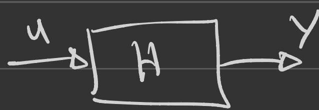
Last time:

- input-output sys.

$$\dot{x} = f(x,u)$$
$$y = h(x,u)$$



assume $f(0,0) = 0$ and $h(0,0) = 0$

- H is passive if $\exists$ a function $V: \mathbb{R}^n \to \mathbb{R}$ s.t.

$$\underbrace{\dot{V}}_{\nabla V(x) f(x,u)} \leq u^T y$$

① input strictly passive $\quad \dot{V} \leq u^T y - \underbrace{u^T \varphi(u)}_{>0 \ \forall u \neq 0}$

② output strictly passive $\quad \dot{V} \leq u^T y - \underbrace{y^T \varphi(y)}_{>0 \ \forall y \neq 0}$

③ strictly passive $\quad \dot{V} \leq u^T y - \underbrace{W(x)}_{>0}$ $\quad \forall x \neq 0$

Today: ① relation between passivity and stability
② passivity for memoryless sys.
③ feedback sys stability analysis by passivity.

Relation between passivity and stability:

Lemma (6.6 in Khalil)

① passive $\xRightarrow{u=0}$ stable

② strictly passive $\xRightarrow{u=0}$ A.S.

③ Output s.p. + $\underbrace{\text{zero-state obs}}_{\substack{\text{if } y(t)=0\ \forall t \\ \text{then } x(t)=0\ \forall t}}$ $\xRightarrow{u=0}$ A.S.

- Moreover, when V is radially unbdd, we have GAS.

proof of ③:

- when $u=0$, we have $\dot{V} \leq -y^T\varphi(y) \leq 0$

- By LaSalle's principle, $x(t) \to M \subseteq E = \{x \in \mathbb{R}^n ; \dot{V}(x) = 0\}$.

$x(t) \in M\ \forall t \Rightarrow \dot{V} = 0\ \forall t \Rightarrow y(t) = 0\ \forall t \xRightarrow{\text{zero-state obs.}} x(t) = 0\ \forall t$

Example:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a x_1^3 - k x_2 + u$$
$$y = x_2$$

with $a, k > 0$. Take $V(x) = \frac{1}{2} x_2^2 + \frac{1}{4} a x_1^4$. Then

$$\dot{V} = \nabla V(x) f(x,u) = -k x_2^2 + u x_2 = -k y^2 + u y$$

output s.p.

- It is also zero-state obs.:

$$y(t) = 0 \Rightarrow x_2(t) = 0 \Rightarrow \dot{x}_2(t) = 0 \Rightarrow x_1(t) = 0$$

- Moreover, V is radially unbdd $\rightsquigarrow$ GAS.

Lemma: (lemma 6.5)

- output strictly passive with $\dot{V} \leq u^T y - \delta \|y\|^2$

implies input-output L_2-stable with L_2-gain smaller then $\frac{1}{\delta}$.

proof:

$$\dot{V} \leq u^T y - \delta \|y\|^2$$

$$\leq \frac{1}{2\delta}\|u\|^2 + \frac{\delta}{2}\|y\|^2 - \delta\|y\|^2$$

$$\leq \underbrace{\frac{1}{2\delta}}_{\alpha^2}\|u\|^2 - \underbrace{\frac{\delta}{2}}_{\beta^2}\|y\|^2$$

lec. 13

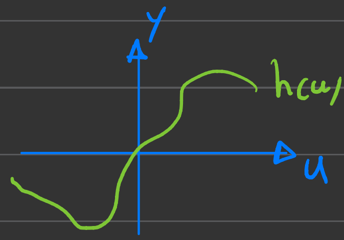
$\Rightarrow$ L_2-stable wit gain smaller then $\frac{\alpha}{\beta} = \frac{1}{\delta}$

Passivity for memoryless or static systems:

- memoryless sys.:

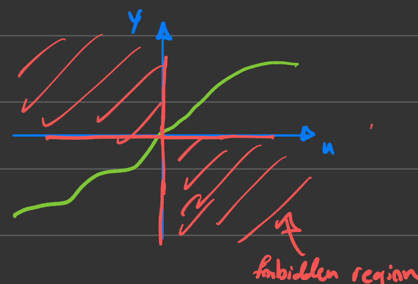
$y = h(u)$ $\quad \xrightarrow{u} \boxed{h(\cdot)} \xrightarrow{y}$

$h(0) = 0$



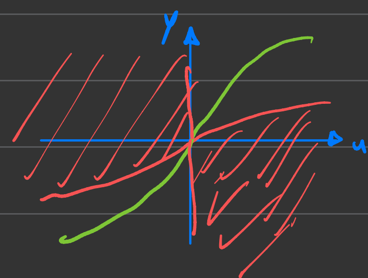
- We use the same def. of passivity, but with $V = 0$.

① passive if $u^T y \geq 0$



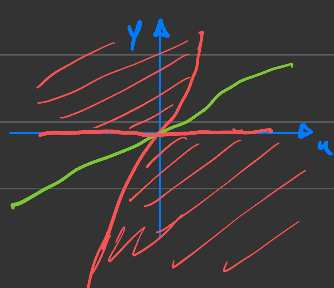
② input s.p. if

$$u^T y \geq u^T \varphi(u)$$



③ output s.p. if

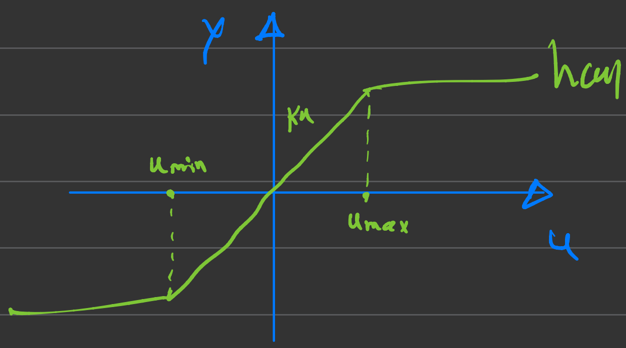
$$u^T y \geq y^T \varphi(y)$$



Example :

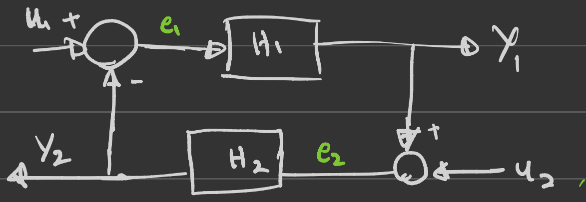
$$h(u) = \begin{cases} Ku & u \in [u_{min}, u_{max}] \\ K u_{max} & u \geq u_{max} \\ K u_{min} & u \leq u_{min} \end{cases}$$

and $K > 0$



input and output strictly passive

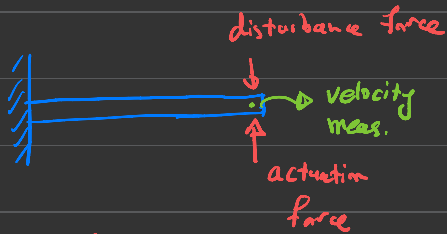
Passivity for feedback connection:

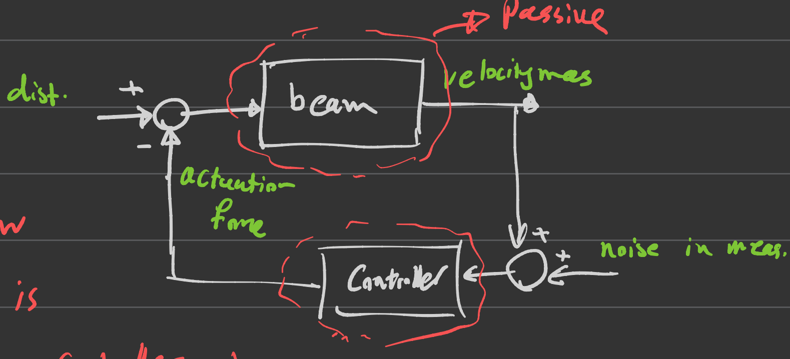


- study stability of feedback sys. when H_1 and H_2 are passive.

Example:

- disturbance rejection in a a flexible beam





We will show feedback sys. is stable when Controller is designed to be passive.

Thm :

- The feedback sys is A.S. if

① both H_1 and H_2 are strictly passive

or

② both H_1 and H_2 are output s.p. and zero-state obs.

or

③ one component is s.p. and the other one is output s.p. and zero-state obs.

proof:

① $\dot{V}_1 \leq e_1^T y_1 - W_1(x)$ $\quad e_1 = u_1 - y_2$

$\dot{V}_2 \leq e_2^T y_2 - W_2(x)$ $\quad e_2 = u_2 + y_1$

let $V = V_1 + V_2 \Rightarrow \dot{V} = \dot{V}_1 + \dot{V}_2 \leq e_1^T y_1 + e_2^T y_2 - W_1(x) - W_2(x)$

$$= u_1^T y_1 + u_2^T y_2 - W_1(x) - W_2(x)$$

$$= \underbrace{[u_1, u_2]^T \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{u^T y} - \underbrace{W(x)}_{W_1 + W_2}$$

$\Rightarrow$ the sys is strictly passive

$\Rightarrow$ A.S when $u = 0$.

② $\dot{V}_1 \leq e_1^T y_1 - y_1^T \varphi(y_1)$

$\dot{V}_2 \leq e_2^T y_2 - y_2^T \varphi(y_2)$

$\Rightarrow \dot{V} = \dot{V}_1 + \dot{V}_2 \leq u^T y - y^T \varphi(y) \Rightarrow$ output s.p.

$y(t) = 0 \Rightarrow y_1(t) = 0, y_2(t) = 0 \Rightarrow x_1(t) = 0, x_2(t) = 0 \Rightarrow x(t) = 0$

$\Rightarrow$ output s.p. and zero-state obs. $\Rightarrow$ A.S.

L_2- input-output stability of feedback sys:

- we showed feedback sys is stable

  when $u = 0$

- Now, we study how much disturbances effect output.

Thm:

- if H_1 and H_2 satisfy

$$\dot{V}_1 \leq u_1^T y_1 - \varepsilon_1 \|u_1\|^2 - \delta_1 \|y_1\|^2$$

$$\dot{V}_2 \leq u_2^T y_2 - \varepsilon_2 \|u_2\|^2 - \delta_2 \|y_2\|^2$$

- Then, feedback sys is input-output stable if

$$\varepsilon_1 + \delta_2 > 0 \quad \text{and} \quad \varepsilon_2 + \delta_1 > 0.$$

→ less restrictive compared to small gain thm.