Control Lyapunov functions:

- Consider the sys. $\dot{x} = f(x,u)$ $(*)$

where $x \in \mathbb{R}^n$; $u \in \mathcal{U} \subseteq \mathbb{R}^m$

$\mathcal{U}$ → Constraint set

Objective: design a stabilizing feedback control law.

Def: a $\underbrace{\text{positive definite}}_{V(x)>0,\ x \neq 0,\ V(0)=0}$ func. $V: \mathbb{R}^n \to \mathbb{R}$ is CLF

for $(*)$ if $$\min_{u \in \mathcal{U}} \underbrace{\dot{V}(x,u)}_{\nabla V(x)^T f(x,u)} < 0 \quad \forall x \neq 0$$

→ there is always an input u that makes the derivative of V along trajectory negative.

- With a CLF, we can define the set of stabilizing inputs u at each point x:

$$\Omega(x) = \{ u \in \mathcal{U};\ \dot{V}(x,u) < 0 \}$$

- With a CLF, we need to choose our control law $u = k(x)$ s.t.

$$k(x) \in \Omega(x) \quad \forall x$$

- With such a choice, $(*)$ becomes AS (GAS if V is radially unbdd)

- For example, we can choose

$$k(x) = \underset{u \in U}{\operatorname{argmin}} \|u\| \quad \text{s.t.} \quad \dot{V}(x,u) \le -\underbrace{W(x)}_{\text{p.d. function.}}$$

→ point-wise min-norm control law

plan for today: analyze CLF method for control affine sys.

Control affine sys.

$$\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2 + \cdots + g_m(x)u_m$$

$$= f(x) + \underbrace{[g_1(x) : g_2(x) : \cdots , g_m(x)]_{n \times m}}_{g(x)} \underbrace{\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}_m}_{u} \qquad (**)$$

- Many systems are control affine $\rightarrow$ legged robots, Segways, lane control

- Checking the CLF condition for a control affine sys. becomes

simple: $\dot{V}(x,u) = \nabla V(x)^T (f(x) + g(x)u)$

$$= \nabla V(x)^T f(x) + \nabla V(x)^T g(x) u$$

$\Rightarrow$ if $\nabla V(x)^T g(x) \neq 0$, then we can find a u that makes $\dot{V}(x,u) < 0$

$\Leftrightarrow$ $\min_u \dot{V}(x,u) < 0$ if $\nabla V(x)^T g(x) \neq 0$

$\Rightarrow$ V is a CLF if $\nabla V(x)^T f(x) < 0$ whenever $\nabla V(x)^T g(x) = 0$

Lemma: V is a CLF for $(**)$ iff

$$\underbrace{\nabla V(x)^T f(x)}_{a(x)} < 0 \quad \text{whenever} \quad \underbrace{\nabla V(x)^T g(x)}_{b^T(x)} = 0$$

- Finding a smooth stabilizing feedback control law for a control affine sys with a CLF becomes simple.

Thm (Sontag)

- If $(**)$ has CLF V, then

$$K(x) = \begin{cases} -\alpha(x)\, b(x) & \text{if } b(x) \neq 0 \\ 0 & \text{else} \end{cases}$$

where $b(x) = \underbrace{[g(x)^T]}_{m\times n} \underbrace{\nabla V(x)}_{n} \in \mathbb{R}^m$, $a(x) = \nabla V(x)^T f(x) \in \mathbb{R}$

$$\alpha(x) = \frac{a(x) + \sqrt{a(x)^2 + \|b(x)\|^4}}{\|b(x)\|^2}$$

is C^1 away from 0 and makes $(**)$ A.S. (GAS if V is radially unbdd)

Proof: need to verify $\dot{V}(x, K(x)) < 0$

$$\dot{V}(x,u) = \underbrace{\nabla V(x)^T f(x)}_{a(x)} + \underbrace{\nabla V(x)^T g(x)}_{b(x)^T} u$$

$$= a(x) + b(x)^T u$$

$u = K(x)$ if $b(x) \neq 0$

$$\Rightarrow \quad \dot{V}(x, K(x)) = a(x) + b(x)^T(-\alpha(x)\, b(x))$$

$$= a(x) - \alpha(x)\, \|b(x)\|^2$$

$$= -\sqrt{a(x)^2 + \|b(x)\|^4} < 0 \quad \text{when } b(x) \neq 0$$

and

if $b(x) = 0$

$$\dot{V}(x, K(x)) = a(x) < 0 \quad \text{when } b(x) = 0$$

from lemma

$\Rightarrow$ A.S. ✓

- The C^1 property comes from special form of the control

$$\phi(a,b) = \frac{a + \sqrt{a^2+b^2}}{b} \text{ is real analytic}$$

__Example:__

$$\dot{x} = -x^3 + u$$

- This system is stable when $u=0$, but the convergence for $\dot{x} = -x^3$ is very slow!

- We like to design a control law to make the system converge faster. We use 3 methods:

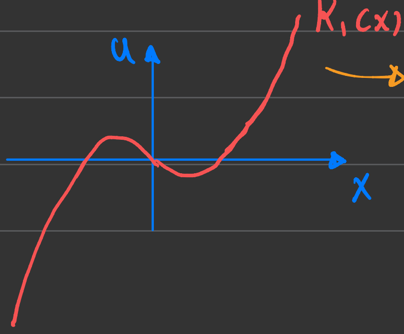
① Feedback linearization: Cancel the nonlinear part and add a stable linear term → does not require CLF

but does not take advantage of the physics of the model

$$u = +x^3 - x =: K_1(x)$$

$$\Rightarrow \dot{x} = -x^3 + u = -x \checkmark$$

stable and fast



→ large control when $|x| \to \infty$

② CLF and pointwise min Control:

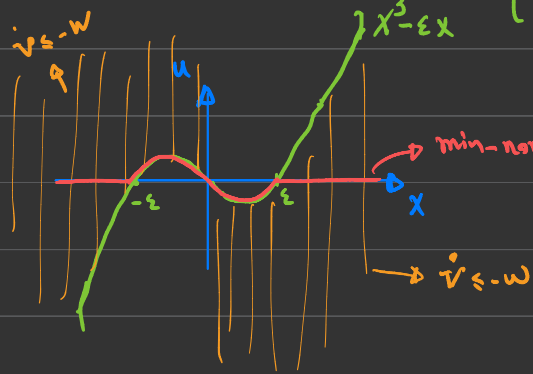
let $V(x) = \frac{1}{2}x^2 \Rightarrow \dot{V}(x,u) = x(-x^3+u) = -x^4 + xu$

It is a CLF because $\forall x \neq 0 \quad \min_u (-x^4 + xu) = -\infty < 0$

Pointwise min Control:

$K_2(x) = \operatorname{argmin} |u|$

s.t. $\underbrace{\dot{V}(x,u)}_{-x^4+xu} \leq -\underbrace{W(x)}_{\varepsilon^2 x^2} \Rightarrow xu \leq x^4 - \varepsilon^2 x^2 \Rightarrow \begin{cases} u \leq x^3 - \varepsilon^2 x & \text{if } x > 0 \\ u \geq x^3 - \varepsilon^2 x & \text{if } x < 0 \\ u \text{ can be anything} & \text{if } x = 0 \end{cases}$



$\Rightarrow K_2(x) = \begin{cases} 0 & \text{if } |x| > \varepsilon \\ x^3 - \varepsilon x & \text{if } |x| \leq \varepsilon \end{cases}$ → Not smooth. → Zero Control when $|X| \to \infty$

and

$\dot{X} = \begin{cases} -\varepsilon x & \text{if } |x| \leq \varepsilon \\ -x^3 & \text{if } |x| > \varepsilon \end{cases}$ → stable and fast if $\varepsilon \approx 1$

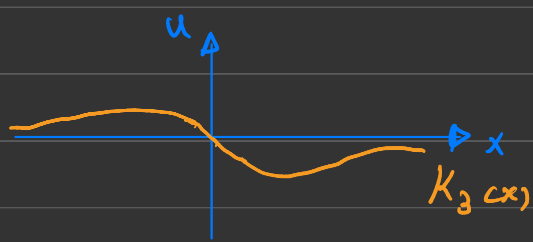
③ CLF and Sontag's formula

$$\dot{V}(x,u) = \underbrace{-x^4}_{a} + \underbrace{x}_{b}\, u$$

$$\alpha(x) = \frac{a(x) + \sqrt{a(x)^2 + b(x)^4}}{b(x)^2} = \frac{-x^4 + \sqrt{x^8 + x^4}}{x^2} = -x^2 + \sqrt{x^4+1}$$

$$\Rightarrow K_3(x) = \begin{cases} -\alpha(x)\, b(x) & \text{if } b(x) \neq 0 \\ 0 & \text{if } b(x) = 0 \end{cases} = x^3 - x\sqrt{x^4+1}$$

$\rightarrow 0$ as $|x| \rightarrow \infty$



and $\dot{x} = -x^3 + K_3(x) = -x\sqrt{x^4+1} \begin{cases} \simeq -x & \text{when } |x| \text{ is small} \\ \simeq -x^3 & \text{when } |x| \rightarrow \infty \end{cases}$

Control Barrier functions: (CBF)

- Consider $\dot{x} = f(x)$.
- Instead of stability, suppose we are interested to ensure safety, by requiring $x(t) \in S \quad \forall t$

Safe set → for example to maintain safe distance from another car in cruise control.

- Suppose, we can express the safe set as super-level set of a function $h: \mathbb{R}^n \to \mathbb{R}$

$$S = \{x ; h(x) \geq 0\}$$

e.g $h(x) = 1 - \|x\|^2$ to ensure $\|x\| \leq 1$

- Then, safety condition is expressed as

$$h(x(t)) \geq 0 \quad \forall t.$$

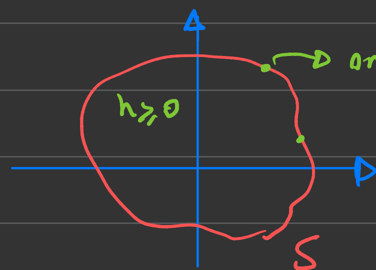
- We can have Lyapunov like results that ensures safety.

If $\dot{h}(x) \geq -\alpha h(x) \quad \forall x$

then $S = \{x ; h(x) \geq 0\}$ is an invariant set

$x(0) \in S \Rightarrow x(t) \in S \quad \forall t.$

proof by comparison lemma:



$$\dot{h} \geq -\alpha h$$

$$\Rightarrow \frac{d}{dt}(-h) \leq -\alpha(-h)$$

$$\Rightarrow -h(x(t)) \leq -e^{-\alpha t} h(x(0))$$

$$\Rightarrow h(x(t)) \geq e^{-\alpha t} h(x(0))$$

$$\Rightarrow h(x(t)) \geq 0 \text{ if } h(x(0)) \geq 0$$

$$\Rightarrow x(t) \in S \text{ if } x(0) \in S.$$

- Similar to CLF, we can construct Control Barrier func.

$$\dot{x} = f(x,u)$$
$$(***) \quad u \in U \subseteq \mathbb{R}^m$$

- h is a CBF for $(***)$ if

$$\max_{u \in U} \dot{h}(x,u) \geq -\alpha h(x) \quad \text{for some positive } \alpha$$

- CBF and CLF are combined for control design.

$$\operatorname*{argmin}_{u \in U} \|u\|$$
$$\dot{V} < 0$$
$$\dot{h} > -\alpha h$$

$\longrightarrow$ see the paper in Canvas Syllabus lec. 18.