Goal: present a recicpi to construct Lyapnov funct. for lin. sys.. Important for analysis of "nearly liner sys"

- It is easy to construct a Lyapunov funct. for a one-dim lin. system

$$\dot{x} = -x, \quad V(x) = x^2 \rightarrow \dot{V}(x) = -2x^2 < 0 \quad \forall x \neq 0$$

- $V(x) = x^2$ works for any stable lin sys. $\dot{x} = -ax$ with $a > 0$

$$\dot{V}(x) = -ax^2 < 0 \quad \forall x \neq 0$$

- However, the construction is not as easy for a multi-dim sys.

- Let's consider a 2-dim example: (mass-spring)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}}_{\text{Hurwitz}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\rightarrow \dot{x}_1 = x_2$

$\rightarrow \dot{x}_2 = -4x_1 - x_2$

Try 1: let $V(x) = x_1^2 + x_2^2 \longrightarrow$ satisfies p.d. condition

$$\dot{V}(x) = 2x_1(x_2) + 2x_2(-x_1 - 2x_2)$$

$$= 2x_1x_2 - 2x_1x_2 - 4x_2^2$$

$$= -4x_2^2 \leq 0$$

$\rightarrow$ but we can not say $\dot{V}(x) < 0 \quad \forall x \neq 0$

Try 2: $V(x) = Ax_1^2 + 2Bx_1x_2 + Cx_2^2$

then,
$$\dot{V}(x) = 2Ax_1x_2 - 2Cx_1x_2 - 4Cx_2^2 + 2Bx_2^2 - 2Bx_1^2 - 4Bx_1x_2$$
$$= (2A - 2C - 4B)x_1x_2 - (4C - 2B)x_2^2 - 2Bx_1^2$$

- we choose A, B, C so that $\dot{V}(x) = -x_1^2 - x_2^2$

$$\Rightarrow B = \frac{1}{2}, \quad C = \frac{1}{2}, \quad A = \frac{3}{2}$$

$$\Rightarrow \dot{V}(x) = -x_1^2 - x_2^2 < 0, \quad \forall x \neq 0$$

- To check if V is p.d.

$$V(x) = \frac{3}{2}x_1^2 + x_1x_2 + \frac{1}{2}x_2^2$$

$x_1x_2 \geq -\frac{1}{4}x_2^2 - x_1^2$

$$\geq \frac{1}{2}x_1^2 + \frac{1}{4}x_2^2 > 0 \quad \forall x \neq 0$$

- This try and error is messy and difficult to check.
- Let's automate the procedure.
- Need to introduce positive quadratic functions.

Quadratic function

- General form of a quadratic function of n variables

wlog, assume $P_{ij} = P_{ji}$

$$f(x) = \sum_{i,j=1}^{n} P_{ij} x_i x_j$$

$$= \underbrace{[x_1, x_2, \ldots, x_n]}_{x^T} \underbrace{\begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ P_{n1} & \cdots & & P_{nn} \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x}$$

$$= x^T P x$$

symmetric $n \times n$ matrix

$P = P^T$

- Example:

$$f(x) = x_1^2 + 2x_2^2 + 3x_1x_2 = x_1^2 + 2x_2^2 + \frac{3}{2}x_1x_2 + \frac{3}{2}x_2x_1$$

$$= [x_1 \quad x_2] \underbrace{\begin{bmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & 2 \end{bmatrix}}_{P} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

positive definite matrices:

$$P \succ 0 \iff \lambda_{min}(P) > 0 \iff x^T P x > 0 \quad \forall x \neq 0$$

- we have the ineq.

$$\lambda_{min}(P) \|x\|_2^2 \leq x^T P x \leq \lambda_{max}(P) \|x\|_2^2 , \quad \forall x$$

- A 2x2 matrix $P = \begin{bmatrix} A & B \\ B & C \end{bmatrix}$ is p.d. iff

$$A > 0 \quad \text{and} \quad AC - B^2 > 0$$

Lyapunov method for lin. systems:

- Consider the n-dim. linear sys. $\dot{x} = Ax$

- let $V(x) = x^T P x$ be a candidate Lyapunov func.

- To conclude GAS, we need

  - V to be p.d. and radially unbdd
    This is true when $P \succ 0$.

  - $\frac{d}{dt} V(x(t)) < 0 \quad \forall\, x \neq 0.$

$$\frac{d}{dt} V(x(t)) = \frac{d}{dt}\left(x(t)^T P x(t)\right)$$

$$= \dot{x}(t)^T P x(t) + x(t) P \dot{x}(t)$$

$$= x(t)^T (A^T P + P A)\, x(t)$$

This is true when $Q := -A^T P + PA \succ 0$

- Therefore, it is enough to find a $P \succ 0$

s.t. $Q = -A^T P - PA \succ 0$.

Thm: (Thm. 4.6 in Khalil)

- If A is Hurwitz, then the eq.

Lyapunov eq. $\boxed{A^T P + PA + Q = 0}$

has a solution $P \succ 0$ for all $Q \succ 0$.

- The solution is

$$P = \int_0^\infty e^{tA^T} Q e^{tA} \, dt$$

- Conversely, if Lyp. eq. has a solution $P \succ 0$, for any $Q \succ 0$ then, A is Hurwitz

A is Harwitz $\Longleftrightarrow$ $x=0$ is GAS $\Longleftrightarrow$ Lyp. eq. has a P.d. solution

① why is P a solution

- let $S(t) = e^{tA^T} Q e^{tA}$. Then

$$\frac{d}{dt} S(t) = A^T S(t) + S(t) A$$

$$\Rightarrow \int_0^T \frac{d}{dt} S(t) = \int_0^T A^T S(t) + S(t) A \, dt$$

$$\Rightarrow S(T) - S(0) = A^T \int_0^T S(t) dt + \int_0^T S(t) dt \, A$$

let $T \to \infty$. if A is Hurwitz, then

$$S(T) = e^{TA^T} Q e^{TA} \to 0 \quad \text{as } T \to \infty$$

and $$\int_0^T S(t) dt = \int_0^T e^{tA^T} Q e^{tA} dt \to P \quad \text{as } T \to \infty$$

- Therefore,

$$0 - Q = A^T P + PA \quad \checkmark$$

② why is $P \succ 0$?

$$x^T P x = \int_0^\infty \underbrace{x^T e^{tA^T}}_{y(t)^T} Q \underbrace{e^{tA} x}_{y(t)} \, dt$$

$$= \int_0^\infty \underbrace{y(t)^T Q y(t)}_{>0} \, dt > 0 \quad \text{when} \quad x \neq 0$$

because $Q \succ 0$
and $y(t) \neq 0$
when $x \neq 0$

Example:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \rightarrow \text{A is Hurwitz}$$

- let $Q = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and find $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$

such that $A^T P + PA + Q = 0$

$$\Rightarrow \begin{bmatrix} -2p_{12}+1 & p_{11}-2p_{12}-p_{22} \\ p_{11}-2p_{12}-p_{22} & 2p_{12}-4p_{22}+1 \end{bmatrix} = 0$$

$$\Leftrightarrow \quad p_{12} = \frac{1}{2}, \quad p_{22} = \frac{1}{2}, \quad p_{11} = \frac{3}{2}$$

$$\Leftrightarrow \quad P = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \rightarrow \text{pos. definite.}$$

Summary:

- Consider the lin sys $\dot{x} = Ax$

$\forall Q \succ 0,\ \exists P \succ 0$ s.t.

$x = 0$ is GAS $\iff$ A is Hurwitz $\iff$ $PA + A^T P + Q = 0$

$\Downarrow$

$V(x) = x^T P x$ is the Lyapunov func.

- We use this Lyapunov func. to analyze stability of nearly lin. systems.