Input-Output Stability

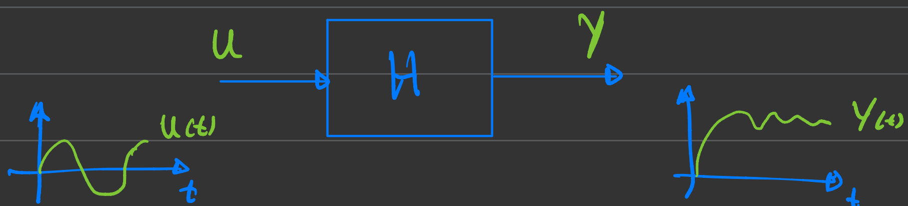
- So far, we studied stability of "autonomous" dyn. sys.

$$\dot{x} = f(x)$$

- Now, in part III, we will consider input and output:

$$H \begin{cases} \dot{x} = f(x, \overset{\text{input}}{u}) \\ \underset{\text{output}}{y} = h(x, u) \end{cases}$$

- This nonlinear system defines a relationship or a map from input signal $u(t)$ to output signal $y(t)$.



- Our goal is to understand this input-output relationship.

- In particular we are looking for this type of

inequalities: $\underbrace{\|Y\|_L \leq \gamma \|u\|_L + \beta}$ $\forall u$ where

norm of Y is controlled by norm of u

- $\|\cdot\|_L$ is a norm for signals.

- γ, β are positive constants.

- A system H that satisfies this ineq. is called L-stable.

- the smallest possible value for γ is called gain.

- Why is this inequality useful:

  ① small disturbance $\to$ small output error

  for example in navigating a robot or cruise control

  ② we can use it to establish stability of interconnected systems. (small gain thm.)

plan:

① define signal norms L_p → we will focus on L_2-norm

② L_2-stability of lin. sys.

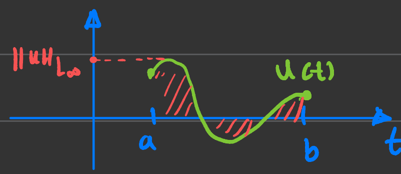
③ L_2-stability of nonlin. sys.. (using Lyapunov function)

④ small gain thm.

Signal norms:

- L_p-norm: extension of $\|\cdot\|_p$-norm for vectors to signals

$$\|x\|_{L_p} = \left[\int_a^b |x(t)|^p \, dt\right]^{\frac{1}{p}}$$



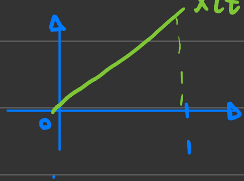
$$\|x\|_{L_\infty} = \sup_{t\in[0,T]} |x(t)|$$

$$x : [a,b] \to \mathbb{R}$$

- Important special case:

$$\|x\|_{L_2} = \left[\int_0^T |x(t)|^2 \, dt\right]^{\frac{1}{2}} \to \text{energy of the signal}$$

Example:



$$\|x\|_{L_1} = \int_0^1 |x(t)| \, dt = \int_0^1 t \, dt = \frac{1}{2}$$

$$\|x\|_{L_2} = \left[\int_0^1 |x(t)|^2 \, dt\right]^{\frac{1}{2}} = \left[\int_0^1 t^2 \, dt\right]^{\frac{1}{2}} = \frac{1}{\sqrt{3}}$$

$$\|x\|_{L_\infty} = \sup_{t\in[0,1]} |x(t)| = \sup_{t\in[0,1]} |t| = 1$$

L_2-Stability of Lin. systems: (SISO)

assume A is Hurwitz

$$\dot{x} = Ax + Bu, \qquad x(0) = 0$$
$$y = Cx + Du$$

- we like to establish the inequality

$$\|y\|_{L_2} \leq \gamma \|u\|_{L_2} + \beta$$

Smallest possible γ is L_2-gain

and find the value of γ.

- we do this by using Laplace-transform and transfer function:

$x(t) \longrightarrow \hat{x}(s) = \int_0^\infty x(t) e^{-st} dt$

$$s\hat{X}(s) = A\hat{X}(s) + B\hat{u}(s)$$
$$\hat{Y}(s) = C\hat{X}(s) + D\hat{u}(s)$$

$$\Longrightarrow \quad \hat{Y}(s) = \underbrace{\left[C(sI-A)^{-1}B + D\right]}_{G(s)} \hat{u}(s)$$

$$\overset{s=j\omega}{\Longrightarrow} \quad \hat{Y}(j\omega) = G(j\omega)\,\hat{u}(j\omega)$$

- Parseval's idenitity: $\boxed{\int_0^\infty |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{x}(j\omega)| \, d\omega}$

- Using the PI:

$$\|y\|_{L_2}^2 = \int_0^\infty |y(t)|^2 dt$$

$$= \int_{-\infty}^{+\infty} |\hat{y}(j\omega)|^2 dt$$

$$= \int_{-\infty}^{+\infty} |G(j\omega)\,\hat{u}(j\omega)|^2 dt$$

$$\leq \sup_w |G(j\omega)|^2 \int_{-\infty}^{+\infty} |\hat{u}(j\omega)|^2 dt$$

$$= \sup_w |G(j\omega)|^2 \, \|u\|_{L_2}^2$$

$$\Rightarrow \quad \|y\|_{L_2} \leq \underbrace{\sup_w |G(j\omega)|}_{\text{maximum value in Bode plot.}} \, \|u\|_{L_2} + \beta$$

if $x_0 \neq 0$

this is actually the smallest possible value, hence L_2-gain

Thm. 5.4 in Khalil.

L_2- stability for nonlin. sys.:

- Consider $\dot{x} = f(x,u)$, $x(0) = x_0$

$$y = h(x,u)$$

- If you find a function $V: \mathbb{R}^n \to \mathbb{R}$ s.t. $V(x) \geq 0 \;\; \forall x$

and $\nabla V(x)^T f(x,u) \leq \alpha^2 \|u\|^2 - \beta^2 \|y\|^2$, $\forall x,u$

$h(x,u)$

- Then

$$\|y\|_{L_2} \leq \frac{\alpha}{\beta} \| u\|_{L_2} + \frac{1}{\beta}\sqrt{V(x_0)}$$

upper bound on L_2-gain.

effect of init. condition

- why?

$$\frac{d}{dt} V(x(t)) = \nabla V(x(t))^T f(x(t), u(t))$$

$$\leq \alpha^2 \|u(t)\|^2 - \beta^2 \|y(t)\|^2$$

- Integrating with respect to time

$$V(X(T)) - V(X_0) \leq \alpha^2 \int_0^T \|u(t)\|^2 dt - \beta^2 \int_0^T \|Y(t)\|^2 dt$$

$$\Rightarrow \int_0^T \|Y(t)\|^2 dt \leq \frac{\alpha^2}{\beta^2} \int_0^T \|u(t)\|^2 dt + \frac{1}{\beta^2} V(X_0) - \frac{1}{\beta^2} V(X(T))$$

let $T \uparrow \infty$

$$\Rightarrow \|Y\|_{L_2}^2 \leq \frac{\alpha^2}{\beta^2} \|u\|_{L_2}^2 + \frac{1}{\beta^2} V(X_0)$$

$\sqrt{a^2+b^2} \leq |a| + |b|$

$$\Rightarrow \|Y\|_{L_2} \leq \frac{\alpha}{\beta} \|u\|_{L_2} + \frac{1}{\beta} \sqrt{V(X_0)}$$

Example :

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a x_1^3 - K x_2 + u$$
$$y = x_2$$

- we like to find $V(x)$ s.t.

$$\nabla V(x)^T f(x,w) \leq \alpha^2 |w|^2 - \beta^2 |y|^2$$

or

$$\underset{a x_1^3}{\frac{\partial V}{\partial x_1}} x_2 + \underset{x_2}{\frac{\partial V}{\partial x_2}} (-a x_1^3 - K x_2 + u) \leq \alpha^2 u^2 - \beta^2 x_2^2$$

- let $V(x) = \frac{1}{4} a x_1^4 + \frac{1}{2} x_2^2$. Then

$$\nabla V(x)^T f(x,w) = \frac{\partial V}{\partial x_1} x_2 + \frac{\partial V}{\partial x_2} (-a x_1^3 - K x_2 + u)$$

$$= (-a x_1^3) x_2 + x_2 (-a x_1^3 - K x_2 + u)$$

$$= -K x_2^2 + x_2 u$$

- We use $u x_2 \leq \frac{1}{2\varepsilon} u^2 + \frac{\varepsilon}{2} x_2^2$ for any $\varepsilon > 0$ to conclude

$$\nabla V(x)^T f(x,u) \leq \frac{1}{2\varepsilon} u^2 - \left(K - \frac{\varepsilon}{2}\right) x_2^2$$

- let $\varepsilon = K$. Then,

$$\nabla V(x)^T f(x,u) \leq \underbrace{\frac{1}{2K}}_{\alpha^2} u^2 - \underbrace{\frac{K}{2}}_{\beta^2} y^2$$

$$\Rightarrow \quad \|y\|_{L_2} \leq \frac{1}{K} \|u\|_{L_2} + \sqrt{\frac{2}{K} V(x_0)}$$

$\Rightarrow$ L_2-gain is smaller than $\frac{1}{K}$.

increasing K reduces the L_2-gain.

Remarks

- L_2-gain for control affine sys. (Thm. 5.5)

- Lyapunov function can be used for L_p-stability (Thm. 5.1)