Mathematical Structuralism, S06

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1 Category Theory (continued)

1.1 Functors and Natural Transformations

Example 1.1. (*Exponentiation*) Let A be a fixed set. Define the assignment $(-)^A : \mathbf{Set} \to \mathbf{Set}$, mapping a set B to $B^A = \{f : A \to B\}$ and a function $f : B \to C$ to a function $f^A : B^A \to C^A$ defined by $f^A(g) = fg$. Then, $(-)^A$ is a functor, generalizing the finite power functor $A \mapsto A^n$ generated by the iteration of the product functor. Similarly, it is possible to define the functor $A^{(-)} : \mathbf{Set}^{op} \to \mathbf{Set}$, mapping a set B to $A^B = \{f : B \to A\}$ and a map $f : B \to C$ to a function $A^f : A^C \to A^B$ defined by $A^f(g) = gf$. Then, $A^{(-)}$ is a functor, generalizing the functor $\bar{P} = 2^{(-)}$. Any combination of the product, the sum, and the functor $(-)^A$, for different fixed sets A such as $F(X) = A_0 \times X^{N_0} + A_1 \times X^{N_1} + \cdots + A_k \times X^{N_k}$ is a polynomial functor. The notion of polynomial functor, though, is more general than this.

Remark 1.2. *(Algebras)* Algebras are sets equipped with some operations that have some properties. For instance, a monoid is a set M with an element e and a binary operation such that the latter is associative and the former is the identity element for the latter. The operational data (not the properties) can be stored in one function $a : F_m(M) \to M$, where $F_m(X) = 1 + X^2$ is a functor, storing the type of the algebra and $a(0, *) = e$ and $a(1, m, n) = mn$, storing the operations. By type we mean the number and the arity of the operations (in the monoid case it is one nullary and one binary operations). Some examples may be helpful here. A group $(G, e, (-)^{-1}, \cdot)$ is a set G with a function $a : F_g(G) \to G$, where $F_g(X) = 1 + X + X^2$, $a(0, *) = e$, $a(1, m) = m^{-1}$ and $a(2, m, n) = mn$; the basic structure of natural numbers, i.e., $(\mathbb{N}, s, 0)$ is a function $a : F_i(\mathbb{N}) \to \mathbb{N}$, where $F_i(X) = 1 + X$, $a(0, *) = 0$ and $a(1, n) = s(n) = n + 1$ and the structure $(\mathbb{W}, s_0, s_1, \epsilon)$ of binary strings can be described by a function $a : F_s(\mathbb{W}) \to \mathbb{W}$, where $F_s(X) = 1 + X + X$,

$a(0,*) = \epsilon$, $a(1,w) = s_0(w) = w0$ and $a(2,w) = s_1(w) = w1$. To have a general notion of algebra, we use a functor $F : \mathbf{Set} \to \mathbf{Set}$ to formalize the type of the algebra and then by an F-algebra, (an algebra of type F), we mean a function $a : F(A) \to A$. This also suggest a generalization for homomorphisms. Generally, a homomorphism is a function that preserves all the operations in the type of the algebra. With our generalization here, an F-algebra homomorphism from the F-algebra $a_A : F(A) \to A$ to the F-algebra $a_B : F(B) \to B$ is a function $f : A \to B$ such that

$$\begin{tikzcd} F(A) \arrow[r, "a_A"] \arrow[d, "F(f)"'] & A \arrow[d, "f"] \\ F(B) \arrow[r, "a_B"'] & B \end{tikzcd}$$

It is easy to check that in the familiar cases it really captures the notion of homomorphism.

Example 1.3. (*Forgetful Functors*) Sometimes, we have a category and we will forget some of the structures that the objects posses and the maps preserve, to think somewhat loosely about the same data that we originally had. Let us provide three examples of such phenomenon. First, the forgetful assignment mapping any group G and any homomorphism $f : G \to H$ in $\mathbf{Grp}$ to themselves in $\mathbf{Set}$, forgetting that there is the group structure there, is a functor. For the second example, take the two forgetful functors from $\mathbf{Set}^{\to}$ to $\mathbf{Set}$, forgetting that a variable set actually varies, by making two snapshots of a variable set in the two possible moments. More precisely, for any $i \in \{0,1\}$, define $p_i : \mathbf{Set}^{\to} \to \mathbf{Set}$, by mapping any $f : A_0 \to A_1$ to A_i and any $\alpha : f \to g$ to $\alpha_i : A_i \to B_i$, where $f : A_0 \to A_1$ and $g : B_0 \to B_1$. Both p_0 and p_1 are functors. Finally, as the third example, define $V : \mathbf{Quiv} \to \mathbf{Set}$, by mapping any quiver to its set of elements and any quiver morphism to its underlying function on vertices. This V is a functor. We can do the same thing to define the edge functor E.

Example 1.4. *(Free Functors)* In some cases, we want to put a structure on an object in a free way, meaning we want it to be free from any unexpected relations. For instance, let X be a set. Then, $F(X)$ as the set of all finite sequences of the elements of X (including the empty sequence) with concatenation is a free-monoid constructed from X. It is a monoid, since concatenation is associative and the empty sequence is an identity. It is free because we add all possible products of the elements of X, and there is no non-trivial relation on the elements of $F(X)$, except what the monoid structure dictates. This assignment F gives rise to a functor **Set** $\to$ **Mon**, mapping any set X to the monoid $F(X)$ and any map $f : X \to Y$ to the homomorphisms $F(f) : F(X) \to F(Y)$ such that $F(f)(\sigma) = f(\sigma_0) \cdots f(\sigma_n)$, for any finite sequence $\sigma = \sigma_0\sigma_1 \cdots \sigma_n$.

Example 1.5. Let $\mathcal{C}$ be a category. Then, the identity functor $id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ mapping any object and morphism to itself is a functor. Moreover, if A is a fixed object in $\mathcal{C}$, the constant assignment $c_A : \mathcal{C} \to \mathcal{C}$, mapping all objects to A and all morphisms to identity is another functor.

Example 1.6. Let $\mathcal{C}$ be a groupoid. Then, the inverse assignment $inv : \mathcal{C} \to \mathcal{C}^{op}$, defined by $inv(A) = A$ and $inv(f) : B \to A$ as $inv(f) = f^{-1}$, for $f : A \to B$, is a functor.

Example 1.7. Let $\mathcal{C}$ be a locally small category. The assignment $Hom_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \to$ **Set**, defined by $Hom_{\mathcal{C}}(A, B) = \{f : A \to B \mid f \in Mor(\mathcal{C})\}$ and $Hom_{\mathcal{C}}(g, h) : Hom_{\mathcal{C}}(A, B) \to Hom_{\mathcal{C}}(C, D)$ as $Hom_{\mathcal{C}}(g, h)(f) = hfg$, for any $f : A \to B$, $g : C \to A$ and $h : B \to D$, is a functor. This functor captures the whole structure of the category $\mathcal{C}$.

Example 1.8. Let $\mathcal{C}$ be a locally small category. For any object A in $\mathcal{C}$, there is a canonical functor $Hom_{\mathcal{C}}(A, -) : \mathcal{C} \to$ **Set**, capturing the behavior of the maps above A. It is defined by $B \mapsto Hom_{\mathcal{C}}(A, B)$ and $Hom(A, f) : Hom_{\mathcal{C}}(A, B) \to Hom_{\mathcal{C}}(A, C)$ as $Hom_{\mathcal{C}}(A, f)(g) = fg$, for any $f : B \to C$. Similarly, there is a canonical functor $y_A = Hom_{\mathcal{C}}(-, A) : \mathcal{C}^{op} \to$ **Set**, capturing the behavior of the maps below A. It is defined by $y_A(B) = Hom_{\mathcal{C}}(B, A)$ and $y_A(f) : Hom_{\mathcal{C}}(C, A) \to Hom_{\mathcal{C}}(B, A)$ as $y_A(f)(g) = gf$, for any $f : B \to C$. These functors are the localized version of the concrete representation we have introduced for the small categories, mapping an object A to $A_* = \{g : C \to A \mid g \in Mor(\mathcal{C})$ and $f : A \to B$ to $f_* : A_* \to B_*$ by $f_*(g) = fg$. The current act of localization has no point except to handle the size issue that in a locally small category the collection A_* is not necessarily a set.

Example 1.9. Let $\mathcal{C}$ be a category and $f : A \to B$ be a morphism. The assignment mapping an object $g : X \to A$ in $\mathcal{C}/A$ to the object $fg : X \to B$ in $\mathcal{C}/B$ and mapping to themselves is a functor from $\mathcal{C}/A$ to $\mathcal{C}/B$. We denote this functor by $f_* : \mathcal{C}/A \to \mathcal{C}/B$.

Example 1.10. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be some categories and $F : \mathcal{D} \to \mathcal{E}$ and $G : \mathcal{C} \to \mathcal{D}$ be two functors. Then, the composition $FG : \mathcal{C} \to \mathcal{E}$ with the canonical definition is also a functor.

Note that all small categories with functors as morphisms constitute a category. We denote this category by **Cat**.

Example 1.11. Let $\mathcal{C}$ be a small category. Then, the assignment mapping an object A to the category $\mathcal{C}/A$ and morphism $f : A \to B$ to the functor $f_* : C/A \to C/B$ is a functor from $\mathcal{C}$ to **Cat**.

Example 1.12. (*Baby Schemes*) Let $\mathcal{R}$ be the category of all subsets R of $\mathbb{C}$, including 1 and closed under addition and multiplication with morphisms as the functions that preserve the element 1 and these two operations. Let $I(\vec{x}) = I(x_0, \ldots, x_n)$ be a set of equations between polynomials in variables $x_0 \ldots, x_n$ with coefficients in $\mathbb{Z}$. For instance, we can take $I(x_0, x_1) = \{x_0^2 + x_1^2 = 1\}$. Define the assignment $V_I : \mathcal{R} \to \mathbf{Set}$ by mapping R to $V_I(R) = \{\vec{r} \in R^{n+1} \mid \text{all equations in } I(\vec{x}) \text{ hold for } \vec{x} = \vec{r}\}$ and any $f : R \to S$ to the function $V_I(f) : V_I(R) \to V_I(S)$ defined by $V_I(f)(\vec{x}) = (f(x_0), \ldots, f(x_n))$. The function $V_I(f)$ is well-defined, because when $\vec{r}$ is the root for an equation, then so is $f(\vec{r})$, simply because f preserves 1, addition and multiplication. This assignment is clearly a functor. It is reasonable to think of V_I as a method to keep track of all the possible realizations (models) of the set of equations in all possible worlds. It is the semantical way to capture the syntactic data $I(\vec{x})$.

Remark 1.13. Note that V_I is not a faithful semantical apparatus. For instance, for the different sets of equations $I(x) = \{x = 0\}$ and $J(x) = \{x^2 = 0\}$, we have $V_I(R) = V_J(R) = \{0\}$.

Example 1.14. *(Fundamental set* Π_0*)* Let **Quiv** be the category of quivers (directed multi-graphs). For any quiver Q, define the equivalence relation $\sim$ on $V(Q)$ by $v \sim w$ iff there exist two paths of edges in $E(Q)$ (including the paths with length zero), one starting from v and ending in w and one starting from w and ending in v. (Why is it an equivalence relation?) Define the assignment $\Pi_0 : \mathbf{Quiv} \to \mathbf{Set}$ on objects by $\Pi_0(Q)$ as the set of equivalence classes in $V(Q)$ and on quiver morphism $f : Q \to Q'$ by $\Pi_0(f)([v]) = [f(v)]$. (Why is it well-defined?) The assignment Π_0 is a functor. It measures how connected the quiver is. It is also possible to use a more refined version in which the functor returns not only the set $\Pi_0(Q)$ but also its underlying order, defined by $[v] \leq [w]$ iff there exists a path from v to w. (Why is it a poset order?) It is not hard to see that $\Pi_0(f)$ also respects this order. Denote this functor by $\Pi_0^d : \mathbf{Quiv} \to \mathbf{Poset}$.

Remark 1.15. Note that Π_0 is not faithful as it sends any two connected quivers to a singleton.

Exercise 1.16. Prove that functors preserve isomorphisms.

Philosophical Note 1.17. Non-faithful functors provide some room to simplify the original object A in a discourse $\mathcal{C}$ to a simpler object $F(A)$ in $\mathcal{D}$. When $F(A)$ is "computable" in a relatively easy way, F can be useful in showing that two given objects in $\mathcal{C}$ are not isomorphic. The strategy is as follows: Assume that an isomorphism $f : A \to B$ exists between two given objects A and B. Then, by the application of the functor F, we must have an isomorphism between $F(A)$ and $F(B)$ in $\mathcal{D}$. Now, compute both $F(A)$ and $F(B)$ and show that they can not be isomorphic. Let us explain the strategy by an example. Consider the following quivers:

$$Q : \begin{tikzcd} \bullet \arrow[d] \arrow[dr, bend left] \arrow[dr, bend right] & \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad Q' : \begin{tikzcd} \bullet \arrow[r] & \bullet \\ \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \end{tikzcd}$$

They are not isomorphic, since the forgetful functor $V : \mathbf{Quiv} \to \mathbf{Set}$ maps Q to a three element set (the set of vertices) and Q' to a four element set. These two sets can not be isomorphic in **Set**. Hence, Q and Q' are not isomorphic as quivers. Note that the functor V is easy to compute and this is the key element that makes it useful here. Moreover, it is important to observe that showing two sets are not isomorphic boils down to an easy cardinality argument. However, as the functor is not faithful, it has its own blind spots. For instance, in the following situation

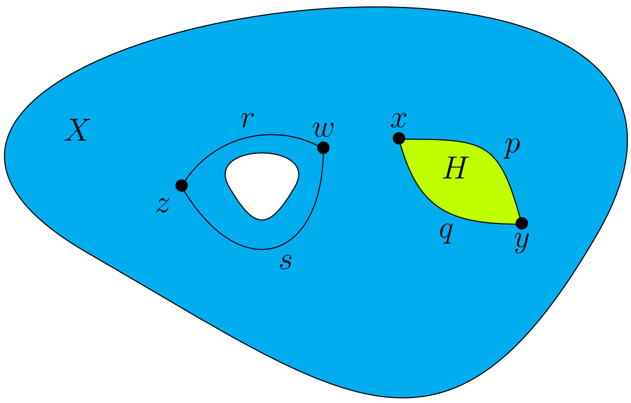
$$P : \begin{tikzcd} \bullet \arrow[d] \arrow[dr] & \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad P' : \begin{tikzcd} \bullet \arrow[d] & \\ \bullet \arrow[r] & \bullet \arrow[ul] \end{tikzcd}$$

both functors V and E are blind to the difference. In such cases, it is reasonable to use more sophisticated functors. But, remember, they must remain relatively easier to handle than the original object. In this case, we use the functor Π_0. Since, $\Pi_0(P)$ is a three element set while $\Pi_0(P')$ is just a singleton, P and P' are not isomorphic as quivers. As the last example, consider the following two quivers:

$$R : \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, bend left] \arrow[dr, bend right] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad R' : \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet \arrow[r] \arrow[ur, bend left] \arrow[ur, bend right] & \bullet \arrow[u] \end{tikzcd}$$

Here, all the three functors V, E and Π_0 agree. However, $\Pi_0^d(R)$ is a lozenge while $\Pi_0^d(R')$ is just a line.

Example 1.18. (*Fundamental Groupoid*) Let **Top** be the category of all topological spaces with continuous functions. For any topological space X, consider the set of paths in X from x to y, i.e., all continuous functions $p : [0,1] \to X$ such that $p(0) = x$ and $p(1) = y$, denoted by $Path_X(x, y)$. Define the equivalence relation $\sim$ on $Path_X(x, y)$ by $p \sim q$ iff there exists a surface in X filling between p and q, i.e., a continuous function $H : [0,1] \times [0,1] \to X$ such that H maps $\{0\} \times [0,1]$ to x, $\{1\} \times [0,1]$ to y and the restrictions of H to $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$ becomes p and q, respectively. (Why is it an equivalence relation?)



In the figure, the image of H is depicted by the green area and hence $p \sim q$, while r and s can not be in the same class as the white hole in the middle prevent any filling 2-surface between r and s. Now, define $\Pi_0(X)$ as the groupoid with the objects as the elements of X, the morphisms from x to y as $Path_X(x, y)$ and composition and identity as the canonical pasting paths to each other and the class of the constant path. (Why is composition well-defined? Why is the constant map the identity morphism?) Define the assignment Π_1 : **Top** $\to$ **Groupoid** on objects by $\Pi_1(X)$ and on a morphism $f : X \to Y$ by the functor $\Pi_1(f)$ defined by $\Pi_1(f)(x) = f(x)$ and $\Pi_1(f)([p]) = f[p]$. (Why is it well-defined?) The assignment Π_1 is a functor. It is possible to simplify the functor Π_1 with some non-canonical choice for a base point. Let X be a space and $x \in X$ be a point in X. Now, restrict the groupoid $\Pi_(X)$ to the object x and the morphisms over x. This is also a functor, usually denoted by π_1, this time from the category of pointed spaces, denoted by $\mathbf{Top}_*$ to the category **Grp**. Both Π_1 and π_1 measure the 2-holes in a space X as Π_0 measured 1-holes. (1-hole means disconnectedness. Right?) For instance, for the space $B_2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ and any possible choice for the base point $a \in B_2$, the group $\pi_1(B_2, a)$ is just a singleton, as any path over a in B_2 can be filled and B_2 (why?) or in other words as B_2 has no holes. At the same time, for the circle $S_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, the group $\pi_1(S_1, a)$ for any base point $a \in S_1$ is $\mathbb{Z}$, as any path over a in S_1 is uniquely determined by the number it goes around S_1. (Why?) These are obvious claims. But intuitively, they are just clear.

Example 1.19. (*Application of the Fundamental Groups*) We want to prove *Brouwer's fixed point theorem* for 2-ball $B_2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ that states that any continuous function $f : B_2 \to B_2$ has a fixed point. For the sake of contradiction, assume f does not have a fixed point. Then, for any $(x, y) \in B_2$, we have $f(x, y) \neq (x, y)$. Define $r : B_2 \to S_1$ in the following way: Take the directed line L, connecting $f(x, y)$ to (x, y) and define $r(x, y)$ as the intersection of L and the border of B_2 which is S_1. By definition, the restriction r to S_1 is the identity function. Therefore, if we denote the inclusion of S_1 in B_2 by $i : S_1 \to B_2$, we have:

$$\begin{tikzcd} S_1 \arrow[r, "i"'] \arrow[rr, bend left, "id_{S_1}"] & B_2 \arrow[r, "r"'] & S_1 \end{tikzcd}$$

Since $\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$ is a functor, we have:

$$\begin{tikzcd} \pi_1(S_1, a) \arrow[r, "\pi_1(i)"'] \arrow[rr, bend left, "id_{\pi_1(S_1,a)}"] & \pi_1(B_2, a) \arrow[r, "\pi_1(r)"'] & \pi_1(S_1, a) \end{tikzcd}$$

which is impossible, as $\pi_1(S_1, a)$ is isomorphic to $\mathbb{Z}$, while $\pi_1(B_2, a)$ is a singleton group.