Mathematical Structuralism

Amir Tabatabai

June 3, 2021

1 Category Theory

Definition 1.0.1. A category $\mathcal{C}$ is the following data:

- a colection of objects, denoted by $ob(\mathcal{C})$,
- a collection of morphisms, denoted by $Mor(\mathcal{C})$,
- for any morphism $f \in Mor(\mathcal{C})$, an object $s(f)$ called the source of f,
- for any morphism $f \in Mor(\mathcal{C})$, an object $t(f)$ called the target of f,
- for any object $A \in ob(\mathcal{C})$, a morphism id_A,
- for any two morphisms $f, g \in Mor(\mathcal{C})$ such that $s(f) = t(g)$, a morphism $f \circ g$,

satisfying the following properties:

- $s(id_A) = t(id_A) = A$,
- $s(f \circ g) = s(g)$ and $t(f \circ g) = t(f)$,
- $f \circ id_A = f = id_B \circ f$, if $s(f) = A$ and $t(f) = B$,
- $f \circ (g \circ h) = (f \circ g) \circ h$.

For any $f \in Mor(\mathcal{C})$, we summarize the data $s(f) = A$ and $t(f) = B$ by $f : A \to B$. For any two objects $A, B \in ob(\mathcal{C})$ by $\mathcal{C}(A, B)$ or $Hom_{\mathcal{C}}(A, B)$, we mean the collection of all morphisms $f : A \to B$. A category is called small if $Mor(\mathcal{C})$ is a set. It is called locally small if $Hom_{\mathcal{C}}(A, B)$ is a set, for any two objects A, B.

$$\begin{tikzcd}
 & & D \\
A \arrow[r, bend left, "f"] \arrow[r, "g" description] \arrow[r, bend right, "h"'] \arrow[urr, bend left, "jf=jg=jh"] \arrow[dr, bend right, "if=ig=ih"'] & B \arrow[ur, "j"] \arrow[d, "i"] & \\
 & C \arrow[uur, dotted, "k"'] &
\end{tikzcd}$$

Philosophical Note 1.0.2. To have some informal interpretation in mind, read objects as the entities of a given discourse and maps as the transformations between them, composition as the composition of the transformations and the identity as the do-nothing transformation. Note that in a category, an object is just an abstract node that bears no information except what is encoded in the maps starting from or ending in the object itself. In this sense, the only way to inspect an object is by using its behaviour in the context of the other objects, other than that, it is just one abstract node.

Example 1.0.3. The collection of all sets as the objects and the usual functions as the morphisms with the usual composition and identity constitutes a category. This category is denoted by **Set**. If we restrict ourselves to the finite sets, then the result is the category **FinSet**.

Example 1.0.4. The collection of all sets as the objects and the binary relations $R \subseteq A \times B$ as the morphisms from A to B, together with the relation composition as the composition and equality as the identity constitutes a category. This category is denoted by **Rel**.

Example 1.0.5. *(Discrete Categories)* A category $\mathcal{C}$ is called discrete if it only has the identity maps. Therefore, any set can be considered as a small discrete category.

$$\begin{tikzcd}
A \arrow[loop above, "id_A"] & B \arrow[loop above, "id_B"] & C \arrow[loop above, "id_C"]
\end{tikzcd}$$

Example 1.0.6. *(Some Finite Categories)* These are some finite categories:

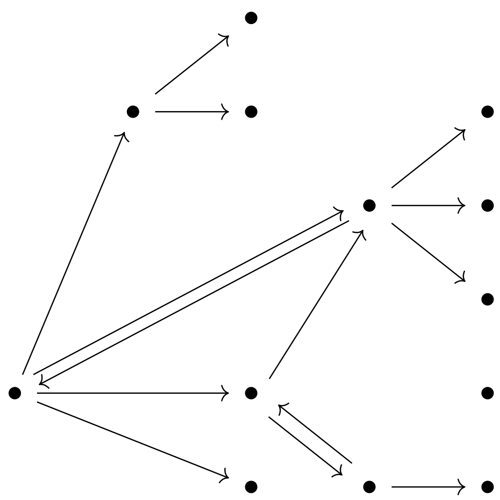
$$\mathbf{0} :$$

$$\mathbf{1}: \quad \bullet$$

$$\mathbf{2}: \quad \bullet \longrightarrow \bullet$$

$$\mathbf{3}: \quad \bullet \longrightarrow \bullet \longrightarrow \bullet$$

Example 1.0.7. *(Preorders)* A small category $\mathcal{C}$ is called a preorder if for any two objects $A, B \in ob(\mathcal{C})$, the collection $Hom_{\mathcal{C}}(A, B)$ has at most one element. Spelling out the definition of a category in this special case, a preorder is actually a set, usually denoted by P with a binary relation $\leq\ \subseteq P \times P$ such that $x \leq x$, for any $x \in P$ and if $x \leq y$ and $y \leq z$ then $x \leq z$. There are many concrete examples of preorders. For instance, the set of integers $\mathbb{Z}$ with its usual order is a preorder. This set with the divisibility relation is another preorder. The prototype example of preorders is a set of subsets of some set X with inclusion as the order.



Remark 1.0.8. It is useful to think of preorders as the shadow of the usual categories, reducing all transformations between two objects to just one transformability between them. In the logical reading, this means that we collapse all the proofs between two statements to one provability map. Hence, in this sense logic can be considered as a special case of categories.

Example 1.0.9. *(Monoids)* A small category $\mathcal{C}$ is called a monoid if it has exactly one object. Spelling out the definition of a category in this special case, a monoid is actually a set, usually denoted by M with a binary operation

$\cdot : M \times M \to M$ and an element $e \in M$ such that $e \cdot x = x \cdot e = x$, for any $x \in M$ and $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, for any $x, y, z \in M$. There are many concrete examples of monoids. For instance, the set of natural numbers $\mathbb{N}$ or $\mathbb{R}^+$ with their usual products are monoids. Moreover, any sets of endomaps of some set X that includes the identity and is also closed under composition is a monoid. This example is the prototype example of monoids.

Philosophical Note 1.0.10. A category is a combination of the two aforementioned extreme cases, a preorder and a monoid. The first handles the existence of different objects in a category and the second addresses different maps between any two objects.

Exercise 1.0.11. Check with all the details that all the previously claimed categories are actually categories.

Exercise 1.0.12. Show that the identity map of a given object is unique.

Now, it is reasonable to see the categorical formalization of some of the notions we talked about in the first session.

Example 1.0.13. *(Euclidean Geometry of the Plane)* The collection of all polygons P in $\mathbb{R}^2$ as the objects and $f_T : P \to Q$ as maps, where f_T is some formal map assigned to a distance preserving function $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T[P] = Q$, together with the usual composition and identity is a category.

Example 1.0.14. *(The Geometry of Maxwell's equations)* The collection of all the subsets U of the set of the lines going through the origin in $\mathbb{R}^5$ as objects and $f_T : U \to V$, where f_T is some formal map assigned to the function $T : \mathbb{R}^5 \to \mathbb{R}^5$ that preserves $[\mathbf{x}, \mathbf{y}] = x_0y_0 + x_1y_1 + x_2y_2 - x_3y_3 - x_4y_4$ and $T[U] = V$, together with the usual composition and identity is a category.

Example 1.0.15. *(Vectors and tensors)* The collection $\{v\}_{v \in \mathbb{R}^n}$ as the objects and $A : v \to w$ as maps, where A is an $n \times n$ invertible matrix such that $Av = w$, together the usual composition and identity is a category. More generally, for any pair (p, q), the collection $\{T\}_{T \in \mathbb{R}^{n^{p+q}}}$ as the objects and $R : T \to S$ as maps, where R is an invertible $n \times n$ matrix R such that

$$S^{i'_1, \cdots, i'_p}_{j'_1, \cdots, j'_q} = \sum_{i_1, \cdots, i_p, j_1, \cdots, j_q} (R^{-1})^{i'_1}_{i_1} \cdots (R^{-1})^{i'_p}_{i_p} \, T^{i_1, \cdots, i_p}_{j_1, \cdots, j_q} \, R^{j_1}_{j'_1} R^{j_1}_{j'_1} \cdots R^{j_q}_{j'_q}.$$

together with the usual composition and identity is a category.

Philosophical Note 1.0.16. A category can be interpreted in two different ways. In its face, any category is just a structured graph interpretable as a syntactic algebraic theory describing the behaviour of some arrows. However, it is also possible to interpret it in a more semantical and geometrical manner. Here, there are two general approaches. The *petit* and the *gros* interpretations. In the first interpretation, we read the objects as an admissible family of models and maps as the structure preserving transformations. This covers the following more specific interpretations:

- *(Logical interpretation)* Objects as the statements and maps as the conditional proofs, i.e., the map $f : A \to B$ is a proof for B, using the assumption A,
- *(Bourbaki interpretation)* Objects as the structures of a given type and morphisms as the structure preserving transformations,
- *(Computer science interpretation)* Objects as the data types and morphisms as the computable transformations.

It is also possible to read the category itself as one huge model whose objects are the admissible parts of the model that are small enough to get observed and its maps are the admissible transformations between the parts. The following is a specific example of such interpretation:

- Objects as the points of a space and maps as the paths between them, i.e., a map $f : A \to B$ is interpreted as a path from A to B.
- Objects as the subspaces of a space and morphisms as the spatial maps between them.
- Objects as the linear subspaces of a linear space and morphisms as the linear maps between them.

Example 1.0.17. For instance, a monoid is just a syntactical entity consisting of a set together with a fixed element and a binary product satisfying some properties. The interpretation reads the one object of the category as a concrete set X, the morphisms as a set of concrete functions over X and the identity and composition as their usual concrete counterparts. In this sense, the interpretation tries to *realize* the abstract graph-like category by concrete notions.

Here are two examples of the categories that admit both the petit and gros interpretations:

Example 1.0.18. Let X be an infinite set and $Fin(X)$ be the poset of all finite subsets of X with the inclusion as its partial order. As we have observed, any preorder including $(Fin(X), \subseteq)$ can be transformed to a category. Using the petit interpretation, this category will be read as the category of some sort of models, here the finite sets, while the gros interpretation reads it as the category of the finite approximations of the "huge" set X.

Example 1.0.19. Consider the category **FinVect**, constituting of $\mathbb{R}^n$, for any $n \in \mathbb{N}$, as the objects and the linear maps as the morphisms with the usual identity and composition. This category can be interpreted both as the category of all finite dimensional vector spaces (the models) or as the category of all finite dimensional approximations of an infinite dimensional vector space (the "huge" model).

Example 1.0.20. (*Variable Sets*) The collection of functions

$$\begin{tikzcd} A_1 \\ A_0 \arrow[u, "f"] \end{tikzcd}$$

as the objects and the morphisms $\alpha : f \to g$ as the pair of functions (α_0, α_1), where $\alpha_0 : A_0 \to B_0$ and $\alpha_1 : A_1 \to B_1$ such that $\alpha_1 f = g\alpha_0$:

$$\begin{tikzcd} A_1 \arrow[r, "\alpha_1"] & B_1 \\ A_0 \arrow[u, "f"] \arrow[r, "\alpha_0"] & B_0 \arrow[u, "g"'] \end{tikzcd}$$

with the evident composition and identity, constitutes a category denoted by $\mathbf{Set}^{\to}$. Any object of the category can be interpreted as a *variable set*, varying over the discrete structure of time $\{0 \leq 1\}$. The set A_0 is the set of all the elements available at the moment $t = 0$ and the set A_1 is the set of the elements at the moment $t = 1$. Moving from $t = 0$ to $t = 1$, there are three main possibilities. Either some elements is created or some elements remain intact (up to some name change) or some of the distinct elements in A_0 become equal. These possible scenarios is formalized by a function f. The elements outside the range of f are the new elements in $t = 1$, while the elements in the range come from $t = 0$, with the latter two possible changes. Any map between these variable sets is naturally a pair of two maps, each for each moment of time, respecting the change of the sets through time.

Remark 1.0.21. In the previous example, there is nothing special about the structure $\{0, 1\}$ and it can be replaced by any other preorder or even by any other small category. Generalizing the variable sets in this way leads to very interesting conceptions of the incomplete sets growing over different structures of time. It also leads to some new models of the usual classical set theory. For that matter, it is enough to pick the variable sets and restrict ourselves to a subclass of complete ones. It is not easy to define these complete sets in one line. But to have an intuition, think about the variable sets so complete that in each moment of time, the set in that moment is large enough to have all the *imaginable elements* in that moment. For instance, the Cohen forcing to prove the independence of the axiom of choice is just the result of such a process: First setting a suitable structure to encode the growth of time and then letting the sets vary on that structure to finally harvest all the completed sets as a model of the usual classical set theory.

Example 1.0.22. (*Dynamical Systems*) The collection of functions

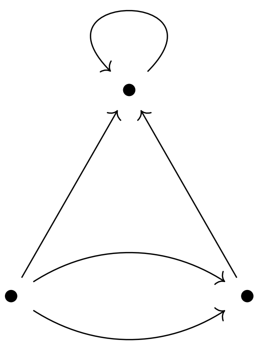
$$\begin{tikzcd} A \arrow[loop above, "f"] \end{tikzcd}$$

as the objects and the morphisms $\alpha : f \to g$ as a function $\alpha : A \to B$ such that $\alpha f = g\alpha$

$$\begin{tikzcd} A \arrow[loop above, "f"] \arrow[r, "\alpha"'] & B \arrow[loop above, "g"] \end{tikzcd}$$

with the evident composition and identity constitutes a category, denoted by $\mathbf{Set}^a$. Any object of this category can be interpreted as a *dynamical system* consisting of a set A and a function $f : A \to A$, encoding the dynamism of the system. Of course, any map between the dynamic systems must be a function from the base sets preserving the dynamism.

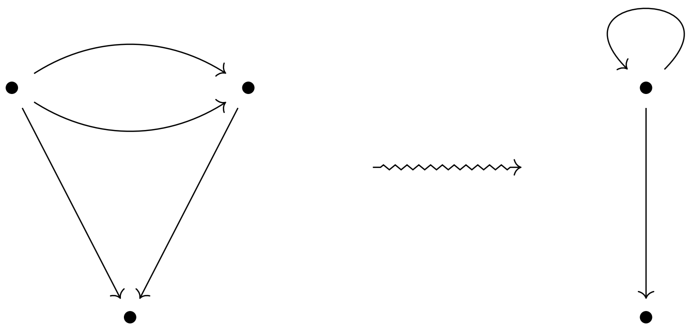
Example 1.0.23. (*Quivers*) Quivers are the directed multi-graphs as in the following figure:



formalized by

$$E \underset{t}{\overset{s}{\rightrightarrows}} V$$

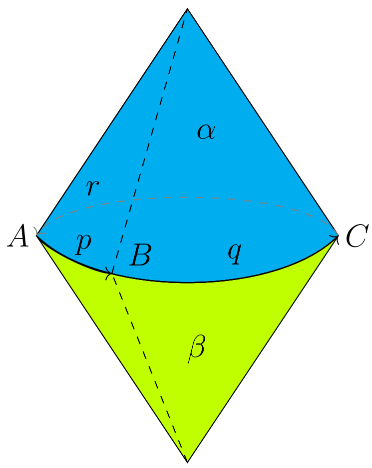
The set V is the set of vertices, the set E is the set of the edges and the two maps $s, t : E \to V$ are to encode the source and the target of any edge. The quiver morphisms then are the pairs of two functions mapping the vertices and the edges of the quivers, respecting the sources and the targets as in:



Formally, the quiver morphisms are the pairs of two functions $\alpha_V : V_0 \to V_1$ and $\alpha_E : E_0 \to E_1$ commuting with the source and the target functions, i.e., $\alpha_V s_0 = s_1 \alpha_E$ and $\alpha_V t_0 = t_1 \alpha_E$:

$$\begin{tikzcd}
E_0 \arrow[r, shift left, "s_0"] \arrow[r, shift right, "t_0"'] \arrow[d, "\alpha_E"'] & V_0 \arrow[d, "\alpha_V"] \\
E_1 \arrow[r, shift left, "s_1"] \arrow[r, shift right, "t_1"'] & V_1
\end{tikzcd}$$

Example 1.0.24. (*2-quivers*) How to formalize the higher-order geometrical version of quivers as in the following figure?



It is easy to follow the formalization of the quivers again: A set V of the vertices, a set E of edges, and another set T of triangles with two maps $s, t : E \to V$ to encode the source and the target of any edge and three *face maps* $f, g, h : T \to E$, to record the different faces of a triangle.

$$\begin{tikzcd} T \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & E \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & V \end{tikzcd}$$

In the figure, $V = \{A, B, C\}$, $E = \{f, g, h\}$, $T = \{\alpha, \beta\}$, canonical sources and targets and $f(\alpha) = f(\beta) = p$, $g(\alpha) = g(\beta) = q$ and $h(\alpha) = h(\beta) = r$. For morphisms, it is enough to have a triple $(\alpha_V, \alpha_E, \alpha_T)$ such that $\alpha_V : V_0 \to V_1$, $\alpha_E : E_0 \to E_1$ and $\alpha_T : T_0 \to T_1$ commuting with the source, the target and the face functions, i.e., the following diagram becomes commutative:

$$\begin{tikzcd} T_0 \arrow[r, shift left=2, "f_0"] \arrow[r, "g_0" description] \arrow[r, shift right=2, "h_0"'] \arrow[d, "\alpha_T"'] & E_0 \arrow[r, shift left, "s_0"] \arrow[r, shift right, "t_0"'] \arrow[d, "\alpha_E"] & V_0 \arrow[d, "\alpha_V"] \\ T_1 \arrow[r, shift left=2, "f_1"] \arrow[r, "g_1" description] \arrow[r, shift right=2, "h_1"'] & E_1 \arrow[r, shift left, "s_1"] \arrow[r, shift right, "t_1"'] & V_1 \end{tikzcd}$$

Leaving the many examples we had, we are ready to introduce the first categorical notion. We have seen that any map $f : A \to B$ can be interpreted as a transformation, changing the object A to the object B. Given this interpretation, one natural question is that when this transformation is reversible. Here is the categorical formulation:

Definition 1.0.25. A map $f : A \to B$ is called an isomorphism, if there exists a morphism $g : B \to A$ such that $fg = id_B$ and $gf = id_A$. This g is called *an* inverse of f.

Exercise 1.0.26. Prove that the inverse of a map is unique. Hence, it is well-defined to denote the inverse of f by f^{-1}.

Exercise 1.0.27. Prove that $id_A : A \to A$ is an isomorphism and if $f : A \to B$ and $g : B \to C$ are isomorphisms, then so is $g \circ f : A \to C$.

Exercise 1.0.28. Prove that in **Set**, the isomorphisms are the bijective maps. What are the isomorphisms in posets, monoids, $\mathbf{Set}^{\to}$ and $\mathbf{Set}^{a}$?

Definition 1.0.29. *(Groupoids and Groups)* A groupoid is a category whose morphisms are all isomorphisms. A group is a groupoid with just one object. Spelling out the definition in this special case, a group is a monoid, usually denoted by G, such that for any $x \in G$, there exists $y \in G$ such that $x \cdot y = y \cdot x = e$.

Example 1.0.30. The category of all sets and bijective maps as morphisms with the usual composition and identity is a groupoid.

All the Examples 1.0.13, 1.0.14, 1.0.15 are groupoids.

Example 1.0.31. The prototype example of groups is a set of invertible functions over some set X that includes the identity and is closed under composition and inversion.

Philosophical Note 1.0.32. Groupoids can be interpreted as the formalizations of equality, where $f : A \to B$ is read as a proof or witness to show why A is equal to B. With this interpretation, it is easy to see that the group axioms are the natural conditions the reflexivity, the symmetry and the transitivity of the equality induce on the witnesses.

Definition 1.0.33. A function $f : G \to H$ is called a group homomorphism if it preserves the product. The category of groups and homomorphisms is denoted by **Grp**.

1.1 A digression: the representation theorems and the baby Erlangen program

1.1.1 Representation theorems

We have explained that how any category can be interpreted as the collection of the different ways that we can inspect a huge model. Is it possible to make this interpretation more formal? Let us begin with the two easy cases: the posets and the monoids. In theses case, we should ask if any poset is a poset of subsets of a concrete set and if any monoid is a monoid of concrete functions over a concrete set. The answer in both cases is positive.

Theorem 1.1.34. *(Cayley's Representation Theorem) Any monoid (group) is isomorphic to a monoid (group) of concrete functions over a concrete set.*

Proof. Let M be a monoid. Define the set X as the monoid itself and consider N as the set of all functions $f_m : X \to X$ defined by $f_m(x) = mx$, for $m \in M$. It is easy to see that $f_e = id$, since e is the left identity and $f_{mn} = f_m \circ f_n(x)$, since the product is associative. Hence, the map $F : M \to N$ defined by $F(m) = f_m$ is a homomorphism. By definition, F is clearly onto. It is also one to one, because if $F(m) = F(n)$, then $f_m = f_n$ which implies $f_m(e) = f_n(e)$. Hence, by the fact that e is also the right identity, we have $m = n$. For groups, note that if M is also a group, then $f_{m^{-1}} = f_m^{-1}$. Therefore, the set N is also a group. $\square$

Theorem 1.1.35. *Any poset is isomorphic to a poset of subsets of a concrete set.*

Proof. Let $(P, \leq)$ be a poset. Set X as the set of all the subsets of P of the form $I_a = \{x \in P \mid x \leq a\}$. Define $F : P \to X$ by $F(a) = I_a$. Note that if $a \leq b$ then $F(a) \subseteq F(b)$, because if $x \leq a$, then $x \leq b$, by the transitivity of the order. F is clearly onto. It is also one to one, because if $F(a) = F(b)$, then $I_a = I_b$. By reflexivity, $a \leq a$. Hence, $a \in I_a = I_b$. Therefore, $a \leq b$. By a similar argument, $b \leq a$. Therefore, by anti-symmetry $a = b$. This means the inverse function $G : X \to P$ sending I_a to a is well-defined. To show that G preserves the order, we have to show that if $F(a) \subseteq F(b)$, then $a \leq b$. $\square$

Remark 1.1.36. It is worth mentioning that the previous theorems need and also use all the conditions in the definition of a monoid and a poset, respectively. Therefore, they imply that the conditions are necessary and sufficient to capture the abstract behaviour of a family of functions over a set, including the identity and being closed under composition and a set of subsets of a given set, respectively.

As the next natural step, we generalize the previous two cases to any category:

Theorem 1.1.37. *Any small category is "isomorphic" to a category of concrete sets with concrete functions.*

Proof. Let $\mathcal{C}$ be category. To any object A of $\mathcal{C}$ assign the set $A_* = \{g : B \to A \mid g \in Morph(\mathcal{C})\}$ and to any map $f : A \to B$, the function $f_* : A_* \to B_*$ defined by $f_*(g) = fg$. Now consider the category $\mathcal{D}$ consisting of A_* as objects and $f_* : A_* \to B_*$ as morphisms. Then, defining $F : \mathcal{C} \to \mathcal{D}$ by sending A to A_* and $f : A \to B$ to $f_* : A_* \to B_*$ we can reach an isomorphism. It is easy to see that F preserves composition and identity. F is also one-to-one on objects and morphisms. For objects the claim is obvious. For morphisms, if $f, g : A \to B$ and $f_* = g_* : A_* \to B_*$, then since $id_A \in A_*$ we have $f_*(id_A) = g_*(id_A)$, which implies $f = g$. Now, it is easy to define the converse of F and check that it the respects identity and the composition. $\square$

Now, it is natural to extend the previous representation theorems to all categories to see if it is possible to represent any category as a category of sets together with some concrete functions as morphisms? This time the answer is negative and its proof is beyond the scope of this section. However, it is worth mentioning that this negative result seriously affects the universal applicability of Bourbaki's set-based approach to structures.

1.1.2 Baby Erlangen program

So far, we have seen that any monoid (group) is actually a monoid (group) of concrete functions over a concrete set. Therefore, any group is a group of transformations over some set. Now, following Klein's Erlangen program, it is reasonable to ask that given the group of transformations, what different geometries it may be possible.

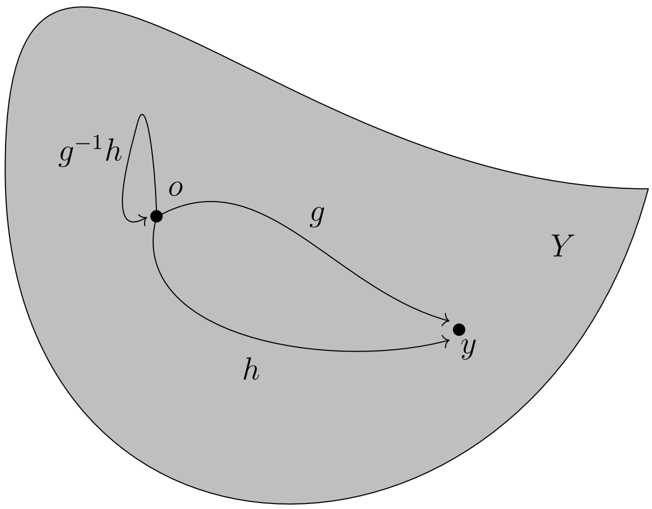
Definition 1.1.38. Let X be a set and $Aut(X)$ be the group of all permutations of X, i.e., the bijections from X to itself. A homomorphism from $F : G \to Aut(X)$ is called an action of G on X. Sometimes, for simplicity, we write gx for $F(g)(x)$. Two actions $F : G \to Aut(X)$ and $F' : G \to Aut(Y)$ are called isomorphic if there exists a bijection $\phi : X \to Y$ such that $F(g)\phi = \phi F'(g)$, for any $g \in G$.

Example 1.1.39. The trivial example of an action of the group G is the action of G on itself, defined by $F : G \to Aut(G)$, where $F(g) = f_g$ and $f_g(x) = gx$. For a more sophisticated example, let us do the trivial example in a modular manner. Let N be a subset of G closed under some operations that we meet later. Then, we call two elements $f, g \in G$ congruent modulo N if $f^{-1}g \in N$. It is reasonable to expect that the congruence to be an equivalence relation and if we denote the set of the equivalence classes by G/N, the function $G \to Aut(G/N)$ defined by $g[h] = [gh]$ becomes an action. These expectations are not automatically true. To make them true, N must be closed under product, inverse and all the operations in the form $x \mapsto g^{-1}(x)g$, for any $g \in G$.

Example 1.1.40. The trivial example of an action of the group G is the action of G on itself, defined by $F : G \to Aut(G)$, where $F(g) = f_g$ and

$f_g(x) = gx$. For a more sophisticated example, let us do the trivial example in a modular manner. Let N be a subset of G closed under some operations that we meet later. Then, we call two elements $f, g \in G$ congruent modulo N if $f^{-1}g \in N$. It is reasonable to expect that the congruence to be an equivalence relation and if we denote the set of the equivalence classes by G/N, the function $G \to Aut(G/N)$ defined by $g[h] = [gh]$ becomes an action. These expectations are not automatically true. To make them true, N must be closed under product, inverse and all the operations in the form $x \mapsto g^{-1}xg$, for any $g \in G$.

In group theory literature, there is a characterization theorem, stating that any G-action *is* the "disjoint union" of the G-actions introduced in Example 1.1.40. We will repeat the usual argument here. Let $F : G \to Aut(X)$ be a G-action. Define the reachability relation R on X by $(x, y) \in R$, if there exists $g \in G$ such that $gx = y$. It is not hard to prove that the relation R is an equivalence relation, using the fact that G is actually a group. Each equivalence class inherits a G-action from the original G-action F. The reason simply is that if x is an element in the class and $g \in G$, the result of the action, namely gx, is in the same class as x. Finally, we will show that each of these restricted G-actions on the equivalence classes is isomorphic to a G-action of the type introduced in the Example 1.1.40. Let Y be one of these classes. Set an arbitrary element $o \in Y$:



Define $N = \{g \in G \mid go = o\}$. It is easy to see that N has the required closure properties, namely the closure under product, inverse and the operations

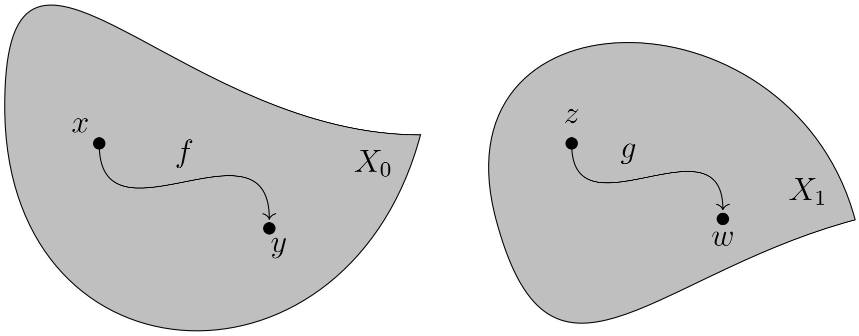
$x \mapsto g^{-1}xg$, for any $g \in G$. Define $\phi : G/N \to Y$ by $\phi([g]) = go$. The function is well-defined and one-to-one, because $\phi([g]) = \phi([h])$ iff $go = ho$ iff $g^{-1}ho = o$ iff $g^{-1}h \in N$ iff $[g] = [h]$. It is not hard to see that ϕ is an isomorphism between the G-actions. The important thing is that the function is surjective, because any y in the class is reachable from o and hence $go = y$, for some $g \in G$.

Remark 1.1.41. Note that the above construction has some unsatisfactory elements. First, some of its parts are chosen in a non-canonical manner, like the element $o \in Y$. These choices do not affect the construction, but makes the construction tricky at best and resistant to generalizations at worst. Secondly, although the notion of action is equivalently meaningful for monoids, the above construction seriously uses the fact that G is a group and hence it does not suggest any way to handle the monoid case, as well.

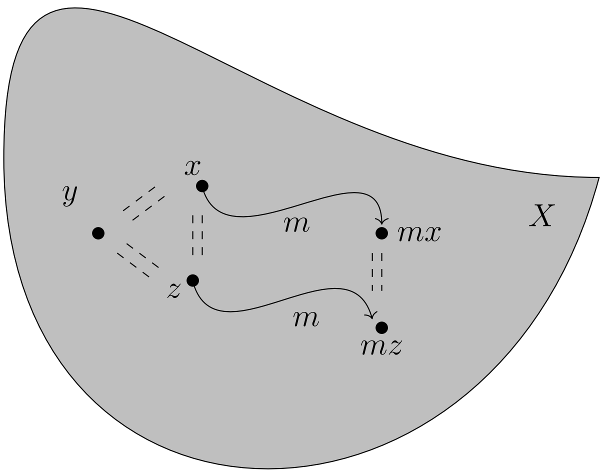
To overcome the issues mentioned above, let us provide a characterization theorem again. This time, we use the canonical approach consisting of simple, intuitive and justifiable steps that uses no ingredient except what it is essentially required. As a result, this time, everything is more transparent so much so that we can even address the case of monoids.

Definition 1.1.42. Let X be a set and $End(X)$ be the monoid of all functions on X. A homomorphism $F : M \to End(X)$ is called an action of M on X or an M-action, for short. Sometimes, we write mx for $F(m)(x)$, for simplicity. Two M-actions $F : M \to End(X)$ and $F' : M \to End(Y)$ are called isomorphic, if there exists a bijection $\phi : X \to Y$ such that $F(m)\phi = \phi F'(m)$, for any $m \in M$.

The trivial example of an M-action is the action of M on itself, defined by $F : M \to End(M)$, where $F(m) = f_m$ and $f_m(x) = mx$. To provide a characterization theorem, we will introduce two methods to construct the new M-actions from the old. First, the "*disjoint union*". Let $\{F_i : M \to End(X_i)\}_{i \in I}$ be a family of M-actions. Define $X = \sum_{i \in I} X_i = \{(i, x) \mid i \in I, x \in X_i\}$ with the fibrewise M-action $m(i, x) = (i, mx)$. This is clearly an M-action:



The second method, the "*quotient*" operation, picks one M-action and glue some of its elements together to get a new one. More precisely, let $F : M \to End(X)$ be an M-action and $R \subseteq X \times X$ be a set of the pairs of the elements of X that we want to glue to each other. It is possible to provide the minimal M-action in which these intended equalities are *forced* to hold. It is enough to define the equivalence relation $\sim$ as the least equivalence E, extending the relation R and respecting the M-action, i.e., if $(x, y) \in E$ then $(mx, my) \in E$, for any $m \in M$. (Why does such an equivalence relation exist?) Then, define Y as the set of the equivalence classes with respect to $\sim$ and define $m[x] = [mx]$. (Why is it well-defined, i.e., independent of the representative of the classes?)



To prove that any action is constructible from the basic action via disjoint union and quotient operations, let $F : M \to End(X)$ be an arbitrary M-action. Then, define Z as the quotient of the disjoint union $Y = \sum_{x \in X} M$ by the set $\{((x,m),(y,n)) \in Y^2 \mid mx = ny\}$ and define $\phi : Z \to X$ by $\phi[(x,m)] = mx$. It is clearly well-defined and one-to-one. It is also surjective since $\phi([(x,e)]) = x$. It will be easy to define the converse function and show that it is an M-action.

Now, again, it is a natural question that if it is possible to generalize the aforementioned characterization to any small category. The answer is again positive. But we first need the right notions of an action (realization) and isomorphism between these actions (realizations), for categories. The first is called a *functor* and the second is *natural isomorphism.* We will spend some time on these notions to set the scene to provide a characterization theorem for the small categories.

1.2 new categories from the old

Example 1.2.43. (*Opposite Category*) Let $\mathcal{C}$ be a category. By its dual (opposite), $\mathcal{C}^{op}$, we mean a category with same collection of objects and morphisms as of $\mathcal{C}$ with the source and the target assignment swapped and $f \circ' g = g \circ f$:

$$\mathcal{C} : \quad \begin{tikzcd} & A \arrow[dl, "g"'] \arrow[dr, "h=f\circ g"] & \\ B \arrow[rr, "f"'] & & C \end{tikzcd} \qquad \mathcal{C}^{op} : \quad \begin{tikzcd} & A & \\ B \arrow[ur, "g"] & & C \arrow[ul, "h=g\circ' f"'] \arrow[ll, "f"] \end{tikzcd}$$

Exercise 1.2.44. Show that if $f : A \to B$ is an isomorphism in $\mathcal{C}$, then it is also an isomorphism in $\mathcal{C}^{op}$. Use this fact to show that the dual of a groupoid is also a groupoid.

Example 1.2.45. (*Arrow Category*) Let $\mathcal{C}$ be a category. Then, by the arrow category $\mathcal{C}^{\to}$, we mean the category with the morphisms of $\mathcal{C}$ as the objects and the pair of morphisms $(\alpha_0, \alpha_1) : f \to g$ of $\mathcal{C}$ as the morphism, where $\alpha_0 : A_0 \to B_0$ and $\alpha_1 : A_1 \to B_1$ such that $g\alpha_0 = \alpha_1 f$:

$$\begin{tikzcd}
A_0 \arrow[r, "\alpha_0"] \arrow[d, "f"'] & B_0 \arrow[d, "g"] \\
A_1 \arrow[r, "\alpha_1"'] & B_1
\end{tikzcd}$$

Example 1.2.46. (*Slice Category*) Let $\mathcal{C}$ be a category and A be an object. Then, by the *slice* category or *over* category $\mathcal{C}/A$, we mean the category with the morphisms $f : B \to A$ of $\mathcal{C}$ with the target A as the objects and $\alpha : f \to g$ as the morphism, where $\alpha : B \to C$ is a morphism in $\mathcal{C}$ such that $g\alpha = f$:

$$\begin{tikzcd}
B \arrow[rr, "\alpha"] \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\
& A &
\end{tikzcd}$$

As a concrete example, note that in any poset $(P, \leq)$, the slice P/a is just P restricted to the elements less than or equal to a. As another example, observe that $\mathbf{Set}/\{0, 1\}$ is actually the category of partitioned sets into two parts, i.e., (A, A_0, A_1), where $A = A_0 \cup A_1$ and $A_0 \cap A_1 = \emptyset$ and functions, i.e., $f : (A, A_0, A_1) \to (B, B_0, B_1)$, where $f : A \to B$ is a function and $f[A_i] \subseteq B_i$, for any $i \in \{0, 1\}$. The reason simply is that any map $m : A \to \{0, 1\}$ is nothing but the partition of A into $m^{-1}(0)$ and $m^{-1}(1)$ and any commutative triangle means respecting these parts.

Example 1.2.47. (*Coslice Category*) Let $\mathcal{C}$ be a category and A be an object. Then, by the *coslice* category or *under* $A/\mathcal{C}$, we mean the category with the morphisms $f : A \to B$ of $\mathcal{C}$ with the source A as the objects and $\alpha : f \to g$ as the morphism, where $\alpha : B \to C$ is a morphism in $\mathcal{C}$ such that $g = \alpha f$:

$$\begin{tikzcd}
& A \arrow[dl, "f"'] \arrow[dr, "g"] & \\
B \arrow[rr, "\alpha"'] & & C
\end{tikzcd}$$

As a concrete example, note that in any poset $(P, \leq)$, the coslice a/P is just P restricted to the elements greater than or equal to a. As another example, observe that $\{0\}/\mathbf{Set}$ is actually the category of pointed sets, i.e., (A, a), where $a \in A$ and pointed functions, i.e., $f : (A, a) \to (B, b)$, where $f : A \to B$ is a function and $f(a) = b$. The reason simply is that any map $\{0\} \to A$ is nothing but the choice of an element and any commutative triangle means respecting this chosen element.

Example 1.2.48. (*Product of Categories*) Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. Then by $\mathcal{C} \times \mathcal{D}$, we mean the category with the pairs (C, D) as the objects, where C and D are the objects of $\mathcal{C}$ and $\mathcal{D}$, respectively and the pair $(f, g) : (C, D) \to (E, F)$ as the morphisms, where $f : C \to E$ and $g : D \to F$ are morphisms in $\mathcal{C}$ and $\mathcal{D}$, respectively:

$$(C, D) \xrightarrow{(f,g)} (E, F)$$

Note that this construction generalizes the product of monoids and groups on the one hand and the product of sets on the other.

Example 1.2.49. (*Coproduct of Categories*) Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. Then by $\mathcal{C} + \mathcal{D}$, we mean the category with $(0, C)$ and $(1, D)$ as the objects, where C and D are the objects of $\mathcal{C}$ and $\mathcal{D}$, respectively and $(0, f) : (0, C) \to (0, C')$ and $(1, g) : (1, D) \to (1, D')$ as the morphisms, where $f : C \to C'$ and $g : D \to D'$ are morphisms in $\mathcal{C}$ and $\mathcal{D}$, respectively. Note that this construction generalizes the disjoint union of sets.

1.3 Functors and Natural Transformations

To find the natural formalization of realizations for categories, note that a realization of a monoid (or a group) is an assignment that maps the only abstract object of the monoid (or the group) to a concrete set and any abstract morphism (i.e., the elements of the monoid) to a concrete function on the set, respecting the identity and the composition:

Definition 1.3.50. (*Functors*) Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. By a functor $F : \mathcal{C} \to \mathcal{D}$, we mean a pair of two assignments F_0 and F_1, such that F_0 maps any object A of $\mathcal{C}$ to an object of $\mathcal{D}$, denoted by $F_0(A)$ and F_1 maps any morphism f of $\mathcal{C}$ to a morphism in $\mathcal{D}$, denoted by $F_1(f)$, respecting the source, the target, the identity and the composition operations:

$$
\begin{tikzcd}
& A \arrow[loop above, "id_A"] \arrow[dl, "h"'] \arrow[dr, "i"] & {} \arrow[rr, squiggly, "F"] & & {} & F(A) \arrow[loop above, "F(id_A)=id_{F(A)}"] \arrow[dl, "F(h)"'] \arrow[dr, "F(i)"] & \\
B \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] & & C & & F(B) \arrow[rr, "F(f)=F(g)"'] & & F(C)
\end{tikzcd}
$$

Usually, for simplicity, one drops the subscript in F_0 and F_1 and denote both by F.

Philosophical Note 1.3.51. It is possible to interpret a functor $F : \mathcal{C} \to \mathcal{D}$ as a way to interpret the discourse $\mathcal{C}$ in the discourse $\mathcal{D}$, as a way to realize $\mathcal{C}$ in $\mathcal{D}$, as a $\mathcal{C}$-indexed family in $\mathcal{D}$ or a $\mathcal{C}$-variable object in $\mathcal{D}$. For instance, any M-action $F : M \to End(X)$ is a functor $M \to \mathbf{Set}$, realizing the only abstract object of M by X and the morphisms of M by real functions over X, according to F. As another example, it is possible to see that any variable set in $\mathbf{Set}^{\to}$ is actually a functor from $\mathbf{2}$ to $\mathbf{Set}$, realizing the abstract graph

$$\mathbf{2} : \quad \bullet \longrightarrow \bullet$$

by the concrete sets and functions. Similarly, any quiver is a functor from the category

$$\bullet \rightrightarrows \bullet$$

to $\mathbf{Set}$, realizing the abstract points as concrete sets of vertices and edges and abstract arrows as concrete source and target functions. For an example of the other interpretation, we have already seen that any object in $\mathbf{Set}^{\to}$ can be read as a variable set over the structure of time, encoded by the category $\mathbf{2}$.

Example 1.3.52. Any homomorphism between two monoids is a functor. Any order-preserving map between two posets is a functor. It is worth mentioning that functors are the right common generalization of composition- and order-preserving maps.

Example 1.3.53. The assignment mapping any set A to its powerset $P(A)$ and any function $f : A \to B$ to the function $P(f) : P(A) \to P(B)$, defined by $P(f)(S) = f[S] = \{f(a) \mid a \in S\}$ is a functor from **Set** to itself. Similarly, the functor $P^\circ : \mathbf{Set}^{op} \to \mathbf{Set}$, mapping any set A to its powerset $P(A)$ and any function $f : A \to B$ to the function $P^\circ(f) : P(B) \to P(A)$, defined by $P^\circ(f)(S) = f^{-1}(S) = \{a \in A \mid f(a) \in S\}$ is a functor.

Example 1.3.54. The assignment mapping any object (A, B) in $\mathbf{Set} \times \mathbf{Set}$ to $A \times B$ and any morphism $(f, g) : (A, B) \to (C, D)$ of $\mathbf{Set} \times \mathbf{Set}$ to the function $f \times g : A \times B \to C \times D$ defined by $[f \times g](a, b) = (f(a), g(b))$ is a functor.

Example 1.3.55. The assignment mapping any object (A, B) in $\mathbf{Set} \times \mathbf{Set}$ to $A + B = \{(0, a) \mid a \in A\} \cup \{(1, b) \mid b \in B\}$ and any morphism $(f, g) : (A, B) \to (C, D)$ of $\mathbf{Set} \times \mathbf{Set}$ to the function $f + g : A + B \to C + D$ defined by $[f + g](0, a) = (0, f(a))$ and $[f + g](1, b) = (1, g(b))$ is a functor.

Example 1.3.56. (*Exponentiation*) Let A be a fixed set. Define the assignment $(-)^A : \mathbf{Set} \to \mathbf{Set}$, mapping a set B to $B^A = \{f : A \to B\}$ and a function $f : B \to C$ to a function $f^A : B^A \to C^A$ defined by $f^A(g) = fg$. Then, $(-)^A$ is a functor, generalizing the finite power functor $A \mapsto A^n$ generated by the iteration of the product functor. Similarly, it is possible to define the functor $A^{(-)} : \mathbf{Set}^{op} \to \mathbf{Set}$, mapping a set B to $A^B = \{f : B \to A\}$ and a map $f : B \to C$ to a function $A^f : A^C \to A^B$ defined by $A^f(g) = gf$. Then, $A^{(-)}$ is a functor, generalizing the functor $P^\circ = 2^{(-)}$. Any combination of the product, the sum, and the functor $(-)^A$, for different fixed sets A such as $F(X) = A_0 \times X^{N_0} + A_1 \times X^{N_1} + \cdots + A_k \times X^{N_k}$ is a polynomial functor. The notion of polynomial functor, though, is more general than this.

Remark 1.3.57. *(Algebras)* Algebras are sets equipped with some operations that have some properties. For instance, a monoid is a set M with an element e and a binary operation such that the latter is associative and the former is the identity element for the latter. The operational data (not the properties) can be stored in one function $a : F_m(M) \to M$, where $F_m(X) = 1 + X^2$ is a functor, storing the type of the algebra and $a(0, *) = e$ and $a(1, m, n) = mn$, storing the operations. By type we mean the number and the arity of the operations (in the monoid case it is one nullary and one binary operations). Some examples may be helpful here. A group $(G, e, (-)^{-1}, \cdot)$ is a set G with a function $a : F_g(G) \to G$, where $F_g(X) = 1 + X + X^2$, $a(0, *) = e$, $a(1, m) = m^{-1}$ and $a(2, m, n) = mn$; the basic structure of natural numbers, i.e., $(\mathbb{N}, s, 0)$ is a function $a : F_i(\mathbb{N}) \to \mathbb{N}$, where $F_i(X) = 1 + X$, $a(0, *) = 0$ and $a(1, n) = s(n) = n + 1$ and the structure $(\mathbb{W}, s_0, s_1, \epsilon)$ of binary strings can be described by a function $a : F_s(\mathbb{W}) \to \mathbb{W}$, where $F_s(X) = 1 + X + X$, $a(0, *) = \epsilon$, $a(1, w) = s_0(w) = w0$ and $a(2, w) = s_1(w) = w1$. To have a general notion of algebra, we use a functor $F : \mathbf{Set} \to \mathbf{Set}$ to formalize the type of the algebra and then by an F-algebra, (an algebra of type F), we mean a function $a : F(A) \to A$. This also suggest a generalization for homomorphisms. Generally, a homomorphism is a function that preserves all the operations in the type of the algebra. With our generalization here, an F-algebra homomorphism from the F-algebra $a_A : F(A) \to A$ to the F-algebra $a_B : F(B) \to B$ is a function $f : A \to B$ such that

$$\begin{tikzcd} F(A) \arrow[r, "a_A"] \arrow[d, "F(f)"'] & A \arrow[d, "f"] \\ F(B) \arrow[r, "a_B"'] & B \end{tikzcd}$$

It is easy to check that in the familiar cases it really captures the notion of homomorphism.

Example 1.3.58. (*Forgetful Functors*) Sometimes, we have a category and we will forget some of the structures that the objects posses and the maps preserve, to think somewhat loosely about the same data that we originally had. Let us provide three examples of such phenomenon. First, the forgetful assignment mapping any group G and any homomorphism $f : G \to H$ in **Grp** to themselves in **Set**, forgetting that there is the group structure there, is a functor. For the second example, take the two forgetful functors from $\mathbf{Set}^{\to}$ to **Set**, forgetting that a variable set actually varies, by making two snapshots of a variable set in the two possible moments. More precisely, for any $i \in \{0, 1\}$, define $p_i : \mathbf{Set}^{\to} \to \mathbf{Set}$, by mapping any $f : A_0 \to A_1$ to A_i and any $\alpha : f \to g$ to $\alpha_i : A_i \to B_i$, where $f : A_0 \to A_1$ and $g : B_0 \to B_1$. Both p_0 and p_1 are functors. Finally, as the third example, define $V : \mathbf{Quiv} \to \mathbf{Set}$, by mapping any quiver to its set of elements and any quiver morphism to its underlying function on vertices. This V is a functor. We can do the same thing to define the edge functor E.

Example 1.3.59. *(Free Functors)* In some cases, we want to put a structure on an object in a free way, meaning we want it to be free from any unexpected relations. For instance, let X be a set. Then, $F(X)$ as the set of all finite sequences of the elements of X (including the empty sequence) with concatenation is a free-monoid constructed from X. It is a monoid, since concatenation is associative and the empty sequence is an identity. It is free because we add all possible products of the elements of X, and there is no non-trivial relation on the elements of $F(X)$, except what the monoid structure dictates. This assignment F gives rise to a functor $\mathbf{Set} \to \mathbf{Mon}$, mapping any set X to the monoid $F(X)$ and any map $f : X \to Y$ to the homomorphisms $F(f) : F(X) \to F(Y)$ such that $F(f)(\sigma) = f(\sigma_0) \cdots f(\sigma_n)$, for any finite sequence $\sigma = \sigma_0 \sigma_1 \cdots \sigma_n$.

Example 1.3.60. Let $\mathcal{C}$ be a category. Then, the identity functor $id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ mapping any object and morphism to itself is a functor. Moreover, if A is a fixed object in $\mathcal{C}$, the constant assignment $\Delta_A : \mathcal{C} \to \mathcal{C}$, mapping all objects to A and all morphisms to identity is another functor.

Example 1.3.61. Let $\mathcal{C}$ be a groupoid. Then, the inverse assignment $inv : \mathcal{C} \to \mathcal{C}^{op}$, defined by $inv(A) = A$ and $inv(f) : B \to A$ as $inv(f) = f^{-1}$, for $f : A \to B$, is a functor.

Example 1.3.62. Let $\mathcal{C}$ be a locally small category. The assignment $Hom_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$, defined by $Hom_{\mathcal{C}}(A, B) = \{f : A \to B \mid f \in Mor(\mathcal{C})\}$ and $Hom_{\mathcal{C}}(g, h) : Hom_{\mathcal{C}}(A, B) \to Hom_{\mathcal{C}}(C, D)$ as $Hom_{\mathcal{C}}(g, h)(f) = hfg$, for any $f : A \to B$, $g : C \to A$ and $h : B \to D$, is a functor. This functor captures the whole structure of the category $\mathcal{C}$.

Example 1.3.63. Let $\mathcal{C}$ be a locally small category. For any object A in $\mathcal{C}$, there is a canonical functor $Hom_{\mathcal{C}}(A, -) : \mathcal{C} \to \mathbf{Set}$, capturing the behavior of the maps above A. It is defined by $B \mapsto Hom_{\mathcal{C}}(A, B)$ and $Hom(A, f) : Hom_{\mathcal{C}}(A, B) \to Hom_{\mathcal{C}}(A, C)$ as $Hom_{\mathcal{C}}(A, f)(g) = fg$, for any $f : B \to C$. Similarly, there is a canonical functor $y_A = Hom_{\mathcal{C}}(-, A) : \mathcal{C}^{op} \to \mathbf{Set}$, capturing the behavior of the maps below A. It is defined by $y_A(B) = Hom_{\mathcal{C}}(B, A)$ and $y_A(f) : Hom_{\mathcal{C}}(C, A) \to Hom_{\mathcal{C}}(B, A)$ as $y_A(f)(g) = gf$, for any $f : B \to C$. These functors are the localized version of the concrete representation we have introduced for the small categories, mapping an object A to $A_* = \{g : C \to A \mid g \in Mor(\mathcal{C})$ and $f : A \to B$ to $f_* : A_* \to B_*$ by $f_*(g) = fg$. The current act of localization has no point except to handle the size issue that in a locally small category the collection A_* is not necessarily a set.

Example 1.3.64. Let $\mathcal{C}$ be a category and $f : A \to B$ be a morphism. The assignment mapping an object $g : X \to A$ in $\mathcal{C}/A$ to the object $fg : X \to B$ in $\mathcal{C}/B$ and mapping to themselves is a functor from $\mathcal{C}/A$ to $\mathcal{C}/B$. We denote this functor by $f_* : \mathcal{C}/A \to \mathcal{C}/B$.

Example 1.3.65. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be some categories and $F : \mathcal{D} \to \mathcal{E}$ and $G : \mathcal{C} \to \mathcal{D}$ be two functors. Then, the composition $FG : \mathcal{C} \to \mathcal{E}$ with the canonical definition is also a functor.

Note that all small categories with functors as morphisms constitute a category. We denote this category by **Cat**. The same is true for the category of all small groupoids that we denote by **Groupoid**.

Example 1.3.66. Let $\mathcal{C}$ be a small category. Then, the assignment mapping an object A to the category $\mathcal{C}/A$ and morphism $f : A \to B$ to the functor $f_* : \mathcal{C}/A \to \mathcal{C}/B$ is a functor from $\mathcal{C}$ to **Cat**.

Exercise 1.3.67. Prove that functors preserve isomorphisms, i.e., if $F : \mathcal{C} \to \mathcal{D}$ is a functor and $f : A \to B$ is an isomorphism in $\mathcal{C}$, then $F(f) : F(A) \to F(B)$ is an isomorphism in $\mathcal{D}$.

Definition 1.3.68. A functor $F : \mathcal{C} \to \mathcal{D}$ is called faithful if for any $f \neq g : A \to B$, we have $F(f) \neq F(g) : F(A) \to F(B)$. In other words, F is faithful if $F : Hom_{\mathcal{C}}(A, B) \to Hom_{\mathcal{D}}(F(A), F(B))$ is one-to-one. It is called full if any $h : F(A) \to F(B)$ is equal to $F(f)$ for some $f : A \to B$. In other words, F is full if $F : Hom_{\mathcal{C}}(A, B) \to Hom_{\mathcal{D}}(F(A), F(B))$ is surjective.

Example 1.3.69. An order-preserving map $f : (P, \leq_P) \to (Q, \leq_Q)$ between two posets is always faithful. It is full iff it is an order-embedding, i.e., $a \leq_P b$ iff $f(a) \leq_Q f(b)$. A homomorphism between two monoids is faithful iff it is one-to-one and it is full iff it is surjective. The forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$ is faithful but not full.

Exercise 1.3.70. The product functor $(-) \times (-) : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ is faithful but not full.

Exercise 1.3.71. Let $f : A \to B$ be a morphism in a category $\mathcal{C}$. When is $f_* : \mathcal{C}/A \to \mathcal{C}/B$ faithful? When is it full?

Exercise 1.3.72. Let $\mathcal{C}$ be a locally small category. Is $Hom_{\mathcal{C}}(-, -) : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$ full or faithful?

Exercise 1.3.73. Let $\mathcal{C}$ be a locally small category. Is $Hom_{\mathcal{C}}(-, -) : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$ full or faithful?

Exercise 1.3.74. Let $F : \mathcal{C} \to \mathcal{D}$ be a full and faithful functor. Show that if $F(A)$ and $F(B)$ are isomorphic, then so are A and B.

Example 1.3.75. (*Baby Schemes*) Let $\mathcal{R}$ be the category of all subsets R of $\mathbb{C}$, including 1 and closed under addition and multiplication with morphisms as the functions that preserve the element 1 and these two operations. Let $I(\vec{x}) = I(x_0, \ldots, x_n)$ be a set of equations between polynomials in variables $x_0 \ldots, x_n$ with coefficients in $\mathbb{Z}$. For instance, we can take $I(x_0, x_1) = \{x_0^2 +$

$x_1^2 = 1\}$. Define the assignment $V_I : \mathcal{R} \to \mathbf{Set}$ by mapping R to $V_I(R) = \{\vec{r} \in R^{n+1} \mid \text{all equations in } I(\vec{x}) \text{ hold for } \vec{x} = \vec{r}\}$ and any $f : R \to S$ to the function $V_I(f) : V_I(R) \to V_I(S)$ defined by $V_I(f)(\vec{x}) = (f(x_0), \ldots, f(x_n))$. The function $V_I(f)$ is well-defined, because when $\vec{r}$ is the root for an equation, then so is $\vec{f(r)}$, simply because f preserves 1, addition and multiplication. This assignment is clearly a functor. It is reasonable to think of V_I as a method to keep track of all the possible realizations (models) of the set of equations in all possible worlds. It is the semantical way to capture the syntactic data $I(\vec{x})$.

Remark 1.3.76. Note that V_I is not a full and faithful semantical apparatus. For instance, for the different sets of equations $I(x) = \{x = 0\}$ and $J(x) = \{x^2 = 0\}$, we have $V_I(R) = V_J(R) = \{0\}$.

Example 1.3.77. *(Fundamental set Π_0)* Let **Quiv** be the category of quivers (directed multi-graphs). For any quiver Q, define the equivalence relation $\sim$ on $V(Q)$ by $v \sim w$ iff there exist two paths of edges in $E(Q)$ (including the paths with length zero), one starting from v and ending in w and one starting from w and ending in v. (Why is it an equivalence relation?) Define the assignment $\Pi_0 : \mathbf{Quiv} \to \mathbf{Set}$ on objects by $\Pi_0(Q)$ as the set of equivalence classes in $V(Q)$ and on quiver morphism $f : Q \to Q'$ by $\Pi_0(f)([v]) = [f(v)]$. (Why is it well-defined?) The assignment Π_0 is a functor. It measures how connected the quiver is. It is also possible to use a more refined version in which the functor returns not only the set $\Pi_0(Q)$ but also its underlying order, defined by $[v] \leq [w]$ iff there exists a path from v to w. (Why is it a well-define poset order?) It is not hard to see that $\Pi_0(f)$ also respects this order. Denote this functor by $\Pi_0^d : \mathbf{Quiv} \to \mathbf{Poset}$.

Remark 1.3.78. Note that Π_0 is not full and faithful as it sends any two connected quivers to a singleton. The same also holds for Π_0^d.

Philosophical Note 1.3.79. *Non-full-and-faithful functors* provide some room to simplify the original object A in a discourse $\mathcal{C}$ to a simpler object $F(A)$ in $\mathcal{D}$. When $F(A)$ is "computable" in a relatively easy way, F can be useful in showing that two given objects in $\mathcal{C}$ are not isomorphic. The strategy is as follows: Assume that an isomorphism $f : A \to B$ exists between two given objects A and B. Then, by the application of the functor F, we must have an isomorphism between $F(A)$ and $F(B)$ in $\mathcal{D}$. Now, compute both $F(A)$ and $F(B)$ and show that they can not be isomorphic. The basic version of this argument is when we find an "easy-to-check" property P such that it is invariant under the given isomorphisms and A and B disagree on this property P. For instance, to prove that the two groups $(\mathbb{Z}, +)$ and $(\mathbb{Q}, +)$

are not isomorphic, it is enough to observe that the latter has the property $P = \forall x \exists y (x = y + y)$, while the former lacks it. Note also that P is a group-theoretic property, meaning it is invariant under all group isomorphisms. This argument is a special kind of the argument above, using a groupoid $\mathcal{C}$ of objects together with their isomorphisms and the functor $P : \mathcal{C} \to \{0, 1\}$ to capture the invariant-under-isomorphism property P, where $\{0, 1\}$ is a discrete category encoding true and false values.
It is also possible to have more complex examples, using more sophisticated categories for $\mathcal{D}$. For instance, consider the following quivers:

$$Q : \quad \begin{tikzcd} \bullet \arrow[d] \arrow[dr, bend right] \arrow[dr, bend left] & \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad\qquad Q' : \quad \begin{tikzcd} \bullet \arrow[r] & \bullet \\ \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \end{tikzcd}$$

They are not isomorphic, since the forgetful functor $V : \mathbf{Quiv} \to \mathbf{Set}$ maps Q to a three element set (the set of vertices) and Q' to a four element set. These two sets can not be isomorphic in $\mathbf{Set}$. Hence, Q and Q' are not isomorphic as quivers. Note that the functor V is easy to compute and this is the key element that makes it useful here. Moreover, it is important to observe that showing two sets are not isomorphic boils down to an easy cardinality argument. However, as the functor is not faithful, it has its own blind spots. For instance, in the following situation

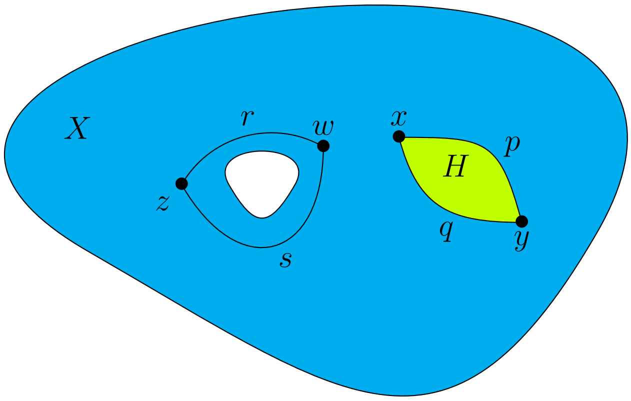
$$P : \quad \begin{tikzcd} \bullet \arrow[d] \arrow[dr] & \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad\qquad P' : \quad \begin{tikzcd} \bullet \arrow[d] & \\ \bullet \arrow[r] & \bullet \arrow[ul] \end{tikzcd}$$

both functors V and E are blind to the difference. In such cases, it is reasonable to use more sophisticated functors. But, remember, they must remain relatively easier to handle than the original object. In this case, we use the functor Π_0. Since, $\Pi_0(P)$ is a three element set while $\Pi_0(P')$ is just a singleton, P and P' are not isomorphic as quivers. As the last example, consider the following two quivers:

$$R: \quad \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, bend left] \arrow[dr, bend right] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad R': \quad \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet \arrow[r] \arrow[ur, bend left] \arrow[ur, bend right] & \bullet \arrow[u] \end{tikzcd}$$

Here, all the three functors V, E and Π_0 agree. However, $\Pi_0^d(R)$ is a lozenge while $\Pi_0^d(R')$ is just a line.
As another example, consider the category $\mathcal{R}$ of Example 1.3.75. To show that $\mathbb{Q}$ and $\mathbb{R}$ are not isomorphic in $\mathcal{R}$, it is enough to consider the forgetful functor $F : \mathcal{R} \to \mathbf{Set}$, since $F(\mathbb{Q})$ is countable, while $F(\mathbb{R})$ is uncountable and they can not be isomorphic as sets. However, to show that $\mathbb{R}$ and $\mathbb{C}$ are not isomorphic in $\mathcal{R}$, the forgetful functor does not work, as the underlying sets have equal cardinality. In this case, it is useful to have the more refined functor V_I, for $I(x) = \{x^2 + 1 = 0\}$. Here, we have $V_I(\mathbb{R}) = \emptyset$, while $V_I(\mathbb{C}) = \{i, -i\}$ and these two sets are not isomorphic.

Example 1.3.80. (*Fundamental Groupoid* Π_1) Let **Top** be the category of all topological spaces with continuous functions. For any topological space X, consider the set of paths in X from x to y, i.e., all continuous functions $p : [0,1] \to X$ such that $p(0) = x$ and $p(1) = y$, denoted by $Path_X(x, y)$. First, note that it is again possible to define the functor $\Pi_0 : \mathbf{Top} \to \mathbf{Set}$ by setting $\Pi_0(X)$ as X up to the equivalence $\sim$ defined by $x \sim y$ if there exists a path in X from x to y. The function $\Pi_0(f)$ is also defined canonically as before. The functor Π_0 measures how connected the space X can be. Now, to define another functor, lift these considerations one level up, i.e., define the equivalence relation $\sim$ on $Path_X(x, y)$ by $p \sim q$ iff there exists a surface in X filling between p and q, i.e., a continuous function $H : [0,1] \times [0,1] \to X$ such that H maps $\{0\} \times [0,1]$ to x, $\{1\} \times [0,1]$ to y and the restrictions of H to $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$ becomes p and q, respectively. (Why is it an equivalence relation?)



In the figure, the image of H is depicted by the green area and hence $p \sim q$, while r and s can not be in the same class as the white hole in the middle prevents any surface between r and s. Now, define $\Pi_0(X)$ as the groupoid with the objects as the elements of X, the morphisms from x to y as $Path_X(x, y)$ and composition and identity as the canonical pasting paths to each other and the class of the constant path. (Why is composition well-defined? Why is the constant map the identity morphism?) Define the assignment $\Pi_1 : \mathbf{Top} \to \mathbf{Groupoid}$ on objects by $\Pi_1(X)$ and on a morphism $f : X \to Y$ by the functor $\Pi_1(f)$ defined by $\Pi_1(f)(x) = f(x)$ and $\Pi_1(f)([p]) = f[p]$. (Why is it well-defined?) The assignment Π_1 is a functor. It is possible to simplify the functor Π_1 with some non-canonical choice for a base point. Let X be a space and $x \in X$ be a point in X. Now, restrict the groupoid $\Pi_(X)$ to the object x and the morphisms over x. This is also a functor, usually denoted by π_1, this time from the category of pointed spaces, denoted by $\mathbf{Top}_*$ to the category $\mathbf{Grp}$. Both Π_1 and π_1 measure the 2-holes in a space X as Π_0 measured 1-holes. (1-hole means disconnectedness. Right?) For instance, for the space $B_2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ and any possible choice for the base point $a \in B_2$, the group $\pi_1(B_2, a)$ is just a singleton, as any path over a in B_2 can be filled and B_2 (why?) or in other words as B_2 has no holes. At the same time, for the circle $S_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, the group $\pi_1(S_1, a)$ for any base point $a \in S_1$ is $\mathbb{Z}$, as any path over a in S_1 is uniquely determined by the number it goes around S_1. (Why?) These are not obvious claims. But intuitively, they are just clear.

Philosophical Note 1.3.81. It is possible to interpret any topological space X as a set with multiplicities, any path $p : x \to y$ as a *proof* of equality between x and y, any surface between two paths $p, q : x \to y$ as a *proof* of equality between p and q and so on. With this interpretation, while $\Pi_0(X)$ computes the set of distinct elements of X, the functor $\Pi_1(X)$ computes the distinct proofs between two equal elements.

Example 1.3.82. (*Application of the Fundamental Groups*) We want to prove *Brouwer's fixed point theorem* for 2-ball $B_2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ that states that any continuous function $f : B_2 \to B_2$ has a fixed point. For the sake of contradiction, assume f does not have a fixed point. Then, for any $(x, y) \in B_2$, we have $f(x, y) \neq (x, y)$. Define $r : B_2 \to S_1$ in the following way: Take the directed line L, connecting $f(x, y)$ to (x, y) and define $r(x, y)$ as the intersection of L and the border of B_2 which is S_1. By definition, the restriction r to S_1 is the identity function. Therefore, if we denote the inclusion of S_1 in B_2 by $i : S_1 \to B_2$, we have:

$$\begin{tikzcd} S_1 \arrow[r, "i"'] \arrow[rr, bend left, "id_{S_1}"] & B_2 \arrow[r, "r"'] & S_1 \end{tikzcd}$$

Since $\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$ is a functor, if we pick an arbitrary $a \in S_1$, we have:

$$\begin{tikzcd} \pi_1(S_1, a) \arrow[r, "\pi_1(i)"'] \arrow[rr, bend left, "id_{\pi_1(S_1,a)}"] & \pi_1(B_2, a) \arrow[r, "\pi_1(r)"'] & \pi_1(S_1, a) \end{tikzcd}$$

which is impossible, as $\pi_1(S_1, a)$ is isomorphic to $\mathbb{Z}$, while $\pi_1(B_2, a)$ is a singleton group.

Remark 1.3.83. In almost all the applications of the fuctors we have seen so far, except maybe the previous example, the only thing we used was the fact that the functors from one discourse to the other preserve isomorphisms, as they are expected to preserve the corresponding notion of "sameness". Following such observations, one may find it tempting to restrict category theory to groupoids as the formalization of a discourse equipped with its notion of sameness. The previous example is just a simple instance to show that this temptation is somewhat naive. Morphisms and not just isomorphisms are important to capture the behavior of an object and it is useful if we know how to transfer them from one discourse to another.

Definition 1.3.84. (*Natural Transformations*) Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. By a natural transformation $\alpha : F \Rightarrow G$, depicted as

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathcal{D} \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$$

we mean an assignment mapping any object of $\mathcal{C}$ to a morphism $\alpha_C : F(C) \to G(C)$ in $\mathcal{D}$ such that for any morphism $f : A \to B$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd} F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\alpha_B"'] & G(B) \end{tikzcd}$$

Philosophical Note 1.3.85. If we read functors $F, G : \mathcal{C} \to \mathcal{D}$ as two $\mathcal{C}$-variable objects in $\mathcal{D}$, then any natural transformation $\alpha : F \Rightarrow G$ is a transformation between these variable objects. Naturally, any transformation between variable objects must specify the way we change the object $F(C)$ to the object $G(C)$ in $\mathcal{D}$, for each parameter $C \in ob(\mathcal{C})$. These changes can not be arbitrary. They must respect the changes in parameter in $\mathcal{C}$:

$$\begin{tikzcd} & B \arrow[dr, "g"] & & & & F(B) \arrow[dr, "F(g)"] \arrow[dd, dotted, "\alpha_B"] & \\ A \arrow[ur, "f"] & & C & \overset{F}{\rightsquigarrow} \quad \underset{G}{\rightsquigarrow} & F(A) \arrow[ur, "F(f)"] \arrow[dd, dotted, "\alpha_A"] & & F(C) \arrow[dd, dotted, "\alpha_C"] \\ & & & & & G(B) \arrow[dr, "G(g)"] & \\ & & & & G(A) \arrow[ur, "G(f)"] & & G(C) \end{tikzcd}$$

Philosophical Note 1.3.86. Let us read two functors $F, G : \mathcal{C} \to \mathcal{D}$ as two *construction methods* that read an object in $\mathcal{C}$ and transform it to an object in $\mathcal{D}$. When can we call F and G "equal" as two methods of construction? Of course we do not want to restrict ourselves to the very strict equality that demands the functors to be equal both on the objects and the morphisms. This is just too restrictive. For instance, consider $F, G : \mathbf{Set} \to \mathbf{Set}$ as $F(A) = A \times \{0\}$ and $G(A) = A \times \{1\}$. In this case, although F and G are not strictly equal, they must be considered as the same methods of construction, as they are only different up to an isomorphism. Using this criterion, one natural candidate for the intended equality between F and G is the existence of an isomorphism between $F(A)$ and $G(A)$, for any object A in $\mathcal{C}$. However, it is clear that any random assignment of isomorphisms between $F(A)$ and $G(A)$ does not work. The isomorphisms must be assigned in a *uniform* way, as we want F and G to be equal as two methods of constructions not two mere structureless assignments. This uniformity demands the isomorphisms to be somewhat independent of the choice of the object A. Of course, one may object that the isomorphisms clearly depend on the object A (the source and the target of the isomorphism, for instance), but at same time it is intuitively meaningful to talk about the constructions that apply the *same* method to *different* objects. An example may be more illuminating. Consider the canonical isomorphism $s_{A,B} : A \times B \to B \times A$ defined by $s_{A,B}(a, b) = (b, a)$ that shows the order in the product of two sets is not important. This map clearly depends on the choice of A and B, but at the same time it is defined in a uniform way of "swapping the elements in a pair" which dos not use the sets in an essential way. Natural transformations is historically developed for the sole purpose of capturing this very intuition of uniformity.

Example 1.3.87. The assignment $s : id_{\mathbf{Set}} \Rightarrow P$ defined by $s_A : A \to P(A)$ as $s_A(a) = \{a\}$ is a natural transformation. It is natural simply because if $f : A \to B$ maps $a \in A$ to $f(a) \in B$, then $P(f)$ maps $\{a\}$ to $f[\{a\}] = \{f(a)\}$.

$$\begin{tikzcd} A \arrow[r, "\{-\}"] \arrow[d, "f"'] & P(A) \arrow[d, "{f[-]}"] \\ B \arrow[r, "\{-\}"'] & P(B) \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, mapsto, "\{-\}"] \arrow[d, mapsto, "f"'] & \{a\} \arrow[d, mapsto, "{f[-]}"] \\ f(a) \arrow[r, mapsto, "\{-\}"'] & \{f(a)\} \end{tikzcd}$$

Example 1.3.88. The assignment $i : id_{\mathbf{Set}} \Rightarrow (P^\circ)^\circ$ defined by $i_A : A \to PP(A)$ as $i_A(a) = \{S \subseteq A \mid a \in S\}$ is a natural transformation:

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[d, "f"'] & P(P(A)) \arrow[d, "P^\circ(P^\circ(f))"] \\ B \arrow[r, "i_B"'] & P(P(B)) \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, mapsto, "i_A"] \arrow[d, mapsto, "f"'] & \{S \subseteq A \mid a \in S\} \arrow[d, mapsto, "P^\circ(P^\circ(f))"] \\ f(a) \arrow[r, mapsto, "i_B"'] & \{T \subseteq B \mid f(a) \in T\} \end{tikzcd}$$

Note that $P^\circ(P^\circ(f))(\mathcal{S}) = (P^\circ(f))^{-1}(\mathcal{S}) = \{T \subseteq B \mid f^{-1}(T) \in \mathcal{S}\}$ which maps $\{S \subseteq A \mid a \in S\}$ to $\{T \subseteq B \mid f(a) \in T\}$.

Example 1.3.89. Let $Ex : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set} \times \mathbf{Set}$ be the exchange functor, i.e, $Ex(A, B) = (B, A)$ and $Ex(f, g) = (g, f)$ and $(-) \times (-) : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ be the product functor. Then, the assignment $s : (-) \times (-) \Rightarrow [(-) \times (-)] \circ Ex$ defined by $s_{(A,B)} : A \times B \to B \times A$ as $s_{A\times B}(a, b) = (b, a)$ is a natural transformation:

$$\begin{tikzcd}
A \times B \arrow[r, "s_{(A,B)}"] \arrow[d, "f\times g"'] & B \times A \arrow[d, "g\times f"] \\
C \times D \arrow[r, "s_{(C,D)}"'] & D \times C
\end{tikzcd}
\qquad
\begin{tikzcd}
(a, b) \arrow[r, mapsto, "s_{(A,B)}"] \arrow[d, mapsto, "f\times g"'] & (b, a) \arrow[d, mapsto, "g\times f"] \\
(f(a), g(b)) \arrow[r, mapsto, "s_{(C,D)}"'] & (g(b), f(a))
\end{tikzcd}$$

Exercise 1.3.90. Prove that $\alpha : ((-) \times (-)) \times (-) \Rightarrow (-) \times ((-) \times (-))$ defined by $\alpha_{A,B,C} : (A \times B) \times C \to A \times (B \times C)$ such that $\alpha_{A,B,C}((a, b), c) = (a,(b, c))$ is a natural transformation.

Example 1.3.91. The assignment $(-)^{-1}(1) : Hom(-, 2) \Rightarrow P^{\circ}(-)$ defined by $f \mapsto f^{-1}(1)$ is a natural transformation:

$$\begin{tikzcd}
Hom(B, 2) \arrow[r, "(-)^{-1}(1)"] \arrow[d, "(-)\circ f"'] & P(B) \arrow[d, "f^{-1}(-)"] \\
Hom(A, 2) \arrow[r, "(-)^{-1}(1)"'] & P(A)
\end{tikzcd}
\qquad
\begin{tikzcd}
g \arrow[r, mapsto, "(-)^{-1}(1)"] \arrow[d, mapsto, "(-)\circ f"'] & g^{-1}(1) \arrow[d, mapsto, "f^{-1}(-)"] \\
gf \arrow[r, mapsto, "(-)^{-1}(1)"'] & (gf)^{-1}(1)
\end{tikzcd}$$

Example 1.3.92. The assignment $(-)(0) : Hom(1, -) \Rightarrow id_{\mathbf{Set}}$ defined by $g \mapsto g(0)$ is a natural transformation:

$$\begin{tikzcd}
Hom(1, A) \arrow[r, "(-)(0)"] \arrow[d, "f\circ(-)"'] & A \arrow[d, "f"] \\
Hom(1, B) \arrow[r, "(-)(0)"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
g \arrow[r, mapsto, "(-)(0)"] \arrow[d, mapsto, "f\circ(-)"'] & g(0) \arrow[d, mapsto, "f"] \\
fg \arrow[r, mapsto, "(-)(0)"'] & fg(0)
\end{tikzcd}$$

Similarly, for the category of groups, we have:

$$\begin{tikzcd}
Hom(\mathbb{Z}, G) \arrow[r, "(-)(1)"] \arrow[d, "f\circ(-)"'] & U(G) \arrow[d, "f"] \\
Hom(\mathbb{Z}, H) \arrow[r, "(-)(1)"'] & U(H)
\end{tikzcd}
\qquad
\begin{tikzcd}
g \arrow[r, mapsto, "(-)(1)"] \arrow[d, mapsto, "f\circ(-)"'] & g(1) \arrow[d, mapsto, "f"] \\
fg \arrow[r, mapsto, "(-)(1)"'] & fg(1)
\end{tikzcd}$$

where U : **Grp** $\to$ **Set** is the forgetful functor, $(-)(1) : Hom(\mathbb{Z}, -) \Rightarrow U$ defined as $g \mapsto g(1)$. We also have the same phenomenon in $\mathbf{Vec}_{\mathbb{R}}$, i.e.,

$$\begin{tikzcd} Hom(\mathbb{R}, V) \arrow[r, "(-)(1)"] \arrow[d, "f\circ(-)"'] & U(V) \arrow[d, "T"] \\ Hom(\mathbb{R}, W) \arrow[r, "(-)(1)"'] & U(W) \end{tikzcd} \qquad \begin{tikzcd} g \arrow[r, mapsto, "(-)(1)"] \arrow[d, mapsto, "T\circ(-)"'] & g(1) \arrow[d, mapsto, "T"] \\ Tg \arrow[r, mapsto, "(-)(1)"'] & Tg(1) \end{tikzcd}$$

Example 1.3.93. The assignment $! : \Delta_1 \Rightarrow Hom(-, 1)$ defined by $!_A(0) = cons_A$ is a natural transformation, where $cons_A : A \to 1$ is the constant function, mapping everything to zero:

$$\begin{tikzcd} 1 \arrow[r, "!_B"] \arrow[d, "id_1"'] & Hom(B, 1) \arrow[d, "(-)\circ f"] \\ 1 \arrow[r, "!_A"'] & Hom(A, 1) \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r, mapsto, "!_B"] \arrow[d, mapsto, "id_1"'] & y \mapsto 0 \arrow[d, mapsto, "(-)\circ f"] \\ 0 \arrow[r, mapsto, "!_A"'] & x \mapsto 0 \end{tikzcd}$$

Exercise 1.3.94. Prove that $\alpha : p_1 \Rightarrow Hom(-, -)$ defined by $\alpha_{A,B} : A \to Hom(B, A)$ such that $\alpha_{A,B}(a) = cons_{A,B,a}$ is a natural transformation, where $p_1 : \mathbf{Set}^{op} \times \mathbf{Set} \to \mathbf{Set}$ is the projection on the second element functor and $cons_{A,B,a} : B \to A$ maps every element in B to a.

Example 1.3.95. The assignment $(*, id_{(-)}) : id_{\mathbf{Set}} \Rightarrow 1 \times (-)$ defined by $a \mapsto (*, a)$ is a natural transformation:

$$\begin{tikzcd} A \arrow[r, "(*,id_A)"] \arrow[d, "f"'] & 1 \times A \arrow[d, "id_1 \times f"] \\ B \arrow[r, "(*,id_B)"'] & 1 \times B \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, mapsto, "(*,id_A)"] \arrow[d, mapsto, "f"'] & (*, a) \arrow[d, mapsto, "id_1 \times f"] \\ f(a) \arrow[r, mapsto, "(*,id_B)"'] & (*, f(a)) \end{tikzcd}$$

Example 1.3.96. Let B and C be two fixed sets. Then, the assignment $\alpha : (-)^{B+C} \Rightarrow (-)^B \times (-)^C$ defined by $\alpha_A(g) = (g|_B, g|_C)$ is a natural transformation:

$$\begin{tikzcd} A^{B+C} \arrow[r, "\alpha_A"] \arrow[d, "f\circ(-)"'] & A^B \times A^C \arrow[d, "f\circ(-)\times f\circ(-)"] \\ A'^{B+C} \arrow[r, "\alpha_B"'] & A'^B \times A'^C \end{tikzcd} \qquad \begin{tikzcd} g \arrow[r, mapsto, "\alpha_A"] \arrow[d, mapsto, "f\circ(-)"'] & (g|_B, g|_C) \arrow[d, mapsto, "f\circ(-)\times f\circ(-)"] \\ fg \arrow[r, mapsto, "\alpha_B"'] & (fg|_B, fg|_C) \end{tikzcd}$$

Exercise 1.3.97. Prove that $\alpha : (-)^{(-)+(-)} \Rightarrow (-)^{(-)} \times (-)^{(-)}$ defined by $\alpha_{A,B,C}(g) = (g|_B, g|_C)$ is a natural transformation.

Exercise 1.3.98. Prove that $\alpha : ((-) \times (-))^{(-)} \Rightarrow (-)^{(-)} \times (-)^{(-)}$ defined by $\alpha_{A,B,C} : (A \times B)^C \to A^C \times B^C$ such that $\alpha_{A,B,C}(g) = (p_0 \circ g, p_1 \circ g)$ is a natural transformation, where $p_0 : A \times B \to A$ and $p_1 : A \times B \to B$ are the projection functions.

Exercise 1.3.99. Prove that $\alpha : Hom((-), (-)^{(-)}) \Rightarrow Hom(- \times -, -)$ defined by $\alpha_{A,B,C} : Hom(A, C^B) \to Hom(A \times B, C)$ such that $\alpha_{A,B,C}(g) = \hat{g}$ is a natural transformation, where $\hat{g} : A \times B \to C$ maps (a, b) to $g(a)(b)$.

Philosophical Note 1.3.100. Looking inside the world of categories, there are three sorts of data: First, the categories as the nodes or the zero-dimensional data; the functors between the categories as the edges or the 1-dimensional data and finally, natural transformations as the surfaces or the 2-dimensional data. In this sense, the world of categories is at least 2-dimensional in some intuitive sense. One may ask to go further to define morphisms between natural transformations, morphisms between these morphisms and so on. This is possible. However, these higher level data trivialize after the second level of natural transformations and this is the reason why category theory stops here. The world of categories is somehow like the 2-dimensional plane. Of course, it is possible to find "three dimensional" spaces in the plane. It is just enough to map the three dimensional cube inside the plane by a continuous map. The important thing, though, is that as the plane is too restricting for such a map, one dimension of the cube would collapse, inevitably. Hence, the three dimensional spaces inside the plane are all degenerate, which implies that there is no need to keep track of them.
To make the comparison with the topological space more precise, let us restrict ourselves to categories and isomorphisms as the morphisms. With this restriction we can eliminate the direction from the picture. Now, this 2-dimensional groupoid has different connective components; the sub-groupoids in which all objects are connected to each other. There are some possible scenarios. Some of these components are just zero-dimensional points. The others are divided in 1-dimensional spaces (with or without holes) and 2-dimensional spaces (with or without holes). We will complete this picture by providing some examples of these spaces.

Example 1.3.101. (*Non-natural transformations*) Let α be an assignment of a map $\alpha_A : A \to A$ to any set A. Then, $\alpha : id_{\mathbf{Set}} \Rightarrow id_{\mathbf{Set}}$ is a natural transformation iff $\alpha_A = id_A$. It is clear that $\alpha_A = id_A$ is a natural transformation. For the converse, assume α is a natural transformation and consider the following commutative diagram:

$$\begin{tikzcd}
\{0\} \arrow[r, "\alpha_{\{0\}}"] \arrow[d, "\hat{a}"'] & \{0\} \arrow[d, "\hat{a}"] \\
A \arrow[r, "\alpha_A"'] & A
\end{tikzcd}$$

where $a \in A$ and $\hat{a}(0) = a$. It is clear that $\alpha_{\{0\}} = id_{\{0\}}$. Hence, $\alpha_A \circ \hat{a} = \hat{a}$. Applying both sides on 0, we have $\alpha_A(a) = a$. Hence, $\alpha_A = id_A$.

Exercise 1.3.102. Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is an isomorphism iff there exists a functor $G : \mathbf{Set} \to \mathbf{Set}$ such that $F \circ G = G \circ F = id_{\mathbf{Set}}$ and in this situation $\mathcal{C}$ and $\mathcal{D}$ is called isomorphic. Then, prove that if $\mathcal{E}$ and $\mathcal{F}$ are two categories isomorphic to **Set** and $F_0, F_1 : \mathcal{E} \to \mathcal{F}$ be two isomorphisms, then there exists exactly one natural transformation from F to F'.

Philosophical Note 1.3.103. Reading the groupoid of categories as a 2-dimensional space, Exercise 1.3.102 implies that the connective component of the category **Set** is 1-dimensional, as the level of natural transformations collapses in this component. Moreover, it has no holes, as any space between two isomorphisms can be filled by a natural transformation. Therefore, it may be reasonable to think of this component as a straight line.

Example 1.3.104. Let G and H be two groups and $F, F' : G \to H$ be two group homomorphisms. Then, a natural transformation $\alpha : F \Rightarrow F'$, by definition is an element $\alpha_* = h \in H$ such that $F(g)h = hF'(g)$, for any $g \in G$.

$$\begin{tikzcd}
F(*) \arrow[r, "\alpha_*"] \arrow[d, "F(g)"'] & F'(*) \arrow[d, "F'(g)"] \\
F(*) \arrow[r, "\alpha_*"'] & F'(*)
\end{tikzcd}$$

Exercise 1.3.105. Prove that there is no natural transformation from $id_{(\mathbb{Z},+)}$ to $-id_{(\mathbb{Z},+)}$.

Exercise 1.3.106. Let $\mathcal{C}$ be a category. By the center of $\mathcal{C}$, denoted by $Z(\mathcal{C})$, we mean the class of all natural transformation $\alpha : id_{\mathcal{C}} \Rightarrow id_{\mathcal{C}}$. Show that for any group G considered as a category, $Z(G)$ corresponds to the set $\{g \in G \mid \forall h \in G\ gh = hg\}$. Use this characterization to show that for any non-trivial abelian groups G and H, if $G \simeq H$ and $F : G \to H$ is an isomorphism, there are at least two different natural transformations over F. Moreover, find a group G such that between any two isomorphisms over G, there is at most one morphism.

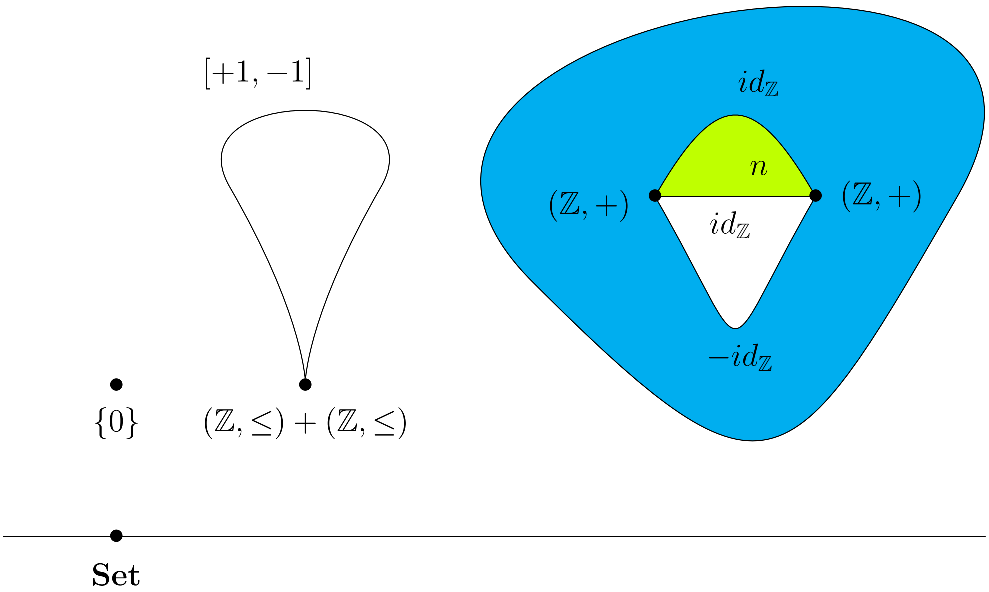
Example 1.3.107. Let $G \simeq H$ be two groups and $Z(G) = Z(H) = \{e\}$. Then, for any group homomorphisms $F, F' : G \to H$, there is at most one natural transformation $\alpha : F \Rightarrow F'$. To prove this claim, let $\alpha, \beta : F \Rightarrow F'$ be two natural transformations. Then, $\alpha_* = i \in H$ and $\beta_* = j \in H$ such that for any $g \in G$, we have $F(g)i = iF'(g)$ and $F(g)j = jF'(g)$. We claim that $i^{-1}j \in Z(H)$. Let $h \in H$ be an arbitrary element. Then, there exists $g \in G$ such that $F'(g) = h$. Hence, $F(g) = ihi^{-1} = jhj^{-1}$. Therefore, $i^{-1}jh = hi^{-1}j$, for any $h \in H$. Hence, $i^{-1}j \in Z(H)$ which mean that $ij^{-1} = e$ and hence $i = j$.

Philosophical Note 1.3.108. Exercises 1.3.106 implies that the connective component of any non-trivial abelian group is truly 2-dimensional, while 1.3.107 ensures that the connective component of a group G, where $Z(G) = \{e\}$ is 1-dimensional. This may explain why the abelian groups are easier to work with, or more generally, why the groups become more complex, as soon as their centers start to shrink.

Example 1.3.109. Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two posets and $F, G : (P, \leq_P) \to (Q, \leq_Q)$ be two poset morphisms. Then, a natural transformation $\alpha : F \Rightarrow G$ is necessarily unique, as there is at most one map from $F(p)$ to $G(p)$, for any $p \in P$. This unique natural transformation exists iff $F(p) \leq_Q G(p)$, for any $p \in P$: The similar thing happens if we replace $(P, \leq_P)$ with any other category.

Exercise 1.3.110. Consider the poset $(\mathbb{Z}+\mathbb{Z}, \leq)$, where $\leq$ is the usual order on each component. Then, take the isomorphism $F = [+1, -1] : (\mathbb{Z} + \mathbb{Z}, \leq) \to (\mathbb{Z} + \mathbb{Z}, \leq)$ define by $F(0, a) = a + 1$ and $F(1, b) = b - 1$. Prove that there is no natural transformations $\alpha : id_{(\mathbb{Z}+\mathbb{Z},\leq)} \Rightarrow F$ and $\beta : F \Rightarrow id_{(\mathbb{Z}+\mathbb{Z},\leq)}$.

Philosophical Note 1.3.111. Consider the groupoid of all locally small categories with isomorphism as the morphisms. Then, the previous considerations imply that the "*topological*" picture of this category must be like:



where n in the green area means that for any number $n \in \mathbb{Z}$, there is one surface there and the discrete category $\{0\}$ is depicted as a zero-dimensional object as there is no non-trivial isomorphism over $\{0\}$.

Example 1.3.112. Let G be a group. Recall that a G-action is a group homomorphism from G to $Aut(X)$, where $Aut(X)$ is the group of all bijections on the set X. A morphism between two G-actions is a function $\phi : X \to Y$ such that $\phi(F(g)(x)) = F'(g)(\phi(x))$, for any $g \in G$ and $x \in X$. Then, any G-action is just a functor $G \to \mathbf{Set}$ and any morphism between two G-actions is a natural transformation:

$$\begin{tikzcd}
X = F(*) \arrow[r, "\phi"] \arrow[d, "F(g)"'] & F'(*) = Y \arrow[d, "F'(g)"] \\
X = F(*) \arrow[r, "\phi"'] & F'(*) = Y
\end{tikzcd}$$

Example 1.3.113. Let $(-)^* : \mathbf{Vec}_{\mathbb{R}} \to \mathbf{Vec}_{\mathbb{R}}^{op}$ be the functor mapping V to $V^* = \{T : V \to \mathbb{R} \mid T \text{ is linear}\}$ and $S : V \to W$ to $(-) \circ S : W^* \to V^*$. Then, the assignment $i : id_{\mathbf{Vec}_{\mathbb{R}}} \Rightarrow ((-)^*)^*$ defined by $i_V(v) : Hom(V, \mathbb{R}) \to \mathbb{R}$

as $i_V(v)(T) = T(v)$ is a natural transformation:

$$\begin{tikzcd}
V \arrow[r, "i_V"] \arrow[d, "T"'] & V^{**} \arrow[d, "T^{**}"] \\
W \arrow[r, "i_V"'] & W^{**}
\end{tikzcd}
\qquad
\begin{tikzcd}
v \arrow[r, mapsto, "i_V"] \arrow[d, mapsto, "T"'] & {[S \mapsto S(v)]} \arrow[d, mapsto, "T^{**}"] \\
T(v) \arrow[r, mapsto, "i_W"'] & {[R \mapsto R(T(v))]}
\end{tikzcd}$$

because, if we spell out the definition of $T^{**} : V^{**} \to W^{**}$, we see $T^{**}(F)(f) = F(f \circ T)$, where $F \in Hom(V, \mathbb{R}) \to \mathbb{R}$ and $f \in Hom(W, \mathbb{R})$.

Remark 1.3.114. It is well-known that any finite-dimensional vector space V is isomorphic to its dual V^*, using the map $\alpha_V(v) = \hat{v}$, where $\hat{v}(w) = \langle w, v \rangle$ in which $\langle -, - \rangle$ is the usual inner product. This transformation is not natural, simply because the functors $id_{\mathbf{Vec}_{\mathbb{R}}} : \mathbf{Vec}_{\mathbb{R}} \to \mathbf{Vec}_{\mathbb{R}}$ and $(-)^* : \mathbf{Vec}_{\mathbb{R}} \to \mathbf{Vec}_{\mathbb{R}}^{op}$ don't have the same codomain. One may find this reason quite artificial, as the map i_V seems quite natural, indeed. To address this issue, let us restrict ourselves to the subcategory of $\mathbf{Vec}_{\mathbb{R}}$, where all morphism are isomorphisms. Denote this subcategory by $i\mathbf{Vec}_{\mathbb{R}}$. Then, it is possible to make the directions right, using the functor $inv : i\mathbf{Vec}_{\mathbb{R}} \to i\mathbf{Vec}_{\mathbb{R}}^{op}$ that fixes the objects and maps any isomorphism to its inverse. Now, we have the following *possibly* natural transformation:

$$\begin{tikzcd}
i\mathbf{Vec}_{\mathbb{R}} \arrow[r, bend left=40, "inv"{name=U}] \arrow[r, bend right=40, "(-)^*"'{name=D}] & i\mathbf{Vec}_{\mathbb{R}}^{op}
\arrow[from=U, to=D, Rightarrow, "\alpha"]
\end{tikzcd}$$

However, it is still not natural, as if we check the naturality condition, it requires:

$$\begin{tikzcd}
W \arrow[r, "\alpha_W"] \arrow[d, "T^{-1}"'] & W^* \arrow[d, "(-)\circ T"] \\
V \arrow[r, "\alpha_V"'] & V^*
\end{tikzcd}
\qquad
\begin{tikzcd}
w \arrow[r, mapsto, "\alpha_W"] \arrow[d, "T^{-1}"'] & {[u \mapsto \langle u, w \rangle]} \arrow[d, "(-)\circ T"] \\
T^{-1}(w) \arrow[r, mapsto, "\alpha_V"'] & {[v \mapsto \langle v, T^{-1}(w) \rangle]}
\end{tikzcd}$$

meaning, $\langle T(v), w \rangle = \langle v, T^{-1}(w) \rangle$, which is not the case. One can easily check that this equation holds for any $T : V \to W$ that preserves the inner product. Therefore, if we restrict the categories more to invertible linear maps that preserve inner product (*orthogonal transformations*), then our assignment α finally will be a natural transformation. Note that this restricted category actually captures the Euclidean geometry as it works with maps that respect distance and angle. Therefore, we can read the naturality of α as "angles are natural in Euclidean geometry, while they are not in linear world".

Example 1.3.115. (*No-deleting theorem*) There is only one natural transformation $\alpha : (-) \times (-) \to pr_1$. This natural transformation is the trivial $\alpha_{B,A} = \emptyset$. First, it is clear that this assignment is a natural transformation. Conversely, assume such an α exists. Then, we have the following commutative diagram:

$$\begin{tikzcd} B \times A \arrow[r, "\alpha_{B,A}"] \arrow[d, "S\times R"'] & A \arrow[d, "R"] \\ B \times A \arrow[r, "\alpha_{B,A}"'] & A \end{tikzcd}$$

for any set A and B and any relations $R \subseteq A^2$ and $S \subseteq B^2$. Set $R = \{(a, a) \mid a \in A\}$ and $S = \emptyset$. Then, $\alpha_{B,A}$ must be empty. It is useful to check why the usual projection function does not work in this case. If we spell out all the details, the reason boils down to the fact that the relations can be partial.

Example 1.3.116. (*No-cloning theorem*) There is only one natural transformation $\alpha : id_{\mathbf{Rel}} \to (-)^2$, where $(-)^2 : \mathbf{Rel} \to \mathbf{Rel}$ is defined by $A \mapsto A^2$ on objects and $R \mapsto R \times R$ on morphisms. This natural transformation is the trivial $\alpha_A = \emptyset$. First, it is clear that this assignment is a natural transformation. Conversely, assume that $\alpha : id_{\mathbf{Rel}} \to (-)^2$ is a natural transformation. Then, we have the following commutative diagram:

$$\begin{tikzcd} \{0\} \arrow[r, "\alpha_{\{0\}}"] \arrow[d, "R"'] & \{0\}^2 \arrow[d, "R^2"] \\ A \arrow[r, "\alpha_A"'] & A^2 \end{tikzcd}$$

for any set A and any relation $R \subseteq A \times \{0\}$. First, note that $\alpha_{\{0\}} \subseteq \{0\} \times \{0\}^2$. As $\{0\} \times \{0\}^2$ has just one element, then either $\alpha_{\{0\}} = \emptyset$ or $\alpha_{\{0\}} = \{0\} \times \{0\}^2$. The first case implies that $\alpha_A = \emptyset$, for every A. If R_A is non-empty, then there exists $(a, (b, c)) \in \alpha_A$ for some $a, b, c \in A$. Define $R = \{(0, a)\}$. Then, $(0, (b, c)) \in \alpha_a \circ R$, while $R^2 \circ \alpha_{\{0\}}$ is empty. Hence, $\alpha_A = \emptyset$. For the second case, we prove that $\alpha_{\{0\}} = \{0\} \times \{0\}^2$ is impossible. First, set $R = \{(0, a)\}$. Then, $R^2 \circ \alpha_{\{0\}} = \{(0, (a, a))\}$. Therefore, $\alpha_A \circ R = \{(0, (a, a))\}$ which means $(a, (a, a)) \in \alpha_A$. Therefore, $\{(a, (a, a)) \mid a \in A\} \subseteq \alpha_A$. It is easy to prove that α_A can not have any other element and hence $\alpha_A = \{(a, (a, a)) \mid a \in A\}$. Now, set $A = \{0, 1\}$ and $R = \{0\} \times A$. We have $\alpha_A \circ R = \{(0, (0, 0)), (0, (1, 1))\}$. But, $R^2 \circ \alpha_{\{0\}} = \{0\} \times A^2$ which is a contradiction. It is useful to check why the usual function $a \mapsto (a, a)$ does not work. If we spell out all the details, the reason boils down to the fact that the relations can be multi-valued.

Philosophical Note 1.3.117. The simplest category that encodes the quantum behavior is **Rel**, in which sets encode the set of states and relations encode the non-deterministic processes that change one state to another. In this sense, the previous two theorems are the baby version of the *entanglement* phenomenon in quantum theory by which we know it is impossible to clone or delete a quantum bit of information. The reason for the simplest case of **Rel** may be explained by the fact that relations can be partial or multi-valued and this makes the elements of the set somewhat entangled to each other. The more advanced version states that there is no natural transformation $\alpha_V : V \to V \bigotimes V$ or $\beta_{V,W} : V \bigotimes W \to V$ on vector spaces. For the real version, replace vector spaces by Hilbert spaces and linear maps by bounded linear maps.

Exercise 1.3.118. Let $List : \mathbf{Set} \to \mathbf{Set}$ be the functor mapping any set X to the set of all finite sequences of the elements of X and mapping any function $f : X \to Y$ to the function $List(f) : List(X) \to List(Y)$ defined by $List(f)(\sigma_0 \cdots \sigma_n) = f(\sigma_0) \cdots f(\sigma_n)$. Show that the assignment $i : \Delta_1 \to List$ defined by $i_X : \{0\} \to List(X)$ as $i_X(0) = \epsilon$ is a natural transformation, where ϵ is the sequence with the length zero. Moreover, show that the assignment $m : List \times List \to List$ defined by $m_X : List(X) \times List(X) \to List(X)$ as the concatenation operation is a natural transformation.

Example 1.3.119. Let $\mathcal{B}$ be the groupoid of finite sets and bijections. Define $Aut : \mathcal{B} \to \mathbf{Set}$ as the functor mapping any set X to the set of all bijections on X and mapping a bijection $f : X \to Y$ to the function $f \circ (-) \circ f^{-1} : Aut(X) \to Aut(Y)$. Moreover, define $Ord : \mathcal{B} \to \mathbf{Set}$ as the functor mapping any set X to the set of all finite sequences of the elements of X in which any element of X occurs exactly once. For the morphisms, map a bijection $f : X \to Y$ to the function $Ord(f) : Ord(X) \to Ord(Y)$ defined as $Ord(f)(\sigma_0 \cdots \sigma_n) = f(\sigma_0) \cdots f(\sigma_n)$. Then, there is no natural transformation $\alpha : Aut \to Ord$. Because, if there is such a transformation, then:

$$\begin{array}{ccc} Aut(X) & \xrightarrow{\alpha_X} & Ord(X) \\ {\scriptstyle f\circ(-)\circ f^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle Ord(f)} \\ Aut(X) & \xrightarrow[\alpha_X]{} & Ord(X) \end{array}$$

for any set X and any bijection $f : X \to X$. Set X as a set with at least two elements and $f : X \to X$ as a non-identity bijection. Now, apply the diagram on $id_X \in Aut(X)$. We have $\alpha_X(f id_X f^{-1}) = Ord(f)(\alpha_X(id_X))$ which means $\alpha_X(id_X) = Ord(f)(\alpha_X(id_X))$. This implies that $\alpha_X(id_X)$ is a list of all the elements of X that does not change under the application of f. Hence, f must be the identity function which is a contradiction. Note that in this example, although for any finite set X, the sets $Ord(X)$ and $Aut(X)$ are isomorphic, there is no natural transformation between Aut and Ord as construction methods. Specially, it means that the isomorphisms between $Ord(X)$ and $Aut(X)$ is not natural in X.

Example 1.3.120. Let $\mathcal{R}_{in}$ be the category $\mathcal{R}$ of Example 1.3.75, restricted to injective homomorphism. Let $GL_n : \mathcal{R}_{in} \to \mathbf{Grp}$ be the functor mapping any object R to the group of all invertible $n \times n$ matrices with entries in R and any morphism $f : R \to S$ to $GL_n(f) : GL_n(R) \to GL_n(S)$ defined as $GL_n(f)(A) = f[A]$, where $f[A]$ is the result of the application of f on all the entries of A. Note that $GL_n(f)(A)$ is well-defined, because, if A is invertible, then so is $f[A]$. The reason is that if $f(det(A)) = det(f[A]) = 0 = f(0)$, then $det(A) = 0$, as f is injective which implies that A is not invertible. Moreover, note that the assignment $det : GL_n \Rightarrow GL_1$ is a natural transformation. The reason is that the determinant of a matrix is a polynomial in the entries of the matrix and hence it is preserved by the morphisms of $\mathcal{R}$:

$$\begin{array}{ccc} GL_n(R) & \xrightarrow{det_R} & R \\ {\scriptstyle GL_n(f)}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ GL_n(S) & \xrightarrow[det_S]{} & S \end{array}$$

Example 1.3.121. Let $U : \mathbf{Mon} \to \mathbf{Set}$ and $F : \mathbf{Set} \to \mathbf{Mon}$ be the forgetful and the free functors, respectively. Then, the assignments $i : id_{\mathbf{Set}} \Rightarrow UF$ mapping a set A to the function $i_A : X \to UF(X)$ defined by $i_A(x) = x$ is a natural transformation. Similarly, the assignments $p : FU \Rightarrow id_{\mathbf{Mon}}$ mapping a monoid M to the homomorphism $p_M : FU(M) \to M$ defined by $p_M(\sigma_0 \cdots \sigma_n) = \sigma_0 \times \cdots \times \sigma_n$ is a natural transformation.

Example 1.3.122. Let $\mathcal{C}$ be a category and $f : A \to B$ be a map. Then, the assignment $y_f : Hom(-, A) \to Hom(-, B)$ defined by $(y_f)_C = f \circ (-)$ is a natural transformation:

$$\begin{tikzcd}
Hom(D,A) \arrow[r, "f\circ(-)"] \arrow[d, "(-)\circ g"'] & Hom(D,B) \arrow[d, "(-)\circ g"] \\
Hom(C,A) \arrow[r, "f\circ(-)"'] & Hom(C,B)
\end{tikzcd}$$

Example 1.3.123. Let $\mathcal{C}$ be a category, $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then, the assignment $\alpha : F \Rightarrow F$ defined by $\alpha_C = id_{F(C)}$ is a natural transformation, because:

$$\begin{tikzcd}
F(C) \arrow[r, "id_{F(C)}"] \arrow[d, "F(f)"'] & F(C) \arrow[d, "F(f)"] \\
F(D) \arrow[r, "id_{F(D)}"'] & F(D)
\end{tikzcd}$$

Example 1.3.124. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, $F, G, H : \mathcal{C} \to \mathcal{D}$ be three functors and $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ be two natural transformations, then $\beta \circ \alpha : F \Rightarrow H$, defined by $(\beta \circ \alpha)_C = \beta_C \alpha_C$ is a natural transformation:

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, bend left=50, "F"{name=F}] \arrow[rr, "G"{description, name=G}] \arrow[rr, bend right=50, "H"'{name=H}] & & \mathcal{D}
\arrow[from=F, to=G, Rightarrow, "\alpha"']
\arrow[from=G, to=H, Rightarrow, "\beta"']
\end{tikzcd}$$

Because in the following diagram:

$$\begin{tikzcd}
F(C) \arrow[r, "\alpha_C"] \arrow[d, "F(f)"'] & G(C) \arrow[r, "\beta_C"] \arrow[d, "G(f)"] & H(C) \arrow[d, "H(f)"] \\
F(D) \arrow[r, "\alpha_D"'] & G(D) \arrow[r, "\beta_D"'] & H(D)
\end{tikzcd}$$

if both of the squares commute, the bigger rectangular also commutes.

Definition 1.3.125. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. Then, the functors from $\mathcal{C}$ to $\mathcal{D}$ as the objects together with the natural transformations as the morphisms constitutes a category. This category is denoted by $\mathcal{D}^{\mathcal{C}}$ and is called a functor category.

Remark 1.3.126. Note that if $\mathcal{C}$ and $\mathcal{D}$ are both small categories, then $\mathcal{D}^{\mathcal{C}}$ is also small. If $\mathcal{C}$ is small and $\mathcal{D}$ is locally small, then $\mathcal{D}^{\mathcal{C}}$ is locally small. But if $\mathcal{C}$ and $\mathcal{D}$ are both locally small, there is no reason for $\mathcal{D}^{\mathcal{C}}$ to be locally small and it is usually not the case.

Example 1.3.127. The category of variable sets $\mathbf{Set}^{\rightarrow}$, the category of dynamical spaces $\mathbf{Set}^{\circlearrowright}$ and the category of G-actions are the functor categories $\mathbf{Set}^{\mathbf{2}}$, $\mathbf{Set}^{\mathbf{S}}$ and $\mathbf{Set}^{G}$, respectively, where $\mathbf{S}$ is the following category:

$$\begin{tikzcd} \bullet \arrow[loop above] \end{tikzcd}$$

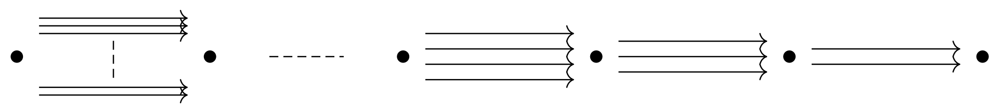
Note that $\mathbf{Set}^{\circlearrowright}$ is just $\mathbf{Set}^{(\mathbb{N},+)}$. As two other examples, note that $\mathcal{C}^{\rightarrow}$ is the functor category $\mathcal{C}^{\mathbf{2}}$ and if we consider the set $n = \{0, \cdots, n-1\}$ as a discrete category, $\mathcal{C}^n$ is essentially the same as the category $\mathcal{C} \times \mathcal{C} \times \cdots \mathcal{C}$, where the number of $\mathcal{C}$'s is n.

Philosophical Note 1.3.128. There is a philosophical shift in considering functor categories, as it treats functors or more philosophically "construction methods" as the objects of the discourse, themselves.

Example 1.3.129. The category of quivers is the functor category $\mathbf{Set}^{\rightrightarrows}$. In a similar way, the category of 2-quivers is $\mathbf{Set}^{\Delta_2^{nd}}$, where Δ_2^{nd} is the following category:

$$\bullet \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; \bullet \;\rightrightarrows\; \bullet$$

Similarly, we can imagine the category of n-quivers as $\mathbf{Set}^{\Delta_n^{nd}}$, where Δ_n^{nd} is the following category:



with $n+1$ objects and $i+1$ primitive morphisms between the ith and $i+1$th objects, counted from the right. What is the category of ∞-quivers?

The previous examples lead to a general notion of diagram. Intuitively, a diagram is a set of objects together with a set of morphisms between them in a given category $\mathcal{C}$. More formally, though:

Definition 1.3.130. Let $\mathcal{J}$ and $\mathcal{C}$ be two categories. Then, a functor $D : \mathcal{J} \to \mathcal{C}$ is called a diagram in $\mathcal{C}$ with shape $\mathcal{J}$ or a $\mathcal{J}$-diagram in $\mathcal{C}$. Therefore, the functor category $\mathcal{C}^{\mathcal{J}}$ is called the category of diagrams in $\mathcal{C}$ with shape $\mathcal{J}$.

Example 1.3.131. (*Algebra*) What is an algebraic construction, only using the algebraic concepts? It is reasonable to assume that an algebraic construction, whatever it is, must be available for all the algebras in consideration and it must respect the algebraic maps. In this sense, if we choose the category $\mathcal{R}$ as the world of algebra, then the functor category $\mathbf{Set}^{\mathcal{R}}$ can be considered as the world of all algebraic constructions. In this category we have all V_I's (the roots of the polynomial equations in I). In this sense, V_I may be considered as the extension of the set of integers by the roots of the given polynomials in I. This mindset is the extension of the usual approach of extending the number systems by adding the solutions of the equations and hence we can think of $\mathbf{Set}^{\mathcal{R}}$ as the *ultimate completion* of the algebra $\mathbb{Z}$. Interestingly, there are more algebraic notions than what we get by adding the roots of polynomials. For instance, the functor $\mathbb{P} : \mathcal{R} \to \mathbf{Set}$ defined by $\mathbb{P}(R) = \{L \subseteq R^2 \mid L \text{ is a line}\}$ and $\mathbb{P}(f)(L) = f(L)$ is a functor, where by a line $L \subseteq R^2$ we mean the set of the roots of a linear equation $ax+by = 0$, for $a, b \in R$ and by $f(L)$ we mean the line define by the equation $f(a)z + f(b)w = 0$. We have to check that $\mathbb{P}$ is well-defined, as the equation of a line is not uniquely determined by the line itself. However, it is easy to see that the equations $ax + by = 0$ and $cx + dy = 0$ define the same line iff $(a, b) = \lambda(c, d)$, for some $\lambda \in R$. This proves that $\mathbb{P}$ is well-defined. The functor $\mathbb{P}$ corresponds to the projective space $\mathbb{P}(\mathbb{Z})$, which is again a completion of $\mathbb{Z}$ by adding the *points at infinity* it lacks.

Example 1.3.132. (*Topology*) Let $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be the unit circle with its usual topology. First, let us show that it is impossible to find a continuous way to compute the angle between the point $a \in \mathbb{S}^1$ as a vector and the positive part of the x-axis. Formally, it means that there is no continuous function $\Theta : \mathbb{S}^1 \to \mathbb{R}$ such that:

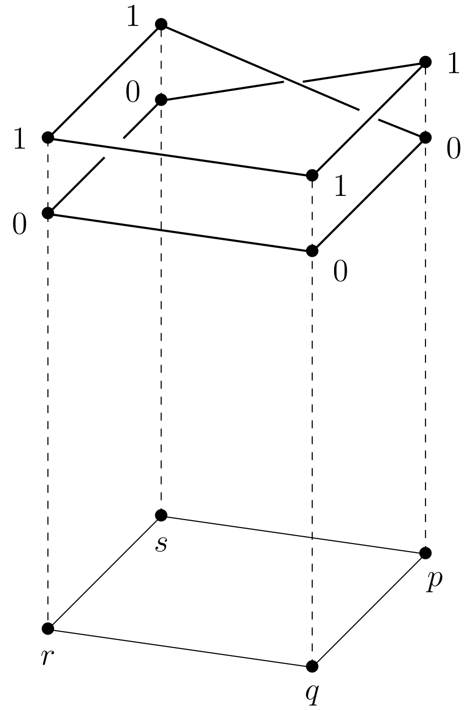
$$\begin{tikzcd} \mathbb{S}^1 \arrow[r, "\Theta"'] \arrow[rr, bend left, "id_{\mathbb{S}^1}"] & \mathbb{R} \arrow[r, "p"'] & \mathbb{S}^1 \end{tikzcd}$$

where $p : \mathbb{R} \to \mathbb{S}^1$ is the continuous function $p(\theta) = (cos\theta, sin\theta)$, mapping an angle to its corresponding point. The reason is again the argument we used for Brouwer's fixed point theorem. Since $\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$ is a functor, if we pick an arbitrary $a \in \mathbb{S}^1$, we have:

$$\begin{tikzcd} \pi_1(\mathbb{S}^1, a) \arrow[r, "\pi_1(\Theta)"'] \arrow[rr, bend left, "id_{\pi_1(S_1,a)}"] & \pi_1(\mathbb{R}, \Theta(a)) \arrow[r, "\pi_1(p)"'] & \pi_1(\mathbb{S}^1, a) \end{tikzcd}$$

which is impossible, as $\pi_1(\mathbb{S}^1, a)$ is isomorphic to $\mathbb{Z}$, while $\pi_1(\mathbb{R}, \Theta(a))$ is a singleton group.
Although, we just provided a proof, it feels paradoxical that a such continuous map does not exist. The reason is that if we restrict ourselves to a local neighborhood U of a point on $\mathbb{S}^1$, there is clearly a continuous angle map on U and since the continuity is a local notion, we expect to have a continuous map in the end. What is wrong? The problem is that the angle is continuous as long as we consider it as a multi-valued function. Let us explain why by Starting from $(1, 0)$ and moving along the circle counterclockwise. If we set the angle zero at the beginning, then it continuously grows from zero to 2π. Reaching the starting point again, if we want to remain continuous, the angle should be 2π which is impossible, as it has been set to zero before. The space is too *twisted* to have a continuous single-valued angle. To capture the true nature of the angle function, we must accept that it really is a multi-valued function, defined as an assignment mapping the point $a \in \mathbb{S}^1$ to the set $\{\theta \in \mathbb{R} \mid p(\theta) = a\}$. Now, based on the argument we had, we expect Θ to be continuous. But, what does it mean to have a continuous set-valued function? Here is an idea. For the usual functions, we can observe that they are continuous iff their restrictions to the subspaces of the space can be *glued* together. We can use the same idea here to say that a set-valued function is continuous if its restrictions to the subspaces of the space can be glued together in a reasonable generalized sense. For now, our machinery is not mature enough to talk about this gluing notion. However, we are ready to appreciate the fact that this generalized notion of continuity, whatever it is, needs the set-valued angle function to be defined on all subspaces of the space $\mathbb{S}^1$ and not just on the points. In our case, the natural definition is $\Theta : P(\mathbb{S}^1)^{op} \to \mathbf{Set}$ defined by $\Theta(X) = \{f : X \to \mathbb{R} \mid pf = id\}$. This Θ is a functor, if we map the inclusion function in $P(\mathbb{S}^1)$ to the restriction function in $\mathbf{Set}$. Hence, it is reasonable to think of the category $\mathbf{Set}^{P(\mathbb{S}^1)^{op}}$ as the world of all multi-valued functions inside of which the world of continuous multi-valued functions exists.

Example 1.3.133. (*Logic*) Let $\Phi = \{p \leftrightarrow q, q \leftrightarrow r, r \leftrightarrow s, s \leftrightarrow \neg p\}$ be a set of formulas. Clearly, Φ is inconsistent and has no models. Similar to the previous example, here again, the situation is a bit paradoxical. First, the set is *locally consistent* in the following sense: for any proper subset X of the set $\{p, q, r, s\}$, the part of Φ that constructed only from the atoms in X, denoted by Φ_X, is consistent. Secondly, if a valuation does not satisfy the whole set, it must behave inconsistent at some atom, where it must be forced to both zero and one. Hence, the inconsistency must be a local notion, while the set is locally consistent. To see how it is similar to the previous example, let us try to find a model for Φ. If we set the value $a \in \{0, 1\}$ for the atom p, then to satisfy Φ, the atoms q, r and s must have the same value a. Then, reaching p again, we can see that it must have the value $1 - a$ to remain consistent while the value has been set to a. The set Φ is too *twisted* to have a single-valued model:



Again, one can say that Φ has a model, but this model is multi-valued. To capture that multi-valued nature, we must use functors again. Define the generalized model, not only on points, but also on all subsets. We have $V : P(\{p, q, r, s\})^{op} \to \mathbf{Set}$ defined by $V(X) = \{v : X \to \{0, 1\} \mid v \text{ satisfies } \Phi_X\}$. This is again a functor. Hence, it is reasonable to think of $\mathbf{Set}^{P(\{p,q,r,s\})^{op}}$ as the world of all generalized models for the formulas constructing from these atoms.

Philosophical Note 1.3.134. One might object that as joyful as the previous approach to the inconsistencies is, it is simply empty, as it is actually impossible to have a real inconsistency in the real world. First, in our weak defence, it is worth mentioning that in practice, it usually happens that we have some local mistakes in some extremely huge database and we obviously do not want to get rid of the whole dataset because of a local mistake probably even in some other far way parts of our database. This twisted valuations is a natural way to handle such locally consistent yet globally inconsistent database. In our strong defence, though, these inconsistencies really happen in the nature and even better, the previous example is the logical version of a real situation. More precisely, assume that p, q, r and s are four quantum bits in a way that $\{p, q\}$, $\{q, r\}$, $\{r, s\}$ and $\{s, p\}$ are co-measurable, while it is impossible to measure all the quantum bits altogether. One may object that this does not solve the problem, as we can measure any two co-measurable bits to see that the value of p must be both zero and one. There are two ways to explain that. First, that the quantum bits and hence the physical quantities do not have any objective value, independent from the context and the measurements we do to observe them. Therefore, in different measurements, the quantum bit value may become zero or one. More provocatively, we can solve the inconsistency by saying that *the objective real world does not exist*. The second approach, though is that to accept the new generalized valuations as some sort of new reality. In this apparently better scenario, we might say that our usual models for reality are insufficient and we must simply model the world by these multi-valued quantities. The price to pay is now the non-locality of the reality, as these new models are global and twisted.

Example 1.3.135. (*Set Theory*) One of the prominent foundational paradigms in mathematics is Brouwerian intuitionism. Among many other things, the paradigm believes that mathematics is just a mental story told by a creative subject to herself and like any other story, this story is also changing through time by adding new constructions and proving new properties. In this sense, the truth in mathematics is temporal and dynamic and hence can be characterized by our variable sets in $\mathbf{Set}^{\mathcal{C}}$, where $\mathcal{C}$ is a suitable category that encodes the growth of time. There are many valid formalizations of this notion of time. For instance, the simplest formalization that comes to mind is the set of all natural numbers and its usual order encoding the instances and the arrow of time. However, in this example we focus on Brouwer's own understanding of time:

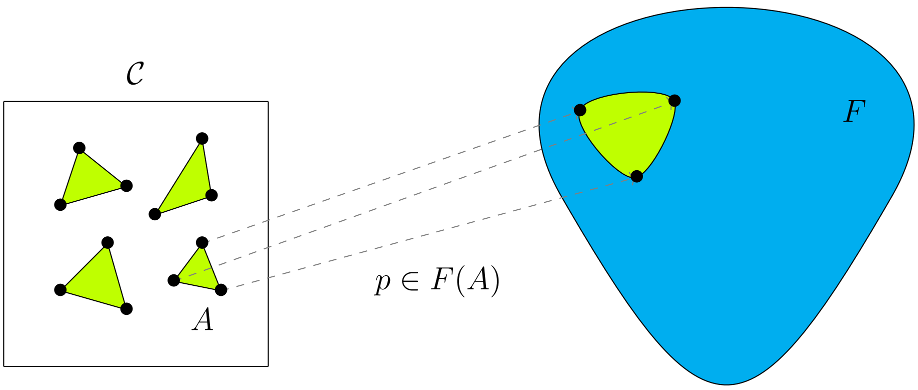
> *This perception of a move of time may be described as the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the twoity thus*

born is divested of all quality, it passes into the empty form of the common substratum of all twoities. And it is this common substratum, this empty form, which is the basic intuition of mathematics. [**?**]

To formalize this notion of time, we use $[n] = \{0, 1, \dots, n-1\}$, for $n \geq 0$, as the objects to represent the nth moment of time and for any $n \leq m$, we define the morphisms from $[n]$ to $[m]$ as a function $f : [m] \to [n]$ where $f(i) = i$, for any $i < n$. The equation $f(i) = j$ represents the creation process of the moments by encoding the fact that the moment i has been created from the moment j. Therefore, the condition $f(i) = i$ just says that when we are at the nth moment, the moment $i < n$ is fixed throughout the creation process and only the moments greater than or equal to n are newly created. As it is expected, the category $\mathbf{Set}^{\mathcal{C}}$ leads to an interesting intuitionistic dynamic version of sets. What is surprising, though, is the fact that some of these variable growing sets are in some sense *completed* and the category of these completed sets satisfies all classical axioms of set theory except the axiom of choice. Hence, it can serve as a model to prove the unprovability of the axiom of choice from **ZF**.

1.4 Baby Erlangen extended

How to interpret the objects of the category $\mathbf{Set}^{\mathcal{C}^{op}}$? We saw that a functor $F : \mathcal{C}^{op} \to \mathbf{Set}$ is a $\mathcal{C}^{op}$-variable set or a realization of the category $\mathcal{C}^{op}$ using the usual concrete sets. Now, we add another interpretation to the league. Interpret $\mathcal{C}$ as the category of some sort of interesting yet simple objects and then read a functor $F : \mathcal{C}^{op} \to \mathbf{Set}$ as an ideal object identifiable by the set of the "maps" going from the simple object A in $\mathcal{C}$ to the ideal object F. Note that the category $\mathcal{C}$ is considered to be too small with too simple objects to have the ideal object F and hence the set $F(A)$ of "maps" from A to F is not a priori meaningful. However, whatever these sets are, they must behave in a functorial way and hence it is reasonable to think of any functor as the way we describe the ways the category of lenses in $\mathcal{C}$ looks inside of F:



To have an intuitive example, we can think of $\mathcal{C}$ as the category with a single object $\mathbb{R}$ and continuous functions over it. Then, we can interpret $\mathcal{C}$ as the category consisting of one flat one-dimensional line and the new ideal object as the circle $\mathbb{S}^1$ that is not flat and hence lives outside of $\mathcal{C}$. However, as the circle is locally homeomorphic with $\mathbb{R}$, we can identify it by the continuous functions from $\mathbb{R}$ to it. In other words, if I know all possible maps from $\mathbb{R}$ to $\mathbb{S}^1$, then I know the space $\mathbb{S}^1$.
Now, as we interpret a functor F as a generalized ideal object $\mathcal{C}$-object, it is reasonable to replace even the simple objects of $\mathcal{C}$ by the functors that capture their behavior. In other words, if functors are ideal objects, the usual objects must be among them. as well. This is what the Yoneda functor does:

Definition 1.4.136. (*Yoneda functor*) Let $\mathcal{C}$ be a locally small category. Define the Yoneda functor $y : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{op}}$ by $y_A = Hom(-, A)$ on objects and on the morphism $f : A \to B$ by $y_f : Hom(-, A) \to Hom(-, B)$, where $(y_f)_C : Hom(C, A) \to Hom(C, B)$ defined by $(y_f)_C(g) = fg$. A functor $F : \mathcal{C}^{op} \to \mathbf{Set}$ is called representable, if there exists an object A in $\mathcal{C}$ such that $F \cong y_A$.

Theorem 1.4.137. *The Yoneda functor is actually a functor.*

Proof. First, recall that the map y_f is a natural transformation, for any map $f : A \to B$, as we have:

$$\begin{tikzcd}
Hom(D, A) \arrow[r, "(y_f)_D = f\circ(-)"] \arrow[d, "Hom(g,A)=(-)\circ g"'] & Hom(D, B) \arrow[d, "Hom(g,B)=(-)\circ g"] \\
Hom(C, A) \arrow[r, "(y_f)_C = f\circ(-)"'] & Hom(C, B)
\end{tikzcd}$$

Now, to prove that y is a functor, we need to show that $y_{id} = id$ and $y_{fg} = y_f y_g$. Both claim are clear by the definition of the Yoneda functor on morphisms. $\square$

Remark 1.4.138. Changing $\mathcal{C}$ to $\mathcal{C}^{op}$, it is equally natural to have the dual functor $y^{(-)} : \mathcal{C}^{op} \to \mathbf{Set}^{\mathcal{C}}$, defined by $y^A = Hom(A, -)$ and $(y^f)_C(g) = gf$. It is also customary to call a functor $F : \mathcal{C} \to \mathbf{Set}$ representable if $F \cong y^A$, for some object A in $\mathcal{C}$.

Example 1.4.139. The functors $id_{\mathbf{Set}} : \mathbf{Set} \to \mathbf{Set}$ and $P^\circ : \mathbf{Set}^{op} \to \mathbf{Set}$ are representable, because $id_{\mathbf{Set}} \cong Hom(1, -)$ and $P^\circ \cong Hom(-, \{0, 1\})$.

Example 1.4.140. The forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ is representable, because $U \cong Hom(1, -)$, where $1 = \{0\}$ is the trivial topological space. Also, the functor $\mathcal{O} : \mathbf{Top}^{op} \to \mathbf{Set}$ defined on objects by $\mathcal{O}(X)$ as the set of the open subsets of X and on morphisms by $\mathcal{O}(f) = f^{-1}$, is representable, because $\mathcal{O} \cong Hom(-, S)$, where S is the *Serpienski space* that is the space $\{0, 1\}$ with the opens $\{\emptyset, \{1\}, \{0, 1\}\}$.

Example 1.4.141. The forgetful functor $U : \mathbf{Mon} \to \mathbf{Set}$ is representable, because $U \cong Hom(\mathbb{N}, -)$. Similarly, the forgetful functors $V : \mathbf{Grp} \to \mathbf{Set}$ and $W : \mathbf{Vec}_{\mathbb{R}} \to \mathbf{Set}$ are representable, because $V \cong Hom(\mathbb{Z}, -)$ and $W \cong Hom(\mathbb{R}, -)$.

Example 1.4.142. Let A and B be two fixed sets. The functor $Hom(A, -) \times Hom(B, -) : \mathbf{Set} \to \mathbf{Set}$ is representable, because $Hom(A, -) \times Hom(B, -) \cong Hom(A + B, -)$.

Example 1.4.143. Let G and H be two fixed groups. The functor $Hom(-, G) \times Hom(-, H) : \mathbf{Grp}^{op} \to \mathbf{Set}$ is representable, because $Hom(-, G) \times Hom(-, H) \cong Hom(-, G \times H)$.

Example 1.4.144. The functor $T_n : \mathbf{Grp} \to \mathbf{Set}$ mapping any group G to $\{x \in G \mid x^n = e\}$ and any homomorphism to its appropriate restriction is representable, because $T_n \cong Hom(\mathbb{Z}_n, -)$.

Example 1.4.145. Let U and V be two fixed vector spaces. Then, the functor $Bilin_{U,V} : \mathbf{Vec}_{\mathbb{R}} \to \mathbf{Set}$ defined by $Bilin_{U,V}(W) = \{T : U \times V \to W \mid T \text{ is bilinear}\}$ and composition, is representable, because $Bilin_{U,V} \cong Hom(U \otimes V, -)$.

Philosophical Note 1.4.146. The last example has some special illuminating role. Pedagogically, tensor product with its relatively complex construction is hard to grasp for the newcomers. To solve this issue, sometimes it is helpful to replace its detailed uninformative construction with the functor it represents, namely $Bilin_{U,V}$. This is a point in usual Borbaki-style algebra that we need to make a shift from what the objects actually *are* to what they practically *do*. We can safely pretend that the only thing that we know about the tensor product $U \otimes V$ is that it is a vector space with the property that the linear maps going out of it *naturally* correspond to the bilinear maps going out from $U \times V$. This technique of replacing the huge construction of an object with what it does is the simplest example of what we can call the structuralism in action.

Highlighting the importance of representables, it is now natural to ask if there is a criterion to check whether a given functor is representable or not. We approach this problem slowly. First, three examples:

Example 1.4.147. The functor $\Delta_2 : \mathbf{Set} \to \mathbf{Set}$ mapping all objects to $2 = \{0, 1\}$ and all morphisms to identity is *not* representable. Because, if $\Delta_2 \cong Hom(A, -)$, then since $Hom(A, B \times C) \cong Hom(A, B) \times Hom(A, C)$, we must have $\Delta_2(B \times C) \cong \Delta_2(B) \times \Delta_2(C)$ which means $\{0, 1\} \times \{0, 1\} \cong \{0, 1\}$.

Example 1.4.148. Let G and H be two groups such that there are at least two homomorphisms from G to H. Then, the functor $Hom(- \times G, H) : \mathbf{Grp}^{op} \to \mathbf{Set}$ is *not* representable. Because, if $Hom(- \times G, H) \cong Hom(-, K)$, then since $Hom(\{e\}, K)$ has just one element, the set $Hom(\{e\} \times G, H) \cong Hom(G, H)$ must have one element which is impossible by assumption.

Example 1.4.149. The functor $Sub : \mathbf{Grp}^{op} \to \mathbf{Set}$ mapping a group to the set of its subgroups and a morphism to the inverse image is *not* representable. Because, if $Sub(-) \cong Hom(-, K)$, then since $Hom(G \oplus H, K) \cong Hom(G, K) \times Hom(H, K)$, we have to have $Sub(G \oplus H) \cong Sub(G) \times Sub(H)$. The last statement is false, because $Sub(\mathbb{Z}_2) \times Sub(\mathbb{Z}_2)$ has exactly four elements while $Sub(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ has at least five elements including all the elements of $Sub(\mathbb{Z}_2) \times Sub(\mathbb{Z}_2)$ plus the subgroup $\{(0, 0), (1, 1)\}$.

In the general situation, there is a criteria to check the representability of a functor, imitating what we saw in the previous two examples. The main idea is that the Hom functor preserves some sort of construction (in our examples product, the "smallest possible" object, and the direct sum, respectively) and if a functor is representable, it must preserve these structures, as well. We will introduce these structures to see when this preservation can be even sufficient for representability. For now, let us focus our main story of interpreting functors as ideal objects.

So far, we have provided a way to interpret the objects of $\mathcal{C}$ as the ideal objects embodied as functors. Now, we have two things to check. First, we have to make sure that the behavior of these new copies in their new world is the same as their behavior in their original world. This means that we have to show that the Yoneda functor is a full and faithful functor, also called an embedding. Secondly, if $F : \mathcal{C}^{op} \to \mathbf{Set}$ is an ideal object and if $F(A)$ encodes the set of all "maps" from A to F, then moving to the new world of $\mathbf{Set}^{\mathcal{C}^{op}}$ where there is a copy of A, namely y_A, and also there is a well-defined notion of map from this copy to F, stored in $Hom(y_A, F)$, we expect to have a canonical isomorphism between $Hom(y_A, F)$ and $F(A)$. This expectation is fortunately a theorem and it is called the Yoneda lemma. We first prove this lemma and then we will use it to prove the fullness and faithfulness of $y : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{op}}$.

Theorem 1.4.150. *(The Yoneda lemma) The functors $Hom(y_{(-)}, -) : \mathcal{C}^{op} \times \mathbf{Set}^{\mathcal{C}^{op}} \to \mathbf{Set}$ and $(-)(-) : \mathcal{C}^{op} \times \mathbf{Set}^{\mathcal{C}^{op}} \to \mathbf{Set}$ are naturally isomorphic via the maps $\alpha_{A,F} : Hom(y_A, F) \to F(A)$ and $\bar{\alpha}_{A,F} : F(A) \to Hom(y_A, F)$ defined by $\alpha_{A,F}(\beta) = \beta_A(id_A)$ and $[\bar{\alpha}_{A,F}(p)]_C(f) = F(f)(p)$. Specially, $Hom(y_A, F) \cong F(A)$, natural in A and F.*

Proof. We have to show that α and $\bar{\alpha}$ are natural transformations and for each A and F the maps $\alpha_{(A,F)}$ and $\bar{\alpha}_{A,F}$ are the inverse of each other in $\mathbf{Set}$. For the first, note that $\beta = \bar{\alpha}_{A,F}(p)$ is a natural transformation because

$$\begin{tikzcd} Hom(D,A) \arrow[r, "\beta_D"] \arrow[d, "Hom(g,A)"'] & F(D) \arrow[d, "F(g)"] \\ Hom(C,A) \arrow[r, "\beta_C"'] & F(C) \end{tikzcd}$$

But $F(g)\beta_D(f) = F(g)F(f)(p) = F(gf)(p)$. For naturality, we just check the naturality for α. The naturality of $\bar{\alpha}$ will be the result of the fact that it is the pointwise inverse of α. For α, we have to show that for any map $f : B \to A$ and any $\gamma : F \Rightarrow G$:

$$\begin{tikzcd} Hom(y_A,F) \arrow[r, "\alpha_{(A,F)}"] \arrow[d, "Hom(y_f,\gamma)"'] & F(A) \arrow[d, "G(f)\gamma_A = \gamma_B F(f)"] \\ Hom(y_B,G) \arrow[r, "\alpha_{(B,G)}"'] & G(B) \end{tikzcd}$$

It is not hard to prove the commutativity of the diagram and we will leave this tiresome task to the reader. For the second part, note that any $\beta \in Hom(y_a, F)$ is uniquely determined by $\beta_A(id_A)$. The reason is the following naturality diagram, for a map $f : C \to A$:

$$\begin{tikzcd}
Hom(A,A) \arrow[r, "\beta_A"] \arrow[d, "Hom(f,A)"'] & F(A) \arrow[d, "F(f)"] \\
Hom(C,A) \arrow[r, "\beta_C"'] & F(C)
\end{tikzcd}$$

which implies that for any $f : C \to A$, we have $\beta_C(f) = F(f)(\beta_A(id_A))$. This shows that $\alpha_{\bar{A},F}\alpha_{A,F}(\beta) = \bar{\alpha}_{A,F}(\beta_A(id_A)) = \beta$ as both $\bar{\alpha}(\beta_A(id_A))$ and β on C and f are $\beta_C(f) = F(f)(\beta_A(id_A))$. For the converse, we simply have $\alpha_{A,F}\bar{\alpha}_{A,F}(p) = F(id_A)(p) = p$. □

Corollary 1.4.151. *(The Yoneda embedding) The functor $y : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{op}}$ is full and faithful.*

Proof. By the Yoneda lemma, the map $\bar{\alpha}_{A,y_B} : y_B(A) = Hom(A,B) \to Hom(y_A, y_B)$ is a natural isomorphism. Computing $\bar{\alpha}_{A,y_B}$, we see that

$$[\bar{\alpha}_{A,y_B}(f)]_C(g) = y_B(f)(g) = fg = [y_f]_C(g),$$

for any C and $g : C \to A$. Hence, $\bar{\alpha}_{A,y_B}(f) = y_f$. Therefore, the map $y_{(-)} : Hom(A,B) \to Hom(y_A, y_B)$ is a bijection which means that $y : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{op}}$ is a full and faithful functor. □

Philosophical Note 1.4.152. Note that this embedding is a representation theorem stating that any abstract category can be seen as a category of variable sets. This is useful, as the category $\mathbf{Set}^{\mathcal{C}^{op}}$ is a category of sets with set-like behavior. Hence, whenever we want to investigate something about $\mathcal{C}$, we can embed it into $\mathbf{Set}^{\mathcal{C}^{op}}$ to have enough set-theoretic machinery. Then, if we finally reach a representable functor, we can come back to the original category we started with.

Corollary 1.4.153. *(Uniqueness of the representing object) $y_A \cong y_B$ iff $A \cong B$. The same holds for $y^{(-)}$.*

Proof. Since the Yoneda functor is full and faithful and for any such functor F, we have $F(A) \cong F(B)$ iff $A \cong B$, the claim follows. □

Philosophical Note 1.4.154. From the philosophical point of view, the uniqueness of the representing object means that the relative data of an object is enough to identify it. Therefore, whenever it is convenient, we forget the object and work with its functor.

Example 1.4.155. Using the relative behavior of tensor products, we prove that it is commutative, i.e., $U \otimes V \cong V \otimes U$ and $\mathbb{R} \otimes V \cong V$. We have

$$Hom(U \otimes V, W) \cong Bilin_{U,V}(W) \cong Bilin_{V,U}(W) \cong Hom(V \otimes U, W)$$

natural in W. Hence, $y^{U \otimes V} \cong y^{V \otimes U}$ which implies $U \otimes V \cong V \otimes U$. With the same line of reasoning, we have

$$Hom(\mathbb{R} \otimes V, W) \cong Bilin_{\mathbb{R},V}(W) \cong Hom(V, W)$$

natural in W. Hence, $y^{\mathbb{R} \otimes V} \cong y^{V}$ which implies $\mathbb{R} \otimes V \cong V$.

We will see more applications later, but first, we want to use our new machinery to define some new categorical objects by identifying the relative behavior that we expect them to have. Then, the uniqueness of the representing object ensures that the defined object is unique up to isomorphism. To that purpose, it is convenient to provide an equivalent characterization of the representable functors by one of the core notions of category theory, namely the *universality*.

Theorem 1.4.156. *(Universal elements) A functor $F : \mathcal{C}^{op} \to$ **Set** is representable iff there exists an object A in $\mathcal{C}$ and an element $a \in F(A)$ such that for any object B and any element $b \in F(B)$, there exists a unique $f : B \to A$ such that $F(f)(a) = b$. The object A and the element $a \in F(A)$ are called the universal object and the universal element, respectively.*

Proof. By Yoneda lemma, an element $a \in F(A)$ corresponds to the natural transformation $\beta : y_A \Rightarrow F$, defined by $\beta_C(f) = F(f)(a)$. Note that β is a natural isomorphism iff β_C is an isomorphism for all C. The latter is exactly what the universality condition says. $\square$

Philosophical Note 1.4.157. If we read F as a structured set, then $a \in F(A)$ may be interpreted as the *generic point* of *the generic structure* that can act as all structures and all elements generically.

Remark 1.4.158. Note that the universal pair (A, a) if exists is unique up to isomorphism, i.e., if both (A, a) and (B, b) are universal for F, then there exists an isomorphisms $f : B \to A$ such that $F(f)(a) = b$. Why?

Example 1.4.159. For the functor $P^\circ : \mathbf{Set}^{op} \to \mathbf{Set}$, the universal element is $\{1\} \in P(\{0,1\})$. The universality condition states that any set $U \in P(X)$ is obtainable by applying $P^\circ(f) = f^{-1}$ on $\{1\}$, for a unique $f : X \to \{0,1\}$. This unique function is the characteristic function of U in X.

Example 1.4.160. For the forgetful functors $U : \mathbf{Mon} \to \mathbf{Set}$, $V : \mathbf{Grp} \to \mathbf{Set}$ and $W : \mathbf{Vec}_{\mathbb{R}} \to \mathbf{Set}$, the universal elements are $1 \in U(\mathbb{N}) = \mathbb{N}$, $1 \in V(\mathbb{Z}) = \mathbb{Z}$ and $1 \in W(\mathbb{R}) = \mathbb{R}$. We just explain the case of monoids. The reason is that for any element $m \in U(M)$, there exists a unique monoid homomorphism $f : \mathbb{N} \to M$ such that $U(f)(1) = f(1) = m$.

Example 1.4.161. For the functor $Hom(A,-) \times Hom(B,-) : \mathbf{Set} \to \mathbf{Set}$, the universal element is $(i_0, i_1) \in Hom(A, A+B) \times Hom(B, A+B)$, where $i_0 : A \to A+B$ is defined by $i_0(a) = (0,a)$ and $i_1 : B \to A+B$ is defined by $i_1(b) = (1,b)$. The reason is that for any set C and any element $(f,g) \in Hom(A,C) \times Hom(B,C)$, there exists a unique map $h : A+B \to C$ such that $[Hom(A,h) \times Hom(B,h)](i_0,i_1) = (hi_0, hi_1) = (f,g)$, i.e.,

$$\begin{tikzcd}
 & D & \\
A \arrow[ur, "f"] \arrow[r, "i_0"'] & A+B \arrow[u, "h"] & B \arrow[ul, "g"'] \arrow[l, "i_1"]
\end{tikzcd}$$

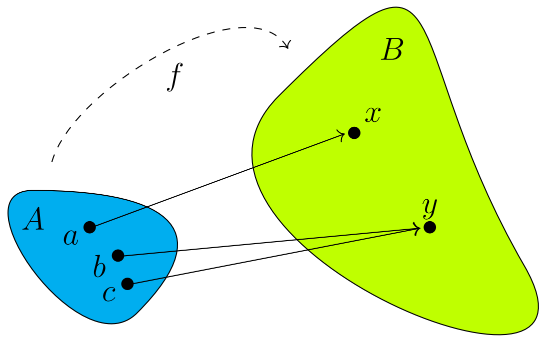
Example 1.4.162. For the functor $Hom(-,G) \times Hom(-,H) : \mathbf{Grp}^{op} \to \mathbf{Set}$, the universal element is $(p_0,p_1) \in Hom(G \times H, G) \times Hom(G \times H, H)$, where p_0 and p_1 are the projections. The reason is that for any group K and any element $(f,g) \in Hom(K,G) \times Hom(K,H)$, there exists a unique map $h : K \to G \times H$ such that $[Hom(h,G) \times Hom(h,H)](p_0,p_1) = (p_0h, p_1h) = (f,g)$, i.e.,

$$\begin{tikzcd}
 & K \arrow[dl, "f"'] \arrow[d, "h"] \arrow[dr, "g"] & \\
G & G \times H \arrow[l, "p_0"] \arrow[r, "p_1"'] & H
\end{tikzcd}$$

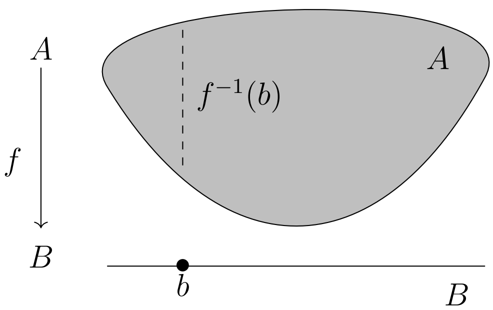
Example 1.4.163. For the functor $Bilin_{U,V} : \mathbf{Vec}_{\mathbb{R}} \to \mathbf{Set}$, the universal element is $i \in Bilin(U \otimes V)$, where $i : U \times V \to U \otimes V$ is defined by the bilinear function $i(u,v) = u \otimes v$. The universality condition says that for any element $f : Bilin_{U,V}(W) = \{f : U \times V \to W \mid f \text{ is bilinear}\}$, there exists a unique linear map $g : U \otimes V \to W$ such that $Bilin(g)(i) = gi = f$, i.e.,

$$\begin{tikzcd}
U \times V \arrow[d, "i"'] \arrow[dr, "f"] & \\
U \otimes V \arrow[r, "g"'] & W
\end{tikzcd}$$

Philosophical Note 1.4.164. There are two ways to interpret a function $f : A \to B$ in **Set**. First, as an A-indexed element of B or simply an A-element of B, reading a parameter $a \in A$ to output $f(a) \in B$:



Here we are labelling the elements of B by A. In the second interpretation, we read a map $f : A \to B$ as a B-indexed family of subsets of A, a B-subset of A or just a fibration over B, mapping $b \in B$ to the set (fiber) $f^{-1}(b) \subseteq A$:



Here we are stacking the elements of A by B. Thanks to Yoneda embedding, it is reasonable to lift these interpretations to any arbitrary category, by interpreting objects as variable sets and morphisms as variable functions. This way, we can interpret a map $f : A \to B$ as some sort of A-element of B, reading a parameter $a : X \to A$ to output $fa : X \to B$ or as some sort of B-part of A or a fibration over B, reading a parameter $b : X \to B$ to output the fiber $\{a : X \to A \mid fa = b\}$.

These two interpretations are useful in different settings. Usually, in a category, we have some small simple known objects and to know any arbitrary object A, we investigate the maps to/from A from/to these simple objects. For instance, in geometry, we investigate a geometrical object by the maps *from* the Euclidean cubes or the higher dimensional balls *into* it, while in algebra, we study an algebraic object by more relations we can put on its elements transforming the algebra *to* simpler algebras of the same kind. These two dual approaches is what distinguish geometrical from algebraic way of thinking. In some cases, it is possible to see both of the approaches at the same time. For instance, living in **Set**, as $\{0\}$ and $\{0, 1\}$ are simple, we can study X *geometrically* by all the maps going from $\{0\}$ to X, i.e., its elements, while investigating X by the maps from X to $\{0, 1\}$ is the *algebraic* study of X via the boolean algebra of its subsets. A similar situation happens in algebraic geometry, logic and functional analysis. In the first, we can study a polynomial equation either by working in the polynomial algebra modulo the equation or by the zeros the equation has in some choice of simple rings such as algebraically closed fields. In logic we have syntax versus semantics and in functional analysis we can study a topological space either by looking inside the topology or by working with its function algebra as the world of measurable quantities over the space.

Finally, note that using these two interpretations, if we interpret A as our interesting object in a category $\mathcal{C}$, the slice category $\mathcal{C}/A$ is the category of all fibration over A, while the coslice category $A/\mathcal{C}$ is the category of A-enhanced objects having a distorted copy of A inside.

Now, we are ready to define some categorical constructions by representability or equivalently by universality.

Definition 1.4.165. An object A is called terminal if it represents the functor $\Delta_1 : \mathcal{C}^{op} \to \mathbf{Set}$, i.e., $Hom(B, A) \cong \{0\}$, natural in B. Equivalently, A is terminal if for any B, there exists a unique map form B to A. Since this object is unique up to isomorphism, we denote it by 1.

Example 1.4.166. In categories **Set**, **Grp**, **Ab**, $\mathbf{Vect}_{\mathbb{R}}$ and **Cat**, the terminal object exits and is $\{0\}$, interpreted respectively. In a poset $(P, \leq)$, the terminal object is by definition an element $a \in P$ such that for any $b \in P$, we have $b \leq a$. Hence, the terminal object is the greatest element of the poset. Any non-trivial monoid as a category does not have a terminal object, because if the only object of a monoid is terminal, then there must be exactly one morphism over that object.

Example 1.4.167. In the category $\mathcal{C}/A$, the terminal object is $id_A : A \to A$, as for any object $g : B \to A$, there is exactly one morphisms $g : B \to A$ such that $id_A g = f$ and that morphism is f itself.

$$\begin{tikzcd} B \arrow[rr, "g"] \arrow[dr, "f"'] & & A \arrow[dl, "id_A"] \\ & A & \end{tikzcd}$$

In $\mathbf{Set}/A$ the terminal object $id_A : A \to A$ corresponds to the fibration $a \mapsto \{a\}$.

Example 1.4.168. In the category $\mathbf{Set}^{\mathcal{C}^{op}}$, the terminal object is $\Delta_1 : \mathcal{C}^{op} \to \mathbf{Set}$, as for any functor $F : \mathcal{C}^{op} \to \mathbf{Set}$, there is exactly one natural transformation $\alpha : F \Rightarrow \Delta_1$, where $\alpha_C : F(C) \to \{0\}$ maps everything to 0.

Definition 1.4.169. Let A and B be two objects. An object C together with a natural isomorphism $\alpha : Hom(-, C) \cong Hom(-, A) \times Hom(-, B)$ is called a product of A and B. Equivalently, C together with two morphisms $p_0 : C \to A$ and $p_1 : C \to B$ is called a product if for any object D and any morphisms $f : D \to A$ and $g : D \to B$, there exists a unique map $h : D \to C$ such that:

$$\begin{tikzcd} & D \arrow[dl, "f"'] \arrow[d, dashed, "h"] \arrow[dr, "g"] & \\ A & C \arrow[l, "p_0"] \arrow[r, "p_1"'] & B \end{tikzcd}$$

The product of A and B is denoted by $A \times B$. It is possible to extend products from the binary case to any arbitrary family. More precisely, if I is a set and $\{A_i\}_{i \in I}$ is a family of objects in $\mathcal{C}$, by their product we mean an object C together with a natural isomorphism $\alpha : Hom(-, C) \cong \Pi_{i \in I} Hom(-, A_i)$. Equivalently, it is an object C with maps $p_i : C \to A_i$ such that for any other family of maps $f_i : D \to A_i$, there exists a unique map $h : D \to C$ such that $p_i h = f_i$, for any $i \in I$. The product of $\{A_i\}_{i \in I}$ is denoted by $\Pi_{i \in I} A_i$.

Example 1.4.170. In categories **Set**, **Top**, **Grp**, **Ab**, $\mathbf{Vect}_{\mathbb{R}}$ and **Cat**, the product is the usual product. In a poset $(P, \leq)$, the product of a family $\{a_i\}_{i \in I}$ is by definition the greatest lower bound of $\{a_i\}_{i \in I}$ i.e., an element c such that $c \leq a_i$ for all $i \in I$ and for any $d \in P$ if $d \leq a_i$ for all $i \in I$ then $d \leq c$. For the prototype posets, namely posets of subsets of X with inclusion, if they are closed under arbitrary intersection, the intersection of a family of subsets will be the product of the subsets. Products in posets are usually called meets and denoted by $\bigwedge$ or for finite families with $\wedge$. For the unique object $*$ in a non-trivial finite monoid as a category, even the binary product $* \times *$ does not exists, because if it does, it must be $*$ and we must have:

$$M = Hom(*, *) \cong Hom(*, * \times *) \cong Hom(*, *) \times Hom(*, *) = M \times M$$

which is impossible.

Philosophical Note 1.4.171. When one sees the product topology for the first time, one may wonder why such a topology and its bias towards using only finite proper opens in the basis elements $\Pi_{i \in I} U_i$ is natural. Here is the answer. The product together with this topology is *the* product. For us, behaving as a product has a clear structural meaning and the object that represents this behavior may incarnate in many different forms in the different contexts. In **Top** this topology is what we have to use to have the product. Its construction, though, is secondary to what it must perform.

Example 1.4.172. (*Pullback*) What is a binary product of two objects $f : B \to A$ and $g : C \to A$ in $\mathcal{C}/A$? It is an object $h : D \to A$ and two morphism $p_0 : D \to B$ and $p_1 : D \to C$ such that:

$$\begin{tikzcd}
 & D \arrow[dl, "p_0"'] \arrow[dd, "h"] \arrow[dr, "p_1"] & \\
B \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\
 & A &
\end{tikzcd}$$

and for any other object $e : E \to A$ and any morphisms q from e to f and r

from e to g, there exists a unique map from E to D such that:

$$
\begin{tikzcd}
 & E \arrow[dl, bend right, "q"'] \arrow[d, dashed] \arrow[dr, bend left, "r"] & \\
 & D \arrow[dl, "p_0"'] \arrow[dr, "p_1"] \arrow[dd, "h"] & \\
B \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\
 & A &
\end{tikzcd}
$$

Usually people write this data as:

$$
\begin{tikzcd}
E \arrow[dr, dashed] \arrow[drr, bend left, "r"] \arrow[ddr, bend right, "q"'] & & \\
 & D \arrow[r, "p_1"] \arrow[d, "p_0"'] & C \arrow[d, "g"] \\
 & B \arrow[r, "f"'] & A
\end{tikzcd}
$$

and call the square a pullback square, p_0 a pullback of g along f and p_1 a pullback of f along g. The pullback is also called the fiber product as it is actually the product in the category of fibrations over A. Sometimes, the object D itself is loosely called the pulllback and it is denoted by $B \times_A C$.

Example 1.4.173. All pullbacks exist in the category **Set**. More precisely, for the two functions $f : B \to A$ and $g : C \to A$, the pullback is $B \times_A C = \{(b, c) \in B \times C \mid f(b) = g(c)\}$ with the projection maps. Reading the data as fibrations, the fiber corresponding to $B \times_A C$ over $a \in A$ is nothing but $f^{-1}(a) \times g^{-1}(a)$ that is the pointwise product of fibers.

Example 1.4.174. In the category $\mathbf{Set}^{\mathcal{C}^{op}}$, the product of $E : \mathcal{C}^{op} \to \mathbf{Set}$ and $F : \mathcal{C}^{op} \to \mathbf{Set}$ is defined pointwise, i.e., $(E \times F)(A) = E(A) \times F(A)$ and $(E \times F)(f) = E(f) \times F(f) : E(B) \times F(B) \to E(A) \times F(A)$, for any $f : A \to B$ in $\mathcal{C}$. The projections $p_0 : E \times F \Rightarrow E$ and $p_1 : E \times F \Rightarrow F$ are also defined pointwise, i.e., $(p_0)_C : E(C) \times F(C) \to E(C)$ by projection on the first element and similarly for p_1.

Example 1.4.175. (*Non-existence of terminal objects and binary products*) For an easier example, consider the poset $(\mathbb{N}, \leq)$. This poset has no greatest element and hence no terminal object. For product, take the poset $(P, \subseteq)$ of all infinite subsets of $\mathbb{N}$. Then, the product (meet) of the set E of the even numbers and O of the odd numbers does not exists, as there is no infinite set below both of them. For a more interesting example, take the category of fields. This category has no terminal object, because if F is terminal, for any other field E, there must be a map from E to F. However, any map between two fields is one-to-one and hence F must have the maximum cardinality between all fields which is impossible. The binary product also does not exist. For instance, if the field $F = \mathbb{Q} \times \mathbb{Z}_p$ exists, then it has two maps one into $\mathbb{Q}$ and one into $\mathbb{Z}_p$. Since $p \cdot 1 = 0$ in $\mathbb{Z}_p$ and the maps are one-to-one, we must have $p \cdot 1 = 0$ in F and hence in $\mathbb{Q}$ which is impossible. Restricting fields to a fixed characteristic p can not solve the problem. It is enough to pick a field F with a non-identity endomorphism $e : F \to F$. (For $p = 0$, pick $F = \mathbb{C}$ and $e(z) = \bar{z}$ and for a prime p, pick F as a filed with p^2 elements and $e(x) = x^p$. In the latter case, e is not identity as the equation $x^p = x$ has at most p roots while the field has p^2 elements). Then, we claim that $F \times F$ does not exist. If it does, call it K. Then, by the universal property of the product, there is $h : F \to K$ such that:

$$\begin{tikzcd}
& F \arrow[dl, "id_F"'] \arrow[d, dashed, "h"'] \arrow[dr, "id_F"] & \\
F & K \arrow[l, "p_0"] \arrow[r, "p_1"'] & F
\end{tikzcd}$$

Since $p_0 h = p_1 h = id_F$, both p_0 and p_1 are surjective. Since p_0 and p_1 are also one-to-one, they are bijections and hence h is a bijection. Since $p_0 h = p_1 h$, we have $p_0 = p_1$. Now, by the universal property of the product again, there must be $h' : F \to K$ such that:

$$\begin{tikzcd}
& F \arrow[dl, "id_F"'] \arrow[d, dashed, "h'"'] \arrow[dr, "e"] & \\
F & K \arrow[l, "p_0"] \arrow[r, "p_1"'] & F
\end{tikzcd}$$

But as $p_0 = p_1$ and $e \neq id_F$, this is impossible.

Definition 1.4.176. An object A is called initial if it corepresents the functor $\Delta_1 : \mathcal{C} \to \mathbf{Set}$, i.e., $Hom(A, B) \cong \{0\}$, natural in B. Equivalently, A

is initial if for any object B, there exists a unique map from A to B. The initial object is denoted by 0.

Example 1.4.177. In the category **Set** the initial object is the empty set. In **Grp** and $\mathbf{Vect}_{\mathbb{R}}$ it is $\{0\}$. In **Cat** it is the empty category. In a poset $(P, \leq)$, the initial object is by definition the least element. Any non-trivial monoid as a category does not have an initial object, because if the only object of a monoid is initial, then there must be exactly one morphism over that object.

Example 1.4.178. In the category $A/\mathcal{C}$, the initial object is $id_A : A \to A$, as for any object $f : A \to B$, there is exactly one morphisms $g : A \to B$ such that $g \circ id_A = f$. The morphism is f itself:

$$\begin{tikzcd} & A \arrow[dl, "id_A"'] \arrow[dr, "f"] & \\ A \arrow[rr, "g"'] & & B \end{tikzcd}$$

Example 1.4.179. In the category $\mathbf{Set}^{\mathcal{C}^{op}}$, the initial object is $\Delta_\emptyset : \mathcal{C}^{op} \to$ **Set** as for any functor $F : \mathcal{C}^{op} \to \mathbf{Set}$, there is exactly one natural transformation $\alpha : \Delta_\emptyset \Rightarrow F$, that is defined by $\alpha_C : \emptyset \to F(C)$, where α_C is the empty function.

Definition 1.4.180. Let A and B be two objects. An object C together with a natural isomorphism $\alpha : Hom(C, -) \cong Hom(A, -) \times Hom(B, -)$ is called a coproduct of A and B. Equivalently, C together with two morphisms $i_0 : A \to C$ and $i_1 : B \to C$ is called a coproduct if for any object D and any morphisms $f : A \to D$ and $g : B \to D$, there exists a unique map $h : C \to D$ such that:

$$\begin{tikzcd} & D & \\ A \arrow[ur, "f"] \arrow[r, "i_0"'] & C \arrow[u, dashed, "h"] & B \arrow[ul, "g"'] \arrow[l, "i_1"] \end{tikzcd}$$

The coproduct is denoted by $A + B$. It is possible to extend coproducts from the binary case to any arbitrary family. More precisely, if I is a set and $\{A_j\}_{j \in J}$ is a family of objects in $\mathcal{C}$, by their coproduct, we mean an object C together with a natural isomorphism $\alpha : Hom(C, -) \cong \Pi_{j \in J} Hom(A_j, -)$.

Equivalently, it is an object C with maps $i_j : A_j \to C$ such that for any other family of maps $f_j : A_j \to D$, there exists a unique map $h : C \to D$ such that $hi_j = f_j$, for any $j \in J$. The coproduct of $\{A_j\}_{j \in J}$ is denoted by $\Sigma_{j \in J} A_j$.

Example 1.4.181. In the category **Set**, the coproduct is the disjoint union with its injection functions. In **Ab** and $\mathbf{Vect}_{\mathbb{R}}$, coproduct equals to the product. In **Cat**, the coproduct is the coproduct we saw before. In a poset $(P, \leq)$, the coproduct of a family $\{a_i\}_{i \in I}$ is by definition the least upper bound of $\{a_i\}_{i \in I}$ i.e., an element c such that $a_i \leq c$, for all $i \in I$ and for any $d \in P$ if $a_i \leq d$, for all $i \in I$ then $c \leq d$. For the prototype posets, namely posets of subsets of X with inclusion, if they are closed under arbitrary union, the union of a family of subsets will be the coproduct of the subsets. Coproducts in posets are usually called joins and denoted by $\bigvee$ or for finite families with $\vee$. For the unique object $*$ in a non-trivial finite monoid as a category, the coproduct $* + *$ does not exists, because if it does, it must be $*$ and we must have:

$$M = Hom(*, *) \cong Hom(* + *, *) \cong Hom(*, *) \times Hom(*, *) = M \times M$$

which is impossible.

Philosophical Note 1.4.182. When one sees the finite product in **Ab** for the first time, it may be confusing why one notion has two names, diresct sum and direct product. Later, seeing the general case, one can see the difference in general that collapses in the finite case. However, one may still wonder why we need the finiteness condition in the definition of the direct sums? Similar to what we saw for product topology, we have the same thing here. The direct sum is *the* coproduct in **Ab**. For us, behaving as a coproduct has a clear structural meaning and the object that represents this behavior may incarnate in many different forms in the different contexts. In **Ab** this group is what we have to use to have the coproduct. Its construction, though, is secondary to what it must perform.

Example 1.4.183. (*Pushout*) What is a coproduct of two objects $f : A \to B$ and $g : A \to C$ in $A/\mathcal{C}$? It is an object $h : A \to D$ and two morphism $i_0 : B \to D$ and $i_1 : C \to D$ such that:

$$\begin{tikzcd}
 & A \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "h"] & \\
B \arrow[dr, "i_0"'] & & C \arrow[dl, "i_1"] \\
 & D &
\end{tikzcd}$$

and for any other object $e : A \to E$ and any morphisms q from f to e and r from g to e, there exists a unique map from D to E such that:

$$\begin{tikzcd}
 & A \arrow[dl, "f"'] \arrow[d, "h"] \arrow[dr, "g"] & \\
B \arrow[r, "i_0"'] \arrow[dr, bend right, "q"'] & D \arrow[d, dashed] & C \arrow[l, "i_1"] \arrow[dl, bend left, "r"] \\
 & E &
\end{tikzcd}$$

Usually people write this data as:

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "i_1"] \arrow[ddr, bend left, "r"] & \\
B \arrow[r, "i_0"'] \arrow[drr, bend right, "q"'] & D \arrow[dr, dashed] & \\
 & & E
\end{tikzcd}$$

and call the square a pushout square, i_1 a pushout of f along g and i_0 a pushout of g along f. The pushout is also called the cofiber coproduct as it is dual to fiber product. Sometimes, the object D itself is loosely called the pushout and it is denoted by $B +_A C$.

Example 1.4.184. All pushouts exist in the category **Set**. More precisely, for the two functions $f : A \to B$ and $g : A \to C$, the pushout is $B +_A C = B + C/\sim=$, where $\sim$ is the lest equivalence relation generated by $\{f(a) = g(a) \mid a \in A\}$ with the injection maps. Reading the data as A-enhanced sets, the pushout is nothing but the disjoint union of B and C in which the two copies of A are glued together. The same is also true for the category **Top** where $B + C/\sim$ is equipped with the quotient topology, i.e., the topology where U is open in $B + C/\sim$ if either $i_0^{-1}(U)$ is open in B and $i_1^{-1}(U)$ is open in C. As a concrete example, when $A = \{0\}$, the pushout is the notion of coproduct in the category of pointed spaces. For instance, $\mathbb{S}^1$ is the pushout of $i : \{0, 1\} \to [0, 1]$ along $i : \{0, 1\} \to [0, 1]$, where i is the inclusion function:

$$\begin{tikzcd}
\{0,1\} \arrow[r, "i"] \arrow[d, "i"'] & {[0,1]} \arrow[d] \\
{[0,1]} \arrow[r] & \mathbb{S}^1
\end{tikzcd}$$

In **Ab**, the pushout is $B \oplus C/N$, where N is the subgroup generated by $f(a) - g(a)$'s for any $a \in A$. In **CRing**, it is $B \otimes_A C$, considering B and C as A-algebras via the maps $f : A \to B$ and $g : A \to C$.

Example 1.4.185. One can thinks of pushouts as scalar extensions (cobase change) in the algebraic world as the dual of the geometric base change operation. For instance, if we have an algebras structure over a field K such as $M_n(K)$, then changing the field of scalaras from K to a greater field $L \supseteq K$ is the pushout

$$\begin{tikzcd}
K \arrow[r, hook, "i"] \arrow[d, "a \mapsto aI_n"'] & L \arrow[d, "a \mapsto aI_n"] \\
M_n(K) \arrow[r] & M_n(K) \otimes_K L = M_n(L)
\end{tikzcd}$$

Philosophical Note 1.4.186. For the newcomers in topology, the quotient topology is something complex and mysterious. The structural way of thinking makes it simpler by proposing that it is *the* gluing in the category of **Top**. The quotient topology is just secondary to the pushout role it plays. The same holds for tensor product of A-algebras. They are just the gluing of rings as A-enhanced objects.

Philosophical Note 1.4.187. Structural way of thinking is useful as it shows that gluing of pointed spaces and tensor product of A-algebras for the fixed A are the same thing. Moreover, we can see that this construction is dual to the fiber product of topological spaces. Does it mean that something geometric lives in **CRing**, dully, where tensor product plays the role of fiber product?

Example 1.4.188. In the category $\mathbf{Set}^{\mathcal{C}^{op}}$, the coproduct of $E : \mathcal{C}^{op} \to \mathbf{Set}$ and $F : \mathcal{C}^{op} \to \mathbf{Set}$ is defined pointwise, i.e., $(E + F)(A) = E(A) + F(A)$ and $(E + F)(f) = E(f) + F(f) : E(B) + F(B) \to E(A) + F(A)$, for any $f : A \to B$ in $\mathcal{C}$. The injections $i_0 : E \Rightarrow E + F$ and $i_1 : F \Rightarrow E + F$ are also defined pointwise, i.e., $(i_0)_C : E(C) \to E(C) + F(C)$ by usual set injection and similarly for i_1.

Remark 1.4.189. (*Duality*) Note that a terminal object in $\mathcal{C}$ is an initial object in $\mathcal{C}^{op}$ and the same also holds for the pair product/coproduct and pullback/pushout. In this sense, these pairs of notions are dual to each other or in its slogan form they are the same thing, *reversing the arrows.*

Example 1.4.190. (*Non-existence of initial objects and binary coproducts*) For an easier example, consider the poset $(\mathbb{Z}, \leq)$. This poset has no least element and hence no initial object. For coproduct, take the poset $(P, \subseteq)$ of all subsets of $\mathbb{N}$ whose complement is infinite. Then, the coproduct (join) of the set E of the even numbers and O of the odd numbers does not exists, as the only subset above both of them is $\mathbb{N}$ whose complement is finite. For a more interesting example, take the category of fields. This category has no initial object, because, for any other field E, there must be a map from F to E. As any map between two fields is one-to-one, the characteristics of E and F equals which excludes all E's with different characteristics. The binary product also does not exist for the same reason. Restricting fields to a fixed characteristic p can not solve the problem. The reason is similar to what we had for products before.

Definition 1.4.191. Let $\mathcal{C}$ be a category with products and A and B be two objects. An object C together with a natural isomorphism $\alpha : Hom(-, C) \cong Hom(- \times A, B)$ is called an exponentiation of B to A. Equivalently, an exponentiation of B to A is an object C together with a morphism $ev : C \times A \to B$ such that for any $f : D \times A \to B$, there exists a unique $g : D \to C$ such that:

$$\begin{tikzcd}
D \times A \arrow[d, "g \times id_A"'] \arrow[dr, "f"] & \\
C \times A \arrow[r, "ev"'] & B
\end{tikzcd}$$

The exponentiation is denoted by B^A.

Example 1.4.192. In the category **Set**, the exponential is $B^A = \{f : A \to B\}$ with the morphism $ev : B^A \times A \to B$ by $ev(f, a) = f(a)$. In **Cat**, the exponential category is defined by $\mathcal{D}^{\mathcal{C}}$ as the functor category and $ev : \mathcal{D}^{\mathcal{C}} \times \mathcal{C} \to \mathcal{C}$ by $ev(F, A) = F(A)$ and $ev(\alpha, f) = \alpha_B F(f) = G(f)\alpha_A$, for any $f : A \to B$ and $\alpha : F \Rightarrow G$. The last equality is because of the naturality of α. In a poset $(P, \leq)$, the exponentiation is by definition the least element c such that $c \wedge a \leq b$ i.e., an element c such that $c \wedge a \leq b$ and for any $d \in P$ if $d \wedge a \leq b$ then $d \leq c$. For the prototype posets, namely posets of subsets of X with inclusion, if they are closed under arbitrary union and finite intersections, the exponentiation of two subsets U and V are $V^U = \bigcup\{W \in P \mid W \cap U \subseteq V\}$. Exponential objects in posets are called Heyting implications and denoted by $\to$.

We saw how to define categorical constructions by representability. Here, we show how these constructions are functorial.

Theorem 1.4.193. *Let $F : \mathcal{C}^{op} \times \mathcal{D} \to \mathbf{Set}$ be a functor such that for any object D in $\mathcal{D}$, the functor $F(-, D)$ is representable. Then, there exists a unique (up to natural isomorphism) functor $G : \mathcal{D} \to \mathcal{C}$ such that $Hom(C, G(D)) \cong F(C, D)$, natural in C and D.*

Proof. Since for any D, the functor $F(-, D) : \mathcal{C}^{op} \to \mathbf{Set}$ is representable, there is an object $G(D)$ in $\mathcal{C}$ such that $Hom(C, G(D)) \cong_{\alpha_{C,D}} F(C, D)$, natural in C. For maps, if $f : D \to E$ is a map in $\mathcal{D}$, we define $G(f)$ as the unique morphism whose Yoneda is $y_{G(f)} = \alpha^{-1}_{C,E} F(id_C, f)\alpha_{C,D}$:

$$\begin{tikzcd}
y_{G(D)} \arrow[r, "\alpha_{C,D}"] \arrow[d, "y_{G(f)}"'] & F(C, D) \arrow[d, "F(id_C, f)"] \\
y_{G(E)} \arrow[r, "\alpha_{C,E}"'] & F(C, E)
\end{tikzcd}$$

It is easy to see that G is a functor and $\alpha_{C,D}$ is also natural in D. For uniqueness, assume there are G and H have the property. Then,

$$Hom(C, G(D)) \cong F(C, D) \cong Hom(C, H(D))$$

Hence, $y_{G(D)} \cong y_{H(D)}$, natural in D. By Yoneda embedding, we have $G(D) \cong H(D)$, natural in D. $\square$

Remark 1.4.194. Dually, if $F : \mathcal{C} \times \mathcal{D} \to \mathbf{Set}$ is a functor such that for any object D in $\mathcal{D}$, the functor $F(-, D)$ is corepresentable, there exists a unique (up to natural isomorphism) functor $G : \mathcal{D} \to \mathcal{C}^{op}$ such that $Hom(G(D), C) \cong F(C, D)$, natural in C and D.

As an application, we can see that products, coproducts and exponentials define functors. For products, it is enough to set $F : \mathcal{C}^{op} \times \mathcal{C} \times \mathcal{C} \to \mathbf{Set}$ as $F(X, A, B) = Hom(X, A) \times Hom(X, B)$ to reach $G(A, B) = A \times B$ as the product functor. The case for coproduct is similar. For the exponential functor, set $F : \mathcal{C}^{op} \times \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$ as $F(X, A, B) = Hom(X \times A, B)$ to

reach $G(A, B) = B^A$ as the exponential functor.
It is always possible to provide the functor by the universal behavior that is usually tiresome. Let's do it once for product as it has some pedagogical value. Assume $f : A \to C$ and $g : B \to D$ are two morphisms and we want to define $f \times g : A \times B \to C \times D$. By the universal property of $C \times D$, it is enough to provide two maps from $A \times B \to C$ and $A \times B \to D$ and the we will have our map automatically. For these two maps, pick:

$$\begin{tikzcd}
 & A \times B \arrow[dl, "fp_0"'] \arrow[d, "f\times g"] \arrow[dr, "gp_1"] & \\
C & C \times D \arrow[l, "p_0"] \arrow[r, "p_1"'] & D
\end{tikzcd}$$

We will rewrite the previous diagram as

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \times B \arrow[l, "p_0"'] \arrow[r, "p_1"] \arrow[d, "f\times g"] & B \arrow[d, "g"] \\
C & C \times D \arrow[l, "p_0"] \arrow[r, "p_1"'] & D
\end{tikzcd}$$

to have a more suggestive shape in our later computation. Now, we have to show that product is a functor. For that matter, assume $i : C \to E$ and $j : D \to F$ and we have to show that $(fi) \times (gj) = (f \times g) \circ (i \times j)$. We have

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \times B \arrow[l, "p_0"'] \arrow[r, "p_1"] \arrow[d, "f\times g"] & B \arrow[d, "g"] \\
C \arrow[d, "i"'] & C \times D \arrow[l, "p_0"] \arrow[r, "p_1"'] \arrow[d, "i\times j"'] & D \arrow[d, "j"] \\
E & E \times F \arrow[l, "p_0"] \arrow[r, "p_1"'] & F
\end{tikzcd}$$

Since all the small squares commute, the outer two vertical rectangular also commutes, meaning

$$\begin{tikzcd}
 & A \times B \arrow[dl, "(if)p_0"'] \arrow[d, "f\times g"] \arrow[dr, "(jg)p_1"] & \\
 & C \times D \arrow[d, "i\times j"] & \\
E & E \times F \arrow[l, "p_0"] \arrow[r, "p_1"'] & F
\end{tikzcd}$$

But by definition, there is only one vertical map that makes the diagram commutative, i.e., $(f \times g) \circ (i \times j)$. Hence, $(f \times g) \circ (i \times j) = (fi) \times (gj)$. The proof for $id \times id = id$ is similar.

Example 1.4.195. (*Yoneda lemma as a computational tool*) In any category with binary product and terminal object, we have $A \times 1 \cong A$, natural in A. As we saw before, we have to show that these two objects have the same behavior. We have

$$Hom(X, A \times 1) \cong Hom(X, A) \times Hom(X, 1) \cong Hom(X, A)$$

Hence, $y_{A\times 1} \cong y_A$ which by Yoneda lemma implies $A \times 1 \cong A$. Similarly, it is possible to prove that product is symmetric, i.e., $A \times B \cong B \times A$ and it is associative, i.e., $A \times (B \times C) \cong (A \times B) \times C$.
Again, it is possible to do the same thing by the universal property. To prove that $A \times 1 \cong A$, we must provide two maps, one from A to $A \times 1$ and one from $A \times 1$ to A such that they become each other's inverses. For these two maps, pick $f = p_0 : A \times 1 \to A$ and $g = \langle id_A, !\rangle : A \to A \times 1$. The latter is the unique map that makes the following commutative:

$$\begin{tikzcd}
 & A \arrow[dl, "id_A"'] \arrow[d, "\langle id_A, !\rangle"] \arrow[dr, "!"] & \\
A & A \times 1 \arrow[l, "p_0"] \arrow[r, "p_1"'] & 1
\end{tikzcd}$$

It is clear that $fg = p_0\langle id_A, !\rangle = id_A$. For the converse, consider the following diagram

$$\begin{tikzcd}
A \arrow[d, "id_A"'] & A \times 1 \arrow[l, "p_0"'] \arrow[r, "p_1"] \arrow[d, "p_0"] & 1 \arrow[d, "!"] \\
A \arrow[d, "id_A"'] & A \arrow[l, "id_A"'] \arrow[r, "!"] \arrow[d, "\langle id_A, ! \rangle"] & 1 \arrow[d, "!"] \\
A & A \times 1 \arrow[l, "p_0"] \arrow[r, "p_1"'] & 1
\end{tikzcd}$$

It is easy to see that all small squares are commutative and hence the outer two vertical rectangular must be commutative, meaning

$$\begin{tikzcd}
 & A \times 1 \arrow[ddl, "p_0"'] \arrow[d, "p_0"] \arrow[ddr, "!"] & \\
 & A \arrow[d, "\langle id_A, ! \rangle"] & \\
A & A \times 1 \arrow[l, "p_0"] \arrow[r, "p_1"'] & 1
\end{tikzcd}$$

But the only vertical map that makes the diagram commutative is $id_{A\times 1}$. Hence, $\langle id_A, ! \rangle p_0 = id_{A\times 1}$.

Example 1.4.196. (*Yoneda lemma as a computational tool*) In any category with coproduct, product and exponentiation, we have $A \times (B + C) \cong A \times B + A \times C$, natural in A, B and C. To show that these two objects have the same behavior, note that

$$Hom(A\times(B+C), D) \cong Hom((B+C), D^A) \cong Hom(B, D^A)\times Hom(C, D^A) \cong$$

$$Hom(A \times B, D) \times Hom(A \times C, D) \cong Hom(A \times B + A \times C, D)$$

Hence, $y^{A\times(B+C)} \cong y^{A\times B+A\times C}$ which by Yoneda lemma implies $A\times(B+C) \cong A \times B + A \times C$.

Example 1.4.197. Let $(Sub(\mathbb{R}^2), \subseteq)$ be the poset of all linear subspaces of $\mathbb{R}^2$. In this poset, all joins and meets exist. Meets are just intersections and joins are the linear subspaces generated by the unions. However, we do not have the equality $M \times (N + K) = M \times K + N \times K$ and hence the category does not have all exponentials. To show the failure of the equality, set M, N and K as three distinct lines going through the origin in $\mathbb{R}^2$. It is clear that $N + K = \mathbb{R}^2$ and hence $M \times (N + K) = M \cap (N + K) = M$, while $M \times N = M \times K = \{0\}$ and $\{0\} + \{0\} = \{0\} \neq M$.

Example 1.4.198. (*Non-existence of the exponential objects*) Let $\mathcal{C}$ be a non-preorder category with the initial and terminal objects where $0 \cong 1$. Then, $\mathcal{C}$ does not have all exponentials, because if it does, then we must have

$$Hom(A, B) \cong Hom(1 \times A, B) \cong Hom(1, B^A) \cong Hom(0, B^A)$$

But the last set has exactly one element. Hence $Hom(A, B)$ must have exactly one element, for any choice of A and B, which is a contradiction. As a consequence, the categories **Grp**, **Ab** and $\mathbf{Vec}_{\mathbb{R}}$ don't have all exponential objects.

Exercise 1.4.199. It seems that in **Ab**, the object H^G consisting of all homomorphisms from G to H with the pointwise addition is the exponential object of H by G. Find what is missing here.

Philosophical Note 1.4.200. (*Convenient category of spaces*) The category **Top** does not have all the exponentials and this fact makes the category somewhat cumbersome to work with. One way to overcome this issue is moving to a convenient category of topological spaces that includes a copy of all the tame interesting topological spaces like CW-complexes while having good properties including the closure under products and exponentiation. Steenrod proposed a list of such good properties for such a category. However,

> *It is also known that these propositions do not hold in the category of all Hausdorff spaces. In fact arguments have been given that which imply that there is no convenient category in our sense.*

However, Steenrod himself introduced such a category. He explains the apparent mismatch by:

> *The arguments are based on a blind adherence to the customary definitions of the standard operations. These definitions are suitable for the category of Hausdorff spaces, but they need not be for a subcategory. The categorical viewpoint enables us to defrost these definitions and bend them a bit.*

In fact, the customary definition that needs to change is the construction of the product. In Steenrod's category, a subcategory of the category of all Hausdorff spaces, all the products exist but its topology is far from the usual product topology and the adherence to this usual topology is what made the others blind to find the right category. With a bit of provocation, let's conclude that history also suggests the priority of the *relative* behavior of the entities to their *absolute* constructions.

Example 1.4.201. (*Yoneda lemma as a tool to define functors: subobject classifier*) Consider the functor $Sub : (\mathbf{Set}^{\mathcal{C}^{op}})^{op} \to \mathbf{Set}$ mapping a functor $F : \mathcal{C}^{op} \to \mathbf{Set}$ to the set of all sub-functors of F and a natural transformation to the pre-image function. This functor is representable by a functor $\Omega : \mathcal{C}^{op} \to \mathbf{Set}$, i.e., $Sub(F) \cong Hom(F, \Omega)$. Let's guess this functor. Using the Yoneda lemma, we know that $\Omega(A)$ must be equivalent to $Hom(y_A, \Omega)$. However, we expect $Hom(y_A, \Omega)$ to be equivalent to $Sub(y_A)$. Therefore, we can define $\Omega(A)$ as $Sub(y_A)$ and check if it really works, i.e., if $Sub(F) \cong Hom(F, \Omega)$, natural in F. We will not present the details here, but it fortunately holds.

Philosophical Note 1.4.202. Note that Ω plays the role of $\{0, 1\}$ in **Set**. Therefore, it is reasonable to say that the object Ω is the variable set of the truth values of the new world $\mathbf{Set}^{\mathcal{C}^{op}}$. More precisely, let F be a variable set. Then, any map from F to Ω is a characteristic map of a variable subsets of F assigning truth values to the "elements" of F, according to the way that the subfunctor sits inside F. Such an Ω with this behavior is called a subobject classifier.

Example 1.4.203. (*Yoneda lemma as a tool to define functors: exponential object*) The category $\mathbf{Set}^{\mathcal{C}^{op}}$ has all exponential objects. To prove that, we use the Yoneda lemma again. Let $E, F : \mathcal{C}^{op} \to \mathbf{Set}$ be two functors. We need to define F^E such that $Hom(E \times X, F) \cong Hom(X, F^E)$. Again set $X = y_A$. Then, we have to have $Hom(E \times y_A, F) \cong Hom(y_A, F^E)$. But $Hom(y_A, F^E)$ must be equivalent to $F^E(A)$, by Yoneda lemma. Therefore, it is enough to define F^E by $F^E(A) = Hom(E \times y_A, F)$. The only thing to check is that if F^E satisfies the more general $Hom(E \times X, F) \cong Hom(X, F^E)$. Again, we will not present the details here, but it fortunately holds.

Definition 1.4.204. Let $f, g : A \to B$ be two morphisms. Define the functor $Eq_{f,g} : \mathcal{C}^{op} \to \mathbf{Set}$ by $Eq_{f,g}(X) = \{i : X \to A \mid fi = gi\}$ and $Eq_{f,g}(j) = (-) \circ j$. By the equalizer of f and g, we mean the object C together with the natural isomorphism $Hom(X, C) \cong Eq_{f,g}(X)$. Equivalently, the equalizer of f and g is the object C together with a map $h : C \to A$ such that $fh = gh$, i.e.,

$$\begin{tikzcd} C \arrow[r, "h"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

and for any other map $i : X \to A$ such that $fi = gi$, there exists a unique map $j : X \to C$ such that

$$\begin{tikzcd} X \arrow[d, dashed, "j"'] \arrow[dr, "i"] & & \\ C \arrow[r, "h"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

It is called the equalizer of f and g, as it equalizes f and g.

Example 1.4.205. In a poset as there is at most one map between any two objects, the equalizer of any pair $f, g : A \to B$ exists and it is $id_A : A \to A$. In any groupoid, any two maps $f, g : A \to B$ has the equalizer iff they are equal and the equalizer is again $id_A : A \to A$. More generally, the equalizer of two equal maps always exists and it is the identity of the source object.

Example 1.4.206. In **Set**, any two maps have the equalizer. Let $f, g : A \to B$ be two functions. It is easy to see that the set $C = \{x \in A \mid f(x) = g(x)\}$ together with the inclusion $i : C \to A$ is the equalizer. The same also works for **Grp**, **Ab** and $\mathbf{Vec}_{\mathbb{R}}$, in which C inherits the algebraic structure of A. For **Grp**, note that the equalizer of $f : G \to H$ and the constant map $c_e : G \to H$ mapping everything to e_H is exactly the kernel of f. More generally, if a category has a zero object (when $0 \cong 1$), then the kernel of a map $f : A \to B$ may be defined as the equalizer of f and $0_{A,B} : A \to 1 \cong 0 \to B$, where the maps $A \to 1$ and $0 \to B$ are the unique maps provided by the universal properties of 0 and 1.

Example 1.4.207. In $\mathbf{Set}^{\mathcal{C}^{op}}$ any two maps have the equalizer and it is computed pointwise. Let $\alpha, \beta : F \Rightarrow G$ be two natural transformations. Define the functor $H : \mathcal{C}^{op} \to \mathbf{Set}$ on objects by $H(A) = \{x \in F(A) \mid \alpha_A(x) = \beta_A(x)\}$ and on morphism $f : B \to A$ by $H(f) = F(f)|_{H(A)} : H(A) \to H(B)$. It is easy to check that H is a functor, the canonical inclusion $i_A : H(A) \to F(A)$ is a natural transformation and the whole data is the equalizer of α and β.

Theorem 1.4.208. *Let $\mathcal{C}$ be a category that has the terminal object. Then, $\mathcal{C}$ has all pullbacks iff it has all binary products and all equalizers.*

Proof. If a category has the terminal object and all pullbacks, then it has the binary product, computed as the pullback:

$$\begin{tikzcd} C \arrow[r, "p_1"] \arrow[d, "p_0"'] & B \arrow[d, "!"] \\ A \arrow[r, "!"'] & 1 \end{tikzcd}$$

To prove that C, p_0 and p_1 is the product, note that if we have $f : D \to A$ and $g : D \to B$, then as there is only one map from D to 1, we have $!f =!g$, and as the square is a pullback, there exists a unique map $h : D \to C$ such that:

$$\begin{tikzcd} D \arrow[dr, dashed, "h"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\ & C \arrow[r, "p_1"] \arrow[d, "p_0"'] & B \arrow[d, "!"] \\ & A \arrow[r, "!"'] & 1 \end{tikzcd}$$

Now, we prove that all equalizers exist. Let $f, g : A \to B$ be two maps. Consider the following pullback:

$$\begin{tikzcd} C \arrow[r, "p_0"] \arrow[d, "p_1"'] & A \arrow[d, "\langle f,g \rangle"] \\ B \arrow[r, "\langle id_B, id_B \rangle"'] & B \times B \end{tikzcd}$$

We claim that $p_0 : C \to A$ is the equalizer. First, as the square is commutative, we have $fp_0 = gp_0$. Moreover, if there is a map $i : D \to A$ such that $fi = gi$, then we have

$$\begin{tikzcd} D \arrow[drr, bend left, "i"] \arrow[ddr, bend right, "fi=gi"'] & & \\ & C \arrow[r, "p_0"] \arrow[d, "p_1"'] & A \arrow[d, "\langle f,g \rangle"] \\ & B \arrow[r, "\langle id_B, id_B \rangle"'] & B \times B \end{tikzcd}$$

As the square is a pullback, there is a map $j : D \to C$ such that $p_0 j = i$, i.e.,

$$\begin{tikzcd}
D \arrow[dr, dashed, "j"] \arrow[drr, bend left, "i"] \arrow[ddr, bend right, "fi=gi"'] & & \\
& C \arrow[r, "p_0"] \arrow[d, "p_1"'] & A \arrow[d, "\langle f,g \rangle"] \\
& B \arrow[r, "\langle id_B, id_B \rangle"'] & B \times B
\end{tikzcd}$$

The only thing remains to prove it the uniqueness of this j. If there is $k : D \to C$ such that $p_0 k = i$, then it is easy to see that k we have:

$$\begin{tikzcd}
D \arrow[dr, dashed, "k"] \arrow[drr, bend left, "i"] \arrow[ddr, bend right, "fi=gi"'] & & \\
& C \arrow[r, "p_0"] \arrow[d, "p_1"'] & A \arrow[d, "\langle f,g \rangle"] \\
& B \arrow[r, "\langle id_B, id_B \rangle"'] & B \times B
\end{tikzcd}$$

and as the square is the pullback, we have $k = j$.
Conversely, if the binary products and the equalizers exist, then pullback also exists. Let $f : A \to C$ and $g : B \to C$ be two maps. Then, consider the equalizer:

$$\begin{tikzcd}
D \arrow[r, "e=\langle e_0, e_1 \rangle"] & A \times B \arrow[r, shift left, "fp_0"] \arrow[r, shift right, "gp_1"'] & C
\end{tikzcd}$$

we claim that the diagram

$$\begin{tikzcd}
D \arrow[r, "e_1"] \arrow[d, "e_0"'] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

is a pullback. It is clearly commutative. To show the universality, if there is

$i : E \to A$ and $j : E \to B$ such that

$$\begin{tikzcd}
E \arrow[drr, bend left, "j"] \arrow[ddr, bend right, "i"'] & & \\
& D \arrow[r, "e_1"] \arrow[d, "e_0"'] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

Then, we have

$$\begin{tikzcd}
E \arrow[dr, "\langle i,j \rangle"] & & \\
D \arrow[r, "e=\langle e_0,e_1 \rangle"] & A \times B \arrow[r, shift left, "fp_0"] \arrow[r, shift right, "gp_1"'] & C
\end{tikzcd}$$

which by the fact that $e : D \to A \times B$ is equalizer, there exists a map $h : E \to D$ such that

$$\begin{tikzcd}
E \arrow[d, "h"'] \arrow[dr, "\langle i,j \rangle"] & & \\
D \arrow[r, "e=\langle e_0,e_1 \rangle"] & A \times B \arrow[r, shift left, "fp_0"] \arrow[r, shift right, "gp_1"'] & C
\end{tikzcd}$$

which implies

$$\begin{tikzcd}
E \arrow[drr, bend left, "j"] \arrow[ddr, bend right, "i"'] \arrow[dr, "h"] & & \\
& D \arrow[r, "e_1"] \arrow[d, "e_0"'] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

The uniqueness of $h : E \to D$ is easy. $\square$

Example 1.4.209. In the previous theorem, the existence of the terminal object is essential. For instance, if G is a non-trivial group as we observed before, if $g \neq h$, then they do not have equalizer. But all pullbacks in this category exist. The reason simply is that for any elements $g, h \in G$, the square

$$\begin{tikzcd} * \arrow[r, "g^{-1}"] \arrow[d, "h^{-1}"'] & * \arrow[d, "g"] \\ * \arrow[r, "h"'] & * \end{tikzcd}$$

is a pullback, because it commutes and for any other $i, j \in G$ such that $gj = hi$, we have

$$\begin{tikzcd} * \arrow[dr, dashed, "gj=hi"] \arrow[drr, bend left, "j"] \arrow[ddr, bend right, "i"'] & & \\ & * \arrow[r, "g^{-1}"] \arrow[d, "h^{-1}"'] & * \arrow[d, "g"] \\ & * \arrow[r, "h"'] & * \end{tikzcd}$$

The map $gj = hi$ is clearly unique.

Definition 1.4.210. Let $f, g : A \to B$ be two morphisms. Define the functor $CoEq_{f,g} : \mathcal{C} \to \mathbf{Set}$ by $CoEq_{f,g}(X) = \{i : B \to X \mid if = ig\}$ and $CoEq_{f,g}(j) = j \circ (-)$. By the coequalizer of f and g, we mean the object C together with the natural isomorphism $Hom(C, X) \cong CoEq_{f,g}(X)$. Equivalently, the coequalizer of f and g is the object C together with a map $h : B \to C$ such that $hf = hg$, i.e.,

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] & C \end{tikzcd}$$

and for any other map $i : B \to X$ such that $if = ig$, there exists a unique map $j : C \to X$ such that $jh = i$, i.e.,

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] \arrow[dr, "i"'] & C \arrow[d, dashed, "j"] \\ & & X \end{tikzcd}$$

It is called the coequalizer as it is the dual of the equalizer.

Example 1.4.211. In a poset the coequalizer of any pair $f, g : A \to B$ exists and it is $id_B : B \to B$. In any groupoid, any two maps $f, g : A \to B$ has the coequalizer iff they are equal and the coequalizer is again $id_B : B \to B$. More generally, the coequalizer of two equal maps always exists and it is the identity of the target object. In **Set**, any two maps have the coequalizer. Let $f, g : A \to B$ be two functions. It is easy to see that the set $C = B/\sim$ together with the canonical projection $p : B \to C$ mapping b to $[b]$ is the coequalizer, where $\sim \subseteq B \times B$ is the least equivalence relation extending $\{(b, c) \in B \times B \mid \exists a \in A \ b = f(a) \text{ and } c = g(a)\}$. More specifically, if $R \subseteq B \times B$ is an equivalence relation, then B/R is just the coequalizer of $p_0, p_1 : R \to B$, where p_0 and p_1 are the projections. In **Top** the same construction works, except that we need the quotient topology. For instance, the coequalizer of the two ends of the interval $[0, 1]$ is $\mathbb{S}^1$:

$$\begin{tikzcd} \{0\} \arrow[r, shift left, "0\mapsto 1"] \arrow[r, shift right, "0\mapsto 0"'] & {[0,1]} \arrow[r] & \mathbb{S}^1 \end{tikzcd}$$

For **Ab**, the coequalizer of $f, g : G \to H$ is the group $H/Im(f - g)$. Note that the cokernel of $f : G \to H$, i.e., $H/Im(f)$ is the coequalizer of f and $0 : G \to H$, where 0 is the map that sends everything to 0_H. More generally, if a category has a zero object, then the cokernel of a map $f : A \to B$ may be defined as the coequalizer of f and $0_{A,B} : A \to 1 \cong 0 \to B$.

Example 1.4.212. In **Cat**, the coequalizer of the functors $F, G : \mathbf{1} \to \mathbf{2}$ mapping the only object of $\mathbf{1}$ to the objects of $\mathbf{2}$ is the category $(\mathbb{N}, +)$ and the map $P : \mathbf{2} \to (\mathbb{N}, +)$, mapping objects to the only object of $(\mathbb{N}, +)$ and the only non-trivial map of $\mathbf{2}$ to the map $1 \in \mathbb{N}$:

$$\begin{tikzcd} \mathbf{1} \arrow[r, shift left, "*\mapsto\dagger"] \arrow[r, shift right, "*\mapsto *"'] & \mathbf{2} \arrow[r] & (\mathbb{N}, +) \end{tikzcd}$$

Similarly, for canonical functors $F, G : \mathbf{1} \to \mathcal{I}$, where

$$\mathcal{I} : \qquad \begin{tikzcd} * \arrow[r, bend left] & \dagger \arrow[l, bend left] \end{tikzcd}$$

the coequalizer is $(\mathbb{Z}, +)$ together with the map $Q : \mathcal{I} \to (\mathbb{Z}, +)$, mapping the objects to the only object of $(\mathbb{Z}, +)$ and the two non-trivial maps of $\mathcal{I}$ to 1 and -1:

$$\begin{tikzcd} \mathbf{1} \arrow[r, shift left, "*\mapsto\dagger"] \arrow[r, shift right, "*\mapsto *"'] & \mathcal{I} \arrow[r] & (\mathbb{Z}, +) \end{tikzcd}$$

Reading $\mathcal{I}$ as the categorical version of the topological space $[0, 1]$, this coequalizer in **Cat** is reminiscent of the coequalizer

$$\{0\} \underset{0 \mapsto 0}{\overset{0 \mapsto 1}{\rightrightarrows}} [0,1] \longrightarrow \mathbb{S}^1$$

in **Top**. Can we conclude that $(\mathbb{Z}, +)$ is the categorical version of the circle $\mathbb{S}^1$? Does it related to the fact that the fundamental group of $\mathbb{S}^1$ is $(\mathbb{Z}, +)$?

Example 1.4.213. In $\mathbf{Set}^{\mathcal{C}^{op}}$ any two maps has the coequalizer and it is computed pointwise. More precisely, let $\alpha, \beta : F \Rightarrow G$ be two natural transformations. It is easy to see that the the functor H defined by $H(A) = G(A)/R(A)$, where $R(A)$ is the least equivalence relation extending $\{(x, y) \in G(A) \mid \exists z \in F(A)\ \alpha_A(z) = x \text{ anbd } \beta_A(z) = y\}$ and for any $f : B \to A$ by $H(f) : H(A) \to H(B)$ as the canonical map induced by $G(f)$. It is easy to check that this H is a functor, the natural projection $p_A : G(A) \to H(A)$ is a natural transformation and the whole data is the coequalizer of α and β.

Philosophical Note 1.4.214. (*The Duality Principle*) Let ϕ be a statement about a category, purely written in the language of objects, arrows, identity and composition, using identity, boolean operations and quantifiers over objects and morphisms. We are also allowed to use parameter, meaning names for some given objects and arrows. For instance, the fact that $p_0 : C \to A$ and $p_1 : C \to B$ is the product of A and B is written as:

$$\forall f : D \to A\ \forall g : D \to B\ \exists!\ h : D \to C\ [(p_0 \circ h = f) \wedge (p_1 \circ h = g)]$$

with parameters $p_0 : C \to A$ and $p_1 : C \to B$. Then, by the dual statement of ϕ, denoted by ϕ^{op}, we mean the result of flipping all the arrows in ϕ and then changing $f \circ g$ by $g \circ f$, everywhere including in the parameters. For instance, the dual of the above statement is:

$$\forall f : A \to D\ \forall g : B \to D\ \exists!\ h : C \to D\ [(h \circ p_0 = f) \wedge (h \circ p_1 = g)]$$

for parameters $p_0 : A \to C$ and $p_1 : B \to C$. It is clear that the statement ϕ is true in $\mathcal{C}$ iff ϕ^{op} is true in $\mathcal{C}^{op}$. Now, as the opposite of any category is also a category, it is clear that if a statement ϕ is true in all categories, its dual also holds for all categories. Why?

Theorem 1.4.215. *Let $\mathcal{C}$ be a category that has the initial object. Then, $\mathcal{C}$ has all pushouts iff it has all binary coproducts and all coequalizers.*

Proof. Use the duality principle. □

Now, we are ready to address the general case of limits. Let $F : \mathcal{J} \to \mathcal{C}$ be a diagram (functor). Define a *cone over* F with the summit X as a natural transformation $\alpha : \Delta_X \Rightarrow F$. Spelling out, a cone over F with the summit X is an assignment $\{i_A : X \to F(A)\}_{A \in \mathcal{J}}$ such that $F(f)h_A = h_B$, for any $f : A \to B$, i.e.,

$$\begin{tikzcd}
& & X \arrow[ddll, "i_{A_0}"'] \arrow[d, "i_{A_1}"] \arrow[ddr, "i_{A_2}"] \arrow[drr, "i_{A_3}"] & & \\
& & F(A_1) \arrow[rr, "F(f_3 f_2)"] \arrow[dr, "F(f_2)"'] & & F(A_3) \\
F(A_0) \arrow[urr, "F(f_1)"] \arrow[rrr, "F(f_2 f_1)"'] & & & F(A_2) \arrow[ur, "F(f_3)"'] &
\end{tikzcd}$$

By the cone functor over F, we mean the functor $Cone_F : \mathcal{C}^{op} \to \mathbf{Set}$ defined by $Cone_F(X)$ as the set of all cones over F with summit X and for a map $j : B \to A$ by $Con_F(j) = j \circ (-)$.

Example 1.4.216. Let $F : \mathbf{0} \to \mathcal{C}$ be the functor from the empty category to $\mathcal{C}$. Then, for any object X, there is exactly one cone over F with the summit X and hence $Cone_F = \Delta_{\{0\}}$. For any functor $F : \mathbf{1} + \mathbf{1} \to \mathcal{C}$, a cone over F with summit X is just the pair of two maps $f_0 : X \to F(0)$ and $f_1 : X \to F(1)$:

$$\begin{tikzcd}
& X \arrow[dl, "i_0"'] \arrow[dr, "i_1"] & \\
F(0) & & F(1)
\end{tikzcd}$$

For more examples, define the following categories:

$$\mathcal{J}: \quad \begin{tikzcd} & \dagger \arrow[d] \\ \bullet \arrow[r] & * \end{tikzcd} \qquad\qquad \mathcal{K}: \quad \begin{tikzcd} \bullet \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & * \end{tikzcd}$$

Then a cone over $F : \mathcal{J} \to \mathcal{C}$ with summit X is the tuple of three maps i_*, $i_\bullet$ and $i_\dagger$, such that:

$$\begin{tikzcd} X \arrow[r, "i_\dagger"] \arrow[d, "i_\bullet"'] \arrow[dr, "i_*"] & F(\dagger) \arrow[d] \\ F(\bullet) \arrow[r] & F(*) \end{tikzcd}$$

It is easy to see that the map i_* is uniquely determined by the maps $i_\bullet$ and $i_\dagger$ and hence there is no need to keep its data. Therefore, w.l.o.g, we can say that a cone over F with summit X is a pair of maps $i_\bullet$ and $i_\dagger$, such that:

$$\begin{tikzcd} X \arrow[r, "i_\dagger"] \arrow[d, "i_\bullet"'] & F(\dagger) \arrow[d] \\ F(\bullet) \arrow[r] & F(*) \end{tikzcd}$$

For a functor $F : \mathcal{K} \to \mathcal{C}$, a cone with summit X is a pair of maps $i : \bullet$ and i_* such that$i_* = F(f_0)i_\bullet$ and $i_* = F(f_1)i_\bullet$, i.e.,

$$\begin{tikzcd} X \arrow[d, "i_\bullet"'] \arrow[dr, "i_*"] & \\ F(\bullet) \arrow[r, shift left, "F(f_0)"] \arrow[r, shift right, "F(f_1)"'] & F(*) \end{tikzcd}$$

Again, it is easy to see that i_* is uniquely determined by $i_\bullet$. However, this does not mean that we can pick any $i_\bullet$ as we want. The necessary and sufficient condition for $i_\bullet$ is that $F(f_0)i_\bullet = F(f_1)i_\bullet$:

$$\begin{tikzcd}
X \arrow[d, "i_\bullet"'] & \\
F(\bullet) \arrow[r, shift left, "F(f_0)"] \arrow[r, shift right, "F(f_1)"'] & F(*)
\end{tikzcd}$$

Therefore, w.l.o.g, we can say that a cone over F with summit X is a map $i_\bullet$ such that $F(f_0)i_\bullet = F(f_1)i_\bullet$.
For a functor $F : (\mathbb{N}, \leq)^{op} \to \mathcal{C}$, a cone with summit X is a sequence of maps $\{i_n : X \to F(n)\}_{n\in\mathbb{N}}$ such that:

$$\begin{tikzcd}
 & & & & X \arrow[llll] \arrow[llll, "\cdots"] \arrow[lll, "i_3"] \arrow[ll, "i_2"] \arrow[rr, "i_1"] \arrow[rrrr, "i_0"] & & & & \\
\cdots\cdots \arrow[r] & F(3) \arrow[r] & & F(2) \arrow[rrr] & & & F(1) \arrow[rr] & & F(0)
\end{tikzcd}$$

Can you explain why a cone over $F : (\mathbb{N}, \leq) \to \mathcal{C}$ is not interesting?

Definition 1.4.217. Let $F : \mathcal{J} \to \mathcal{C}$ be a diagram (functor). By the limit of F, we mean an object C together with a natural isomorphism $Hom(X, C) \cong Cone_F(X)$. Equivalently, the limit of F is the object C together with a map $h_A : C \to F(A)$, for any object A in $\mathcal{J}$, such that $F(f)h_A = h_B$, for any

$f : A \to B$, i.e.,

$$
\begin{tikzcd}
 & C \arrow[ddl, "h_{A_0}"'] \arrow[d, "h_{A_1}"] \arrow[ddr, "h_{A_2}"] \arrow[drr, "h_{A_3}"] & & \\
 & F(A_1) \arrow[rr, "F(f_3f_2)"] \arrow[dr, "F(f_2)"'] & & F(A_3) \\
F(A_0) \arrow[ur, "F(f_1)"] \arrow[rr, "F(f_2f_1)"'] & & F(A_2) \arrow[ur, "F(f_3)"'] &
\end{tikzcd}
$$

and for any other maps $i_A : D \to F(A)$, for any object A in $\mathcal{J}$ such that $F(f)i_A = i_B$, for any $f : A \to B$, there exists a unique map $j : D \to C$ such that $h_A j = i_A$, for any A, i.e.,

$$\begin{tikzcd}
D \arrow[rr, dashed, "j"] \arrow[dd, dotted, "i_{A_0}"'] \arrow[drr, dotted, "i_{A_1}"] \arrow[ddrrr, dotted, "i_{A_2}"] \arrow[drrrr, dotted, "i_{A_3}"] & & C \arrow[ddll, "h_{A_0}"'] \arrow[d, "h_{A_1}"] \arrow[ddr, "h_{A_2}"] \arrow[drr, "h_{A_3}"] & & \\
 & & F(A_1) \arrow[rr, "F(f_3 f_2)"] \arrow[dr, "F(f_2)"'] & & F(A_3) \\
F(A_0) \arrow[urr, "F(f_1)"] \arrow[rrr, "F(f_2 f_1)"'] & & & F(A_2) \arrow[ur, "F(f_3)"'] &
\end{tikzcd}$$

A limit is called (finite) small if the category $\mathcal{J}$ is (finite) small. A category is called complete, if it has all small limits and finitely complete, if it has all finite limits.

Example 1.4.218. Let $F : \mathbf{0} \to \mathcal{C}$ be the functor from the empty category to $\mathcal{C}$. The limit of F is the terminal object. For any functor $F : \mathbf{1} + \mathbf{1} \to \mathcal{C}$ the limit is the product of the objects in the image of F. Recall that we had the following categories:

$$\mathcal{J} : \quad \begin{tikzcd} & \dagger \arrow[d] \\ \bullet \arrow[r] & * \end{tikzcd} \qquad\qquad \mathcal{K} : \quad \begin{tikzcd} \bullet \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & * \end{tikzcd}$$

Then, the limit of any functor $F : \mathcal{J} \to \mathcal{C}$ is the pullback of the F-image of the arrows along each other and the limit of any functor $F : \mathcal{K} \to \mathcal{C}$ is the equalizer of the F-image of the two arrows $F(f_0)$ and $F(f_1)$.

Example 1.4.219. Let $F : (\mathbb{N}, \leq)^{op} \to \mathcal{C}$ be a functor. The limit of F is called the *inverse limit* of the family $\{F(n+1 \geq n) : F(n+1) \to F(n)\}_{n\in\mathbb{N}}^{\infty}$:

$$\begin{tikzcd}
 & & & C \arrow[dlll, "\cdots"'] \arrow[dll, "h_3"] \arrow[dl, "h_2"'] \arrow[d, "h_1"] \arrow[dr, "h_0"] & \\
\cdots\cdots \arrow[r] & F(3) \arrow[r] & F(2) \arrow[r] & F(1) \arrow[r] & F(0)
\end{tikzcd}$$

Philosophical Note 1.4.220. (*Completion of Rings*) It is usually helpful to interpret a commutative unital ring R as a ring of some sore of "acceptable" functions from a "space" X to a fixed field F. For instance, we may think of the ring $\mathbb{C}[z]$ as a ring of polynomial functions from the space $\mathbb{C}$ to the field $\mathbb{C}$. Note that in this interpretation, we have no access to the space itself. We know the space through the quantities (functions) we can measure over it and hence we must reconstruct any property of the space from the ring, if it is possible. For instance, a "point" of the "space" may be identified by all the functions that vanish on the point and as F is supposed to be a field, the set of such functions forms a maximal ideal M. Hence, a "point" will be simply a maximal ideal of the ring R. For instance, in our above example, the point $0 \in \mathbb{C}$ is identified by the maximal ideal $\{r \in \mathbb{C}[z] \mid r(0) = 0\}$. Now, what is the value of the function $r \in R$ in the point M? Reading the value r_0 as a constant function, we expect that $r - r_0$ vanishes in the point M. Hence, $r - r_0 \in M$. As r_0 is invariant under any addition of functions that vanishes in M, it is reasonable to set r_0 as the remainder of r modulo M or $r + M \in R/M$. Note that with a similar argument, we can talk about the polynomial approximation of r around M with degree n as the remainder of r modulo M^n or $r + M^n \in R/M^n$.
Now, note that the ring of functions around a point can be incomplete in the sense that we may have a "convergent" sequence of functions whose *limit* does not exist in the original ring R. For instance, think about the sequence of polynomial approximations $\{\Sigma_{i=0}^{n} z^n/n!\}_{n=0}^{\infty}$ of the function e^z around the point $p = 0$. Is it possible to perform such a completion pure algebraically to reach a ring of "analytic functions" around a point? Let's give it a try! An analytic function, what it means, leaves a trace of polynomial approximations in our given ring R exactly as what the elements in R does. The value of the function is stored in R/M, the linear approximation lives in R/M^2 and so on. So any analytic function left the trace of a sequence $\langle r_n + M^n \rangle_n \in \{R/M^n\}_{n=0}^{\infty}$ as its "polynomial" approximations. Note that this sequence must have the property that $p_n(r_{n+1}) = r_n$, where $p_n : R/M^{n+1} \to R/M^n$ is the canonical projection as we expect that by increasing the degree of the approximation,

the partial results remain consistent in their lower degrees. Now, as we believe that an entity is nothing but its behavior, we may identify the analytic functions around M as the ring of these consistent sequences, i.e.,

$$\{\langle r_n + M^n \rangle_n \in \{R/M^n\}_{n=0}^{\infty} \mid \forall n \; p_n(r_{n+1}) = r_n\}$$

How to construct such a ring in pure categorical terms? It is simply the limit of the following diagram:

$$\cdots\cdots \xrightarrow{\quad p_3 \quad} R/M^3 \xrightarrow{\quad p_2 \quad} R/M^2 \xrightarrow{\quad p_1 \quad} R/M$$

A special case of such a situation is the familiar case of p-adic numbers. Let $R = \mathbb{Z}$ and $M = p\mathbb{Z}$. Then, the limit is the ring of p-adic numbers and hence we can interpret any p-adic number as an analytic function around the abstract point $p\mathbb{Z}$ in an abstract space.

Example 1.4.221. (*Solenoids*) Interpreting $\mathbb{S}^1$ as the topological version of the group $(\mathbb{Z}, +)$, we may introduce the topological version of p-adic numbers as the limit of:

$$\cdots\cdots \xrightarrow{\quad (-)^p \quad} \mathbb{S}^1 \xrightarrow{\quad (-)^p \quad} \mathbb{S}^1 \xrightarrow{\quad (-)^p \quad} \mathbb{S}^1$$

where, $(-)^p : \mathbb{S}^1 \to \mathbb{S}^1$ is mapping the point $(cos(\theta), sin(\theta))$ to $(cos(p\theta), sin(p\theta))$. The space is called the p-solenoid.

Example 1.4.222. If $\mathcal{J}$ is the monoid $(\mathbb{N}, +)$, the limit of a functor $F : (\mathbb{N}, +) \to \mathbf{Set}$

$$\begin{tikzcd}
 & C \arrow[dl, "h_*"'] \arrow[dr, "h_*"] & \\
F(*) \arrow[rr, "F(1)"'] & & F(*)
\end{tikzcd}$$

is just the set C of all fixed points of the function $F(1) : F(*) \to F(*)$ together with the inclusion map $h_* : C \to F(*)$.

Example 1.4.223. (*Sheaves*) Let X be a topological space, $\{U_i\}_{i \in I}$ be a family of open subsets and $U = \bigcup_{i \in I} U_i$. Define the functor $C : \mathcal{O}(X)^{op} \to \mathbf{Set}$ on the open subset V of X by $C(V) = \{f : V \to \mathbb{R} \mid f \text{ is continuous}\}$ and on the unique morphism $V \supseteq W$ as the restriction map $|_W : C(V) \to C(W)$. The functor C stores all partial continuous functions over X defined on an open domain. Define P as the set of all $\{i, j\}$'s, where $i, j \in I$ and $F : (P, \subseteq) \to \mathbf{Set}$ as the diagram mapping $\{i, j\}$ to $C(U_i \cap U_j)$ and the only non-trivial morphism $\{i\} \subseteq \{i,j\}$, for $i \neq j$ to the restriction map $|_{U_i\cap U_j} : C(U_i) \to C(U_i \cap U_j)$. Then, $C(U)$ is the limit of the diagram F:

$$\begin{tikzcd}
& C(U) \arrow[dl, "|_{U_1}"'] \arrow[d, "|_{U_2}"] \arrow[dr, "|_{U_3}"] & \\
C(U_1) \arrow[d, "|_{U_1\cap U_2}"'] \arrow[dr, "|_{U_1\cap U_3}"] & C(U_2) \arrow[dl, "|_{U_1\cap U_2}"'] \arrow[dr, "|_{U_2\cap U_3}"] & C(U_3) \arrow[dl, "|_{U_1\cap U_3}"'] \arrow[d, "|_{U_2\cap U_3}"] \\
C(U_1\cap U_2) & C(U_1\cap U_3) & C(U_2\cap U_3)
\end{tikzcd}$$

The diagram is clearly commutative. To show its universality, for any other cone $\{f_i : D \to C(U_i)\}_{i\in I}$:

$$\begin{tikzcd}
D \arrow[rr, dashed, "f"] \arrow[dr, dotted, "f_1"'] \arrow[drr, dotted, "f_2"] \arrow[drrr, dotted, "f_3"] & & C(U) \arrow[dl, "|_{U_1}"'] \arrow[d, "|_{U_2}"] \arrow[dr, "|_{U_3}"] & \\
& C(U_1) \arrow[d, "|_{U_1\cap U_2}"'] \arrow[dr, "|_{U_1\cap U_3}"] & C(U_2) \arrow[dl, "|_{U_1\cap U_2}"'] \arrow[dr, "|_{U_2\cap U_3}"] & C(U_3) \arrow[dl, "|_{U_1\cap U_3}"'] \arrow[d, "|_{U_2\cap U_3}"] \\
& C(U_1\cap U_2) & C(U_1\cap U_3) & C(U_2\cap U_3)
\end{tikzcd}$$

the commutativity of the diagram states that for any $x \in D$, the functions $\{f_i(x) : U_i \to \mathbb{R}\}_{i\in I}$ are consistent on the intersections of their domains and hence we can construct a unique function on U by their union. Set $f(x) : U \to \mathbb{R}$ as this unique function and note that f is continuous. It is easy to see that this f is the only map we can use here.
Note that the main reason behind the argument is that the fact that continuity is a local notion meaning that continuity in a point is determined by the behavior of the function on a small neighbourhood of x. This implies

that if we have a consistent family of continuous functions on some opens, we can glue them together to construct a unique continuous function extending them all. Changing continuity to any other local notion like derivability also works while using global notions like constancy breaks the argument.
To formalize the general situation, let $G : \mathcal{O}(X)^{op} \to \mathbf{Set}$ be a functor. If for any family of opens $\{U_i\}_{i \in I}$ covering U, the set $G(U)$ is the limit of the corresponding functor F, we call G a sheaf over X. We can think of a *sheaf* as a machine to store all the *local* instance of a *local* notion.

Example 1.4.224. A poset is (finitely) complete iff it has all (finite) meets. One direction is clear. For the other direction, let $F : \mathcal{J} \to (P, \leq)$ be a diagram. Then, as in a poset any two maps with the same source and target are equal, we can observe that the meet $\bigwedge_J F(J)$ together with its unique map to all $F(J)$'s is the limit of F. For instance, the poset $(P(X), \subseteq)$ is complete as it has all possible meets.

Example 1.4.225. The category **Set** is complete. To prove that, let $F : \mathcal{J} \to \mathbf{Set}$ be a small diagram. Then, define $C = \{s \in \prod_{A \in \mathcal{J}} F(A) \mid \forall f : A \to B\ [F(f)(s(A)) = s(B)\}$ and $h_A : C \to F(A)$ as the canonical projection on A'th element:

$$
\begin{tikzcd}
 & & C \arrow[dll, "h_{A_0}"'] \arrow[d, "h_{A_1}"] \arrow[drr, "h_{A_2}"] \arrow[rrr, "h_{A_3}"] & & & \\
 & & F(A_1) \arrow[rrr] \arrow[dr, "F(f_2)"] & & & \cdots \\
F(A_0) \arrow[urr, "F(f_1)"] \arrow[rrr, "F(f_2 f_1)"'] & & & F(A_2) \arrow[rr] & & \cdots
\end{tikzcd}
$$

It is easy to see that this data is the limit of F. The same construction with the pointwise algebraic structure also works for **Grp**, **Ab** and $\mathbf{Vec}_{\mathbb{R}}$. For **Top**, we also have the same construction, this time using the product and the subspace topology. Note that the subcategory of all compact Hausdorff spaces is also complete. The reason is simply the combination of the Tychonoff's theorem and the fact that the subspace defined by any number of equalities is compact.

Theorem 1.4.226. *A category is (finitely) complete iff it has all (finite) products and all equalizers.*

Proof. One direction is clear by definition. For the other direction, we use the argument from the previous example. Let $F : \mathcal{J} \to \mathbf{Set}$ be a small diagram. Then, as products of size of $\mathcal{J}$ exists, the product $\prod_{J\in\mathcal{J}} F(J)$ with projections $p_J : P \to F(J)$ exists. Set $P = \prod_{J\in\mathcal{J}} F(J)$. Then, set C and $q : C \to P$ as the equalizer of $\langle p_K\rangle_{J,f}, \langle F(f)p_J\rangle_{J,f} : \prod_{J\in\mathcal{J}} F(J) \to \prod_{J\in\mathcal{J}} \prod_{f:J\to K} F(K)$. The limit will be $\{p_J q : C \to F(J)\}_{J\in\mathcal{J}}$. As $q : C \to P$ is the equalizer, we have $p_K q = F(f)p_J q$. To show that it is the best choice, assume $\{i_J : D \to F(J)\}_{J\in\mathcal{J}}$ has the property $i_K = F(f)i_J$, for any $f : J \to K$:

$$\begin{tikzcd}
 & & C \arrow[d, "q"] & & \\
D \arrow[urr, dashed, "j"] \arrow[rr, "\langle i_J\rangle_{J\in\mathcal{J}}"] \arrow[dd, dotted, "i_{A_0}"'] \arrow[drr, dotted, "i_{A_1}"] \arrow[ddrrr, dotted, "i_{A_2}"] \arrow[drrrr, dotted] & & P = \prod_{J\in\mathcal{J}} F(J) \arrow[ddll, "p_{A_0}"'] \arrow[d, "p_{A_1}"] \arrow[ddr, "p_{A_2}"] \arrow[drr, "p_{A_3}"] & & \\
 & & F(A_1) \arrow[rr] \arrow[dr, "F(f_2)"] & & \cdots \\
F(A_0) \arrow[urr, "F(f_1)"] \arrow[rrr, "F(f_2 f_1)"'] & & & F(A_2) \arrow[r] & \cdots
\end{tikzcd}$$

Therefore, $\langle p_K\rangle_{J,f} \circ \langle i_J\rangle_J = \langle F(f)p_J\rangle_{J,f} \circ \langle i_J\rangle_J$. As $q : C \to P$ is the equalizer, there exists a unique map $j : D \to C$ such that $qj = \langle i_J\rangle_J$. Hence, $p_J qj = i_J$. Uniqueness condition for limit is also easy. $\square$

Now, let us spell out the dual notion of *cones under a diagram* or *cocones* and *colimits*. Let $F : \mathcal{J} \to \mathcal{C}$ be a diagram (functor). Define a *cone under* F with the nadir X as a natural transformation $\alpha : F \Rightarrow \Delta_X$. Spelling out, a cone under F with the nadir X is an assignment $\{i_A : F(A) \to X\}_{A\in\mathcal{J}}$ such that $h_A = h_B F(f)$, for any $f : A \to B$, i.e.,

$$\begin{tikzcd}
F(A_1) \arrow[rrr, "F(f_4)"] \arrow[ddrr, "h_{A_1}"'] & & & F(A_3) \arrow[ddl, "h_{A_3}"'] & \\
& F(A_0) \arrow[ul, "F(f_1)"'] \arrow[rrrr, "F(f_2)"'] \arrow[dr, "h_{A_0}"] & & & F(A_2) \arrow[ul, "F(f_3)"'] \arrow[dll, "h_{A_2}"] \\
& & X & &
\end{tikzcd}$$

By the cone functor under F, we mean the functor $Cone^F : \mathcal{C} \to \mathbf{Set}$ defined by $Cone^F(X)$ as the set of all cones under F with nadir X and for a map $j : A \to B$ by $Con^F(j) = (-) \circ j$.

Example 1.4.227. Let $F : \mathbf{0} \to \mathcal{C}$ be the functor from the empty category to $\mathcal{C}$. Then, for any object X, there is exactly one cone under F with the nadir X and hence $Cone^F = \Delta_{\{0\}}$. For any functor $F : \mathbf{1}+\mathbf{1} \to \mathcal{C}$, a cone under F with nadir X is just the pair of two maps $i_0 : F(0) \to X$ and $i_1 : F(1) \to X$:

$$\begin{tikzcd}
F(0) \arrow[dr, "i_0"'] & & F(1) \arrow[dl, "i_1"] \\
& X &
\end{tikzcd}$$

For more examples, define the following categories:

$$\mathcal{J} : \quad \begin{tikzcd}
* \arrow[r] \arrow[d] & \dagger \\
\bullet &
\end{tikzcd} \qquad\qquad \mathcal{K} : \quad \begin{tikzcd}
\bullet \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & *
\end{tikzcd}$$

Then a cone under $F : \mathcal{J} \to \mathcal{C}$ with nadir X is the tuple of three maps i_*, $i_\bullet$

and $i_\dagger$, such that:

$$\begin{tikzcd}
F(*) \arrow[r] \arrow[d] \arrow[dr, "i_*"] & F(\dagger) \arrow[d, "i_\dagger"] \\
F(\bullet) \arrow[r, "i_\bullet"'] & X
\end{tikzcd}$$

It is easy to see that the map i_* is uniquely determined by the maps $i_\bullet$ and $i_\dagger$ and hence there is no need to keep its data. Therefore, w.l.o.g, we can say that a cone under F with nadir X is a pair of maps $i_\bullet$ and $i_\dagger$, such that:

$$\begin{tikzcd}
F(*) \arrow[r] \arrow[d] & F(\dagger) \arrow[d, "i_\dagger"] \\
F(\bullet) \arrow[r, "i_\bullet"'] & X
\end{tikzcd}$$

For a functor $F : \mathcal{K} \to \mathcal{C}$, a cone under F with nadir X is a pair of maps $i_\bullet$ and i_* such that$i_\bullet = i_* F(f_0)$ and $i_\bullet = i_* F(f_1)$, i.e.,

$$\begin{tikzcd}
\bullet \arrow[r, shift left, "F(f_0)"] \arrow[r, shift right, "F(f_1)"'] \arrow[dr, "i_\bullet"'] & * \arrow[d, "i_*"] \\
& X
\end{tikzcd}$$

Again, it is easy to see that $i_\bullet$ is uniquely determined by i_*. However, this does not mean that we can pick any i_* as we want. The necessary and sufficient condition for i_* is that $i_* F(f_0) = i_* F(f_1)$:

$$\begin{tikzcd}
\bullet \arrow[r, shift left, "F(f_0)"] \arrow[r, shift right, "F(f_1)"'] & * \arrow[d, "i_*"] \\
& X
\end{tikzcd}$$

Therefore, w.l.o.g, we can say that a cone under F with nadir X is a map i_* such that $i_* F(f_0) = i_* F(f_1)$.
For a functor $F : (\mathbb{N}, \leq) \to \mathcal{C}$, a cone under F with nadir X is a sequence of

maps $\{i_n : F(n) \to X\}_{n\in\mathbb{N}}$ such that:

$$\begin{tikzcd}
F(0) \arrow[r] \arrow[drrr, "i_0"'] & F(1) \arrow[r] \arrow[drr, "i_1"] & F(2) \arrow[r] \arrow[dr, "i_2"] & F(3) \arrow[r] \arrow[d, "i_3"] & \cdots\cdots \arrow[dl, "\cdots"'] \\
 & & & X &
\end{tikzcd}$$

Can you explain why a cone under $F : (\mathbb{N}, \leq)^{op} \to \mathcal{C}$ is not interesting?

Definition 1.4.228. Let $F : \mathcal{J} \to \mathcal{C}$ be a diagram (functor). By the colimit of F, we mean an object C together with a natural isomorphism $Hom(C, X) \cong Cone^F(X)$. Equivalently, the colimit of F is the object C together with a map $h_A : F(A) \to C$, for any object A in $\mathcal{J}$, such that $h_A = h_B F(f)$, for any $f : A \to B$, i.e.,

$$\begin{tikzcd}
F(A_1) \arrow[rr, "F(f_4)"] \arrow[ddrr, "h_{A_1}"'] & & F(A_3) \arrow[dd, "h_{A_3}"] & \\
 & F(A_0) \arrow[ul, "F(f_1)"'] \arrow[rr, "F(f_2)"'] \arrow[dr, "h_{A_0}"] & & F(A_2) \arrow[ul, "F(f_3)"'] \arrow[dl, "h_{A_2}"] \\
 & & C &
\end{tikzcd}$$

and for any other maps $i_A : F(A) \to D$, for any object A in $\mathcal{J}$ such that $F(f)i_A = i_B$, for any $f : A \to B$, there exists a unique map $j : C \to D$ such that $jh_A = i_A$, for any A, i.e.,

$$\begin{tikzcd}
F(A_1) \arrow[rr, "F(f_4)"] \arrow[ddrr, "h_{A_1}"'] \arrow[ddrrr, dotted, "i_{A_1}"] & & F(A_3) \arrow[dd, "h_{A_3}"'] \arrow[ddr, dotted, "i_{A_3}"] & \\
& F(A_0) \arrow[ul, "F(f_1)"'] \arrow[rr, "F(f_2)"'] \arrow[dr, "h_{A_0}"] \arrow[drr, dotted, "i_{A_0}"] & & F(A_2) \arrow[ul, "F(f_3)"'] \arrow[dl, "h_{A_2}"] \arrow[d, dotted, "i_{A_2}"] \\
& & C \arrow[r, dashed, "j"'] & D
\end{tikzcd}$$

the category $\mathcal{J}$ is (finite) small. A category is called cocomplete, if it has all small colimits and finitely cocomplete, if it has all finite colimits.

Example 1.4.229. Let $F : \mathbf{0} \to \mathcal{C}$ be the functor from the empty category to $\mathcal{C}$. The colimit of F is the initial object. For any functor $F : \mathbf{1} + \mathbf{1} \to \mathcal{C}$ the limit is the coproduct of the objects in the image of F. Recall that we had the following categories:

$$\mathcal{J}: \quad \begin{tikzcd} * \arrow[r] \arrow[d] & \dagger \\ \bullet & \end{tikzcd} \qquad\qquad \mathcal{K}: \quad \begin{tikzcd} \bullet \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & * \end{tikzcd}$$

Then, the colimit of any functor $F : \mathcal{J} \to \mathcal{C}$ is the pushout of the F-image of the arrows along each other and the colimit of any functor $F : \mathcal{K} \to \mathcal{C}$ is the equalizer of the F-image of the two arrows $F(f_0)$ and $F(f_1)$.

Example 1.4.230. If $\mathcal{J}$ is a group G and $F : G \to \mathbf{Set}$ be a G-action. Then, the colimit of F:

$$\begin{tikzcd}
& C \arrow[dl, "h_*"'] \arrow[dr, "h_*"] & \\
F(*) \arrow[rr, "F(g)"'] & & F(*)
\end{tikzcd}$$

is just the set $F(*)/R$ where $R = \{(x, y) \in F(*) \times F(*) \mid \exists g \in G \; F(g)(x) = y\}$ together with the projection map $h_* : F(*) \to F(*)/R$. The set $F(*)/R$ is actually the set of all orbits.

Example 1.4.231. Any group is a colimit of its finitely-generated subgroups. More formally, let G be a group and $\mathcal{J}$ be the poset of all finitely-generated subgroups of G with the inclusion. Then, if $F : \mathcal{J} \to \mathbf{Grp}$ is the inclusion functor, the colimit of F is G with legs $h_H : H \to G$ as the inclusion homomorphism. It is clear that the diagram is commutative. To show that G is the best choice, assume $\{i_H : H \to K\}_{H \in \mathcal{J}}$ be a cone under F. Then, define $j : G \to K$ by $j(g) = i_{\langle g \rangle}(g)$, where $\langle g \rangle$ is the cyclic group generated by $g \in G$. The map j is a homomorphism, i.e., $j(gg') = j(g)j(g')$. As $\{i_H : H \to K\}_{H \in \mathcal{J}}$ is a cone under F, we have $i_{\langle g \rangle}(g) = i_{\langle g, g' \rangle}(g)$. Similarly, we have $i_{\langle g' \rangle}(g') = i_{\langle g, g' \rangle}(g')$ and $i_{\langle gg' \rangle}(gg') = i_{\langle g, g' \rangle}(gg')$. Since $i_{\langle g, g' \rangle}$ is a homomorphism, we have $i_{\langle g, g' \rangle}(gg') = i_{\langle g, g' \rangle}(g) i_{\langle g, g' \rangle}(g')$. Hence, $i_{\langle gg' \rangle}(gg') = i_{\langle g \rangle}(g) i_{\langle g' \rangle}(g')$. The map i_H is the composition of the inclusion and the map j. The argument is again similar to what we did for proving that j is a homomorphism. The uniqueness of such j is obvious.

Philosophical Note 1.4.232. The main reason why the previous example works is twofold. First, the fact that we are working with algebras (sets equipped with some operators satisfying certain equations) and second that the operators of the algebras (in our example, the product) are finitary. For instance, to show that j preserves the operators, we need to put all the inputs of the operator in one finitely-generated algebra which needs the number of these inputs (the arity of the operator) to be finite. Philosophically speaking, we can say that in the finitary algebraic world (groups, rings, etc) we can *construct* an algebra by their finitely-generated subalgebras and hence understanding the maps *going out* from an algebra reduced to the maps *going out* from some finitely-generated algebras .