect on single-peakedness. To this end, they model deliberation as a process where each voter announces their preferences, and all other voters update their current preference towards that of the announced preference, in doing so they have a bias towards their own preference, as such they try to

[1] This is a test for a corollary. H2 states that the rate of increase of PtS-V decreases. This is not experimentally testable, however since high salience means some sort of deliberation has happened before, they expect this to approximate this affect.

update their preference by minimizing the distance between their current preference and the announced one. This process repeats until all voters have announced their opinion once, which constitutes one "round" of deliberation. The preference a voter adopts when updating must lie between their current profile and the announced profile, which profiles are considered to be "between" is defined by the distance metric used. They considered three distance metrics, Kemeny-Snell (KS) [24], Duddy-Piggins (DP) [12], and Cook-Seiford (CS) [10]. Both KS and DP depend on the judgment set resulting from the voters' preferences, which contains, for each pair of candidates a, b, where $a \neq b$, a proposition $(a \succ b)$ or $\neg(a \succ b)$. The KS distance is then defined as the number of binary swaps a judgment set needs to undergo before it becomes the target judgment set, an example for such a swap would be going from $(a \succ b)$ to $\neg(a \succ b)$. The DP distance is defined on the graph of judgment sets, where 2 sets share an edge if there is no judgment set between them. Since KS and DP share their notion of betweenness, we define their betweenness as follows.

Definition 5: *J-Betweenness*

A judgment set J_i is between preferences J_j and J_k if for every $x, y \in A$, the proposition over x and y in J_i either agrees with the proposition over x and y in J_j or J_k.

From this definition it is clear that this could only result in a voter updating their original opinion in which they have $(a \succ b)$ to a new opinion where $\neg(a \succ b)$ only if the announced opinion contains $\neg(a \succ b)$.

The CS distance is simpler and is simply defined as the number of positions two voters disagree on, and a preference is between two others if for each position it agrees with one of the two preferences.

Each distance has different trade-offs. The CS metric is the simplest, but might exaggerate the distance when there are many candidates, for example if two voters agree on the relative ranking of all but one candidate, which one voter happens to rank first, thereby shifting the opinion of voter 2 right by one, the CS distance would conclude that these voters are in full disagreement, while reasonably one could conclude their opinions do not differ much. The KS distance, using judgment sets instead of raw profiles, captures this more effectively, while still being relatively easy to compute, but in case of many disagreements, it is likely to over count the distance, since the binary changes do not capture logical necessities. For example, swapping $(a \succ b)$ to $\neg(a \succ b)$ must result in $(b \succ a)$ becoming true (in the case of strict preferences), thus one might reasonably conclude this should only count as 1 step. DP improves upon this, Figure 3.1 shows a graph used for the DP distance in the case of 3 candidates. The graph shows the benefit of using the DP distance, as the edges in graphs automatically include logical consequences that the KS distance might not account for. By capturing logical consequences, the DP distances become much harder to compute, mainly through the cost of constructing the full graph of judgment sets, which grows in $f_m = 1 + \sum_{j=1}^{m-1} \binom{m}{j} f_{n-j}$ in the number of vertices, where m is the number of candidates [23]. This can easily be verified by noting that the number of judgments sets over m corresponds to the number of weak preference rankings over m candidates, which is defined as candidates, and a binary choice on each proposition.

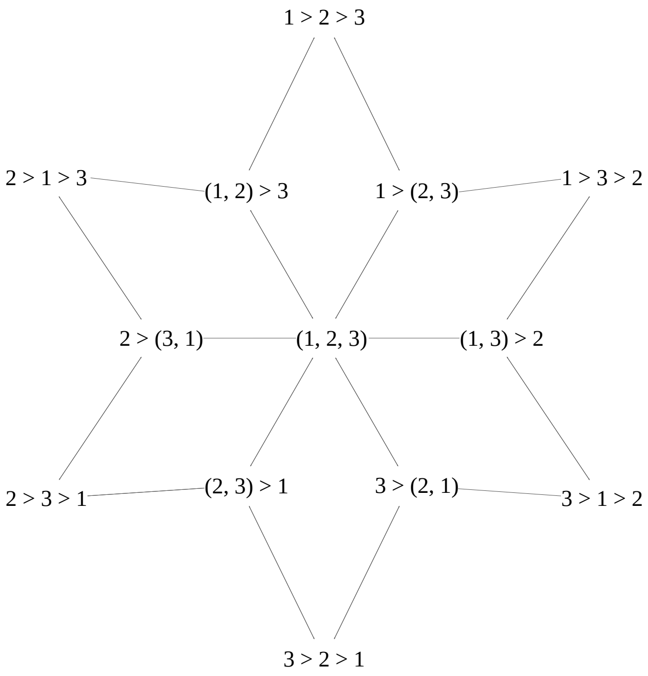


FIGURE 3.1: The graph of judgment sets for all preferences over three candidates, brackets indicate ties. For readability, the corresponding preferences are uses as node labels

Apart from these distances, Rad and Roy define a voter as a tuple of a linear order[2] and a bias $v = \langle r, b \rangle$, with $b \in \mathbb{R}_{[0,1]}$. Finally, a deliberation step $D_s : V^n \to V^n$, where V is the set of all possible voters $(\mathcal{L}(A) \times \mathbb{R}_{[0,1]})^n$ and s being one of the spaces (KS, DP, CS). The deliberation step $D_s(V, v.r)$ returns a fully updated voter set, where each voter has updated their opinion in response to the announced opinion $v.r$. We formulate this procedure in the following program:

[2] We exclude their analysis of preferences containing ties.

```
input  : Set of Voters $V$, metric space $s$
output: Updated set of Voters $V$

$V_\mathrm{u} \leftarrow V$ // Set of unannounced voters (references to $V$)
while $|V_u| > 0$ do
    Select a random $v \in V_\mathrm{u}$
    $V_\mathrm{u} \leftarrow V_\mathrm{u} \setminus \{v\}$
    $V \leftarrow D_s(V, v.r)$ // Update voters based on $v$'s preference
```

Here, we use $v.r$ to denote the preference component of voter $v = \langle r, b\rangle$. The deliberation step $D_s(V, v.r)$ returns a new set of voters, where each voter updates their opinion based on v's preference r, under the influence of the deliberation space s. Each voter updates their preference to a new profile r' that minimizes the weighted distance between their original preference r_i and the announced preference.

$$\sqrt{bd_s(r_i, r')^2 + (1-b)d_s(v.r, r')^2} \tag{3.1}$$

Here b is this voter's bias, and d_s is the distance between two profiles under distance space s.

We present a replication and extension of their work Chapter 6. Furthermore, we present novel (negative) results based on this model in Chapter 4.

While this model effectively captures preference communication, it falls short as a model of meta-agreement in at least two important respects. Firstly, agents do not conceive of anything relating to the structure of the problem. They simply announce their preferences, and all other listen and update accordingly, thereby moving to some sort of substantive agreement. Secondly, the model presupposes that all opinions are equally defensible, and that each voter is equally able to formulate this defense. To address this, we formulate a new model in Chapter 4.

3.4 Deliberative experiments

Empirically, deliberation appears to bring about numerous positive outcomes, both from the perspectives of the electorate and democratic theorists. One example is the Citizens' Initiative Review (CIR), in which randomly selected voters come together to evaluate a policy proposal. Their goal is to collaboratively draft an informational brochure presenting arguments for and against the proposal, aimed at helping the broader public make informed decisions. As part of this process, participants deliberate on the issue and consider possible solutions. CIRs have been successfully implemented in the United States, where they have been shown to enhance voter knowledge and judgment [20]. Their positive effects have also been observed outside the U.S., including in Finland [36].

While this line of research has largely focused on participants' experiences and the development of their attitudes and knowledge, studies that quantitatively map how voters' opinions shift during deliberation have been relatively scarce.

An important exception is a study by Fishkin et al. [17], who conducted the AMERICA IN ONE ROOM experiment, a large-scale deliberative event in which a representative sample of U.S. voters gathered to discuss major policy issues in the lead-up to the 2020 presidential election. Participants completed surveys before and after the event assessing their political knowledge, policy preferences across five issue domains (climate, immigration, the economy, health care, and foreign policy), and partisan affiliation (including intended vote choice and ideological self-identification). A control group completed the same surveys without participating in deliberation. The researchers found that deliberation increased participants' likelihood of voting, improved their opinions of political opponents, and increased support for Joe Biden—especially among moderate and previously disengaged voters. These effects were explained in terms of a "civil awakening", wherein deliberation led to increased self-efficacy and political engagement among previously uninvolved citizens. Notably, these effects persisted for at least a year after the intervention.

While the study did not elicit full preference rankings over all political options, it provides strong evidence for both increased meta-agreement (i.e., alignment on how political differences are framed or understood) and substantive agreement (i.e., convergence of actual issue positions). Participants' opinions tended to shift toward the center, with conservative voters showing the most change. Moderate voters also became more likely to support Biden, suggesting changes in how they conceptualized the candidates' ideological positions.

CHAPTER 4

THEORY

In the model of deliberation by Rad and Roy [32], outlined in Section 3.3, they aim to model deliberation and show that deliberation results in nicely structured profiles which allow for strategy proof voting rules. One important caveat, given by the authors as well, is all participants should honestly and truthfully participate in deliberation. We now provide a formal statement, showing deliberation does not prevent strategic behavior.

Proposition 4.1. The process of deliberation over $|A| \geq 3$ through deterministic deliberation procedure $D : \mathcal{L}(A)^n \to \mathcal{L}(A)^n$, followed by voting with voting rule f cannot be surjective, strategyproof and non-dictatorial.

Proof. Assume, towards a contradiction, such a pair of deliberative procedure (D) and voting rule (f) exists. Any deterministic deliberation procedure D could, in principle, be embedded into a voting rule $f'(\boldsymbol{R}) = f(D(\boldsymbol{R}))$, such that the voting rule simulates D before applying f, which would result in voting rule f' being surjective, strategyproof and non-dictatorial. This is a contradiction, by the Gibbard-Satterthwaite Theorem 2.1.

We extend upon this result, showing the inclusion of biases in voters does not mitigate the negative result. For this, we define BD as follows:

DEFINITION 6: *Biased Deliberation*

A deliberative procedure with biases BD : $\mathcal{L}(A)^n \times \mathbb{R}^n_{[0,1]} \to \mathcal{L}(A)^n$ is an extension on a standard deliberative procedure. BD has access to the bias each voter has towards their own opinion.

We now proceed with a corollary on Proposition 4.1. Towards this, we assume biases are true, in the sense that a voter cannot help but be 'convinced' by the presented profiles as much as their bias allows for this. We think this assumption is weak and natural in the light of the current model. Furthermore, a violation of this assumption would not imply the following corollary to be false, instead the bias itself becomes a point of strategy, allowing voters to pretend to be more hardheaded than they in fact are.

> **Corollary 4.2.** A deliberative procedure with biases, followed by voting with any voting rule f, cannot be surjective, strategyproof and non-dictatorial

The proof of this follows from a reduction of the biased Deliberation BD to general deliberation D.

> *Proof*. Take any election consisting of biased deliberation BD and voting rule f, since biases $\boldsymbol{b}$ are true by assumption, they must be fixed, meaning that $\boldsymbol{b}$ is not reported but some fact of the matter. If this election was immune to strategic manipulation, then a deliberative procedure D could embed this b, and simulate biased deliberation BD, resulting in $D'(\boldsymbol{R}) = \mathrm{BD}(\boldsymbol{R}, \boldsymbol{b})$. As a direct corollary to Proposition 4.1, such a D' cannot be surjective, strategyproof and non-dictatorial, showing a contradiction.

This result is independent of the metric space chosen. From here we now show that even if we take the deliberation procedures on its own, it still not immune to strategic manipulation. For this, we restate strategyproofness as follows:

Definition 7: *Strategyproofness of Deliberation*

A deliberation procedure is strategyproof if there exists no voter i such that there is a profile $\boldsymbol{R}$, in which i misreporting their preference R_i as R'_i results in the profile after deliberation $D(\boldsymbol{R})$ is further from the i's original preference than if they had reported R'_i. This distance is measured as

$$\mathrm{Dist}(R_i, D(\boldsymbol{R})) \geq \mathrm{Dist}_S(R_i, D(\boldsymbol{R}')).$$

Where the Dist function is simply the sum of all distances with distance measure S between R_i and all preferences in $\boldsymbol{R}$.

One important note is that in the final profile, the preferences of voter i might not be the same as it was before the deliberation. That is why the distance is calculated w.r.t. i's original preference. Intuitively this could be read as i misreporting their preference to prevent even their own mind from being changed. Using this definition, we show that the deliberative procedures, under the metric spaces KS, DP, CS are not strategyproof. Stated as follows:

Proposition 4.3. Deliberation, as defined by Rad and Roy [32], under distance measures $KS,\ DP,\ CS$ is not strategyproof, for $n \geq 2$ and $m \geq 3$.

We provide a proof by construction, we show how to do this for the KS and DP distance measures, as they share the same profiles for this proof. The proof for the CS distance measure is laid out in Appendix B.1.

Proof. Assume the following population: we have voter 1 whose bias is 1, and all other voters $j \neq 1$ have bias 0.5. Furthermore, we have $\text{Dist}_S(R_1, R_j) = 2$ for all j. Voter 1 now has the option to report R_1' instead, which has $\text{Dist}_S(R_1', R_j) = 4$ and $\text{Dist}_S(R_1', R_1) = 2$. If voter 1 reports R_1', then all j will update towards 1's true preference, as using equation (3.1) we get $r(R_j, R_1', R_1) = 4$, while $r(R_j, R_1', R_j) = r(R_j, R_1', R_1') = 16$.

Resulting in $\text{Dist}_S(R_1, D(R_1, \boldsymbol{R}_{-1})) = 2(n-1) > \text{Dist}_S(R_1, D(R_1', \boldsymbol{R}_{-1})) = 0$.

Since 1 has a bias of 1, the order of the deliberation has no effect.

We now show that for distance measures KS and DP, there exists these 3 preference orderings such that the necessary profile can be constructed. We use the following profiles:

$$\begin{aligned} R_1' &= a \succ c \succ b \succ \cdots \succ m, \\ R_1 &= a \succ b \succ c \succ \cdots \succ m, \\ R_j &= b \succ a \succ c \succ \cdots \succ m. \end{aligned}$$

As we are only allowing strict preferences, both distance metrics behave the same locally, with the distance of two profiles being 2 whenever one is 1 swap of candidates away from the other. This means that R_i and R_j have a distance of 2, as well as R_1' and R_1 having a distance of 2. In this case, the total distance from R_1' to R_j is simply the sum of the local distances for both distance metrics, thus satisfying our requirements.

These results show it is likely frivolous to attempt to design a strategy proof deliberation procedure of the likes shown. Instead, focus is now brought to modeling 'ideal' deliberation, as laid out in Section 3.2.2. We provide the following mathematical formulations to the four tenants laid out. *Freedom*: voters can report any preference, *Reason*: voters are rational, *Equality*: no voter has special rights, i.e. the axiom of neutrality is satisfied, *Consensus*: voters deliberate with the aim to reach consensus. Which we extend with *Honesty*: Voters represent their true beliefs and preferences only.

4.1 Our Model

In an attempt to model meta-agreement through deliberation, our model needs to make a proper distinction between the 'substantive level' and the 'meta level'. In order to do so, we propose the following, let $\Psi = \{\psi_1, \cdots \psi_k\}$ denote the set of policies that could be implemented. A voter $i \in N$, has support for these policies, represented as a number on an interval over $\mathbb{R}$. At a meta level, a voter has an understanding of which policies are supported by which candidates. This is modelled as a matrix, representing the estimated support for each policy for a candidate, thus voter i has Σ^i, where $\Sigma^i_{j,x}$ represents this voters' estimated support of ψ_j by candidate x.

This model does not explicitly model $D1$, the discovery of a common issue dimension, on the one hand, if the candidates can be reduced to a line, this model should be able to capture this, even if this one line crosses through multiple issue dimension. For example, if all issues are strongly (negatively) correlated on the side of the candidates, but not on the side of the voters, this model allows for the voters to recognize this by properly estimating the candidates' support matrices, while voters themselves can keep an uncorrelated support vector. In the case that the actual issue dimension is simply not included in Ψ, our model would not be able to discover this new dimension, even if human deliberation feasibly could. More straightforwardly, if the measured support is irrelevant to the true issue dimension(s), our model cannot recover the true issue dimension.

Our model adapts the DeGroot learning model, which originally models probability distributions. In that model, a voter is a node in a graph, and deliberation can be modeled as a Markov chain. In our model, we keep voters as nodes on a graph, as well as a Markov chain, however, instead of a probability distribution, a voter has a support vector $S_i \in \mathbb{R}^{|\Psi|}_{[0,1]}$, and estimated support matrix $\Sigma_i \in \mathbb{R}^{|A| \times |\Psi|}_{[0,1]}$.

Note that this does not mean that all policies have to have any (estimated) support, nor that an candidate can only support a specific number of policies, in principle there can be candidates that represent the status quo, and thus do not support any policies, and there can be candidates that are estimated to support all policies. Let $\boldsymbol{S} = [S_1, \ldots, S_n]^T$ denote the population opinion, which has shape $|N| \times |\Psi|$.

In order to extract a ballot from this matrix, we assume a voter ranks the candidates such that the most preferred candidate has the smallest distance between the estimated support matrix for that candidate and her own. We further allow this distance to be weighted, such that a voter may have one or more policies their think are more important.

Next, we define the deliberative procedure in terms of the trust matrix. A deliberative step can be modelled using a transition matrix T, defined as follows:

$$T = \begin{bmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & \ddots & \vdots \\ t_{n1} & \cdots & t_{nn} \end{bmatrix}$$

Here each t_{ij} represents how much voter i trusts the opinion of voter j, in order for this to be a proper stochastic matrix, all rows must sum to one, and have non-negative entries. Although this last requirement could be seen as unrealistic, as a voter might actively distrust another voter and update away from their opinion.

Using this, we can now model the opinions of voters after a deliberative step as a matrix multiplication on some matrix M:

$$M^{(1)} = TM^{(0)} \tag{4.1}$$

Each entry in the matrix then is simply a linear combination of the other entries in that same column in $M^{(0)}$. In the case of $M = \Sigma$, this means that voter i's support vector becomes a linear combination of all support matrices, weighted by the trust in each voter. Deliberation can now be modelled by taking powers of the trust matrix, T^t, representing t deliberation steps. This matrix now represents how much each voter i has learned from the other voters, and can then be used to right multiply both the support and the estimated support matrix to calculate a voter's beliefs after deliberation.

Finally, we provide an example of the first deliberation round in example 4.1, since it is identical for both S and Σ, we only show it for Σ. The example also shows how voters can initially agree on their support for policies, while disagreeing on their preferred candidates, using meta-agreement to come to a consensus.

EXAMPLE 4: *DeGroot deliberation*

We have voters $N = \{1, 2\}$, events $\Psi = \{\psi_1, \psi_2\}$, and candidates $A = \{a, b\}$. The voters both think that $\psi_1 = 1, \psi_2 = 0$, meaning that they fully support the first policy and reject the second, they estimate the support by candidates as:

1	ψ_1	ψ_2
a	0.5	0
b	0.5	1

2	ψ_1	ψ_2
a	1	0.9
b	1	0.1

Interpreting this matrix for both players on ψ_1 shows, voter 2 thinks a and b fully support ψ_1, while voter 1 thinks that a and b support ψ_1 less. We can encode this into the estimated support matrices as follows:

$$\Sigma_1 = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 1 \end{bmatrix} \quad \Sigma_2 = \begin{bmatrix} 1 & 0.9 \\ 1 & 0.1 \end{bmatrix}$$

This results in voter 1 preferring candidate b over candidate a, while voter 2, prefers a. Intuitively, since voter 1 thinks ψ_1 is equally supported by each candidate, while ψ_2 is not supported by a, it makes sense for them to prefer candidate a. Looking at the distances, we see that the absolute distance between voter 1 and candidate a is 0.5, while for candidate b it is 1.5. For voter 2 we see that the distance to a is 0.9, while for candidate b is it 0.1. Thus, voter 2 prefers b to a.
For the deliberation, we assume the following trust matrix:

$$T = \begin{bmatrix} 0.3 & 0.7 \\ 0.2 & 0.8 \end{bmatrix}$$

We get the following updated opinions:

$$\begin{aligned} \mathbf{\Sigma}^{(1)} &= T\mathbf{\Sigma}^{(0)} \\ &= T\begin{bmatrix} \Sigma_1 \Sigma_2 \end{bmatrix}^T \\ &= \begin{bmatrix} (0.3\Sigma_1 + 0.7\Sigma_2) & (0.2\Sigma_1 + 0.8\Sigma_2) \end{bmatrix}^T \\ &= \begin{bmatrix} \begin{bmatrix} 0.85 & 0.63 \\ 0.85 & 0.37 \end{bmatrix} & \begin{bmatrix} 0.9 & 0.72 \\ 0.9 & 0.18 \end{bmatrix} \end{bmatrix}^T \end{aligned}$$

These new estimates are not yet in full consensus, meaning meta-agreement has not yet been reached. Looking at their corresponding ballots, however, shows there is consensus on their most preferred candidate, as they both agree that candidates support ψ_1 equally, while b supports ψ_2 less.

4.1.1 Consensus

Using this model of deliberation, meta-agreement can be seen as some shared estimated support matrix over all policies. If the goal of deliberation is meta-agreement, then the study of interest becomes the dynamics of convergence towards a unified estimate.

We present a summary of results relating to strongly connected graphs, as well as graphs for which there exists only closed and strongly connected subsets of nodes. For other results we refer to Golub and Jackson [22]. Firstly we focus on the strongly connected graphs.

> **Proposition 4.4.** (Golub and Jackson [22]). For a strongly connected matrix T, the following properties are equivalent:
>
> - T is Convergent
> - T is Aperiodic
> - There exists a left eigenvector $\boldsymbol{s}$ for matrix T, with corresponding eigenvalue 1, whose entries sum to one, such that for every P_i, we have
>
> $$\left(\lim_{t\to\infty} T^t \boldsymbol{P}\right)_i = \boldsymbol{sP}$$

This result is positive for studying the convergence dynamics, as no knowledge of the initial distribution is needed to determine convergence, it allows us to simply verify one of these three properties on the network. Though strongly connected graphs might be a strong requirement, in the case of small scale (in person) deliberation, this might be realistic. Fortunately, even outside this setting it might be possible to reach convergence. For this we first define what a closed set of nodes is.

Definition 8: *Closed set of Nodes*

A set of Nodes $C = \{1,...,n\}$ is closed if for each $i, j \in C$ we have $T_{ij} \geq 0$ and for each $i \in C, j \notin C$ we have $T_{ij} = 0$

Using this definition, if each node is part of a closed set, we can form the following proposition

> **Proposition 4.5.** (Golub and Jackson [22]). If for each $i \in N$, i is a member of a closed set in the graph, and each closed set is strongly connected, T is convergent.

4.1.2 Voter Mapping

One might want to expand this model to capture larger scale group dynamics, such as social networks. For this a reasonable approach could be to gather data regarding the opinion of the general population, and to map this onto a graph representing the communication in the population. For this we might want to find a bijection between the voters and the nodes such that the difference between the shortest paths in the graph and the opinion distance is minimized.

We show that mapping voters to a graph as just described is NP-Hard, and the decision variant of the problem to be NP-Complete. We call this problem Distance-based Voter Mapping, and define it as follows.

PROBLEM 1: *δ-DBVM(S)*

Given: $A, B \in S^{n \times n}, k \in \mathbb{R}_{\geq 0}$
Decision: Does there exist some bijection $f : [n] \to [n]$, such that:

$$\delta(A, f(B)) \leq k$$

Here we take $f(B)$ to mean the matrix B' that is created when we take each $B'_{i,j} = B_{f(i),f(j)}$ and δ is some distance function, $\delta : S^{n \times n} \times S^{n \times n} \to \mathbb{R}_{\geq 0}$.

We will be needing the Quadratic assignment problem (QAP), we formulate a decision variant of QAP as follows.

PROBLEM 2: *QAP-Decision*

Given: $A, B \in S^{n \times n}, k \in \mathbb{R}_{\geq 0}$
Decision: Does there exist some bijection $f : [n] \to [n]$, such that:

$$\sum_{i,j} A_{i,j} \cdot B_{f(i),f(j)} \geq k$$

Theorem 4.6. δ-DBVM(S) is NP-Complete for $\delta \in \{\ell_1, \ell_2\}$ and $S = \{0, 1\}^n$

Proof. ($\implies$ NP-Hard) The proof follows from a reduction to the Quadratic Assignment Decision Problem.

Let A be the matrix of pairwise distances between voters, and let B be the matrix of shortest-path distances in the graph G, and k be the δ achieved by the optimal bijection. ℓ_2-DBVM(S) requires finding a bijection f that minimizes the ℓ_2 objective:

$$\sqrt{\sum_{i,j} \left(A_{i,j} - B_{f(i),f(j)}\right)^2}.$$

Since the square root is a strictly increasing function, minimizing the expression above is equivalent to minimizing the sum inside:

$$\sum_{i,j} (A_{i,j} - B_{f(i),f(j)})^2.$$

> Expanding the square gives:
>
> $$\sum_{i,j} A_{i,j}^2 - 2A_{i,j}B_{f(i),f(j)} + B_{f(i),f(j)}^2.$$
>
> The terms $\sum A_{i,j}^2$ and $\sum B_{f(i),f(j)}^2$ are independent of f (the former is fixed, the latter is a permutation of a fixed matrix), so the optimization reduces to:
>
> $$\max_f \sum_{i,j} A_{i,j}B_{f(i),f(j)},$$
>
> which is the standard form of the Quadratic Assignment Decision Problem. Note, $\max_f$ is a consequence of the sum being subtracted from the constants, thus we are still minimizing the total distance.
>
> Now we note that when A and B are in $S = \{0,1\}^{n \times n}$, the ℓ_1 and ℓ_2 norms are identical. We also note that this binary domain would constitute a special instance of QAP, know as 0-1 Max-QAP, and is NP-Hard [29]. Thus solving δ-DBVM(S), on the binary domain, is equivalent to solving 0-1 Max-QAP, and thus NP-Hard. ✓
>
> ($\implies$ NP-Membership) Given any f, we can evaluate the cost of the allocation in $O(n^2)$. ✓

A concern with Theorem 4.6, might be the matrices containing certain patterns that might lead to an easier solution, though this proof concerns itself with the worst-case and thus this possibility of this problem being easier in practice is not issue. For this problem such patterns seem unlikely to be of much help. We show one example to give an intuition for this.

Take the case in which all voters hold one of 2 opinions, thus we can split them into two groups of sizes n, m. Then the mapping algorithm effectively requires finding a partition in the graph, that results in two sub-graphs with exactly n and m nodes each. This is the size-constrained graph partitioning problem, which is NP-Hard.

Thus, given that even under such a strong assumption the problem remains computationally difficult, we suspect that patterns in the data are unlikely to allow for easier exact solutions. This does leave room for approximation algorithms, we do not present an overview of these, however under our constraint of one of the matrices satisfying the triangle inequality, namely the voter distance matrix. There exists a $\frac{2e}{e-1}$-approximation algorithm [29].

Despite these negative results, we attempted to enlist the help of a QAP-solver [37] to find (approximate) solutions, using the Fast Approximate QAP Algorithm [38]. Though, we find the solver does not consistently find better solutions than random assignment, and is unable to handle large enough instances for the experiments presented in the following chapters.

CHAPTER 5

METHODS

This section presents our experimental methodology in three parts. First, we replicate the preference-based deliberation model of Rad and Roy [32] to establish baseline measurements. Second, we develop and validate the adapted DeGroot model using data from the AMERICA IN ONE ROOM experiment. Finally, we apply this validated model to generate synthetic preference profiles and analyze their structural properties.

All experiments are implemented using OCaml and Python. Data sources and ethical considerations are detailed in Appendix A, all code can be found at `https://github.com/amirsahrani/master_thesis`.

5.1 Replication of Rad and Roy [32]

We implement the model as described in Section 3.3. Agents are limited to strict preferences over all candidates. All experiments are done with 3 candidates, and 51 voters. The number of voters is chosen to be an odd number to prevent perfect ties. Each voter receives a random strict preference, created by permuting the candidates. All voters share the same bias factor. The order of deliberation is decided randomly, by shuffling the voters. Bias is varied between 0.45 and 0.99 (exclusive) in steps of 0.1, for each bias factor we run 100 simulations, for a total of 5400 simulations.

We measure evaluations relating to strict preferences, namely the proportion of cyclic profiles, the number of unique profiles and the proximity to single-peakedness by voter deletion (PtS-V), all as also reported by Rad and Roy [32].

Due to the computational complexity of the DP-metric, as well as the calculation of PtS-V, a larger number of candidates is computationally infeasible. Specifically, PtS-V is

NP-complete [16], though it allows for a 2-approximation. We use the method based on an ILP solver, as implemented in `PrefTools` [1].

5.2 Adapted DeGroot

We use the adapted DeGroot model as laid out in Section 4.1 to capture deliberation dynamics. This model requires realistic trust matrices, we propose three mechanisms through which we can construct these.

Knowledge. We consider knowledge as a factor that can influence both the trust a voter places in others and the confidence they have in their own opinion. Let $\boldsymbol{k}$ denote a vector, where each k_i represents a knowledge score for voter i. This score may inform a voter's bias towards their own opinion, under the assumption that greater knowledge increases confidence.

There are two plausible interpretations of how knowledge affects self-bias. On one hand, more knowledgeable voters may be more confident and thus less susceptible to influence. On the other hand, increased knowledge might make voters more aware of the limits of their understanding—capturing the essence of the Dunning-Kruger effect, where individuals with limited knowledge fail to recognize their own ignorance. However, in the context of direct deliberation, we argue that the latter interpretation is less realistic: ideally, as deliberation progresses voters are exposed to new information and opposing viewpoints, making them more aware of the knowledge boundaries of their peers as well as their own. Thereby allowing voters to place more weight on voters more knowledgeable than them.

Regarding trust in others, we follow a similar line of reasoning: voters are more likely to trust peers who exhibit higher levels of knowledge. We assume that expertise becomes apparent during discussion, leading to increased trust in more knowledgeable individuals

Similarity. A voter might trust people more if they are similar to them, in this work we take similarity to mean a similarity in substantive opinion. It is however not hard to conceive of similarity influence trust in other ways such as social status. We calculate the similarity as the ℓ_1-distance between two voters' opinions, normalized by the maximal distance between two voters in their deliberative group.

Ego. Finally, a voter may place greater weight on her own opinion if she is highly trusted by others. This we calculate as the sum of incoming edges in the trust matrix.

Finally, we allow for a bias factor, similar to Rad and Roy, in order to directly affect the self-loop in the trust matrix.

Selecting among these mechanisms is ultimately an empirical question, which we explore in Chapter 6.

Given each notion of trust, we define the full model using matrix operations, including elementwise (Hadamard) products. Hadamard products are entry wise multiplications of matrices. First, we define T_{out} as follows,

$$T_{\text{out}} = A \odot K \odot S \tag{5.1}$$

Here A is the adjacency matrix of shape $n \times n$, without self loops. K is the matrix of knowledge scores of shape $n \times n$ with each row being the vector of knowledge scores $\boldsymbol{k}^T$, such that each $K_{ij} = \boldsymbol{k}_j$. S is the similarity matrix, where $S_{ij} = 1 - d_{ij}$, and d_{ij} is the normalized ℓ_1 distance between voters i and j's opinions. The normalization ensures the maximum distance is 1, so that $S_{ij} \in [0, 1]$.

Next, we compute the vector T_{in}, which represents each voter's internal (self-)trust or bias:

$$T_{\text{in}} = (T_{\text{out}}\boldsymbol{b}) \odot \boldsymbol{k} \odot \boldsymbol{e} \tag{5.2}$$

Here $\boldsymbol{b}$ is a vector of length n containing the bias factor in each entry, and $T_{\text{out}}b$ is a standard matrix-vector product yielding a column vector that captures the total external influence received, scaled by the voter's bias. $\boldsymbol{k}$ is the column vector of knowledge scores, and $\boldsymbol{e}$ is the ego vector, computed as $\boldsymbol{e} = T_{\text{out}}^{\top}\mathbf{1}$, where $\mathbf{1}$ is the all-ones vector. This represents the total trust each voter receives from others

Finally, we construct the full trust matrix T by combining T_{out} with the diagonal matrix formed from T_{in}, and normalize each row so that trust weights sum to 1:

$$T = \text{norm}\left(\text{diag}(T_{\text{in}}) + T_{\text{out}}\right) \tag{5.3}$$

Here + denotes element-wise addition, and norm normalizes the rows of the resulting matrix so that $\sum_j T_{ij} = 1$ for all i.

Once we have the trust matrix T, we model the evolution of both substantive and meta-opinions over t time steps as:

$$S_i^t = T^t S^{(0)} \tag{5.4}$$

$$\Sigma_i^t = T^t \Sigma^{(0)} \tag{5.5}$$

Here equation 5.4 captures the final support of voter i after t times steps, and equation 5.5, captures how they estimate the candidates' support.

In this model, we make the simplifying assumption that trust remains static over time. While this is likely an unrealistic assumption, the absence of data on trust dynamics during deliberation prevents us from empirically modeling its evolution. Although we refrain from speculating in detail on how this assumption could affect the general conclusions, we note it might affect both the rate of convergence, and the equilibrium reached.

Given this formulation, we define an instance of our model through shaping the matrices, such as shown in example Example 5.

EXAMPLE 5: *DeGroot deliberation Instance*

Consider a setting with three voters. Let us define the following matrices:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad K = \begin{bmatrix} 0.5 & 1 & 2 \\ 0.5 & 1 & 2 \\ 0.5 & 1 & 2 \end{bmatrix} \quad S = \begin{bmatrix} 0 & 0.5 & 1 \\ 0.5 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \tag{5.6}$$

Note that the adjacency matrix A has no self loops, K has repeating rows, and S is symmetric, as the similarity of voter i to voter j must be the same as the other way round.

Now suppose we want to create a trust matrix T, that uses knowledge for the outgoing trust, but not the similarity. Uses a constant bias factor of 2, and Ego-based trust, but not self-knowledge. To achieve this, we redefine matrix S as follows, noting that A handles the self-loops,

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{5.7}$$

Taking the element-wise product of these matrices yields:

$$T_{\text{out}} = \begin{bmatrix} 0 & 1 & 2 \\ 0.5 & 0 & 2 \\ 0.5 & 1 & 0 \end{bmatrix}$$

Next, we compute T_{in}:

$$T_{\text{in}} = \begin{bmatrix} 6 \\ 5 \\ 3 \end{bmatrix} \odot \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 12 \end{bmatrix}$$

Here the first vector is the result of each row sum of T_{out} by the bias factor (2), and the second vector is the ego vector, i.e., the sum of the columns in T_{out}.
The trust matrix before normalization is then:

$$\begin{bmatrix} 6 & 1 & 2 \\ 0.5 & 10 & 2 \\ 0.5 & 1 & 12 \end{bmatrix}$$

Which we then normalize to get:

$$T = \begin{bmatrix} \frac{2}{3} & \frac{1}{9} & \frac{2}{9} \\ \frac{1}{25} & \frac{20}{25} & \frac{4}{25} \\ \frac{1}{27} & \frac{2}{27} & \frac{24}{27} \end{bmatrix}$$

5.3 Model Validation

We split the experiments on the adapted DeGroot model into two parts. Firstly, we aim to assess the validity of the model. We use data from the America in One room experiment, focusing on the deliberation group. The control group in this data set shows no significant difference, thus the only sensible trust matrix is the identity matrix. This dataset does not provide full preference rankings over the candidates, instead provides data on voters' opinions on 6 different topics of political discussion, such as climate change and immigration. This data provides knowledge scores for each voter as measured by a set of seven questions relating to governmental institutions. We construct the knowledge scores as the fraction of correct answers. Using these opinions, we assess the validity of the model insofar as it is able to accurately predict the final opinion of voters. This we measure for each voter, as well as for binned groups of voters with similar opinions. The latter measurement replicating the assessment by Fishkin et al. [17], where voters are placed in fixed size bins such that each bin contains voters with similar initial PBS.

We run 5000 simulations, randomizing the independent variables laid out in Section 5.3, excluding `Candidates`, and `Candidate Generator`. We then use an ANOVA to test for the configuration of trust matrices that minimizes the absolute errors in predicted policy based-ideology score. Since the original data provides group numbers for the participants, we also experiment with replicating these groups as opposed to randomly grouping voters together. When using the original groups, the `number of voters parameter` is ignored.

We acknowledge that using the same dataset for both parameter estimation and validation may lead to overfitting. This represents a limitation imposed by data availability, as the AMERICA IN ONE ROOM experiment is the only large-scale deliberation dataset with the necessary pre- / post-measurements/post measurements and knowledge scores to which we had access.

Finally, we use sensitivity analysis to investigate which parameters have the strongest effect on the variance of the model's prediction error. Using Sobol indices and a sample size of 4096, we use the same ranges during the validation and sensitivity analysis. We calculate the first, second, and total order effects. The first order indices refer to their direct effects on the variance of the model, while all other parameters are varied randomly. The second and total order capture this for pairwise interactions of a variable and for all first- and higher-order interactions of a variable, respectively.

5.4 Introducing Meta-Agreement

To incorporate meta-agreement on alternatives, we expand the model by introducing simulated candidates. Candidates are generated in two ways: either by copying the opinion vector of a single randomly selected voter or by averaging the opinions of ten voters sampled with replacement. Each voter's preference ranking over the simulated candidates is then derived based on the ℓ_1 distance between their own opinion vector and that of each candidate.

We conduct 1,000 randomized simulations and evaluate the outcomes using the same metrics as in the Rad and Roy replication. These include the proportion of cyclic profiles, the presence of a Condorcet winner, the number of unique preference profiles, and the proximity to single-peakedness via voter deletion (PtS-V). Additionally, we compute the frequency of Condorcet winners and the proximity to single-peakedness via candidate deletion (PtS-C). PtS-C is an alternative view to PtS-V, focussing on ill-behaved candidates, instead of voters. The frequency of Condorcet winners, when more than three candidates are present, offers insight into the extent of agreement on a single best alternative, even when the majority graph contains cycles. For example, it is possible for one

Parameter	Description	Values
`Number of Voters`	The number of voters in the simulation.	9, 13, . . . ,29*
`Number of Candidates`	The number of candidates to be voted on.	3, 4, 5, 6, 7
`Candidate Generator`	The method used to generate candidates.	Sample, single random voter
`Bias`	The bias all voters have towards their own opinion.	0.8, 1.0, . . . , 2.8*
`Time steps`	The number of deliberation "steps" the voters undergo.	1, 2, . . . , 20
`Group`	Use the original groups.	True/False†
`Similarity`	Distribute trust based on similarity.	True/False*
`Knowledge`	Distribute trust based on knowledge.	True/False*
`Ego`	Scale voters' bias according to the trust other people have in them	True/False*
`Self-Knowledge`	Scale voters' bias according to their knowledge	True/False*

TABLE 5.1: The parameters of the adapted DeGroot model and their values. Parameters marked with an asterisk (*) are randomized during sensitivity analysis. † parameter set to `False` during sensitivity analysis.

candidate to be consistently ranked first by a majority of voters, while the remaining candidates form a Condorcet cycle among themselves.

While PtS-C can be computed in $O(|V| \cdot |C|^3)$ time [31], we rely on the implementation provided by the `PrefTools` library [1], which uses a slower $O(|V| \cdot |C|^5)$ algorithm based on the method from Erdélyi et al. [16].

In the following chapter, we analyze the behavior of these models under both empirical data and controlled simulations, examining their capacity to replicate realistic deliberative outcomes and foster structural agreement.

CHAPTER 6

RESULTS

We first present a full replication and extension of the work by Rad and Roy [32]. Then we present the simulations based on our model of meta-deliberation, as well as the results of the sensitivity analysis on both models.

6.1 Replication

We successfully replicate the results found by Rad and Roy [32]. Figure 6.1 shows for biases less than 0.73, all metrics result in acyclic preferences. We also replicate the behavior of the KS metric, where biases in the range of 0.73-0.85, show that even initially acyclic profiles can become cyclic. This is further illustrated in Figure 6.2, showing that within this range we always observe 3 unique preferences for the KS metric, while DP and CS always have 6 unique preferences, thereby representing all possible preferences. Finally, the proximity to single-peakedness shows a slightly more positive note for the KS metric, showing that while the DP and CS bottom out to the minimum proximity to single-peakedness, KS stays relatively high. However, this should be interpreted cautiously, as it likely reflects the smaller number of unique preferences, and thus the number of voters that need to be removed is at most 1/3.

Through these results, we observe that while the original model does show increase in the proximity to single-peakedness (PtS-V) and discourages cyclic profiles, its outcomes are highly sensitive to both voter bias and the chosen distance metric. In particular, the instability observed with the KS metric across certain bias ranges raises concerns about the robustness and external validity of the approach. Moreover, the model lacks a mechanism for higher-order disagreement or reflection—there is no “meta” level at

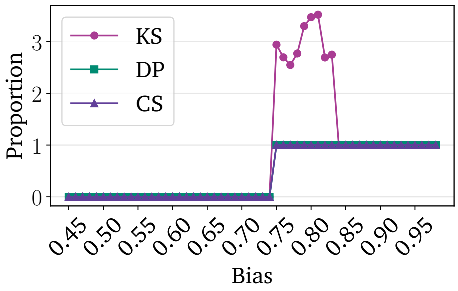


FIGURE 6.1: The proportion of cyclic profiles remaining, 0 indicating that no cyclic profiles were present after deliberation.

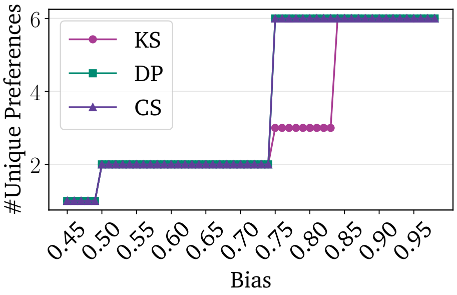


FIGURE 6.2: Number of unique preferences at the final step of deliberation.

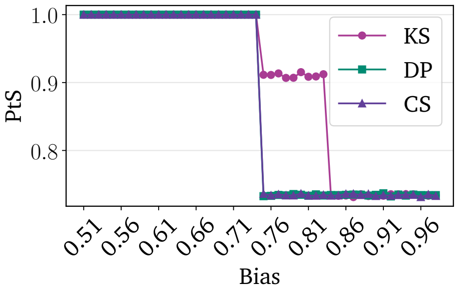


FIGURE 6.3: Proximity to single-peakedness after deliberation. Proximity to single-peakedness as defined in Section 3.3.

which agents evaluate the structure of their preferences. This limitation motivates the development of our own model, which explicitly incorporates meta-deliberation and trust dynamics to better capture the complexities of real-world opinion formation.

6.2 DeGroot Model

The model is calibrated using the data from the AMERICA IN ONE ROOM experiment, which was used to construct the support vectors S_i (each voter's vector of policy opinions), where each element in S_i corresponds to one response in the questionnaire. We follow the original paper, focussing on the 26 most polarizing questions, which we then use to calculate the policy-based ideology score (PBS) as the average these polarizing questions. Low PBS corresponds to more liberal answers, and high PBS indicates more conservative answers.

We remove all participants with missing responses to any pre- or post-deliberation measurements, retaining only participants with complete pre- and post-deliberation data. As a result, only 247 out of the original 523 opinions remain after this selection. This removes a large fraction of participants. However, it limits the number of assumptions we have to make on the opinions of participants. Interpolation of the missing data would likely artificially inflate the accuracy of the model, this might be further exaggerated by the fact that we need to infer preferences over (artificial) candidates.

The support vectors S_i correspond to the participants' reported opinions, based on measured by several policy questions rated from 0 to 10 (inclusive). Each voter's estimated support matrix Σ is generated by adding normally distributed noise($\mu = 0$, $\sigma = 1.37$) to the candidates' true opinions. Ensuring the model does not systematically favor candidates with higher or lower average scores, as otherwise voters would on average be over or underestimating candidates' support. The standard deviation is chosen to match voter PBS distribution before deliberation. The opinions of the candidates are generated as mentioned in Section 5.4, namely by copying the opinion of a voter at random, or by sampling ten voters with replacement.

To generate a deliberation group, we opt for two approaches. Either using the original deliberation groups, selecting a group at random and using the participants from that group. Given the restriction of voters with complete data these groups will tend to be smaller than in the original study, where these groups averaged 13 voters, in our subsection the average is 7. Or we generate new groups by picking n voters uniformly at random without replacement.

To evaluate model performance, we predict each voter's post-deliberation PBS and compare it to the observed data. Additionally, we group voters into ten bins based on their initial PBS and compare the average predicted PBS within each bin to the true bin average. This approach effectively models deliberation the substantive level and thus does not yet incorporate the possibility of *meta-agreement*. However, it allows for the evaluation of the model without assumptions on how to infer the final preferences of the voters, or the opinions of candidates. After this assessment, we investigate the convergence of the model, as well as its sensitivity to the choice of parameters.

Finally, we extend the model to incorporate meta-deliberation through deliberation on the estimated support matrices. Assessing its effect on voters' final preferences, using the metrics introduced in Section 5.4.

6.2.1 Policy-Based Ideology Scores

We first proceed with analyzing the performance of the DeGroot model with respect to substantive agreement. Figure 6.4 shows the PBS of both the deliberation and control group, and the simulation results for both instances, here the results are averaged over all tested configurations of the model. The trust matrix for the control group is generated using the network of citations in physics [34], and is sampled down to the size of the number of voters using the TIES sampling technique [2][1]. As expected the model has high mean absolute error (MAE) when predicting the post deliberation PBS of the control group, as there was no significant change for control group members in the original data. Within the deliberation group, a voter's initial PBS remains a strong indicator of their final PBS. We observe that the models predictions get more accurate after the first time step, with prediction errors increasing over time. This is because the model causes voters to converge too strongly, thereby eliminating most extreme opinions, contrary to the real data. The implications of this depend on the nature of long term deliberation. If, as suggested by Elster [15], deliberation is able to reach full consensus, the model might offer a plausible approximation of this process. However, if full consensus is not typically reached—as is precisely the motivation for incorporating meta-agreement into the model—then the DeGroot model should be seen as overly simplistic in its assumption that individuals converge toward a weighted average of the opinions presented to them.

Figure 6.5 depicts the change in PBS within the deliberation group. In the original data, most changes occur among participants with high initial PBS, who tend to moderate their views. The model, by contrast, predicts the most change among those with low PBS.

One possible explanation for this discrepancy is the correlation between PBS and political knowledge. As shown by Fishkin et al. [17], voters with more extreme PBS also tend to be more knowledgeable. Our filtered dataset supports this, showing a weak negative

[1] To address the issue of assigning voters to nodes in the final sampled graph (see Chapter 4), we used a Fast Approximate Quadratic Assignment Problem solver [38]. However, this approach did not consistently outperform random initialization.

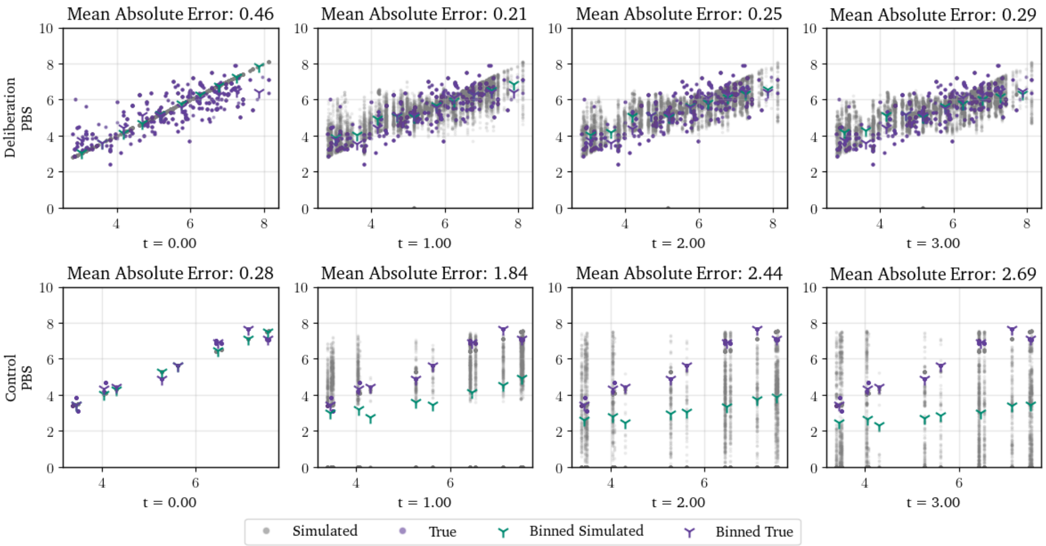


FIGURE 6.4: PBS, purple indicating the PBS after deliberation in the original data, green indicates the results of the simulation in that time step. Large dots indicate the binned data, smaller dots indicate individual voters.

correlation of -0.05 ($p < 0.05$), Figure B.1 in Appendix B shows the distribution of political knowledge across different PBS ranges. Since political knowledge in our sample is skewed toward voters with high PBS, incorporating knowledge-based trust into the model amplifies their influence, resulting in larger prediction errors.

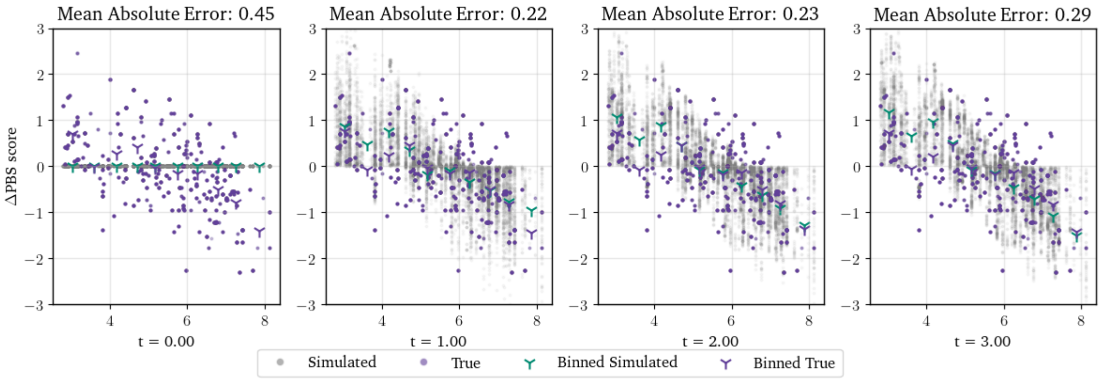


FIGURE 6.5: Change in PBS, relative to the original, pre deliberation, measurement. The control is omitted as there was no significant change.

We note that these positive results appear only when the voters are grouped by their original PBS during the initial 3-4 time steps, thereby giving the model reasonable predictive power over a population of voters. Figure 6.6 shows the progression of errors over time when the error is calculated on a per-individual basis (left), and binned (right). We find the model does not predict the change per individual well, with the original score at t=0 being a better predictor of an individual's final PBS than the model's output during any subsequent time steps. Notably, using the ego-based trust, the model makes smaller prediction errors. When we look at the predictions binned by initial PBS, the model seems to be doing a lot better, again with ego-based trust resulting in the lowest error. Interestingly after the first time step, knowledge-based trust results in the lowest prediction error, after this step ego-based trust outperforms all other kinds of trust.

Furthermore, Figure 6.6 shows that the model predicts individual PBS changes it has higher MAE when knowledge is included in the trust calculation as opposed to Ego, and is equal to similarity. This suggests that political knowledge, at least as measured in this dataset, is a poor predictor of persuasiveness. As he knowledge questions assess factual knowledge of the U.S. government, such as knowing which party holds a Senate majority, knowledge may not correlate well with persuasiveness on specific policy issues such as immigration or the economy.

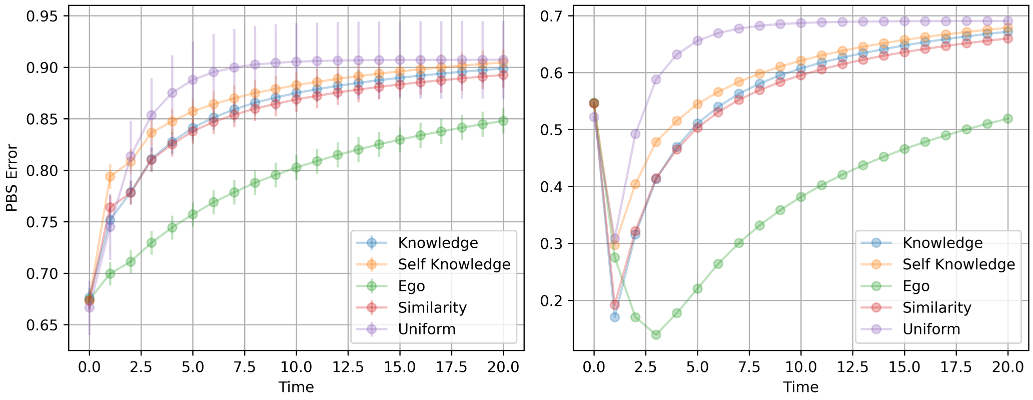


FIGURE 6.6: (Left) Mean absolute prediction error over time for different trust mechanisms, with 95% confidence intervals at each time step. (Right) Mean absolute error binned by voters' initial PBS score. Binning reveals how predictive performance varies across the ideological spectrum.

Further, looking into the change in PBS, Section 6.2.1 divides the change in PBS up into the change on each topic measured. The model predicts the change in PBS for healthcare very well across different trust generation methods, as well as changing the PBS into the right direction for the economy and immigration, still the model predicts roughly half as much change in PBS for immigration for any of the trust generation methods. For the environment and foreign policy, the model predicted an increase in PBS on average, while in reality people decreased their PBS. Comparing different trust generation methods, we see that ego and similarity are quite similar on most topics, but specifically on the economy ego seems to be very accurate, while similarity seems to result in little change at all.

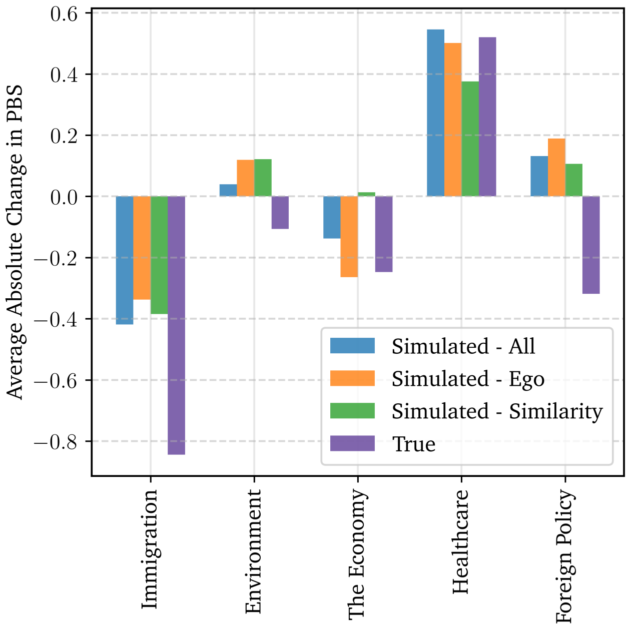


FIGURE 6.7: Change in PBS per topic with bias random between 0.2 and 0.4. This figure compares the observed change in PBS across policy topics (e.g., healthcare, immigration), showing the average change for three trust mechanisms: ego-based, similarity-based, and the model using all trust signals combined.

Figure 6.8 shows the relation between the bias factor and the PBS, showing that the bias does not improve the model's predictive power. As one might expect a bias is "slowing down" the model. Because of this the model is slower to diverge away from the true opinions.

We suspect ego improves predictive accuracy for two reasons. First, by assigning individual-specific biases, the model better reflects heterogeneous deliberative behavior. Second, increased self-bias slows down convergence, preventing the model from over-correcting.

6.2.2 Convergence of Trust Matrices

From Chapter 4, we have seen that in the limit some matrices are convergent, while some are not, in particular if the matrix is aperiodic, it is convergent. As we model the deliberation group as having fully connected matrices, with self-loops, the matrices are aperiodic, and thus convergent. We look at the distance between the estimated support matrix, and the true support matrix, to get a sense of the rate of convergence. The distance is defined as the ℓ_1 norm.

In Figure 6.9, all configurations converge at a similar rate, slowing down the rate of change around t = 15. Since using the original groups leads to generally smaller groups,

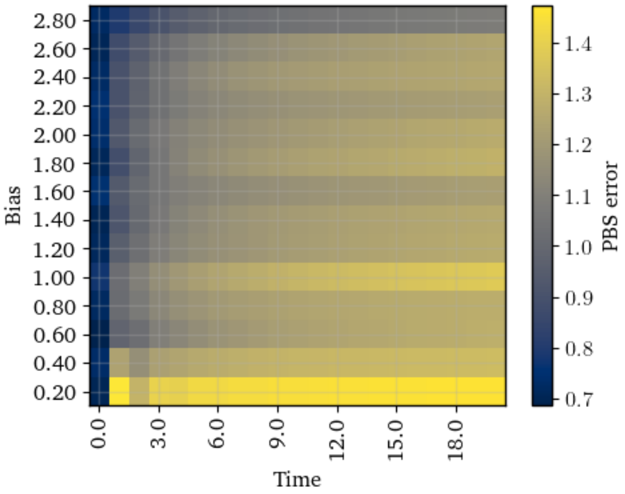


FIGURE 6.8: Mean absolute prediction error as a function of bias and time using ego based trust. The heatmap shows how the PBS error evolves over time for different bias levels. Higher bias slows the rate of opinion change, and thereby prevents the opinions from becoming homogeneous.

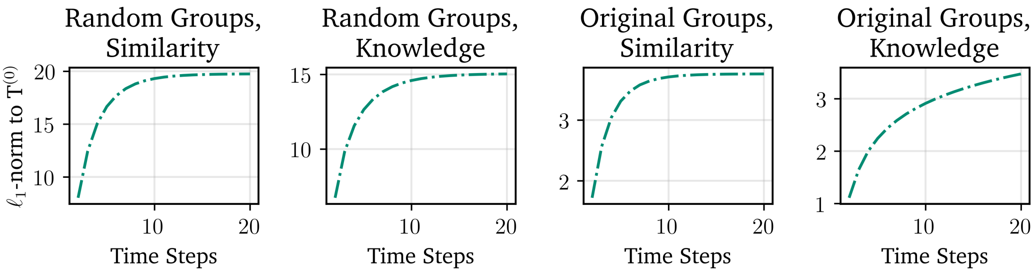


FIGURE 6.9: Convergence of trust matrices, as measured by the ℓ_1-norm between the trust matrix at the start and trust matrix at the current time step.

the mean absolute difference in the matrix is smaller. When using knowledge-based trust there is a lower rate of convergence

6.3 Sensitivity Analysis

We perform sensitivity analysis on the predicted PBS of the model. We do not use the original groups, as this allows us to vary the number of voters. Figure 6.10 shows the sensitivity indices. The first order indices show that the *number of voters* is clearly the biggest factor in the variance of the model. As expected, the *bias* does not directly contribute to the variance in the model. *Knowledge* informed trust and*self knowledge* both are significantly impacting the variance of the model. The second order indices show *number of voters* interacts with *knowledge*, *self knowledge*, and *similarity*, contributing a large portion of their explained total variance induced by the *number of voters*. There is also an interaction between *ego* and *similarity* and *self knowledge*. As for the Total order indices, variables contribute significantly to the variance in the model.

We argue the non-significant first order indices are a result of these parameters not directly incorporating new information into the model, and thus on average they do not affect on the outcome. When these parameters are used in combination parameters that do introduce new information into the model they start to significantly alter the outcome of the model. As partly supported by the second order sensitivity indices

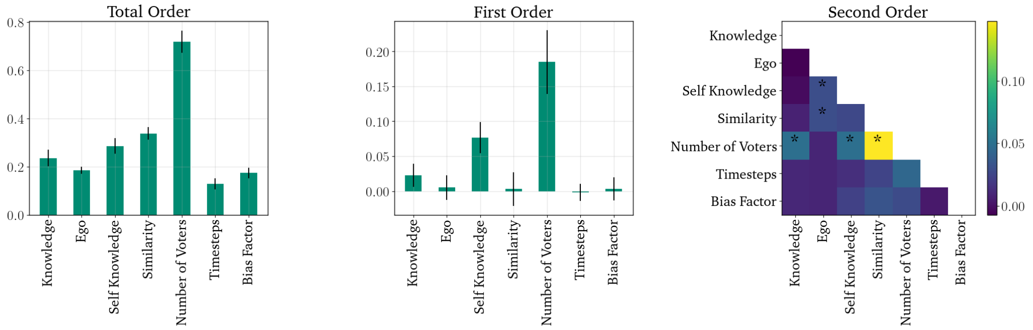


FIGURE 6.10: Sensitivity indices of parameters influencing PBS prediction error. Asterisks in the second-order panel denote statistically significant interactions.

6.4 Adding Meta-Agreement

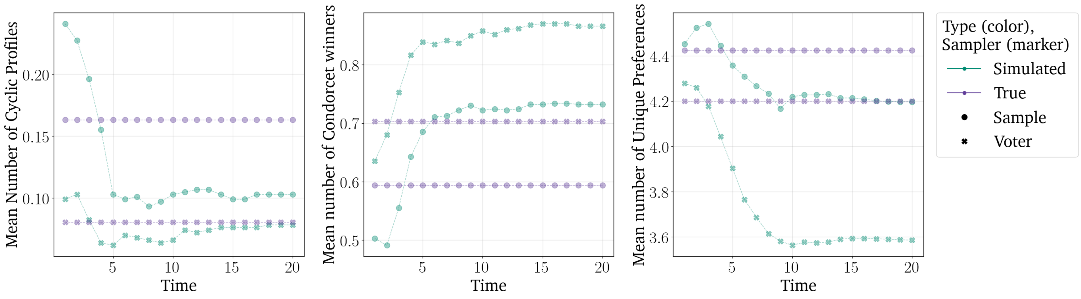


FIGURE 6.11: Proportion of cyclic profiles in the DeGroot model after adding meta-agreement. Lower values indicate more coherent collective preferences.

Firstly, when comparing different voter generation mechanisms, we find that generating a candidate by copying the opinion of a single voter performs best—both in minimizing the number of cyclic profiles and in maximizing the frequency with which a Condorcet winner exists. Though this result may seem unintuitive, we suspect the reason is that pre-deliberation opinions were relatively polarized. As a consequence, constructing candidates as averages of 10 voters tends to produce alternatives that are too similar, making it difficult for anyone to stand out.

In contrast, when copying a single voter's opinion, that candidate is more likely to fall near a large cluster of similar voters, making that candidate closer to the majority. In such cases, that candidate is more likely to become a Condorcet winner. Put simply, averaged candidates tend to represent moderate positions, leading to greater voter indifference between them. In these situations, small errors in perceived support can have disproportionately large effects. Meanwhile, candidates based on a single voter's opinion, especially in a polarized society, are more likely to be distinct and strongly preferred.

Looking at the evaluation metrics used in the model, we observe a pattern similar to that found in the substantive agreement analysis. The simulation initially starts far from the true scores, gradually moves toward them, overshoots, and finally begins to converge.

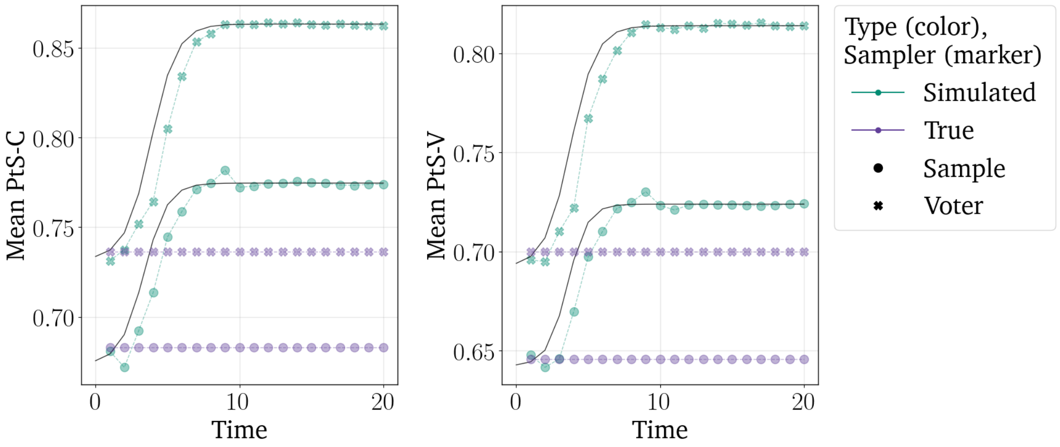


FIGURE 6.12: Proximity to single-peakedness after deliberation via candidate deletion (left) and voter deletion (right). The black line is a fitted sigmoid curve

Figure 6.12 shows similar dynamics across simulation time for both notions of proximity to single-peakedness. Although candidate deletion and voter deletion represent two fundamentally different approaches to measuring this property, they yield a consistent conclusion: voters rapidly become more single-peaked early in the simulation, after which the rate of change slows and eventually plateaus. This behavior is well captured by a sigmoid curve, with an R^2 of 0.997 and 0.993 for the PtS-V and PtS-C respectively. The diminishing rate of change corresponds to the trust matrix stabilizing at its convergent state.

CHAPTER 7

DISCUSSION

7.1 Conclusion

The main goal of the thesis was to get a deeper understanding of deliberation and its effect on preference profiles. To this end we consulted the literature (Chapter 3) laying out various points of view on the goal of deliberation. From this we follow Cohen's [9] four tenants of deliberation; deliberation should be *free, reasoned, equal*, and it should aim to reach *consensus*. In Chapter 4 we show that the deliberative procedure posited by Rad and Roy [32] cannot be strategyproof under classic notions of strategyproofness as well as novel notion of strategyproofness we define. We use this to add one more tenant to Cohen's four, namely *honesty*.

We then set out to mechanically understand deliberation. For this, we introduced the DeGroot learning model, and adapted it to deliberation over opinions. We showed NP-hardness on the δ-DBVM(S) problem, and concluded that using de DeGroot model to model sparse graphs is computationally difficult, if one wants to assign voters to nodes based on some distance metrics.

In Chapter 6 replicated the results by Rad and Roy [32], and we use our adapted DeGroot model to test its predictive power on opinions using the AMERICA IN ONE ROOM dataset [17]. We conclude that though in the first time step the model can do well on the population level, the prediction on the change in opinion for individuals was poor. We also show that this is at least partly explain by the fact that the DeGroot model treats all policies equally. The data showed that some topics had large shifts in opinions, while others showed less. The DeGroot model was unable to capture this.

Using sensitivity analysis, we showed that all parameters affected the final predictions, but interestingly some parameters had non-significant first- and second-order effects. We argue that this is a result of these parameters not introducing new information. As a result, they can only affect the variance of the model by modulating the dynamics induced by the parameters with significant first-order effects.

Finally, we looked at the preference profiles which we simulated based on the opinions from both the data and the simulations. We show, that similar to the population level predictions for the PBS, the profiles based on the simulated and true opinions start looking more similar during the first steps in the simulation. However, after this the model converge too strongly and the profiles of the simulated opinions become too "nice", in the sense that they get closer to being single-peaked and are acyclic more frequently.

These results led us to conclude that the DeGroot learning model was overly simplistic and therefore was unable to adequately explain individual opinion change. As a result it is a bad approximation of what happens during human deliberation. These patterns are also in contradiction to known results in social psychology, where small extreme groups tend to become more extreme [28].

7.2 Discussion

We first present some limitations of these results. We can broadly put these into three categories.

Firstly, given the lack of a complete data source combining pre- and post-deliberation opinions and preference rankings as well as the opinions of these alternatives, we have had to make many assumptions on both the positions represented by the candidates, and well as the method by which voters generate their preference rankings. In terms of generating candidates, our approach is simple, and only assumes that candidates represent the opinions held by the voters. This is however clearly a less rich process than that by which real-world candidates are selected, where these might bring in new opinions or have traits that are desirable, such as being good leaders or well-spoken. In terms of voters creating a ranking over alternatives, we have gone with the assumptions that this is done strictly through distance in opinions, similar to what a political compass test might do. In reality however, voters might be using different and multiple heuristics to order the candidates. Indeed if there are numerous candidates, the ranking might not even be complete. Therefore, distance-based measures will likely diverge from heuristics, such as pre-selecting some list of candidates deemed acceptable.

Secondly, there are some methodological assumptions we made. These mainly relate to the generation of the trust matrices. For all Knowledge, Self-Knowledge, and Similarity the scores were normalized to be between 0 and 1, while the Ego score was not normalized. This results in an asymmetry that allows Ego to increase the values in the trust matrix, where the other parameters could not. This decision was made as we found no clear ceiling with respect to which we could normalize the Ego score. As mentioned in Chapter 6, this might explain why Ego resulted in the lowest error on the Population level.

The same trust matrix was used for substantive and meta deliberation. Though from a modeling perspective this is a pragmatic solution. In reality this assumption seems too strong. This assumption forces someone to be equally willing to change their opinion as they are to change their perception of a candidate's opinion, where, at least intuitively, one might expect more willingness on the latter towards people with dissimilar opinions.

Apart from these limitations in generating the trust matrices, we also note the noise added to the estimates of candidates' opinions is normally distributed. Though this was done to introduce voter uncertainty, over which they could then deliberate, normally distributed noise seems unlikely, especially for voters that hold more extreme positions. Here we might expect that the noise is dependent on the candidates opinions, where candidates that are more similar in opinion to the voters, will be more accurately estimated than dissimilar candidates. For these dissimilar candidates, it might then also be true that this noise is skewed towards the opposite extreme w.r.t. the voter's opinion.

While we opted for the DeGroot model as a more accurate representation of human belief updating than full Bayesian updating, the DeGroot model does have some inherent limitations. Firstly, it does not take into account why people hold certain beliefs, nor does it constrain what kinds of beliefs a voter can hold at the same time. To remedy this, one might consider a framework such as abstract argumentation theory [13], as this is able to model the arguments with the deliberative groups. Though, this is theoretically nice, as it allows for formal description on why opinions and preferences are held, not just their descriptions. From a simulation perspective, such a framework introduces major validity questions. Firstly the framework requires a map on the relation of all arguments, for this one does not only need qualitative data, i.e. reported arguments by participants, but also a method of reliably and accurately transforming these qualitative reports to argumentative graphs. Secondly, the abstract argumentation framework does not pose an updating mechanism, thus the method through which participants would update their believes using this framework is unclear. Secondly, it limits voter's belief updates to linear transformations.

Finally, we address some limitations on the real-world implications of these results. The negative results surrounding strategyproofness in Chapter 4 might be less of an issue in human deliberation, as the dishonest participant could be less convincing defending their dishonest opinion than their true opinion. As a result they might have less total influence than if they had defended their true opinion.

In terms of modeling deliberation, we have now focussed on variables that can clearly be measured. While this might paint a good picture of the quantitative aspects of deliberation, in practice deliberation in humans come with rich interactions affecting their judgement and willingness to listen among other things. If we hope to get an accurate mechanistic model of deliberation, these qualitative aspects of deliberation need to be studied.

7.3 Future work

Based on the limitations of this study, and the literature, we present some areas for future work.

Given the weak performance of the model, a better computational model is needed to understand deliberation and inform the design of deliberative interventions. We propose some extensions to the model, which might better capture human dynamics. Most importantly, it needs to be able to show non-linear affects, and be informed by qualitative descriptions of deliberation. One main improvement of the DeGroot model specifically could be to introduce dynamic trust matrices. When humans deliberate, the amount of trust placed on each person is likely not fixed over time. This can be addressed dynamic trust matrices that update according to voter's familiarity with other voters, and possibly other factors.

Another way in which the trust matrices can be further refined is through introducing topic-dependent trust. As some topics might be more hotly debated, for example as a result of some recent event. These voters could generally be more informed on these topics, and less willing to talk about other topics. This is related to the notion of *Salience* as described by List et al. [26], stating that topics with high salience benefit less from deliberation, as participants have likely received more information on this topic.

Furthermore, any good model will need proper data, as such a study similar to that of Fishkin et al. [17] is needed, where voters are asked not only for their opinion but also their preference order. This could also be a great opportunity to gather qualitative insights into deliberation and the social dynamics thereof. This would also allow for testing participant's knowledge on topics directly, hopefully giving stronger indications of voter's ability to persuade and defend on specific topics.

APPENDIX A

Ethics and Data Management

A new requirement for the thesis is that there must be a short section in which you reflect on the ethical aspects of your project. This requirement is related to one of the final objectives that a graduated student of the Master of Computational Science must meet: "The graduate of the program has insight into the social significance of Computational Science and the responsibilities of experts in this field within science and in society". You don't need to devote an entire chapter to this; a short section or paragraph is sufficient.

I acknowledge that the thesis adheres to the ethical code (https://student.uva.nl/en/topics/ethics-in-research) and research data management policies (https://rdm.uva.nl/en) of UvA and IvI.

The following table lists the data used in this thesis (including source codes). I confirm that the list is complete and the listed data are sufficient to reproduce the results of the thesis. If a prohibitive non-disclosure agreement is in effect at the time of submission "NDA" is written under "Availability" and "License" for the concerned data items.

Short description	**Availability**	**License**
America In One Room	https://doi.org/10.7910/DVN/ERXBAB	CC0 1.0

APPENDIX B

ADDITIONAL MATERIAL

B.1 Extended Proof

We present the following extension to the proof of Proposition 4.3, specifically for the case where the CS distance is used.

> *Proof*. As in the KS and DP cases, we construct profiles R_1, R_j, and R_1' such that:
>
> - $\mathrm{Dist}_{\mathrm{CS}}(R_1, R_j) = 2$ for all $j \neq 1$,
> - $\mathrm{Dist}_{\mathrm{CS}}(R_1, R_1') = 2$,
> - $\mathrm{Dist}_{\mathrm{CS}}(R_1', R_j) = 4$.
>
> Assume voter 1 has a bias of 1, and all other voters $j \neq 1$ have bias 0.5.
>
> Let the profiles be defined as follows:
>
> $$R_1 = a \succ b \succ c \succ \cdots \succ m,$$
> $$R_j = b \succ a \succ c \succ \cdots \succ m,$$
> $$R_1' = a \succ c \succ b \succ \cdots \succ m.$$
>
> Observe that R_1 differs from both R_j and R_1' by a single adjacent transposition, and hence the CS distance between them is 2:
>
> $$\mathrm{Dist}_{\mathrm{CS}}(R_1, R_j) = \mathrm{Dist}_{\mathrm{CS}}(R_1, R_1') = 2.$$

> To compute the CS distance between R'_1 and R_j, consider the rankings of the top three candidates:
>
> $$\text{Positions in } R'_1: \quad a = 1,\ b = 3,\ c = 2,$$
> $$\text{Positions in } R_j: \quad a = 2,\ b = 1,\ c = 3.$$
>
> Then:
>
> $$\text{Dist}_{\text{CS}}(R'_1, R_j) = |1-2| + |3-1| + |2-3| = 1 + 2 + 1 = 4.$$
>
> This satisfies the required conditions: the misreported preference R'_1 increases the distance to other voters while remaining close to the voter's true preference R_1, making strategic manipulation beneficial under this setup.

B.2 Additional Figures

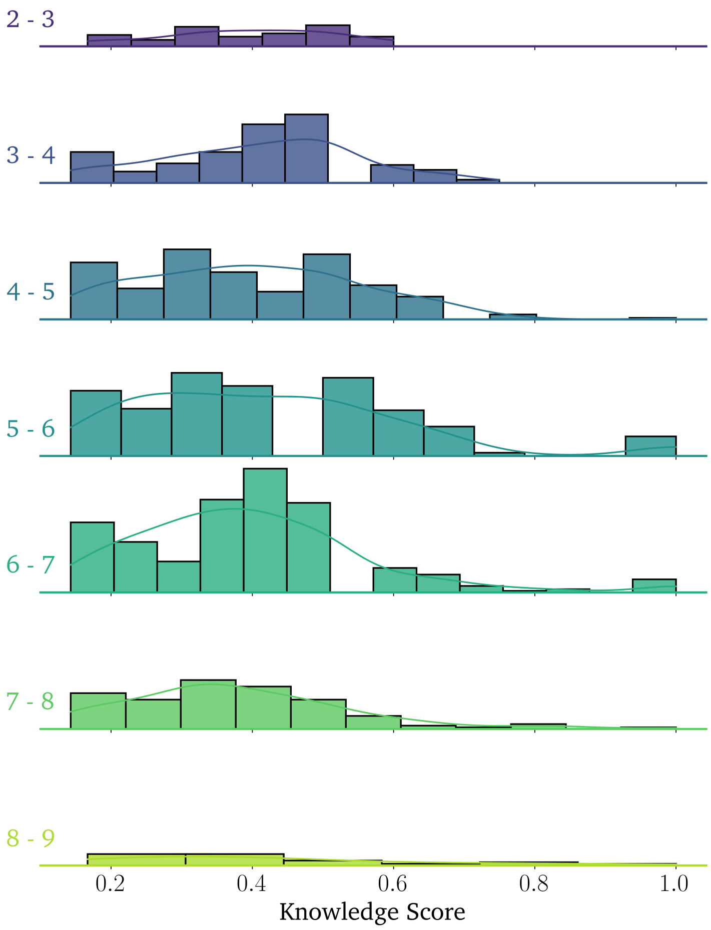


FIGURE B.1: The distribution of knowledge scores for different ranges of policy-based ideology scores.

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