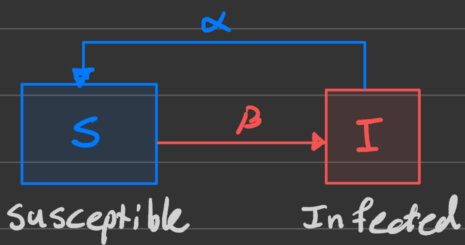
Goal: learn about the phase-portrait method, a visual approach to analyze 1 or 2 dim. dyn. sys. $\dot{x} = f(x)$.

Motivating example: SIS epidemic model

- Suppose we like to study spread of infection in a population.
- The population is divided into two calsses



- We use $S(t)$ and $I(t)$ to denote the number of susceptible and infected individuals at time t.

- In SIS model, the update law for $S(t)$ and $I(t)$ is

$$\dot{S}(t) = -\beta I(t) S(t) + \alpha I(t),$$
$$\dot{I}(t) = \underbrace{\beta}_{\text{infection transmission rate}} I(t) S(t) - \underbrace{\alpha}_{\text{recovery rate}} I(t)$$

- We like to understand long-term behavior of the model

are we all going to get infected?

or we will reach a desease-free state?

- First step, we will simplify this two dimensional model to one-dimension by finding an <u>invariant</u> variable.

$N(t) := I(t) + S(t)$ total population

$\dot{N}(t) = \dot{I}(t) + \dot{S}(t) = \dots = 0$ → total population is constant.

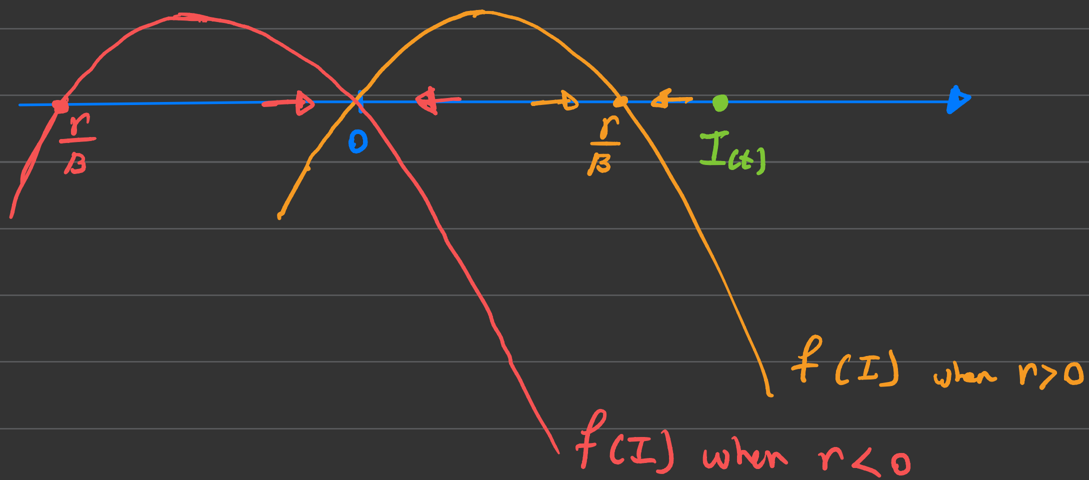
$\Rightarrow N(t) = N(0) =: N$

- We use this to simplify: using $S(t) = N - I(t)$,

$$\dot{I}(t) = \beta I(t)(N - I(t)) - \alpha I(t)$$
$$= I(t)(\underbrace{r}_{\beta N - \alpha} - \beta I(t)) =: f(I(t))$$

logistic eq. ←

- let's try to understand the behavior of the sys.:



- we divide the analysis to two cases:

Case 1: $r>0$

- we draw $f(I)$ for $r>0$.

- there are two points where $f(I)=0$.

$$f(I)=0 \Rightarrow \quad I=0 \quad \text{or} \quad I=\frac{r}{\beta}$$

desease free — endemic

- they are called equilibrium points.

if the sys starts there, it remains there forever.

- the sign of $f(I)$ determines the direction that $I(t)$ moves.

- In this case, $I(t) \to \frac{r}{\beta}$ as $t \to \infty$

- therefore, we call $I = \frac{r}{\beta}$ stable eqlb.

  and $I = 0$ unstable eqlb.

Case 2: $r < 0$

- in this case, the eqlb. point $\frac{r}{\beta}$ is negative

  and does not make sense $\to$ population can not be negative

- $I(t) \to 0$ as $t \to \infty$

  $I = 0$ is stable

  $I = \frac{r}{\beta}$ is unstable

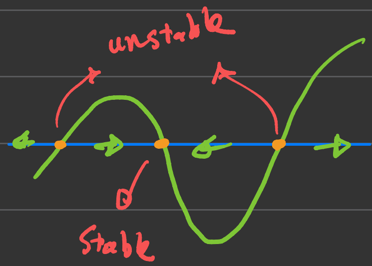
- Let's summarize what we learned so far:

one-dim. system $\dot{x} = f(x)$

eqlb. points: solve for $f(x) = 0$

phase portrait:
method

visual method



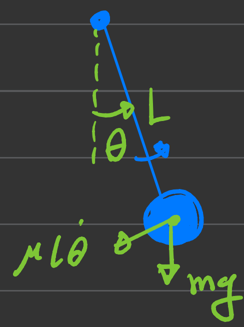
- We can extend the phase portrait to two-dim systems:

my favorite example

Example: (pendelum)

Newton's law:

$$m L^2 \ddot{\theta} = -mgL\sin(\theta) - \mu L \dot{\theta}$$



$$\Rightarrow \ddot{\theta} + \underbrace{\gamma}_{\frac{\mu}{mL}} \dot{\theta} + \underbrace{\omega^2}_{\frac{g}{L}} \sin(\theta) = 0$$

- we will write this as $\dot{x} = f(x)$ for a two-dim x.

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \Rightarrow \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \underbrace{\begin{bmatrix} x_2 \\ -\gamma x_2 - \omega^2 \sin(x_1) \end{bmatrix}}_{f(x)}$$

$$\Rightarrow \underbrace{\dot{x} = f(x)}_{\text{State-space form}} \text{ where } f(x) = \uparrow$$

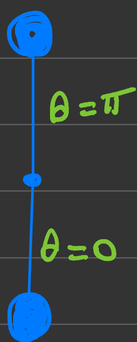
- Equilibrium points:

$$f(x) = 0 \Rightarrow \begin{bmatrix} x_2 \\ -\gamma x_2 - \omega^2 \sin(x_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_2 = 0 \quad \text{and} \quad \sin(x_1) = 0$$

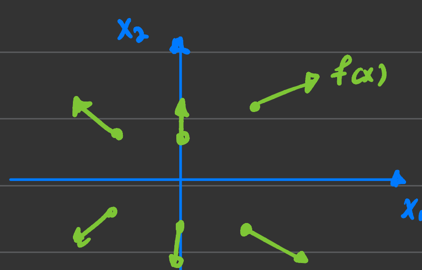
$$\Updownarrow$$

$$x_1 = 0 \quad \text{or} \quad x_1 = \pi$$



the two eqlb. points

- We can understand the behavior of the system by drawing the vector field $f(x)$ in two dim.



Pendulum-Phase-portrait

September 26, 2022

[4]:
```
import numpy as np
import matplotlib.pyplot as plt
```

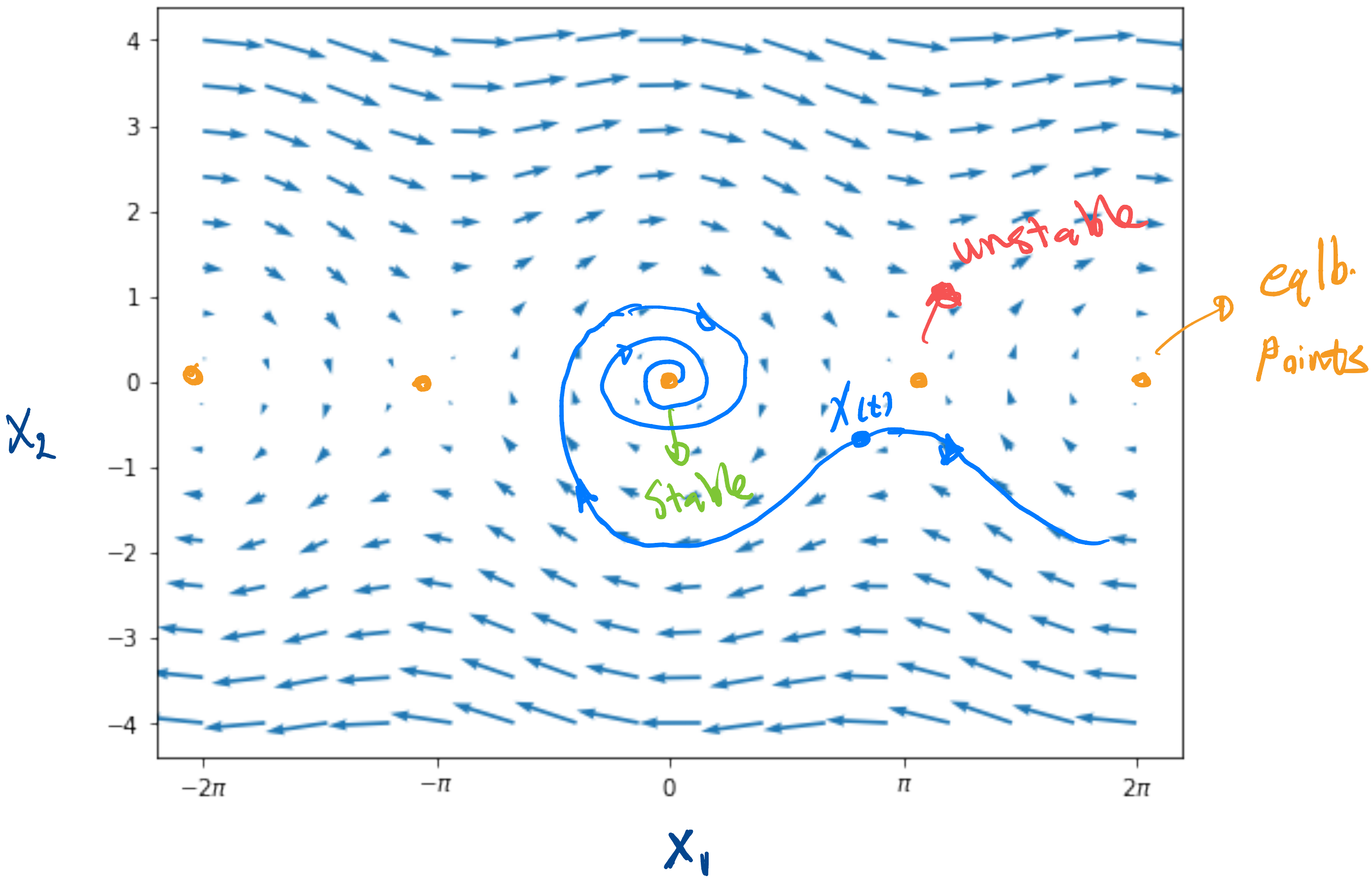
[19]:
```
l1 = 2*np.pi
l2 = 4.0
N = 16
w = 1.0
gamma = 0.1

x1, x2 = np.meshgrid(np.linspace(-l1,l1,N),np.linspace(-l2,l2,N))

f1 = x2
f2 = -w*w*np.sin(x1) - gamma*x2

plt.figure(figsize=(8,6))
plt.quiver(x1,x2,f1,f2,color='C0')
plt.xticks([-2*np.pi,-np.pi,0,np.pi,2*np.
↪pi],[r'$-2\pi$',r'$-\pi$',r'$0$',r'$\pi$',r'$2\pi$'])

plt.show()
```



unstable
eqlb.
Points
X_2
$X(t)$
Stable
X_1
4
3
2
1
0
−1
−2
−3
−4
−2π
−π
0
π
2π

[ ]:

Summary:

① write the sys. dyn. as $\dot{x} = f(x)$

② Solve $f(x) = 0$ for eqlb. points

③ For one and two dim. systems, draw the phase portrait to conclude about system behavior.