Goal: introduce a tool used to study convergence to invariant sets, instead of eqlb. points.

plan: ① motivating example
② invariant sets and its relation to Lyapunov function
③ LaSalle invariance principle

Example: (adaptive Control)

- Suppose your objective is to stabilize the sys.

$$\dot{z} = \underbrace{\bar{\theta}}_{\text{unknown}} z + \underbrace{u}_{\text{Control input}}$$

- If $\bar{\theta}$ was known, you can use $u = -(\bar{\theta}+1)z$ to get

$$\dot{z} = \bar{\theta} z - (\bar{\theta}+1) z = -z \quad \Leftrightarrow \text{stable sys.}$$

- How to stabilize when $\bar{\theta}$ is not known?

- Adaptive control strategy:

$$u = -(\theta+1)z \quad \text{and} \quad \text{adapt } \theta \text{ according to some rule}$$

- for example, let $\dot{\theta} = z^2$ ↝ increases as long as $z \neq 0$

- we end up with the two-dim. state:

$$\dot{x} := \begin{bmatrix} \dot{z} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \bar{\theta} z - (\theta+1) z \\ z^2 \end{bmatrix} =: f(x)$$

- Let's use Lyapunov func. to analyze this:

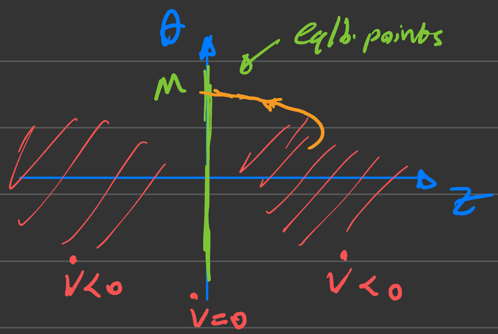
$$V(x) = \frac{1}{2} z^2 + \frac{1}{2} (\theta - \bar{\theta})^2$$

$$\dot{V}(x) = \dots = -z^2 \leq 0$$

- What can we conclude?

- Let's compute the eqlb. points:

$f(x)=0 \Rightarrow z^2=0$ and $\bar{\theta} z-(\theta+1)z=0 \Rightarrow$ $z=0$ and θ can be any number



$M=\{(0,\theta):\theta\in\mathbb{R}\}$

- if the trajectory enters M, it stays in M, because all points in M are eqlb. $\Rightarrow$ M is invariant set

- Also $\dot{V}<0$ outside M and $\dot{V}=0$ on M.

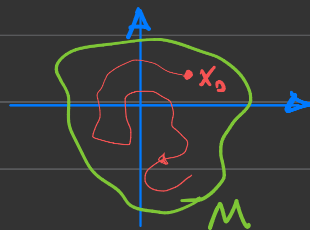
- Can we argue that $X(t)\to M$ as $t\to\infty$? Yes! by LaSalle's invariance principle.

- In order to present the thm. we introduce the notion of invariant sets and their relation to Lyapunov functions.

Invariant set:

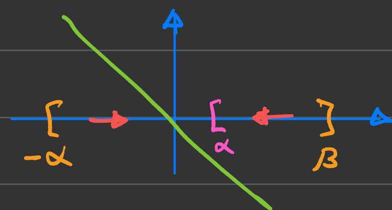
- $M \subseteq \mathbb{R}^n$ is an invariant set for $\dot{x} = f(x)$ if

$$x(0) \in M \implies x(t) \in M, \ \forall t \geq 0$$



Example:

$$\dot{x} = -x$$



- any set of the form $M = [-\alpha, \beta]$ is an invariant set.
  $\wedge$ $\alpha, \beta > 0$

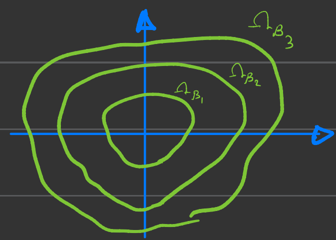
- However $M = [\alpha, \beta]$ is not an invariant set.

- Invariant sets are very useful for safety verification avoiding collisions in navigation.

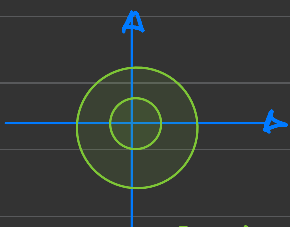
Sub-level sets:

- For $V: \mathbb{R}^n \to \mathbb{R}$, the β sub-level set is

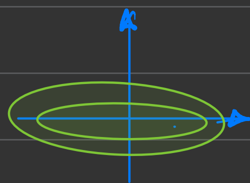
$$\Omega_\beta := \{ x \in \mathbb{R}^n ; \ V(x) \le \beta \}$$



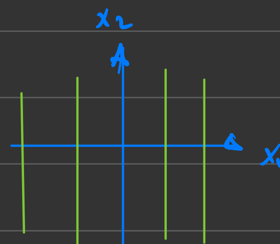
Example:



$V(x) = x_1^2 + x_2^2$



$V(x) = x_1^2 + 10 x_2^2$



$V(x) = x_1^2$

Remark:

- if V is radially unbdd, then sublevel sets are bounded

proof: take any sublevel set Ω_β. Because V is radially unbdd $\exists r > 0$ s.t. $V(x) > \beta$ if $\|x\| > r$. Therefore,

$\|x\| \le r$ if $V(x) \le \beta \Rightarrow \|x\| \le r$ if $x \in \Omega_\beta$

$\Rightarrow \Omega_\beta$ is bounded

Lemma: (sublevel sets of Lyapunov func. is invariant set)

- Consider $\dot{x} = f(x)$ and $V : \mathbb{R}^n \to \mathbb{R}$ s.t.

$$\dot{V}(x) \leq 0 \quad \forall x \in \mathbb{R}^n$$

① Then, all sublevel sets of V are invariant

$$x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta$$

② If, V is radially unbdd, then $x(t)$ is bounded.

proof:

① $x(0) \in \Omega_\beta \Rightarrow V(x(0)) \leq \beta$

$$\frac{d}{dt} V(x(t)) = \dot{V}(x(t)) \leq 0$$

$$\Rightarrow V(x(t)) \leq V(x(0)) \leq \beta \Rightarrow x(t) \in \Omega_\beta$$

② let $\beta = V(x(0))$, $\overset{\text{from part ①}}{\Rightarrow}$ $x(t) \in \Omega_\beta \quad \forall t \geq 0$ $\overset{\Omega_\beta \text{ is bdd because } V \text{ is radially unbdd.}}{\Longrightarrow}$ $x(t)$ is bounded

LaSalle's Thm.: (Thm 4.4)

- Consider $\dot{x} = f(x)$ and assume $x(t) \in \Omega$ → bounded set
- Assume there exists $V: \mathbb{R}^n \to \mathbb{R}$ s.t. $\dot{V}(x) \leq 0 \ \forall x \in \Omega$.
- Let $E = \{ x \in \Omega ; \dot{V}(x) = 0 \}$
- And let M be the largest invariant set in E
- Then $\boxed{x(t) \to M \text{ as } t \to \infty.}$

Remarks

- The Thm. does not assume V to be p.d.
- The bdd. assumption is important
- We can ensure bdd, if V is radially unbdd and $\dot{V}(x) \leq 0$ $\forall x \in \mathbb{R}^n$

Back to the example:

- V is radially unbdd and $\dot{V}(x) \leq 0 \ \forall x$

$\Rightarrow$ $x(t) = (z(t), \theta(t))$ is bdd!

- to compute E:

$$\dot{V}(x) = 0 \Rightarrow z^2 = 0 \Rightarrow z = 0 \Rightarrow E = \{(0,\theta) ; \theta \in \mathbb{R}\}$$

- To compute M:

pick $(0,\theta) \in E$, then as starting point of a trajectory

$$\dot{z} = \bar{\theta} z - (\theta+1) z = 0$$
$$\dot{\theta} = z^2 = 0$$

$\Rightarrow$ trajectory does not move and remains in E

$\Rightarrow$ largest invariant set in E is E $\Rightarrow$ $M = E$

- By LaSalle: $x(t) = (z(t), \theta(t)) \to E = \{(0,\theta) : \theta \in \mathbb{R}\}$ as $t \to \infty$

$\Rightarrow$ $z(t) \to 0$ as $t \to \infty$

Example: (pendelum)

- Consider the pendelum example,

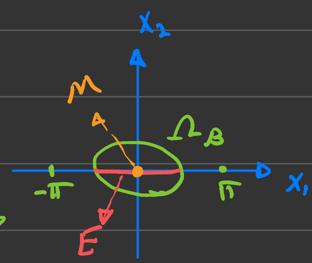
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\gamma x_2 - \omega^2 \sin(x_1) \end{bmatrix}$$

- And take total energy as Lyapunov func.

$$V(x) = \frac{1}{2} x_2^2 + \omega^2 (1 - \cos(x_1))$$

- Then $\dot{V}(x) = -\gamma x_2^2 \leq 0$

$\Rightarrow$ any sublevel set Ω_β is invariant.



- Pick β small enough so that the level set Ω_β is bounded

$\Rightarrow$ $x(t)$ is bounded because $x(t) \in \Omega_\beta \ \forall t$

- We can apply LaSalle's invariance Thm.

- To compute E:

$$E = \{ x \in \Omega_\beta ;\ \dot{V}(x) = 0 \} = \{ x \in \Omega_\beta ;\ x_2 = 0 \}$$

- To compute M:

let $x(t) \in E \quad \forall t \geq 0$

$\Rightarrow x_2(t) = 0 \quad \forall t \geq 0$

$\Rightarrow \dot{x}_2(t) = 0 \quad \forall t \geq 0$

$\Rightarrow \sin(x_1(t)) = 0 \quad \forall t \geq 0$

$\Rightarrow x_1(t) = 0$ or π

but $x_1(t) = \pi$ is excluded because $x(t) \in \Omega_\beta$

$\Rightarrow$ largest invariant set $M = \{(0,0)\}$

$\Rightarrow x(t) \to 0$ as $t \to \infty$

$\Rightarrow x = 0$ is A.S.

Corollary:

- let $\bar{x}=0$ be eqlb. point for $\dot{x}=f(x)$, and V be p.d on D

  open set containing 0

  s.t. $\dot{V}(x) \leq 0 \quad \forall x \in D$

- let $E=\{x \in D;\ \dot{V}(x)=0\}$

- If the only solution that stays in E is $x(t)=0$, then

  $$\bar{x}=0 \text{ is A.S.}$$

- If $D=\mathbb{R}^2$, and V is radially unbdd, then

  $$\bar{x}=0 \text{ is GAS.}$$