Optimal Transportation Methods in Nonlinear Filtering

Presented at
Nonlinear Dynamics and Controls Lab (NDCL)

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University of Washington, Seattle

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Outline

- Introduction to optimal transport
- Application to nonlinear filtering
- Possible collaboration directions

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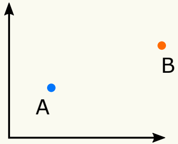
Brief introduction to optimal transportation (OT)
Geometry for probability distributions

Euclidean geometry



Optimal control problem:

$$\min_u \int_0^1 \|u_t\|^2 \mathrm{d}t$$
$$\text{s.t.} \quad \dot{x}_t = u_t,$$
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Optimal trajectory gives us:

- how to interpolate
- a metric (Euclidean distance)

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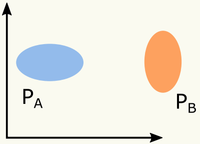

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C. Villani, Topics in Optimal Transportation, 2003

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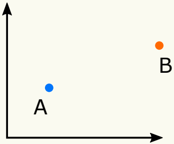

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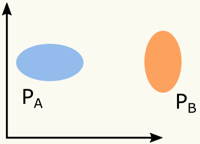

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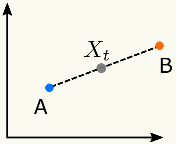

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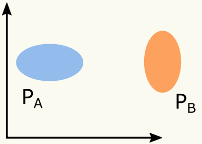

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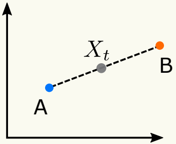

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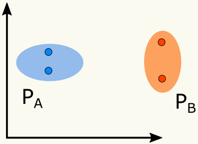

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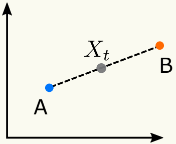

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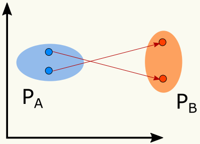

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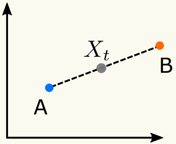

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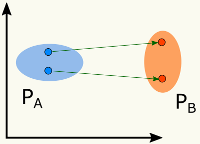

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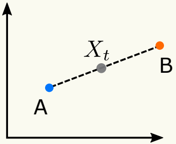

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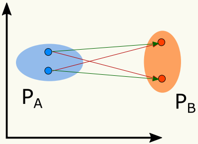

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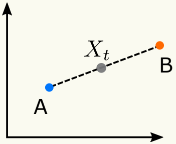

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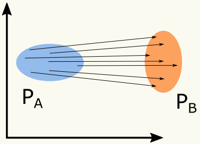

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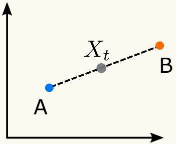

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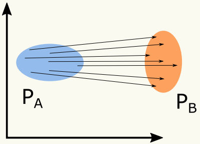

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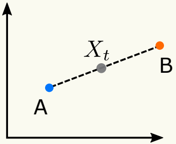

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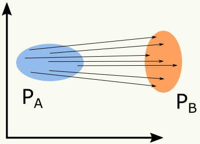

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Why is OT geometry useful?

L_2 average vs OT average

- average by modelling as functions: $\frac{p_1(x)+p_2(x)}{2}$
- average by modelling as prob. dist.: OT average
- interpolating between images
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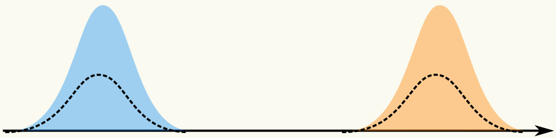

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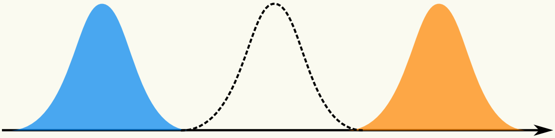

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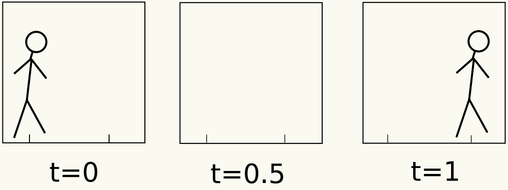

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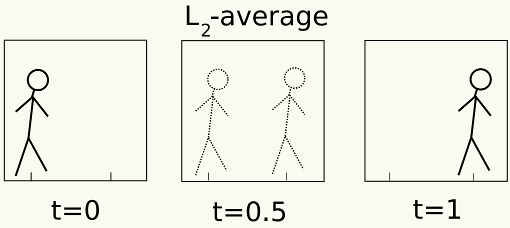

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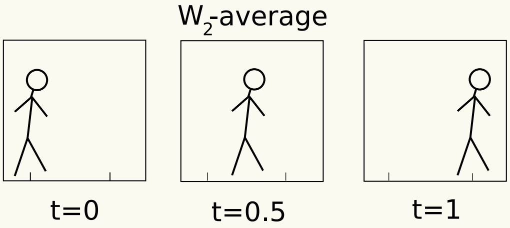

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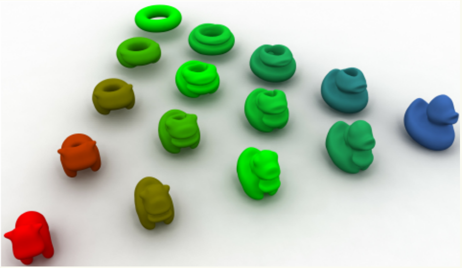

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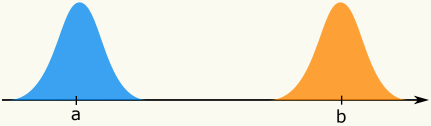

- Kullback-Leibler divergence: $D(p_1 \| p_2) = \dfrac{(a-b)^2}{\sigma^2}$ $\rightarrow$ diverges as $\sigma \rightarrow 0$
- Wasserstein distance : $W_2(p_1, p_2) = |a - b|$

M. Arjovsky, S. Chintala, L. Bottou, Wasserstein GAN, ICML, 2017

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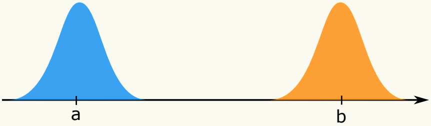

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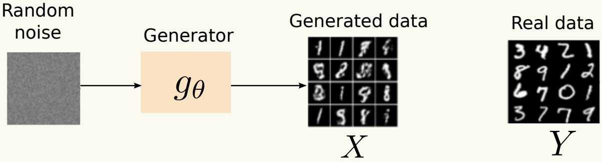
Application: Learning generative models (GAN)



make the distribution of X close to Y: $\min_g \; W(P_X, P_Y)$

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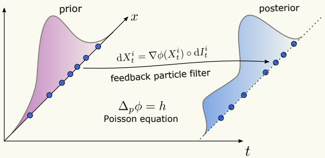
Research overveiw

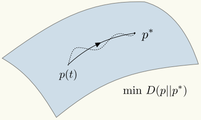
Control & Optimization for probability distributions

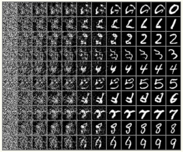
(I) Optimal filtering & control



- Optimal transportation methods in nonlinear filtering: The feedback particle filter, CSM, 2021
- An optimal transport formulation of the ensemble Kalman filter, TAC, 2021

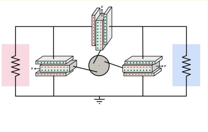
(II) Machine learning

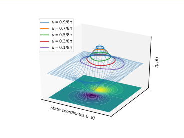




- OT mapping via input-convex neural networks, ICML, 2020
- Scalable computations of Wasserstein barycenter via input convex neural networks, ICML, 2021
- Variational Wasserstein gradient flow, Submitted to ICML, 2022

(III) Stochastic thermodynamics





- Energy harvesting from anisotropic fluctuations, PRE, 2021
- On the relation between information and power in stochastic thermodynamic engines, (L-CSS), 2021
- Maximal power output of a stochastic thermodynamic engine, Automatica, 2021

Common objectives:

- develop efficient and scalable algorithms
- understand fundamental limitations

Theoretical tools:

- optimal transportation
- mean-field optimal control
- statistical learning

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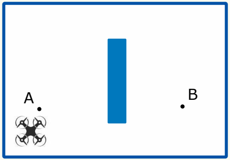
Motivation
Uncertainties are integral to control systems

Sources of uncertainty:

- incomplete model (e.g. disturbances)
- noisy sensors (e.g. LIDAR, IMU)
- augmenting stochastic ML modules
  (e.g. data driven models, object detection)



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- reliable prediction (e.g. to avoid collision)
- decision making and planning under uncertainty

Nonlinear filtering is a principled approach to quantify uncertainty

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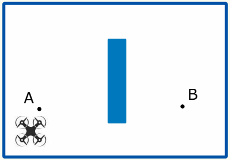

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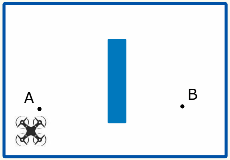

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Nonlinear filtering problem
Mathematical model

$$\begin{tikzcd} {} \arrow[r] & X_{k-1} \arrow[r] \arrow[d] & X_k \arrow[r] \arrow[d] & X_{k+1} \arrow[r] \arrow[d] & {} \\ & Y_{k-1} & Y_k & Y_{k+1} & \end{tikzcd}$$

- X_k is the state (unknown)
- Y_k is the observation
- dynamic and observation model are given

Questions: Given history of observation $Y_{1:k} := \{Y_1, \ldots, Y_k\}$,

- What is the most likely value of X_k?
- What is the probability of $X_k \in A$?
- What is the best m.s.e estimate for X_k?
- ...

Answer: given by the conditional distribution $\pi_k = \mathsf{P}(X_k | Y_{1:k})$ (posterior, belief)

J. Xiong, An introduction to stochastic filtering theory, 2008

Nonlinear filtering problem
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- What is the best m.s.e estimate for X_k? $\int x P(X_k = x | Y_{1:k}) \mathrm{d}x$
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Recursive update for posterior

Given $\pi_k = \mathsf{P}(X_k|Y_{1:k})$, obtain $\pi_{k+1} = \mathsf{P}(X_{k+1}|Y_{1:k+1})$ according to

- Step 1: dynamics update

$$\mathsf{P}(X_k|Y_{1:k}) \xrightarrow{\text{dynamics}} \mathsf{P}(X_{k+1}|Y_{1:k})$$

- Step 2: Bayes law

$$\mathsf{P}(X_{k+1}|Y_{1:k+1}) = \frac{P(Y_{k+1}|X_{k+1})P(X_{k+1}|Y_{1:k}))}{P(Y_{k+1}|Y_{1:k})}$$

As a result:

- π_k has Markov property
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Nonlinear filtering algorithm:

- numerical representation of π_k
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Kalman filter
Linear Gaussian setting

- linear dynamics: $X_{k+1} = AX_k + Be_k$
- linear observation: $Z_k = HX_K + W_k$

Kalman filter: posterior π_k is Gaussian $N(m_k, \Sigma_k)$

$$\text{Update for mean:} \quad m_{k+1} = \underbrace{Am_k}_{\text{dynamics}} + \underbrace{\mathsf{K}_k(Z_k - Hm_k)}_{\text{correction}}$$

$$\text{Update for variance:} \quad \Sigma_{k+1} = \text{(Ricatti equation)}$$

- application in navigation and guidance
- fails to represent multi-modal distributions $\rightarrow$ particle filters

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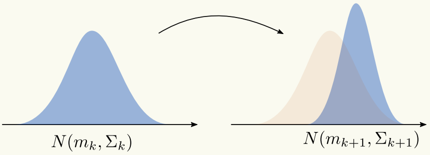

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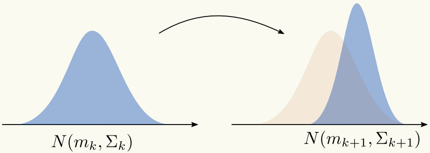

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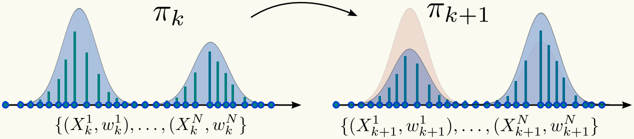
- Approximate π_k with weighted empirical distribution of particles
- Apply the update rule to the particles and weights



- Step 1: update particles according to dynamics
- Step 2: update the weights according to Bayes rule

$$w_{k+1}^i \propto w_k^i P(Z_{k+1}|X_{k+1}^i)$$

Properties:

- exact in the limit as $N \to \infty$
- weight degeneracy $\to$ curse of dimensionality

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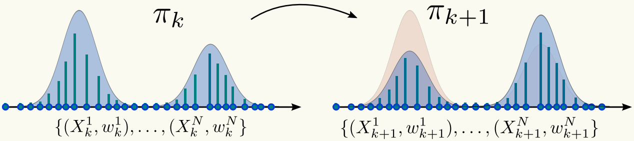

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Particle filters
Monte-Carlo approximation

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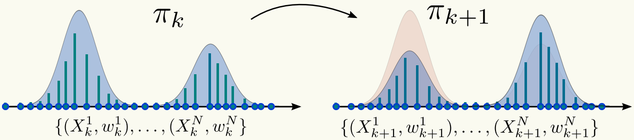

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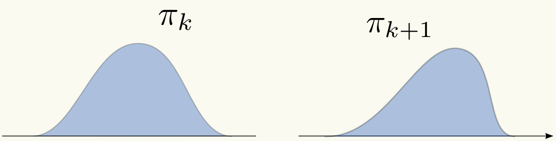
Optimal transportation viewpoint



- approximate π_k with empirical distribution of particles (uniform weights)
- main task of a particle-based algorithm:

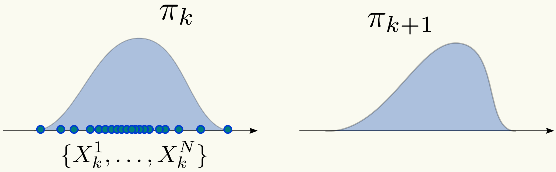
$$\text{given:} \quad \{X_k^1, \ldots, X_k^N\} \sim \pi_k$$
$$\text{generate:} \quad \{X_{k+1}^1, \ldots, X_{k+1}^N\} \sim \pi_{k+1}$$

OT approach: update particles with the optimal transport map from π_k to π_{k+1}

$$X_{k+1}^i = T_k(X_k^i)$$

Question: How to numerically approximate the optimal transport map T_k?

Optimal transportation viewpoint



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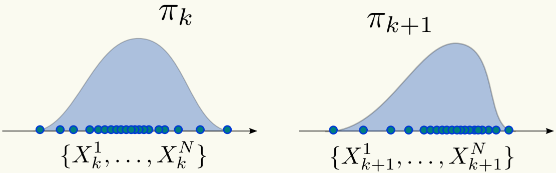

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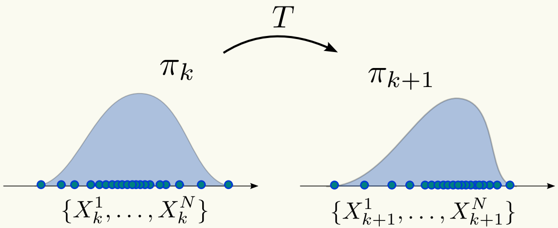

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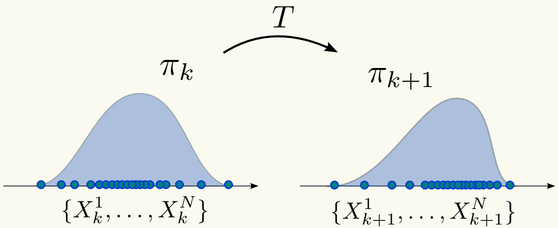

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Optimal transport particle filter

Variational approach: OT map is approximated by solving an optimization problem

$$T(x) = \nabla_x f(x, Y_k), \quad \text{where} \quad f = \arg\min_{f \in \mathcal{F}} \frac{1}{N} \sum_{i=1}^{N} \ell(f(X_k^i, Y_k^i))$$

- this is a stochastic optimization problem
- $\mathcal{F}$ is a subset of functions
- approximation becomes exact when $N = \infty$ and $\mathcal{F}$ is all functions

Computational properties:

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Question: Can this approach mitigate the curse of dimensionality?

Curse of dimensionality
Linear Gaussian setting

- Method 1: OT approach with $\mathcal{F} = \{$quadratic functions$\}$ →Ensemble Kalman filter
- Method 2: standard particle filter
- Compare number of particles to achieve error ϵ:

Question: Does this extend to nonlinear setting? (ongoing work)

S. C. Surace, A. Kutschireiter, J. Pfister, How to avoid the curse of dimensionality: scalability of particle filters . . . , SIAM review, 2019
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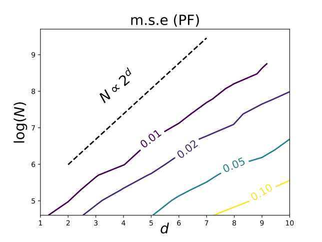

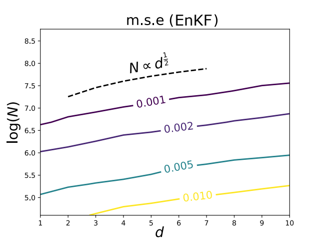

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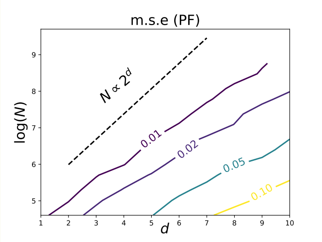

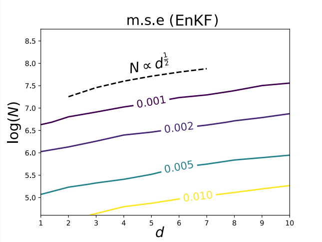

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EnKF: $N \approx O(\frac{d^2}{\epsilon})$

Question: Does this extend to nonlinear setting? (ongoing work)

S. C. Surace, A. Kutschireiter, J. Pfister, How to avoid the curse of dimensionality: scalability of particle filters . . . , SIAM review, 2019
A. Taghvaei, P. G. Mehta, An optimal transport formulation of ensemble Kalman filter, (TAC) 2020

Main feature of OT approach
Flexibility in choosing $\mathcal{F}$

- particle filters treat all filtering problems similarly
- they can not take advantage of possibly existing simplicity or regularity
- OT approach allows flexibility (a knob) by changing optimization domain
- One can trade-off computational effort by the range of problems it can solve

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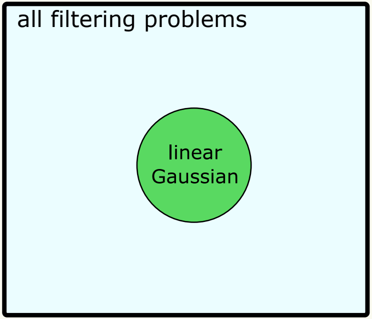

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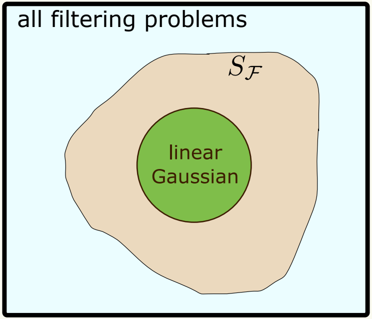

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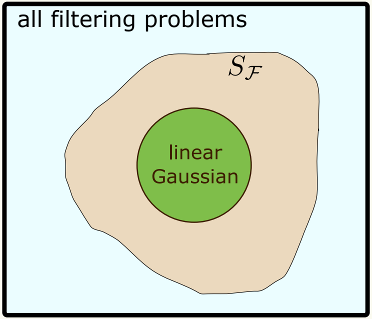

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Summary

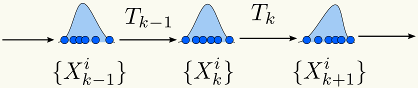
- **Mathematical model:**

$$
\begin{tikzcd}
{} \arrow[r] & X_{k-1} \arrow[r] \arrow[d] & X_k \arrow[r] \arrow[d] & X_{k+1} \arrow[r] \arrow[d] & {} \\
& Y_{k-1} & Y_k & Y_{k+1} &
\end{tikzcd}
$$

- **Nonlinear filtering:** compute the posterior $\pi_k = \mathsf{P}(X_k | Y_{1:k})$

$$
\longrightarrow \pi_{k-1} \longrightarrow \pi_k \longrightarrow \pi_{k+1} \longrightarrow
$$

- **OT approach:**



- **Feature:** variational approach enables flexibility (a knob) by changing optimization domain

Optimal transportation methods in nonlinear filtering: The feedback particle filter, IEEE CSM, 2021

Outline

- Introduction to optimal transport
- Application to nonlinear filtering
- Possible collaboration directions

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Optimal sensing

Nonlinear filtering provides framework to study the effect of sensors on uncertainty:

- sparse sensing: select minimal number of sensors
- how to optimally place sensors
- study biological sensing mechanisms in terms of their effect on uncertainty

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Active perception

- What if we have control?

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- u_k can only depend on the history $\{Y_1, \dots, Y_k\}$
- How to design control policies to reduce uncertainty optimally?
- How to solve optimal control problems

$$\min_u \mathbb{E}[\sum_{k=1}^{T} c(X_k, u_k) + \phi(X_T)]$$

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- Common approach: perform control and estimation independently (separation principle)
- Does not give solutions that might be expensive at first, but reduce uncertainty
- formulate as control problem for the posterior or belief
- use nonlinear filtering algorithms to represent and propagate posterior
- Application: perception aware motion planning

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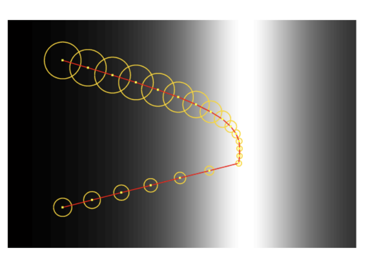

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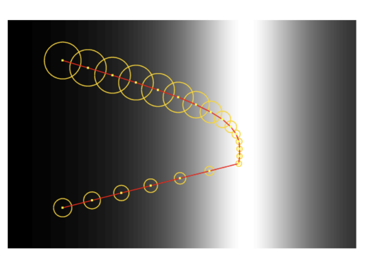

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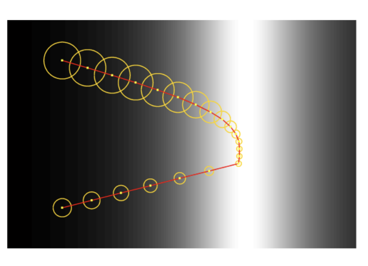

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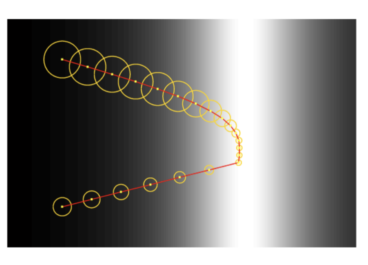

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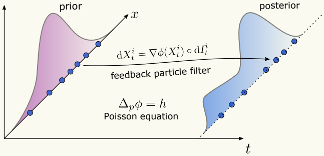
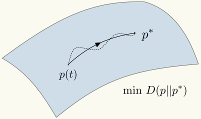
Final slide

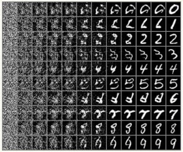
(I) Optimal filtering & control



- Optimal transportation methods in nonlinear filtering: The feedback particle filter, CSM, 2021
- An optimal transport formulation of the ensemble Kalman filter, TAC, 2021

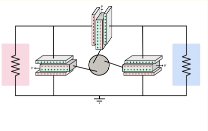
(II) Machine learning

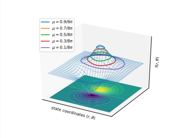




- OT mapping via input-convex neural networks, ICML, 2020
- Scalable computations of Wasserstein barycenter via input convex neural networks, ICML, 2021
- Variational Wasserstein gradient flow, Submitted to ICML, 2022

(III) Stochastic thermodynamics





- Energy harvesting from anisotropic fluctuations, PRE, 2021
- On the relation between information and power in stochastic thermodynamic engines, (L-CSS), 2021
- Maximal power output of a stochastic thermodynamic engine, Automatica, 2021

Possible collaboration:

- Optimal sensing
- Active perception

Thank you for your attention!