Last time: Lyapunov method for lin. sys.

- For a lin sys. $\dot{x} = Ax$, we construct a quadratic Lyapunov function $V(x) = x^T P x$ where P is a p.d. matrix that solves the Lyapunov eq.

$$A^T P + PA + Q = 0 \quad \text{for some } Q \succ 0.$$

we pick this.

- if A is Hurwitz, this eq. has a unique sol. $P \succ 0$.

- The function $V(x) = x^T P x$ serves as Lyapunov function and

$$\dot{V}(x) = x^T (A^T P + PA) x = -x^T Q x < 0 \quad \forall x \neq 0$$

- Today: use this construction to analyze stability of systems that are approximately linear.

- Consider "nearly" lin. sys.

$$\dot{x} = \underbrace{Ax}_{\text{Stable}} + \underbrace{g(t,x)}_{\text{Perturbation}}$$

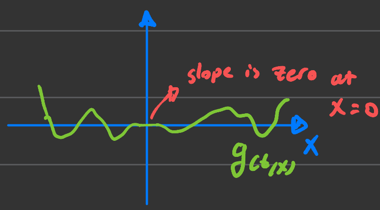
- We like to see if $x=0$ remains stable under perturbation.

- We consider perturbations that are vanishing in the following sense:

$$\boxed{\lim_{\|x\|\to 0} \sup_{t\geq 0} \frac{\|g(t,x)\|}{\|x\|} = 0}$$

What does this mean?

$\forall \varepsilon>0, \exists \delta_\varepsilon>0$ s.t. $\dfrac{\|g(t,x)\|}{\|x\|} < \varepsilon, \quad \forall t\geq 0$ and $\|x\|<\delta_\varepsilon$



- This condition implies that $g(t,0)=0, \ \forall t$. Therefore, $x=0$ remains an eqlb. point.

- In order to study stability of $x=0$ we use a Lyapunov func. for the lin. sys. $\dot{x} = Ax$.

- Let $P \succ 0$ be the solution to

identity matrix

$$A^T P + PA + Q = 0 \quad \text{with} \quad Q = I$$

- Let $\underbrace{V(x) = x^T P x}_{\text{p.d. func. because } P \succ 0}$. Then,

$$\dot{V}(x) = (Ax + g(t,x))^T P x + x^T P (Ax + g(t,x))$$

$$= x^T (\underbrace{A^T P + PA}_{-Q = -I}) x + 2 x^T P g(t,x)$$

from Lyp. eq.

$$= -\|x\|_2^2 + 2 x^T P g(t,x)$$

$|a^T P b| \le \|a\|_2 \|P\|_2 \|b\|_2$

definition of norm

$$\le -\|x\|_2^2 + 2 \|P\|_2 \|x\|_2 \|g(t,x)\|_2$$

$$= -\|x\|_2^2 \Big(1 - 2 \|P\|_2 \underbrace{\frac{\|g(t,x)\|}{\|x\|}}_{\le \varepsilon \text{ if } \|x\| \le \delta_\varepsilon} \Big)$$

vanishing assump.

$$\le -\|x\|_2^2 (1 - 2 \|P\|_2 \varepsilon)$$

- Choose $\varepsilon < \frac{1}{2\|P\|_2}$, then $\dot{V}(x) < 0$ if $\|x\| \leq \delta_\varepsilon$

$\lambda_{max}(P)$

and $x \neq 0$

Lyp. stability result

$\Rightarrow$ $x = 0$ is A.S.

- Moreover, we can also obtain a convergence regime.

$$\lambda_{min}(P)\,\|x\|^2 \leq V(x) \leq \lambda_{max}(P)\,\|x\|_2^2$$

see math prelim. note

- Therefore, from lecture 8:

$$\lim_{t\to\infty} \|x(t)\| = 0 \quad \text{if} \quad \|x(0)\| < \sqrt{\frac{\lambda_{min}(P)}{\lambda_{max}(P)}}\,\delta_\varepsilon$$

ε is anything smaller than $\frac{1}{2\lambda_{max}(P)}$

Convergence regim

- We proved the following result.

Thm:

- Consider $\dot{x} = Ax + g(t,x)$ where A is Hurwitz and $\lim\limits_{\|x\| \to 0} \sup\limits_{t \geq 0} \frac{\|g(t,x)\|}{\|x\|} = 0$.

- Then $x = 0$ is A.S.

Corollary:

- Consider $\dot{x} = f(x)$ and let $\bar{x}$ be an eqb. point.

- If $f(x)$ is C^1 and $A = \frac{\partial f}{\partial x}(\bar{x})$ is Hurwitz, then $\bar{x}$ is A.S.

proof:

- Let $z_{(t)} = x_{(t)} - \bar{x}$ $\Rightarrow$

$$\dot{z} = Az + g(z)$$

where $A = \frac{\partial f}{\partial x}(\bar{x})$ and $g(z) = f(\bar{x}+z) - f(\bar{x}) - \frac{\partial f}{\partial \bar{x}}(\bar{x})\, z$

- if f is C^1, then $\lim\limits_{\|z\| \to 0} \frac{\|g(z)\|}{\|z\|} = 0$ $\Rightarrow$ Conditions of Thm are true, so $z = 0$ or $x = \bar{x}$ is A.S.

why?

by mean value thm, $\underbrace{g(z) - g(0)}_{0} = \underbrace{\frac{\partial g}{\partial z}(tz)}_{\frac{\partial f}{\partial x}(\bar{x}+tz) - \frac{\partial f}{\partial x}(\bar{x})} z$ for some $t \in [0,1]$

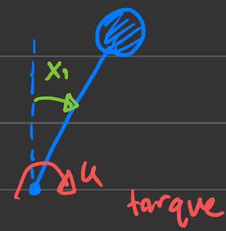
$\Rightarrow \|g(z)\| \leq \|\frac{\partial g}{\partial z}(tz)\| \, \|z\|$

$\Rightarrow \frac{\|g(z)\|}{\|z\|} \leq \underbrace{\|\frac{\partial f}{\partial x}(\bar{x}+tz) - \frac{\partial f}{\partial x}(\bar{x})\|}$

converges to zero as $z \to 0$ because $\frac{\partial f}{\partial x}$ is assumed to be continuous

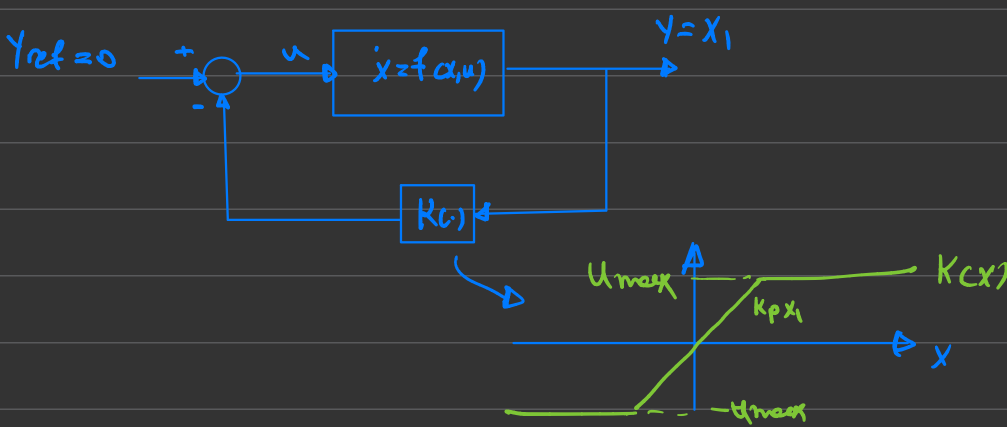
Example : (inverted pendulum)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} x_2 \\ \omega^2 \sin(x_1) - \gamma x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}}_{f(x,u)}$$



- Assume we use a proportional feedback control law to swing up the pendulum with gain K_p

- assume the torque gets saturated at some maximum torque value U_{max}.



- we like to use Lyapunov function method to analyze the stability of this control law.

$$\dot{x} = \underbrace{\begin{bmatrix} x_2 \\ w^2 \sin(x_1) - \gamma x_2 - K(x_1) \end{bmatrix}}_{f(x)}$$

- Linearize around $x=0$:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ w^2 - K_p & -\gamma \end{bmatrix}}_{A = \frac{\partial f}{\partial x}(\bar{x})} x + g(x)$$

where $g(x) = \begin{bmatrix} 0 \\ w^2 \sin(x_1) - w^2 x_1 + K_p x_1 - K(x_1) \end{bmatrix}$

- A is Hurwitz when $K_p > w^2$

- And $\|g(x)\|_2 \leq \frac{w^2}{2} \|x\|_2^2$ if $|x_1| \leq \frac{u_{max}}{K_p}$

because $|\sin(x_1) - x_1| \leq \frac{x_1^2}{2}$ and $|K_p x_1 - K(x_1)| = 0$ if $|x_1| \leq \frac{u_{max}}{K_p}$

- Let $P \succ 0$ be solution to $A^T P + PA + Q = 0$

with $Q = I$.

- Then, $\dot{V}(x) \leq \left(1 - \lambda_{max}(P)\, w^2 \|x\|\right) \|x\|^2 < 0$

if $\|x\| \leq \dfrac{1}{w^2 \lambda_{max}(P)}$ and $|x_1| \leq \dfrac{u_{max}}{K_p}$

$\Longrightarrow \lim_{t \to \infty} \|x(t)\| = 0$ if $\|x(0)\| \leq \sqrt{\dfrac{\lambda_{min}(P)}{\lambda_{max}(P)}}\, r$

where $r = \min\left(\dfrac{1}{w^2 \lambda_{max}(P)}, \dfrac{u_{max}}{K_p}\right)$