Goal: Error analysis of solutions under perturbation

Nominal sys.: $\dot{x} = f(t, x, \alpha)$, $x(0) = x_0$

param.

perturbed sys.: $\dot{y} = f(t, y, \beta) + \underbrace{g(t,y)}_{\text{unmodelled effects.}}$, $y(0) = y_0$

objective: bound the error $\|y(t) - x(t)\|$

in terms of error in params, init. cond., unmodelled dyn.

$\|\alpha - \beta\|$ $\|x_0 - y_0\|$ $\|g(t,y)\|$

plan:

1. Comparison lemma
2. Bellman-Gronwall Lemma
3. Main result

Comparison lemma:

$\alpha(t)$ is a number.

- assume $\alpha(t) \in \mathbb{R}$ satisfies

$$(*) \qquad \dot{\alpha}(t) \le a\alpha(t), \quad \alpha(0) \le 1$$

and we like to bound $\alpha(t)$. How?

- Consider another process $\beta(t)$ which satisfies $(*)$ with equality:

$$(**) \qquad \dot{\beta}(t) = a\beta(t), \quad \beta(0) = 1$$

- The solution is given by $\beta(t) = e^{ta}$

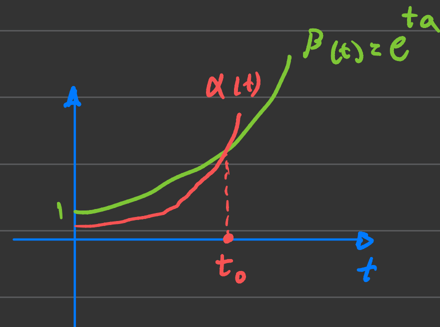
- We can argue that $\alpha(t)$ never exceeds $\beta(t) = e^{ta}$

- If $\alpha(t)$ exceeds $\beta(t)$ at $t=t_0$, then

$$\dot{\alpha}(t_0) > \dot{\beta}(t_0)$$

but

$$\dot{\alpha}(t_0) \overset{(*)}{\le} a\alpha(t_0) = a\beta(t_0) \overset{(**)}{=} \dot{\beta}(t_0)$$



$\rightarrow$ Contradiction!

- Comparison lemma is the generalization of this observation

Lemma:

- Assume $\alpha(t) \in \mathbb{R}$ satisfies

$$\dot{\alpha}(t) \leq f(t, \alpha(t)) \quad \text{and} \quad \alpha(0) \leq \alpha_0$$

- Then, $\alpha(t) \leq \beta(t)$ where $\beta(t)$ solves

$$\dot{\beta}(t) = f(t, \beta(t)), \quad \beta(0) = \alpha_0$$

Example:

① Assume $\dot{\alpha}(t) \leq a\,\alpha(t) + u(t)$, $\alpha(0) \leq \alpha_0$. Then,

$$\alpha(t) \leq \beta(t) = \underbrace{e^{ta}\alpha_0 + \int_0^t e^{(t-s)a}\, u(s)\, ds}_{\text{Solution to } \dot{\beta}(t) = a\beta(t) + u(t),\ \beta(0) = \alpha_0}$$

② Assume $\dot{\alpha}(t) \leq a(t)\,\alpha(t) + u(t)$, $\alpha(0) \leq \alpha_0$, Then

$$\alpha(t) \leq \beta(t) = \underbrace{\overbrace{e^{\int_0^t \alpha(\tau)\,d\tau}}^{\phi(t,0)} \alpha_0 + \int_0^t \overbrace{e^{\int_s^t \alpha(\tau)\,d\tau}}^{\phi(t,s)}\, u(s)\, ds}_{\text{Solution to } \dot{\beta} = a(t)\beta + u(t),\ \beta(0) = \alpha_0},$$

Bellman-Gronwall lemma: (BG)

- From Comparison lemma, we know that

$$(*)\quad \left.\begin{aligned} \dot{\alpha} &\leq a(t)\alpha + u(t) \\ \alpha(0) &\leq \alpha_0 \end{aligned}\right\} \Longrightarrow \alpha(t) \leq \phi(t,0)\alpha_0 + \int_0^t \phi(t,s)\, u(s)\, ds$$

- The BG lemma gives the same conclusion, but under an "integral" version of the assumption:

Lemma:

- if $\alpha(t) \in \mathbb{R}$ satisfies

this is $(*)$ integrated over 0 to t.

$$(**)\quad \alpha(t) \leq \underbrace{\int_0^t a(s)\alpha(s)\, ds}_{\psi(t)} + \underbrace{\alpha_0 + \int_0^t u(s)\, ds}_{\lambda(t)}$$

and $a(s) \geq 0$, then

$$\alpha(t) \leq \phi(t,0)\alpha(0) + \int_0^t \underbrace{\phi(t,s)}_{e^{\int_s^t a(\tau) d\tau}} u(s)\, ds$$

proof :

- Let $\psi(t) = \int_0^t a(s)\alpha(s)\,ds$ and $\lambda(t) = \alpha_0 + \int_0^t u(s)\,ds.$

Then,

$$\dot{\psi}(t) = a(t)\alpha(t) \overset{(**)}{\leq} a(t)\psi(t) + a(t)\lambda(t)$$

Comparison lemma

$$\Longrightarrow \quad \psi(t) \leq \phi(t,0)\underbrace{\psi(0)}_{0} + \int_0^t \phi(t,s)\,a(s)\,\lambda(s)\,ds$$

$\alpha(t) \leq \psi(t) + \lambda(t)$

$$\Longrightarrow \quad \alpha(t) \leq \int_0^t \phi(t,s)\,a(s)\,\lambda(s)\,ds + \lambda(t)$$

integration by parts

$\frac{d}{dt}\phi(t,s) = -a(t)\phi(t,s)$

$$\Longrightarrow \quad \alpha(t) \leq \int_0^t \phi(t,s)\,\underbrace{\dot{\lambda}(s)}_{u(s)}\,ds + \phi(t,0)\underbrace{\lambda(0)}_{\alpha_0}$$

- Special case: assume $a(t) = a$ and $u(t) = u$ are constant.

or, equivalently

$$\alpha(t) \leq \int_0^t a\,\alpha(s)\,ds + \alpha_0 + tu.$$

Then,

$$\alpha(t) \leq \int_0^t e^{(t-s)a} u\,ds + e^{ta}\alpha_0$$

$$= \frac{e^{ta}-1}{a} u + e^{ta}\alpha_0$$

Thm:

- Consider the following two systems

(nominal) $\dot{x} = f(t,x)$, $\qquad x(0) = x_0$

(perturbed) $\dot{y} = f(t,y) + g(t,y)$, $\qquad y(0) = y_0$

- Assume f is globally Lip. unif. in $t \in [0,T]$

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|, \quad \forall x,y, \quad \forall t \in [0,T]$$

and g is unif. bounded

$$\|g(t,y)\| \leq M, \quad \forall y, \quad \forall t \in [0,T]$$

- Then,

$$\|x(t) - y(t)\| \leq e^{Lt}\|y(0) - x(0)\| + \frac{M}{L}(e^{Lt} - 1)$$

proof:

- Subtracting nominal from perturbed eq. yields:

$$\dot{Y}(t) - \dot{X}(t) = f(t, Y(t)) - f(t, X(t)) + g(t, Y)$$

- Integrating from 0 to t:

$$Y(t) - X(t) = Y(0) - X(0) + \int_0^t f(s, Y(s)) - f(s, X(s))\, ds + \int_0^t g(s, Y(s))\, ds$$

- Taking the norm of both sides

$$\|Y(t) - X(t)\| \leq \|Y_0 - X_0\| + \int_0^t L\|Y(s) - X(s)\|\, ds + Mt$$

- Applying the special case of BG lemma to

$\alpha(t) = \|Y(t) - X(t)\|$ with $a = L$ and $u = M$, yields

$$\|Y(t) - X(t)\| \leq e^{tL}\|Y_0 - X_0\| + \frac{M}{L}(e^{tL} - 1)$$

Example:

- Consider the aircraft model

given open-loop control input.

$$\ddot{x} = -\sin(\theta)\, u_1(t) + \varepsilon \cos(\theta)\, u_2(t)$$
$$\ddot{y} = \cos(\theta)\, u_1(t) + \varepsilon \sin(\theta)\, u_2(t)$$
$$\ddot{\theta} = u_2(t)$$

$\longrightarrow$ $\dot{z} = F(t, z, \varepsilon), \quad z(0) = z_0$

with $z = [x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}]^T$

- We like to bound the simulation error when the parameter ε and init. cond. are known up to 1% accuracy.

assumed, real; assumed, real.

$$\frac{|\hat{\varepsilon} - \varepsilon|}{\varepsilon} \leq 0.01, \quad \text{and} \quad \frac{\|\hat{z}_0 - z_0\|}{\|z_0\|} \leq 0.01$$

- We assume

$$|u_1(t)|, |u_2(t)| \leq U_{max}, \quad \forall t \qquad \text{and} \qquad \varepsilon, \hat{\varepsilon} \leq 1$$

- Therefore, we consider the following two systems.

Simulation/nominal
$$\dot{\hat{z}} = \underbrace{F(t,\hat{z},\hat{\varepsilon})}_{f(t,\hat{z})}, \quad \hat{z}(0) = \hat{z}_0$$

real/Perturbed
$$\dot{z} = \underbrace{F(t,z,\varepsilon)}_{f(t,z)+g(t,z)}, \quad z(0) = z_0$$

- Let's write the perturbed sys as

$$\dot{z} = \underbrace{F(t,z,\hat{\varepsilon})}_{f(t,z)} + \underbrace{F(t,z,\varepsilon) - F(t,z,\hat{\varepsilon})}_{g(t,z)}$$

① $f(t,z)$ is globally Lip. in z unif in t:

$$\left\| \frac{\partial f}{\partial z}(t,z) \right\|_\infty \leq \underbrace{1+2U_{max}}_{L}, \quad \forall z$$

$$\Longrightarrow \| f(t,z) - f(t,\hat{z}) \|_\infty \leq L \| z - \hat{z} \|_\infty$$

③ $g(t,y)$ is bounded:

$$\| g(t,y) \|_\infty = \| F(t,z,\varepsilon) - F(t,z,\hat{\varepsilon}) \| \leq U_{max} \, |\varepsilon - \hat{\varepsilon}|$$

- Application of the Thm.:

$$\|Z(t)-\hat{Z}(t)\|_\infty \leq e^{tL}\underbrace{\|Z_0-\hat{Z}_0\|}_{\leq 0.01\|Z_0\|} + \frac{U_{max}\,\underbrace{|\varepsilon-\hat{\varepsilon}|}}{L}\left(e^{tL}-1\right)$$

$$\leq 0.01\left(e^{tL}\|Z_0\| + \frac{U_{max}\,\varepsilon}{L}\left(e^{tL}-1\right)\right)$$

with $L = 1+2U_{max}$