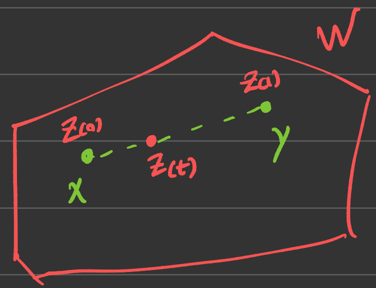
Last time: (subtleties of diff. eq.)

① Examples when a unique solution to $\dot{x} = f(x)$ does not exist.

② Class of Lip. functions.

③ Lemma:



- Assume $f : \mathbb{R}^n \to \mathbb{R}$ is diff. and

$\|\frac{\partial f}{\partial x}(x)\| \leq M \quad \forall x \in W$ where W is a convex set.

Then, $\quad \|f(x) - f(y)\| \leq M \|x - y\|, \quad \forall x, y \in W$

plan:

- We will present the proof of this result.

- Existene & uniqueness result

- motivation for perturbation error analysis. $\to$ Next week.

Proof:

- For any two points $x, y \in W$, define $Z(t) = (1-t)x + ty$, for $t \in [0,1]$

  the line connecting x to y

- $Z(t) \in W$ because W is convex.

- define $g(t) = f(Z(t))$. value of $f(\cdot)$ along the line. $\rightarrow$ $\frac{dg}{dt}(t) = \frac{\partial f}{\partial x}(Z(t))\, \dot{Z}(t) = \frac{\partial f}{\partial x}(Z(t))(y-x)$

- We like to bound $\|f(x) - f(y)\|$. We use the fundamental thm. of calculus

$$\int_0^1 \frac{d}{dt} g(t)\, dt = g(1) - g(0)$$

$$\Rightarrow \quad \int_0^1 \frac{\partial f}{\partial x}(Z(t))(y-x)\, dt = f(y) - f(x)$$

- Taking the norm of both sides:

$$\|f(y) - f(x)\| = \left\| \int_0^1 \frac{\partial f}{\partial x}(Z(t))(y-x)\, dt \right\|$$

$$\overset{\text{triangle ineq.}}{\leq} \int_0^1 \Big\| \underbrace{\frac{\partial f}{\partial x}(Z(t))}_{\text{matrix}} \underbrace{(y-x)}_{\text{vector}} \Big\|\, dt$$

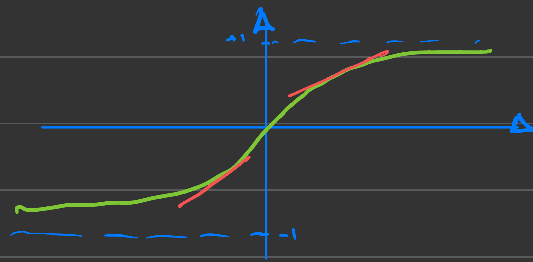
$$\overset{\text{def of matrix norm}}{\leq} \int_0^1 \underbrace{\left\|\frac{\partial f}{\partial x}(Z(t))\right\|}_{\leq L} \|y-x\|\, dt$$

$$\overset{\text{assumption}}{\leq} L\|y-x\| \int_0^1 dt = L\|y-x\| \quad \blacksquare$$

Example:

- $f(x) = \tanh(x)$

$$= \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$



$\Rightarrow f'(x) = 1 - \tanh(x)^2 \Rightarrow |f'(x)| \leq 1, \ \forall x \in \mathbb{R}$

$\Rightarrow f$ is globally Lip

Example:

- Is this funct. Lip?

$$f(x) = \begin{bmatrix} -x_1 + x_1 x_2 \\ x_2 - x_1 x_2 \end{bmatrix}$$

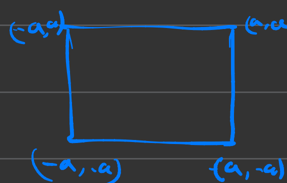
- We will see if we can bound the derivative:

$$\nabla f(x) = \begin{bmatrix} -1 + x_2 & x_1 \\ -x_2 & 1 - x_1 \end{bmatrix}$$

$$\Rightarrow \|\nabla f(x)\|_\infty = \max\{|x_1| + |x_2 - 1|, |x_2| + |x_1 - 1|\}$$

- This is unbdd as $x \to \infty$, but bdd over and bdd convex set:

let $W = [-a, a] \times [-a, a]$



$$\Rightarrow \|\nabla f(x)\|_\infty \leq 1 + 2a$$

$$\Rightarrow \|f(x) - f(y)\|_\infty \leq (1+2a)\|x - y\| \quad \forall x, y \in W$$

$\Rightarrow$ f is locally Lip.

Existence and Uniqueness: (Thm 3.1 Khalil)

- Consider $\dot{x} = f(x)$, $x(0) = x_0$

- Local existence: if f is Lip on $B_r(x_0) = \{x \in \mathbb{R}^n ; \|x - x_0\| \leq r\}$

a unique solution $x(t)$ exists for $[0, \bar{t}]$ for some $\bar{t} > 0$.

- Global existence: if f is globally Lip. then

a unique solution $x(t)$ exists for all $t > 0$.

- The result is extended to time-varying functions $f(t,x)$

as long as the dependence on t is piecewise cont.

Examples:

① $\dot{x} = \sqrt{x}$, $x(0)=0$

$\Downarrow$

Not Lip. around 0 $\Rightarrow$ Thm. does not apply

② $\dot{x} = rx^2$

$\Downarrow$

locally Lip. $\Rightarrow$ a unique sol. exists for finite time

③ $\dot{x} = -\operatorname{sgn}(x)$, $x(0)=0$

$\Downarrow$

Not Lip. around 0 $\Rightarrow$ Thm. does not apply

- Remark: if for some reason, the solution is known to be bdd, then local Lip. implies global existence

$\dot{x} = x^2 \rightarrow$ local Lip $\rightarrow$ local existence

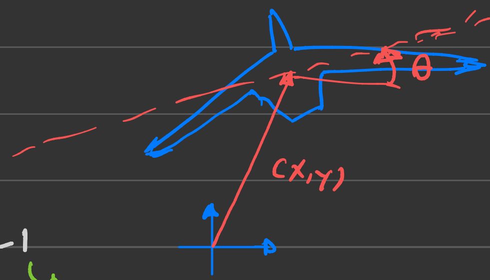
$\dot{x} = -x^2 \rightarrow$ local Lip + bdd sol $\rightarrow$ global existence

Example : (Aircraft)

$$\ddot{X} = -\sin(\theta)\,\underbrace{u_1}_{\text{thrust}} + \underbrace{\varepsilon\cos(\theta)\,u_2}_{\text{coupling}}$$

$$\ddot{Y} = \cos(\theta)\,u_1 + \varepsilon\sin(\theta)\,u_2 \underbrace{-1}_{\text{gravity}}$$

$$\ddot{\theta} = \underbrace{u_2}_{\text{rolling moment}}$$



- State: $Z = [\underbrace{X}_{z_1}, \underbrace{Y}_{z_2}, \underbrace{\theta}_{z_3}, \underbrace{\dot{X}}_{z_4}, \underbrace{\dot{Y}}_{z_5}, \underbrace{\dot{\theta}}_{z_6}]^T$

$$\dot{Z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \\ \dot{z}_5 \\ \dot{z}_6 \end{bmatrix} = \underbrace{\begin{bmatrix} z_4 \\ z_5 \\ z_6 \\ -\sin(z_3)\,u_1 + \varepsilon\cos(z_3)\,u_2 \\ \cos(z_3)\,u_1 + \varepsilon\sin(z_3)\,u_2 - 1 \\ u_2 \end{bmatrix}}_{f(Z,u)}$$

– To check if $f(z,u)$ is Lip. in z, we compute the derivative:

$$\frac{\partial f}{\partial z} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & A_t & 0 & 0 & 0 \\ 0 & 0 & B_t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$A_t = -\cos(z_3)\,u_1(t) - \varepsilon \sin(z_3)\,u_2(t) \Rightarrow |A_t| \leq |u_1(t)| + \varepsilon |u_2(t)|$

$B_t = -\sin(z_3)\,u_1(t) + \varepsilon \cos(z_3)\,u_2(t) \Rightarrow |B_t| \leq |u_1(t)| + \varepsilon |u_2(t)|$

$\Rightarrow$ if $|u_1(t)| \leq M_1$, $|u_2(t)| \leq M_2$, then, f is Lip. in z

with $L = \max(1, M_1 + \varepsilon M_2)$ $\Rightarrow$ global Lip.

$\Rightarrow$ global existence

perturbation error analysis:

- Suppose you want to simulate the aircraft dyn. to repliate reality.
- but you don't know some physical parameters, or Initial cond. accurately.
- The goal is to learn tools that can give guarantees on accuracy of your simulation.
- Mathematically, we have two systems:

(simulation/nominal) $\dot{x} = f(x, t, \underset{\text{Parameters.}}{\alpha})$, $\quad x(0) = x_0$

(real/perturbed) $\dot{y} = f(y, t, \beta) + \underbrace{g(t, y)}_{\text{unmodelled dyn.}}$, $\quad y(0) = y_0$

- We like to bound the difference

$$\| x(t) - y(t) \| \leq \, ? \longrightarrow \text{in terms of Init. error, Param. error, disturbances}$$

- In order to do this, we need two very useful lemmas: ① Comparison lemma ② Bellman-Gronwall (BG) lemma.