Control Systems - 7^{th} Semester - Week 13

Steady State Errors - Chapter 7

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Control Systems Analysis Parameters

Control Systems are judged in terms of performance in three areas:

1. Stability (poles or eigenvalues or RH Table)
2. Transient Response (settling time, overshoot, etc)
3. Steady State Error

In today lecture, we will study steady state error. Before that I explain tracking

The objective of tracking is to track a reference signal, $r(t)$

Control Systems Analysis Parameters

Tracking is typically achieved using negative unity feedback systems.

For example: if $G(s)$ denotes the tf of the system and you want $G(s)$ to track an input signal $r(t)$ or $R(s)$, then use the following configuration:

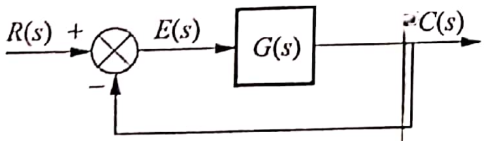

Figure: Negative Unity Feedback System

Tracking Error:

$$E(s) = R(s) - C(s)$$
$$e(t) = r(t) - c(t)$$

Steady State Error

Usually, when an input is applied to a system, tracking error may be present but we want the final value of error to be zero.

$$e(t) = r(t) - c(t)$$

Steady-state error: denoted by e_{ss} and is defined as error when $\lim_{t\to\infty}$. In other words, it is the error when time approaches infinity. We write it $e_{ss}(\infty)$

Steady State Error

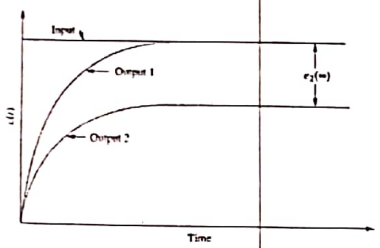

Figure: Steady state error of 2 different systems

- Input signal: step signal
- steady state error of output 1: $e_1(\infty) = 0$
- steady state error of output 2: $e_2(\infty) =$ some finite value

Steady State Error

An important point to consider: System must be stable before computing steady-state error

For unstable systems, steady state error is not defined (or infinity)

Notation: capital or caps E denote error in s domain and small e denotes error in time domain

Steady State Error

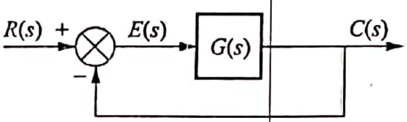

Figure: Negative Unity Feedback System

$$
\begin{aligned}
E(s) &= R(s) - C(s) \\
E(s) &= R(s) - G(s)E(s) \\
E(s) + G(s)E(s) &= R(s) \\
E(s) &= \frac{1}{1+G(s)} R(s)
\end{aligned}
$$

Input Signals

Now, input signals can be of various types but the famous or popular input signals are as follows:

1. Unit Step signal (constant position)

$$r(t) = 1, \quad t \geq 0 \quad \Longrightarrow \quad R(s) = \frac{1}{s}$$

2. Unit Ramp signal (constant velocity)

$$r(t) = t, \quad t \geq 0 \quad \Longrightarrow \quad R(s) = \frac{1}{s^2}$$

3. Unit Parabolic signal (constant acceleration)

$$r(t) = \frac{t^2}{2}, \quad t \geq 0 \quad \Longrightarrow \quad R(s) = \frac{1}{s^3}$$

Steady State Error for Input Signals

Using final value theorem, we can compute the final or steady-state value for any transfer function.

$$e(\infty) = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

Now, we have the following:

$$E(s) = \frac{1}{1+G(s)}R(s)$$

Applying final value theorem, we can state the following:

$$e(\infty) = \lim_{s\to 0}\frac{sR(s)}{1+G(s)}$$

Steady State Error for Input Signals

$$e(\infty) = \lim_{s\to 0}\frac{sR(s)}{1+G(s)}$$

Step Input $R(s) = \frac{1}{s}$

$$e(\infty) = \lim_{s\to 0}\frac{s\frac{1}{s}}{1+G(s)} = \frac{1}{1+\lim_{s\to 0} G(s)}$$

Ramp Input $R(s) = \frac{1}{s^2}$

$$e(\infty) = \lim_{s\to 0}\frac{s\frac{1}{s^2}}{1+G(s)} = \frac{1}{\lim_{s\to 0} sG(s)}$$

Parabolic Input $R(s) = \frac{1}{s^3}$

$$e(\infty) = \lim_{s\to 0}\frac{s\frac{1}{s^3}}{1+G(s)} = \frac{1}{\lim_{s\to 0} s^2G(s)}$$

Example 1

Find steady state error for the 3 inputs given below:

$$r(t) = 5u(t), \qquad r(t) = 5tu(t), \qquad r(t) = 5t^2u(t)$$

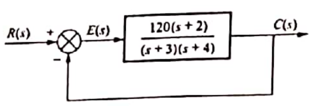

Figure: Schematic to consider for Example 1

Solution: Let us first compute $R(s)$ for each case:

$$r(t) = 5u(t) \Longrightarrow R(s) = \frac{5}{s}$$

$$r(t) = 5tu(t) \Longrightarrow R(s) = \frac{5}{s^2}$$

$$r(t) = 5t^2u(t) = \frac{10}{2}t^2u(t) \Longrightarrow R(s) = \frac{10}{s^3}$$

Example 1 Solution

$$e_{step}(\infty) = \lim_{s\to 0}\frac{s\frac{5}{s}}{1+G(s)} = \frac{5}{1+\lim_{s\to 0} G(s)} = \frac{5}{1+\lim_{s\to 0}\frac{120(s+2)}{(s+3)(s+4)}}$$

$$= \frac{5}{1+\frac{120(2)}{(3)(4)}} = \frac{5}{1+\frac{240}{12}} = \frac{5}{1+20} = \frac{5}{21}$$

$$e_{ramp}(\infty) = \frac{5}{\lim_{s\to 0} sG(s)} = \frac{5}{0\frac{120(2)}{(3)(4)}} = \frac{5}{0} = \infty$$

$$e_{parabola}(\infty) = \frac{10}{\lim_{s^2\to 0} sG(s)} = \frac{10}{0} = \infty$$

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Example 2

Find steady state error for the 3 inputs given below:

$$r(t) = 5u(t), \qquad r(t) = 5tu(t), \qquad r(t) = 5t^2u(t)$$

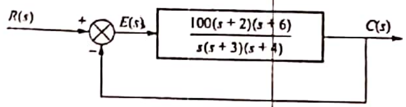

Figure: Schematic to consider for Example 2

Example 2 Solution

$$e_{step}(\infty) = \frac{5}{1+\lim_{s\to 0} G(s)} = \frac{5}{1+\lim_{s\to 0}\frac{100(s+2)(s+6)}{s(s+3)(s+4)}}$$
$$= \frac{5}{1+\frac{100(2)(6)}{(0)(3)(4)}} = \frac{5}{1+\frac{1200}{0}} = \frac{5}{1+\infty} = \frac{5}{\infty} = 0$$

$$e_{ramp}(\infty) = \frac{5}{\lim_{s\to 0} sG(s)} = \frac{5}{\frac{100(2)(6)}{(3)(4)}} = \frac{5}{100} = \frac{1}{20}$$

$$e_{parabola}(\infty) = \frac{10}{\lim_{s\to 0} s^2G(s)} = \frac{10}{0\times 0} = \frac{10}{0} = \infty$$

Error Constants

Now, let us introduce new terms which we call as static error constants.

$$K_p = \lim_{s\to 0} G(s)$$
$$K_v = \lim_{s\to 0} sG(s)$$
$$K_a = \lim_{s\to 0} s^2G(s)$$

Then, we can write the following:

$$e_{step}(\infty) = \frac{1}{1+K_p}$$
$$e_{ramp}(\infty) = \frac{1}{K_v}$$
$$e_{parabola}(\infty) = \frac{1}{K_a}$$

Example 3

Find error constants and steady state errors for the 3 inputs given below:

$$r(t) = u(t), \qquad r(t) = tu(t), \qquad r(t) = t^2u(t)$$

$$G(s) = \frac{500(s+2)(s+5)}{(s+8)(s+10)(s+12)}$$

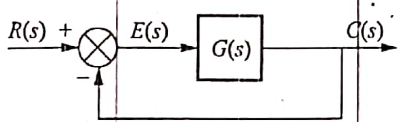

Figure: Schematic to consider for Example 3

Example 3 Solution

$$K_p = \lim_{s\to 0} G(s) = \frac{500 \times 2 \times 5}{8 \times 10 \times 12} = \frac{5000}{960} = 5.208$$
$$K_v = \lim_{s\to 0} sG(s) = 0$$
$$K_a = \lim_{s\to 0} s^2G(s) = 0$$

Then, we can write the following:

$$e_{\text{step}}(\infty) = \frac{1}{1+K_p} = \frac{1}{1+5.028} = 0.161$$
$$e_{\text{ramp}}(\infty) = \frac{1}{K_v} = \infty$$
$$e_{\text{parabola}}(\infty) = \frac{1}{K_a} = \infty$$

Example 4

Continuing with example 3, compute error constants and steady state errors if

$$G(s) = \frac{500(s+2)(s+5)(s+6)}{s(s+8)(s+10)(s+12)}$$

Solution:

$$K_p = \lim_{s\to 0} G(s) = \frac{500 \times 2 \times 5 \times 6}{0 \times 8 \times 10 \times 12} = \frac{5000}{0} = \infty$$
$$K_v = \lim_{s\to 0} sG(s) = 31.25$$
$$K_a = \lim_{s\to 0} s^2G(s) = 0$$

Then, we can write the following:

$$e_{\text{step}}(\infty) = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$
$$e_{\text{ramp}}(\infty) = \frac{1}{K_v} = 0.032$$
$$e_{\text{parabola}}(\infty) = \frac{1}{K_a} = \infty$$

Example 5

Continuing with example 3, compute error constants and steady state errors if

$$G(s) = \frac{500(s+2)(s+4)(s+5)(s+6)(s+7)}{s^2(s+8)(s+10)(s+12)}$$

Solution:

$$K_p = \lim_{s\to 0} G(s) = \infty$$
$$K_v = \lim_{s\to 0} sG(s) = \infty$$
$$K_a = \lim_{s\to 0} s^2G(s) = 875$$

Then, we can write the following:

$$e_{\text{step}}(\infty) = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$
$$e_{\text{ramp}}(\infty) = \frac{1}{K_v} = 0$$
$$e_{\text{parabola}}(\infty) = \frac{1}{K_a} = \frac{1}{875}$$

System Type

Steady state error constants are dependant upon the number of integrators in the system

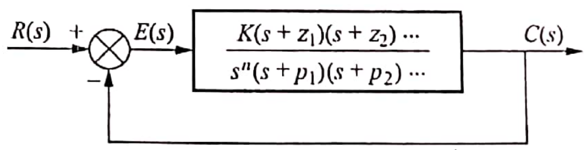

Figure: System Type

System Type: The number of poles at origin (for the open-loop system)

System Type

n	Type	Tracking Capabilites
0	0	Can only track step signals
1	1	Can track step and ramp signals
2	2	Can track step, ramp and parabolic signals

Table: System Type and n

A type 2 or higher order system can track step, ramp and parabolic signals

Summary of Results

	Unit step	unit ramp	unit parabolic
Type 0	$\frac{1}{1+K_p}$	∞	∞
Type 1	0	$\frac{1}{K_v}$	∞
Type 2	0	0	$\frac{1}{K_a}$

Table: Summary of tracking results

Example 6

Continuing with example 3, if

$$G(s) = \frac{1000(s+8)}{(s+7)(s+9)}$$

1. Determine the system type
2. Compute K_p, K_v and K_a
3. Find the steady state errors for unit step, unit ramp and unit parabolic signal

Solution:

1. System type 0
2. $K_p = 127$, $K_v = 0$ and $K_a = 0$
3. $e_{step}(\infty) = 0.0078125$, $e_{ramp}(\infty) = \infty$ and $e_{parabola}(\infty) = \infty$