Linear Systems and Control - Week 5

Step Response of Second Order Systems

Dr. Salman Ahmed

Model - Recalling concepts

A model is representation or abstraction of reality/system.

Who invent model? We, human beings, invent model based on our knowledge.

The following are famous or popular input signals

- Impulse Signal
- Step Signal
- Ramp Signal
- Parabolic Signal

Step Response of First Order System

A first order system has a single pole (irrespective of number of zeros).

Many systems are of first order for example

- velocity of car on road
- control of angular velocity of rotating systems
- an RLC circuit with only one capacitor and no inductor
- an RLC circuit with only one inductor and no capacitor
- fluid flow in a pipeline
- level control in a tank
- pressure control in a gas cylinder

Step Response of First Order System

A first order system without zeros can be written as follows:

$$G(s) = \frac{b}{s+a}$$

The inverse of a is called time constant i.e.

$$\tau = \frac{1}{a}$$

The gain K is also called as dc-gain or steady-state gain of a system.

$$K = \frac{b}{a}$$

Step Response of First Order System

Rise Time: T_r, time taken to reach 90% or 0.9 of final value from 10% or 0.1. Mathematically:

$$T_r = \frac{2.2}{a}$$

Settling Time: T_s, time taken to stay within 2% of its final value (or reach 98% of final value). Mathematically:

$$T_s = \frac{4}{a}$$

First Order Systems Summary

In first order system, we only have 2 parameters: dc gain and time-constant

Varying these two parameters only change the speed or amplitude of step response

Which parameter changes the speed of first order transfer function?

Which parameter changes the amplitude of first order transfer function?

Poles Location of Second Order System

A second order system has 2 poles. So, the following possibilities can occur:

- Both poles are real and equal
- Both poles are real and unequal
- Both poles are complex conjugate

Wait, one more possibility is also there

- Both poles are complex conjugate with real part equal to zero

General Second Order System

A general second order system can be written as follows:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

ω_n is called the natural frequency of second order system and ζ is called damping ratio

ω_n is pronounced as omega n

ζ is pronounced as zeta

General Second Order System

Analyze this second order transfer function and determine ω_n and ζ

$$G(s) = \frac{4}{s^2 + 2s + 4}$$

$\zeta = 0.5$ and $\omega_n = 2$

General Poles of Second Order System

You can apply quadratic formula and compute the poles of transfer function:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The poles of the transfer function are

$$-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

Response Types of Second Order System

Now, we can have the following four possibilities:

- Overdamped response: The system has two real poles which are unequal - in this case $\zeta > 1$
- Critically damped response: The system has two real poles which are equal - in this case $\zeta = 1$
- Underdamped response: The system has two complex-conjugate poles with some real part - in this case $0 < \zeta < 1$
- Undamped response: The system has two imaginary poles with zero real part - in this case $\zeta = 0$

Poles Location of Over Damped Second Order System

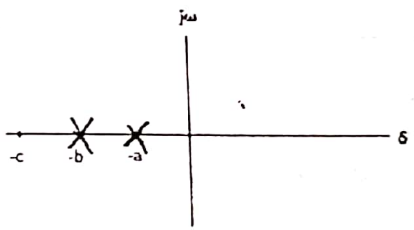

Figure: Over Damped System

Poles Location of Critically Damped Second Order System

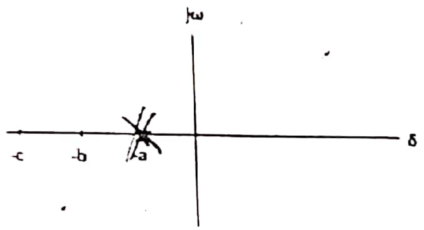

Figure: Critically Damped System

Poles Location of Under Damped Second Order System

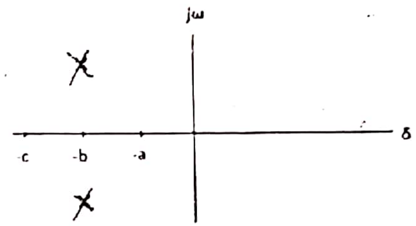

Figure: Under Damped System

Poles Location of Un Damped Second Order System

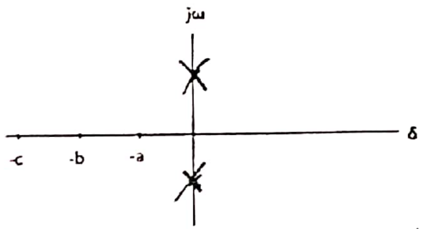

Figure: Un Damped System

Step responses of undamped second order systems

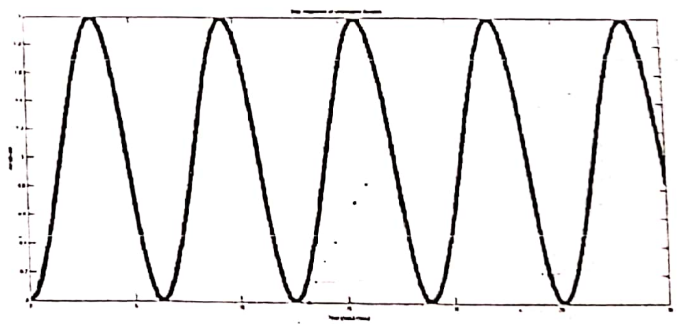

Figure: Step response of undamped system

Step responses of under damped second order systems

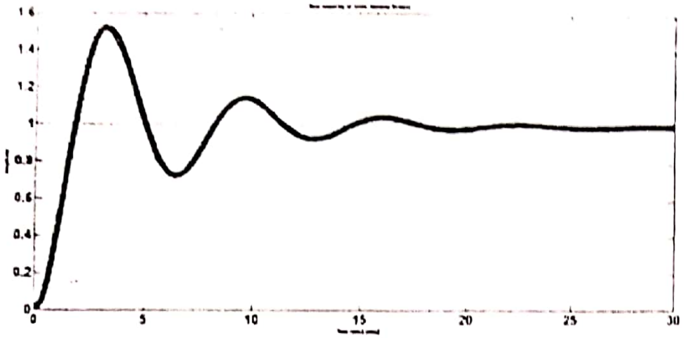

Figure: Step response of under damped system

Step responses of Critically Damped second order systems

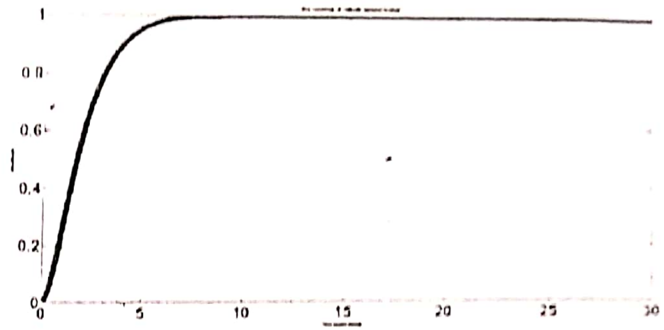

Figure: Step response of Critically damped system

Step responses of over damped second order systems

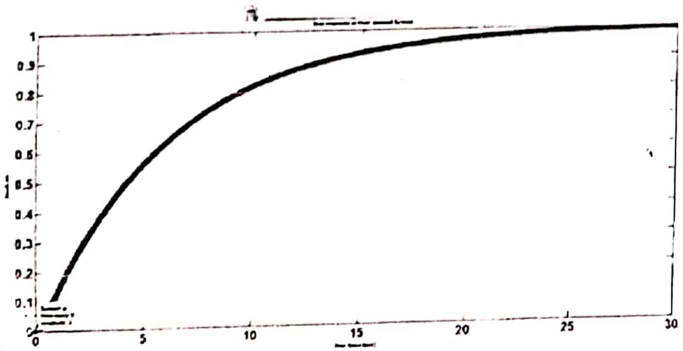

Figure: Step response of over damped system

Step response Analysis

The under damped system has interesting graph which needs more analysis

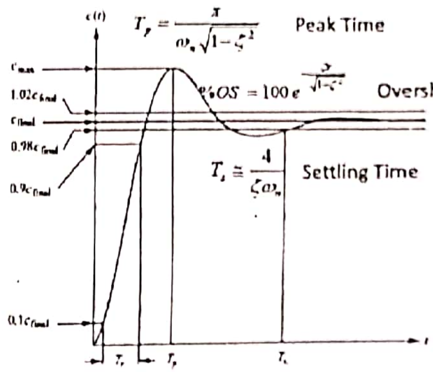

Figure: Step response of under damped system

Second Order System Analysis

You should know (for examination purposes):

- The four types of step responses of second order systems
- Being able to identify from graph, the type of response

Analysis of under damped systems involve many formulae, and we will use MATLAB software to analyze and compute them

Second Order System Analysis - Example

Compute ζ and ω_n for the following transfer function

$$G(s) = \frac{36}{s^2 + 4.2s + 36}$$

Let us compare it with general form of second order systems

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Second Order System Analysis - Example of Mechanical System

Obtain transfer function of this system (assume $m = 3$, $k = 2$ and $b = 8$)

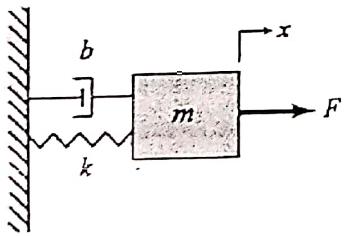

Figure: Obtain transfer function for this system

Second Order System Analysis - Example of Mechanical System

The transfer function is as follows:

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

$$\frac{X(s)}{F(s)} = \frac{1}{3s^2 + 8s + 2}$$

Now what we do, what is ω_n and what is ζ? Let us compare it with more general form.

Second Order System Analysis - Example of Mechanical System

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Now, we have the following:

$$\frac{X(s)}{F(s)} = \frac{1}{3s^2 + 8s + 2}$$

Let us first eliminate the term 3 from this transfer function.

$$\frac{X(s)}{F(s)} = \frac{1/3}{3/3s^2 + 8/3s + 2/3}$$

$$\frac{X(s)}{F(s)} = \frac{1/3}{s^2 + 8/3s + 2/3}$$

Second Order System Analysis - Example of Mechanical System

$$\frac{X(s)}{F(s)} = \frac{1/3}{s^2 + 8/3s + 2/3}$$

Comparing with the standard form, we obtain $\omega_n^2 = 2/3$, which gives us $\omega = 0.8165$. Next, we compute the dc-gain of the transfer function which can be obtained from the numerator part as follows:

$$K\omega_n^2 = \frac{1}{3}$$

$$K \times \frac{2}{3} = \frac{1}{3}$$

which gives us $K = 1/2 = 0.5$. So in the step-response, the steady-state amplitude would be 0.5.

Second Order System Analysis - Example of Mechanical System

Let us determine ζ now, which can be computed as follows:

$$(2)(\zeta)(\omega_n) = \frac{8}{3}$$

$$(2)(\zeta)(0.8165) = \frac{8}{3}$$

which gives us $\zeta = 1.6330$. Now based on ζ, what would be the type of step response (underdamped or overdamped or undamped or critically damped)

Response Types of Second Order System

Now, we can have the following four possibilities:

- Overdamped response: The system has two real poles which are unequal - in this case $\zeta > 1$
- Critically damped response: The system has two real poles which are equal - in this case $\zeta = 1$
- Underdamped response: The system has two complex conjugate poles with some real part - in this case $0 < \zeta < 1$
- Undamped response: The system has two imaginary poles with zero real part - in this case $\zeta = 0$

Second Order System Analysis - Example of Mechanical System

So, for the mechanical system, the response type will be over damped and the poles would be real and unequal. Let us use MATLAB to verify the same.

$$-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

Step Response of Second Order System - MATLAB Analysis

MATLAB code for obtaining step response

```
num = [1] ;

den = [3 8 2] ;

step(num,den)
```


Step Response of Second Order System - MATLAB Analysis

MATLAB code for analyzing step response

```
zeta=1.6330;

omegan=0.8165;

-zeta*omegan + (omegan*sqrt(zeta*zeta-1))

-zeta*omegan - (omegan*sqrt(zeta*zeta-1))
```