Last time:

- input-output systems

$$H \begin{cases} \dot{x} = f(x,u) \\ y = h(x,u) \end{cases}$$

- input-output stability (with finite gain)

Positive constant.

$$\|y\|_L \leq \gamma \|u\|_L + c$$

signal norm e.g. L_2

smallest possible value for γ is called gain.

- L_2-gain of a linear sys.: $\max_{\omega \in \mathbb{R}} \|G(j\omega)\|$

transfer function.

- Lyapunov method for nonlin. sys:

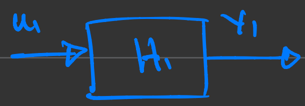
$$\dot{V} \leq \alpha^2 \|u\|^2 - \beta^2 \|y\|^2$$

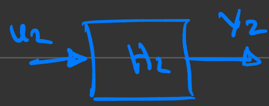
$$\Rightarrow \gamma \leq \frac{\alpha}{\beta}$$

Today: ① small gain thm. ② passivity

Small gain thm :

- Suppose you have two input-output stable systems

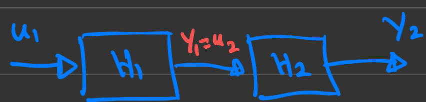




$\|y_1\|_L \leq \gamma_1 \|u\|_L +$ $\qquad\qquad$ $\|y_2\|_L \leq \gamma_2 \|u_2\|_L +$

- We like to study stability when they are connected

① Series Connection :

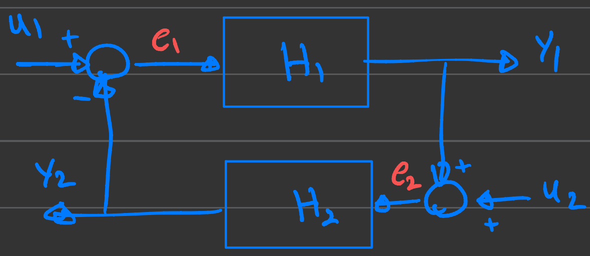


Series Connetion

$$\|y_2\|_L \leq \gamma_2 \|u_2\|_L + \beta_2 = \gamma_2 \|y_1\|_L + \beta_2$$
$$\leq \gamma_2 (\gamma_1 \|u_1\|_L + \beta_1) + \beta_2 = \gamma_1 \gamma_2 \|u_1\|_L + \beta_2 + \gamma_2 \beta_1$$

$\Rightarrow$ series connection is stable with gain $\gamma \leq \gamma_1 \gamma_2$

② feedback Connection.



feedback Connetion

- This case is important in robust Control

- Robust Control is interested in stability analysis under model uncertainty. A common way to represent the uncertainty is a feedback connection where

H_1 represents the "nominal" model for the plant and H_2 represents all the dynamic "uncertainty"

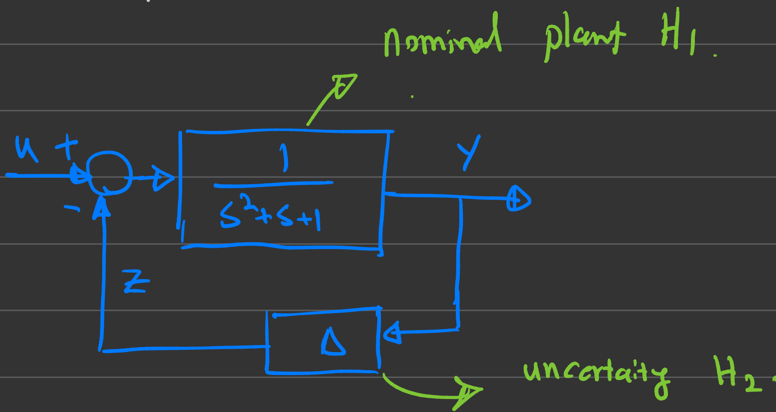
Example:

- Consider $\ddot{y} + \dot{y} + (1+\Delta)y = u$ but assume Δ is unknown and small.

- we can express the system as

$$\ddot{y} + \dot{y} + y = u - z$$

$$z = \Delta y$$



- The stability analysis is simple in linear case

the feedback connection is stable when $|\Delta| < 1$.

- Small gain thm. provides a way to answer this question in non linear setting.

Small gain thm:

- Consider the feedback connection of two input-output stable systems H_1 and H_2.
- Let γ_1, γ_2 be bounds on the gains of H_1 and H_2.
- The feedback sys. $(u_1, u_2) \to (y_1, y_2)$ is input-output stable if $\gamma_1 \gamma_2 < 1$

proof:

- From block diagram we have
$$e_1 = u_1 - y_2$$
$$e_2 = u_2 + y_1$$

- From stability of H_1 and H_2:

$$\|y_1\|_L \leq \gamma_1 \|e_1\|_L + c_1$$

and

$$\|y_2\|_L \leq \gamma_2 \|e_2\|_L + c_2$$

- As a result:

$$\|e_1\|_L = \|u_1 - y_2\|_L \overset{\text{triangle ineq.}}{\leq} \|u_1\|_L + \|y_2\|_L$$

$$\leq \gamma_2 \|e_2\|_L + c_2 + \|u_1\|_L$$

and $$\|e_2\|_L \le \|u_2\|_L + \|y_1\|_L \le \gamma_1 \|e_1\|_L + c_1 + \|u_2\|_L$$

– Combining the inequaliteis:

$$\begin{cases} \|e_1\|_L \le \gamma_1\gamma_2\|e_1\| + \gamma_2 c_1 + \gamma_2\|u_2\|_L + c_2 + \|u_1\|_L \\ \|e_2\|_L \le \gamma_1\gamma_2\|e_2\|_L + \gamma_1 c_2 + \gamma_1\|u_1\|_L + c_1 + \|u_2\|_L \end{cases}$$

$\gamma_1\gamma_2 < 1$

$\rightarrow \gamma_2 c_1 + c_2$

$$\Longrightarrow \begin{cases} \|e_1\|_L \le \dfrac{1}{1-\gamma_1\gamma_2}\left[\|u_1\|_L + \gamma_2\|u_2\|_L + c_1'\right] \\ \|e_2\|_L \le \dfrac{1}{1-\gamma_1\gamma_2}\left[\|u_2\|_L + \gamma_1\|u_1\|_L + c_2'\right] \end{cases}$$

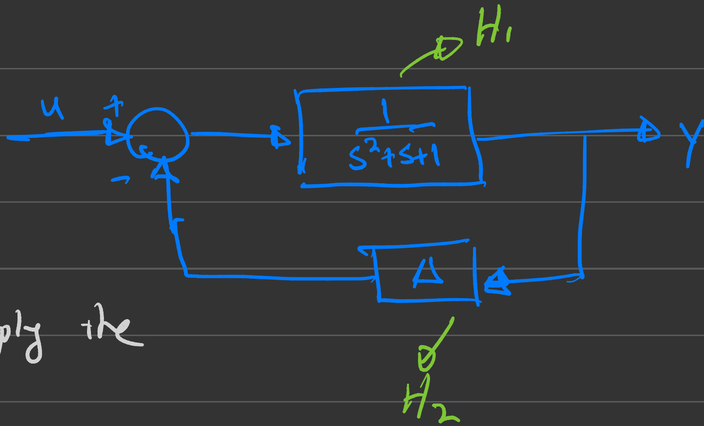
$\rightarrow \gamma_1 c_2 + c_1$

– Therefore

$$\begin{cases} \|y_1\|_L \le \gamma_1\|e_1\|_L + c_1 \le \dfrac{\gamma_1}{1-\gamma_1\gamma_2}\left[\|u_2\|_L + \gamma_1\|u_1\|_L + c_1'\right] + c_1 \\ \|y_2\|_L \le \gamma_2\|e_2\|_L + c_2 \le \dfrac{\gamma_2}{1-\gamma_1\gamma_2}\left[\|u_1\|_L + \gamma_2\|u_2\|_L + c_2'\right] + c_2 \end{cases}$$

$\Longrightarrow$ $(u_1, u_2) \to (y_1, y_2)$ is input-output stable.

Example:



- In order to apply the small gain thm, we need to compute γ_1 and γ_2.

- H_1: $\gamma_1 = \max_{\omega \in \mathbb{R}} |G(j\omega)| = \max_{\omega} \left| \frac{1}{-\omega^2+1+j\omega} \right|$

$$= \max_{\omega} \frac{1}{(1-\omega^2)^2+\omega^2} = \frac{4}{3}$$

- for H_2: $\|y\|_{L_2} = \left(\int y(t)^2 dt \right)^{1/2}$

$$= \left(\int \Delta^2 u(t)^2 dt \right)^{1/2} = |\Delta| \, \|u\|_{L_2} \Rightarrow \gamma_2 = |\Delta|$$

- stable if $\frac{4}{3}|\Delta| < 1$ or $|\Delta| < \frac{3}{4}$

- From lin sys. theory, we know $\ddot{y} + \dot{y} + (1+\Delta) y = u$ is stable when $\Delta > -1$, therefore small gain thm provides conservative bounds.

Passivity:

- passivity allows us to analyze the overall stability of interconnected systems by examining energy exchange between them.

  Power grid

$$\text{passivity} \simeq \text{energy dissipation}$$

- Consider an input-output sys.

$$H \begin{cases} \dot{x} = f(x,u) \\ y = h(x,u) \end{cases} \qquad u \longrightarrow \boxed{H} \longrightarrow y$$

where $f(0,0) = 0$ and $h(0,0) = 0$

- H is <u>passive</u> if there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ s.t.

storage function

$V(x) \geq 0 \quad \forall x$, $V(0) = 0$, and

$$\underbrace{\nabla V(x)^T f(x,u)}_{\dot{V}} \leq \underbrace{u^T h(x,u)}_{u^T y} \qquad \forall x, u$$

① input strictly passive if

$$\dot{V} \leq u^T y - u^T \varphi(u) \quad \text{where} \quad u^T \varphi(u) > 0 \quad \forall u \neq 0$$

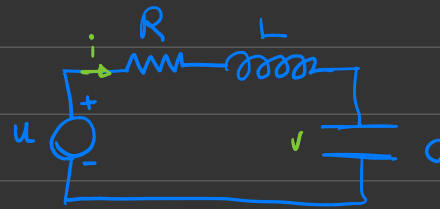
③ output-strictly passive if

$$\dot{V} \leq u^T y - y^T \varphi(u) \quad \text{where} \quad y^T \varphi(y) > 0 \quad \forall y \neq 0$$

③ strictly passive if

$$\dot{V} \leq u^T y - W(x) \quad \text{where} \quad W(x) > 0 \quad \forall x \neq 0$$

Example.



$$L\frac{di}{dt} = -Ri - v + u$$

$$C\frac{dv}{dt} = i$$

$$x = \begin{bmatrix} v \\ i \end{bmatrix}, \qquad y = i = x_2$$

$$\Rightarrow \dot{x} = \begin{bmatrix} \frac{1}{C}x_1 \\ -\frac{R}{L}x_1 - \frac{1}{L}x_2 + \frac{1}{L}u \end{bmatrix} =: f(x,u)$$

- let $V(x) = \frac{1}{2}Cx_1^2 + \frac{1}{2}Lx_2^2$ ← electrical energy

$$\Rightarrow \underbrace{\dot{V}(x,u)}_{\text{change in energy}} = \underbrace{-Rx_2^2}_{\text{dissipation}} + \underbrace{uy}_{\text{energy input}} = -Ry^2 + uy \rightarrow \text{output strictly passive.}$$

passivity means energy input ≥ stored energy

- passive systems are stable when input $u=0$

- strictly passive sys. are A.S.

- output s.p. sys became stable under additional

condition:

- zero-state observability:

$$y(t)=0, \ \forall t \Rightarrow x(t)=0 \ \forall t$$

zero output implies equilibrium

Lemma: output strictly passive + zero state obs. $\Rightarrow$ A.S.

proof: $\dot{V} \leq u^T y - y^T \varphi(y) \xRightarrow{u=0} \dot{V} \leq -y^T \varphi(y) \leq 0$

by La Salle $x(t) \to$ largest inv. set in $\{x : \dot{V}=0\}$

$$\dot{V}=0 \Rightarrow \underset{\forall t}{y(t)=0} \xRightarrow{\text{zero-state obs.}} \underset{\forall t}{x(t)=0} \Rightarrow \text{A.S.}$$

Example:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a x_1^3 - k x_2 + u$$
$$y = x_2$$

with $a, k > 0$. Take $V(x) = \frac{1}{2} x_2^2 + \frac{1}{4} a x_1^4$. Then

$$\dot{V} = \nabla V(x) f(x,u) = -k x_2^2 + u x_2 = \underbrace{-k y^2 + u y}_{\text{output s.p.}}$$

- It is also zero-state obs.:

$$y(t) = 0 \Rightarrow x_2(t) = 0 \Rightarrow \dot{x}_2(t) = 0 \Rightarrow x_1(t) = 0$$

- Moreover, V is radially unbdd $\rightsquigarrow$ GAS.