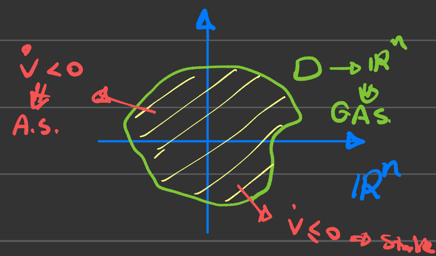
Review from last time:



Thm (Thm. 4.1 and 4.2 in Khalil)

- Assume $\bar{x}=0$ is an eqlb. point for $\dot{x}=f(x)$

- Let $V:\mathbb{R}^n \to \mathbb{R}$ be C^1 and $D\subseteq\mathbb{R}^n$ be open set containing 0.
  continuously diff.

① if $V(0)=0$, $V(x)>0$, $\forall x\in D/\{0\}$ $\longrightarrow$ V is p.d. on D
excluding 0

and $\dot{V}(x)\leq 0$ $\forall x\in D$ $\Rightarrow$ $\bar{x}=0$ is stable

② if, moreover, $\dot{V}(x)<0$ $\forall x\in D/\{0\}$ $\Rightarrow$ $\bar{x}=0$ is AS

③ if, moreover, $D=\mathbb{R}^n$ $\Rightarrow$ $\bar{x}=0$ is GAS
and V is radially unbdd.

plan:

① examples

③ exponential stability and its proof

Examples

$\bar{x}=0$ is eqlb.

① $\dot{x} = -x^3$

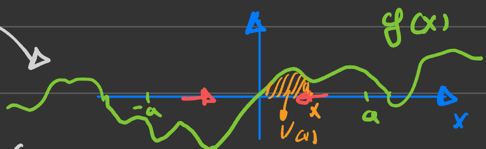
$V(x) = \frac{1}{4}x^4$, $\Rightarrow$ p.d. and radially unbdd

$\dot{V}(x) = V'(x)f(x) = x^3(-x^3) = -x^6 < 0 \quad \forall x \neq 0$

$\Rightarrow \bar{x}=0$ is GAS.

in class, we did $g(x) = \sin(x)$

② $\dot{x} = -g(x)$, where



$g(x)x > 0$ if $|x| \leq a$ and $x \neq 0$

$g(0) = 0$

- From phase portrait, $x=0$ is A.S.

- What is a good Lyapunov function to show $\bar{x}=0$ is AS?

$$V(x) = \int_0^x g(u)\,du \quad \text{and} \quad D = [-a, a]$$

$\Rightarrow V$ is p.d. on $[-a,a]$ and $\dot{V}(x) = -g(x)^2 < 0$ $\forall x \in [-a,a]/\{0\}$

$\Rightarrow x=0$ is A.S.

③ $\dot{x}_1 = -x_1 + x_1 x_2$
$\dot{x}_2 = -x_2$

- eqlb. points : $\dot{x}_2 = 0 \Rightarrow x_2 = 0$
$\dot{x}_1 = 0 \Rightarrow -x_1 + x_1 x_2 = 0$ $\Big\} \Rightarrow x_1 = x_2 = 0$

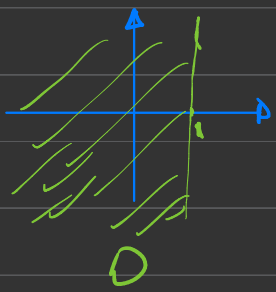
$\Rightarrow \bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is eqlb. point

- Try $V(x) = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 \rightarrow$ p.d. ✓

$\Rightarrow \dot{V}(x) = x_1(-x_1 + x_1 x_2) + x_2(-x_2)$

$= -x_1^2(1 - x_2) - x_2^2 < 0$ if $x_2 < 1$ and $x \neq 0$

- take $D = \{ (x_1, x_2) \in \mathbb{R}^2 ; x_2 < 1 \}$



$\Rightarrow \dot{V}(x) < 0 \quad \forall x \in D / \{0\}$

$\Rightarrow x = 0$ is A.S

- Note that, although we concluded AS., $x = 0$ is actually GAS because $x_2(t) = x_2(0) e^{-t} < 1$ for large t.

④ $\dot{x}_1 = -x_1 - x_2$

$\dot{x}_2 = 2x_1 - x_2^3$ $\Rightarrow \bar{x} = 0$ is an eqlb. point.

Try $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$

$\Rightarrow \dot{V}(x) = -x_1^2 - x_1x_2 + 2x_1x_2 - x_2^4$

$= -x_1^2 + x_1x_2 - x_2^4$ → Not easy to see if it is negative or not

Try $V(x) = x_1^2 + \frac{1}{2}x_2^2$ → p.d. and radially unbdd

$\Rightarrow \dot{V}(x) = -2x_1^2 - 2x_1x_2 + 2x_1x_2 - x_2^4 = -2x_1^2 - x_2^4 < 0 \quad \forall x \neq 0$

$\Rightarrow$ GAS.

proof of Lyapunov thm:

stable

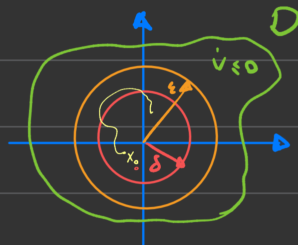
proof of ①: we present the proof under stronger assumption that,

$$a_1\|x\|^2 \overset{\text{I}}{\leq} V(x) \overset{\text{II}}{\leq} a_2\|x\|^2 \quad \forall x \in D$$ → instead of V is p.d. on D

Positive constants

- In order to show $\bar{x}=0$ is stable, we need to show
  $\forall \varepsilon>0,\ \exists \delta_\varepsilon>0$ s.t. $\|x(t)\| \leq \varepsilon$ if $\|x(0)\| \leq \delta_\varepsilon$.

- in order to prove this, we obtain a bound on $V(x(0))$ and use $\dot{V} \leq 0$.



- if $\|x(0)\| \leq \delta_\varepsilon$, then
  $$V(x(0)) \overset{\text{II}}{\leq} a_2\|x(0)\|^2 \leq a_2\delta_\varepsilon^2$$

thm assumption

- also $\frac{d}{dt}V(x(t)) = \dot{V}(x(t)) \leq 0$
  $$\Rightarrow V(x(t)) \leq V(x(0)) \leq a_2\delta_\varepsilon^2$$
  $$\overset{\text{I}}{\Rightarrow} a_1\|x(t)\|^2 \leq V(x(t)) \leq a_2\delta_\varepsilon^2$$
  $$\Rightarrow \|x(t)\| \leq \sqrt{\frac{a_2}{a_1}}\,\delta_\varepsilon, \quad \forall t \geq 0$$

- Therefore, if $\delta_\varepsilon = \sqrt{\frac{a_1}{a_2}}\,\varepsilon$, then $\|x(t)\| \leq \sqrt{\frac{a_2}{a_1}}\,\delta_\varepsilon = \varepsilon,\ \forall t \geq 0$ ⇒ stable!

AS

proof of ②:

- we present the proof under stronger assumption that

$$\dot{V}(x) \leq -a_3 \|x\|^2, \quad \forall x \in D$$

③ Positive Const.

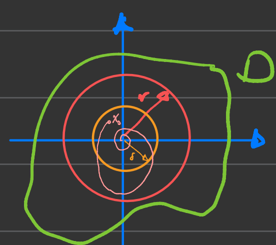
instead of $\dot{V}(x) < 0 \quad \forall x \in D/\{0\}$

- To show AS, we need to show $\exists \delta > 0$ s.t.

$$\lim_{t \to \infty} \|x(t)\| = 0, \quad \text{if} \quad \|x(0)\| \leq \delta.$$

- Let $r > 0$ be the radius of a ball that is contained in D

- let $\delta = \sqrt{\frac{a_1}{a_2}}\, r$



- From part ①, $\|x(t)\| \leq \sqrt{\frac{a_2}{a_1}}\, \delta \leq r \quad \forall t \geq 0$

$$\Rightarrow x(t) \in D \quad \forall t \geq 0$$

- From assumption ③

$$x(t) \in D \Rightarrow \frac{d}{dt} V(x(t)) = \dot{V}(x(t)) \leq -a_3 \|x(t)\|^2 \overset{①}{\leq} -\frac{a_3}{a_2} V(x(t))$$

Comparison lemma $\Rightarrow$ $V(x(t)) \leq e^{-\frac{a_3}{a_2} t} V(x(0)) \leq a_2 e^{-\frac{a_3}{a_2} t} \|x(0)\|^2$

$\overset{①}{\Rightarrow}$ $\underbrace{\|x(t)\|^2 \leq \frac{a_2}{a_1} e^{-\frac{a_3}{a_2} t} \|x(0)\|^2}_{\text{exponential stability.}} \Rightarrow \lim_{t \to \infty} \|x(t)\| = 0$ if $\underbrace{\|x(0)\| \leq \sqrt{\frac{a_1}{a_2}}\, r}_{\text{Convergence regions.}}$

$\Rightarrow$ AS.

proof of 3:

- we present the proof under conditions from 3 but on $D \subset \mathbb{R}^n$

$$\dot{V}(x(t)) \leq -a_3 \|x(t)\|^2, \quad \forall x(0)$$

$$\leq -\frac{a_3}{a_2} V(x(t))$$

$$\Rightarrow V(x(t)) \leq e^{-\frac{a_3}{a_2}t} a_2 \|x(0)\|^2$$

$$\Rightarrow \underbrace{\|x(t)\|^2 \leq \frac{a_2}{a_1} e^{-\frac{a_3}{a_2}t} \|x(0)\|^2}_{\text{globally exp. stable.}}, \quad \forall x(0)$$

- With stronger assumptions (I, II, III) we were able to obtain exp. convergence rates.

- With the weaker assumptions in the thm. statement, it is possible to follow the same procedure to prove stability, AS, GAS, but without exp. convergence rates → Thm 4.1 and 4.2 in book.

Thm: (exp. stability)

- Assume $\bar{x}=0$ is an eqlb. point for $\dot{x}=f(x)$

- Let $V:\mathbb{R}^n \to \mathbb{R}$ be C^1, and $D \subseteq \mathbb{R}^n$ be open set containing 0.

positive constants

① If $a_1\|x\|^2 \leq V(x) \leq a_2\|x\|^2$, and $\dot{V}(x) \leq -a_3\|x\|^2 \ \forall x \in D$ $\Longrightarrow$ $\underbrace{\bar{x}=0 \text{ is exp. stable}}_{\exists \delta>0 \text{ s.t. } \|x(t)\| \leq Ce^{-\lambda t}\|x(0)\| \text{ if } \|x(0)\| \leq \delta}$

② if $D=\mathbb{R}^n$, then $\bar{x}=0$ is $\underbrace{\text{globally exp. stable}}_{\|x(t)\| \leq Ce^{-\lambda t}\|x(0)\|,\ \forall x(0)}$

Positive

Example:

- $\dot{x} = -x + x^3$

- $\bar{x} = 0$ is an eqlb. point

- Try $V(x) = \frac{1}{2}x^2 \longrightarrow$ p.d.

$$\Longrightarrow \dot{V}(x) = -x^2 + x^4$$

$$= -x^2(1-x^2) < 0 \quad \text{if} \quad \underbrace{|x| < 1}_{D} \quad \text{and} \quad x \neq 0$$

$$\Longrightarrow x = 0 \text{ is AS.}$$

- Moreover, it is exp. stable because

$$\underset{a_1}{\frac{1}{2}} x^2 \leq V(x) \leq \underset{a_2}{\frac{1}{2}} x^2$$

$$\dot{V}(x) \leq \underbrace{(1-r^2)}_{a_3} x^2 \quad \text{if } |x| \leq r < 1$$

$$\Longrightarrow \text{exp. stable and } |x(t)| \leq \underset{\sqrt{\frac{a_2}{a_1}}}{1}\, e^{-(1-r^2)t \to \frac{a_3}{2a_2}} |x(0)|, \text{ if } |x(0)| \leq r < 1$$