CIE 227, SIGNALS AND SYSTEMS

Tutorial 4

Fourier Series

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For the continuous-time periodic signal

$$x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right),$$

determine the fundamental frequency ω_0 and the Fourier series coefficients a_k such that

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.$$

Solution

Freq --- gcd

$$w_2 = \frac{2\pi}{3}$$

$$w_3 = \frac{5\pi}{3}$$

$$\frac{\frac{2\pi}{3}}{\frac{\pi}{3}} = 2\ ,,\ \frac{\frac{5\pi}{3}}{\frac{\pi}{3}} = 5$$

$$\gcd(w_2, w_3) = \frac{\pi}{3}$$

$$\therefore w_0 = \frac{\pi}{3}$$

Fourier series coefficient

$$x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right)$$

$$= 2 + 0.5\,e^{j\frac{2\pi}{3}t} + 0.5\,e^{-j\frac{2\pi}{3}t} + \frac{4}{2j}\,e^{j\frac{5\pi}{3}t} - \frac{4}{2j}\,e^{-j\frac{5\pi}{3}t}$$

$$\because w_0 = \frac{\pi}{3}$$

$$= 2 + 0.5\,e^{j2w_0t} + 0.5\,e^{-j2w_0t} + \frac{2}{j}\,e^{j5w_0t} - \frac{2}{j}\,e^{-j5w_0t}$$

$$\because \boldsymbol{FS\ expression \ldots x(t)} = \sum_{-\infty}^{\infty} \boldsymbol{a_k\, e^{jw_0 t}}$$

$$\therefore a_0 = 2$$

$$a_2 = 0.5$$

$$a_{-2} = 0.5$$

$$a_5 = \frac{2}{j}$$

$$a_{-5} = -\frac{2}{j}$$

A continuous-time periodic signal $x(t)$ is real valued and has a fundamental period $T = 8$. The nonzero Fourier series coefficients for $x(t)$ are

$$a_1 = a_{-1} = 2, a_3 = a^*_{-3} = 4j.$$

Express $x(t)$ in the form

$$x(t) = \sum_{k=0}^{\infty} A_k \cos(\omega_k t + \phi_k).$$

Solution

$$k = 1,, a_1 = 2$$

$$k = -1,, a_{-1} = 2$$

$$k = 3,, a_3 = 4j$$

$$k = -3,, a_{-3} = (a_{-3}{}^*)^* = -4j$$

$$\because T_0 = 8$$

$$\therefore f_0 = \frac{1}{8},, w_0 = \frac{2\pi}{8} = \frac{\pi}{4}$$

$$\because \boldsymbol{FS\ expression \ldots x(t)} = \sum_{-\infty}^{\infty} \boldsymbol{a_k\, e^{jw_0 t}}$$

$$= 2\, e^{j\frac{\pi}{4}t} + 2\, e^{-j\frac{\pi}{4}t} + 4j\, e^{j3\frac{\pi}{4}t} - 4j\, e^{-j3\frac{\pi}{4}t}$$

$$= 2 * 2\left(\frac{e^{j\frac{\pi}{4}t} + e^{-j\frac{\pi}{4}t}}{2}\right) + 4j * 2j\left(\frac{e^{j\,3\frac{\pi}{4}t} - e^{-j3\frac{\pi}{4}t}}{2j}\right)$$

$$= 4\cos\left(\frac{\pi}{4}t\right) - 8\sin\left(\frac{3\pi}{4}t\right)$$

$$\because -\sin x = \cos(x + \frac{\pi}{2})$$

$$= 4\cos\left(\frac{\pi}{4}t\right) + 8\cos\left(\frac{3\pi}{4}t + \frac{\pi}{2}\right)$$

3–20 Fourier series coefficients of sum of sinusoids

Solution

PROBLEM:

A periodic signal, $x(t)$, is given by

$$x(t) = 1 + 3\cos(300\pi t) + 2\sin(500\pi t - \pi/4)$$

(a) What is the period of $x(t)$?

(b) Find the Fourier series coefficients of $x(t)$.

period

Solution

a)

$$f_1 = 150$$

$$f_2 = 250$$

gcd (150, 250) = 50

$$\therefore f_0 = 50\,,,\, t_0 = \frac{1}{50} = 0.02\ sec$$

b)

$$\because \boldsymbol{FS\ expression \ldots x(t) = \sum_{-\infty}^{\infty} a_k\, e^{jw_0 t}}$$

$$\mathbf{x(t) = 1 + 3\cos(300\pi t) + 2\sin\left(500\,\pi t - \frac{\pi}{4}\right)}$$

$$\because \sin x = \cos(x - \frac{\pi}{2})$$

$$= 1 + 3\cos(300\pi t) + 2\cos\left(500\,\pi t - \frac{\pi}{4} - \frac{\pi}{2}\right)$$

$$= 1 + 3\cos(300\pi t) + 2\cos\left(500\,\pi t - \frac{3\pi}{4}\right)$$

Using Euler's formula

$$\cos x = \frac{e^{jx} + e^{-jx}}{2}$$

$$x(t) = 1 + \frac{3}{2}\left(e^{j\,300\,\pi t} + e^{-j300\pi t}\right) + \frac{2}{2}\left(e^{j\left(500\,\pi t - 3\frac{\pi}{4}\right)} + e^{-j\left(500\,\pi t - 3\frac{\pi}{4}\right)} \right)$$

$$= 1 + \frac{3}{2}\left(e^{j\,300\pi\, t} + e^{-j300\pi t}\right) + \left(e^{j\,500\pi t}\; e^{-j\,3\frac{\pi}{4}} + e^{-j\,500\pi t}\; e^{j\,3\frac{\pi}{4}} \right)$$

$$= 1 + \frac{3}{2}\left(e^{j\,2\pi(150)\,t} + e^{-j2\pi(150)t}\right) + \left(e^{j\,2\pi(250)t}\; e^{-j\,3\frac{\pi}{4}} + e^{-j\,2\pi(250)t}\; e^{j\,3\frac{\pi}{4}} \right)$$

$\therefore\, The\ fourier\ series\ cofficient\ of\ x(t)$

$$a_0 = 1$$

$$a_3 = a_{-3} = \frac{3}{2}$$

$$a_5 = e^{-j\,3\frac{\pi}{4}}$$

$$a_{-5} = e^{j\,3\frac{\pi}{4}}$$

3–36 Determine Fourier Series for a Sum of Cosine Signals

Solution

PROBLEM:

A periodic signal, $x(t)$, is given by

$$x(t) = 2 + \cos(250\pi t - \pi) + 2\sin(750\pi t)$$

$2 \times 125 \qquad 2 \times 375$

(a) What is the period of $x(t)$?

(b) Find the Fourier series coefficients of $x(t)$ for $-6 \le k \le 6$.

Fourier Series | cosine signals

a)

$$f_1 = 125$$

$$f_2 = 375$$

gcd (125, 375) = 125

$$\therefore f_0 = 125\,,\,,t_0 = \frac{1}{125} = 8\,m\,sec$$

b)

$$\mathbf{x(t) = 2 + \cos(250\,\pi t - \pi) + 2\sin(750\,\pi t)}$$

$$\because \sin x = \cos(x - \frac{\pi}{2})$$

$$x(t) = 2 + \cos(250\,\pi t - \pi) + 2\cos\left(750\,\pi t - \frac{\pi}{2}\right)$$

Using Euler's formula

$$= 2 + \frac{1}{2}\left(e^{j\,250\,\pi t - \pi} + e^{-j250\pi t - \pi}\right) + \frac{2}{2}\left(e^{j\left(750\,\pi t - \frac{\pi}{2}\right)} + e^{-j\left(750\,\pi t - \frac{\pi}{2}\right)}\right)$$

$$= 2 + \frac{1}{2}\left(e^{j\,2\pi(125)t}\, e^{-j\pi} + e^{-j2\pi(125)t-} e^{j\pi}\right) + \left(e^{j\,2\pi(375)t}\, e^{-j\frac{\pi}{2}} + e^{-j\,2\pi(375)t}\, e^{j\frac{\pi}{2}} \right)$$

$\therefore The\ fourier\ series\ cofficient\ of\ x(t)$

$$a_0 = 2$$

$$a_1 = \frac{1}{2} e^{-j\,\pi}$$

$$a_{-1} = \frac{1}{2} e^{j\,\pi}$$

$$a_3 = e^{-j\frac{\pi}{2}}$$

$$a_{-3} = e^{j\frac{\pi}{2}}$$

For all other k ,,,,

$$a_x = zero$$

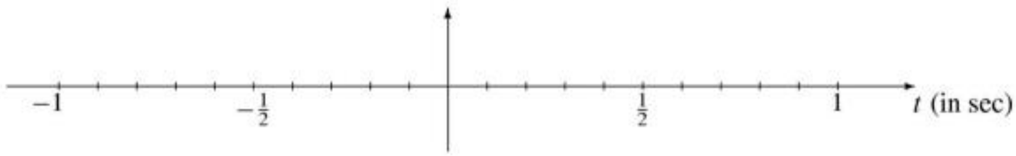
3–60 Fourier Series Integral for Specific Signal

Solution

PROBLEM:

Suppose that a periodic signal is defined (over one period) as: $x(t) = \begin{cases} 1 & \text{for } 0.7 < t < 0.8 \\ -1 & \text{for } 0 < t < 0.7 \end{cases}$

(a) Assume that the period of $x(t)$ is 0.8 s. Sketch $x(t)$ over the ENTIRE range $-1 \leq t \leq 1$ s.



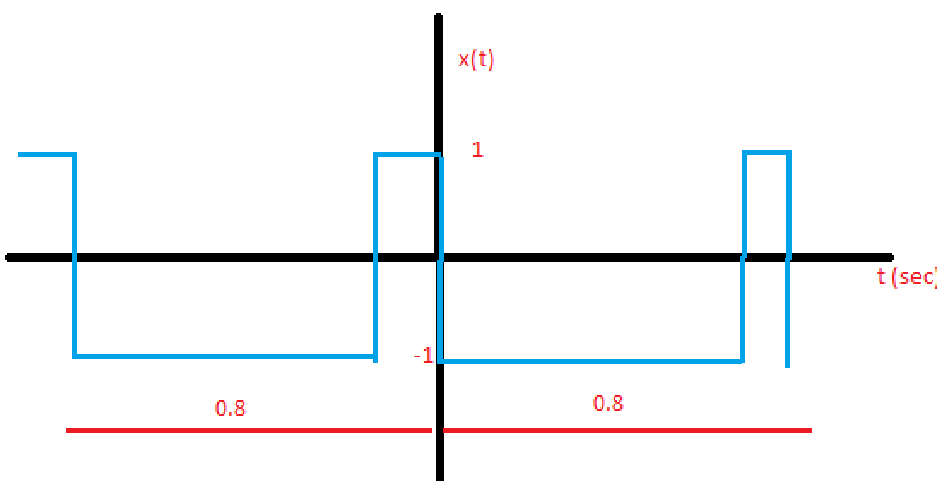
(b) Write the general Fourier integral expression for the coefficient a_k in terms of the specific signal $x(t)$ defined above. *Set up all the specifics of the integrals (e.g., limits of integration), but do not evaluate the integrals. All parameters in the integrals should have numeric values.*

(c) Evaluate the Fourier integral below. Simplify your answer and express it in **polar form.**

$$\frac{1}{4}\int_{-0.5}^{0.5} \cos(\pi t) e^{-j2\pi(2)t/4} dt$$

Fourier Series   Periodic Signal

a)



b)

$$a_k = \frac{1}{T}\int x(t)\, e^{-j\,k w_0 t}\, dt$$

$$\because t_0 = 0.8\,, \qquad \therefore f_0 = \frac{1}{0.8}\,,,\, w_0 = \frac{2\pi}{0.8}$$

$$a_k = \frac{1}{0.8}\left(\int_0^{0.7} (-1) e^{-j\,k\frac{2\pi}{0.8}t}\, dt + \int_{0.7}^{0.8} (1) e^{-j\,k\frac{2\pi}{0.8}t}\, dt\right)$$

c)

$$\boldsymbol{\frac{1}{4}\int_{-0.5}^{0.5} \cos(\pi t)\, e^{-j\,2\pi\,(2)\frac{t}{4}}\, dt}$$

$$= \frac{1}{4}\int_{-0.5}^{0.5} \frac{1}{2}\left(e^{j\pi t} + e^{-j\pi t}\right) e^{-j\,2\pi\,(2)\frac{t}{4}}\, dt$$

$$= \frac{1}{8}\int_{-0.5}^{0.5} \left(e^{j\pi t} e^{-j\,\pi\,t} + e^{-j\pi t} e^{-j\,\pi\,t}\right) dt$$

$$= \frac{1}{8}\int_{-0.5}^{0.5} \left(1 + e^{-j2\pi t}\right) dt$$

$$= \frac{1}{8}\left(t + \frac{e^{-j2\pi t}}{-j2\pi}\right)$$

$$= \frac{1}{8}\left((0.5 + 0.5) + \frac{1}{-j2\pi}\left(e^{-j2\pi\, 0.5} - e^{j2\pi\, 0.5}\right)\right)$$

$$= \frac{1}{8}\left(1 - \frac{1}{j2\pi}\left(e^{-j\pi} - e^{j\pi}\right)\right)$$

$$\because e^{-j\pi} = -1\ \&\ e^{j\pi} = -1$$

$$= \frac{1}{8}\left(1 - \frac{1}{j2\pi}(-1 + 1)\right)$$

$$= \frac{1}{8}$$

3–68 Compute the DC Component of a Periodic Signal

Solution

PROBLEM:

A periodic signal $x(t) = x(t + T_0)$ is described *over one period*, $0 \le t \le T_0$, by the equation

$$x(t) = \begin{cases} t & 0 \le t \le t_c \\ 0 & t_c < t \le T_0 \end{cases}$$

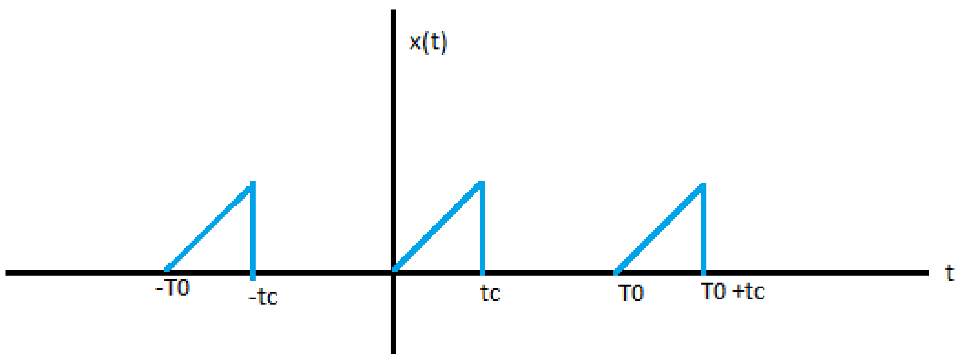
where $0 < t_c < T_0$.

(a) Sketch the periodic function $x(t)$ for $-T_0 < t < 2T_0$ for the specific case $t_c = \frac{1}{2}T_0$.

(b) Determine the D.C. coefficient of the Fourier Series, a_0. Once again, use the specific case of $t_c = \frac{1}{2}T_0$.

DC | Periodic | Fourier Series

a)



b)

$$a_k = \frac{1}{T}\int x(t)\, e^{-j\,kw_0 t}\, dt$$

$$\because k = 0 \;\therefore e^{-j\,0w_0 t} = 1$$

$$a_0 = \frac{1}{T_0}\int x(t)\, dt$$

$$= \frac{1}{T_0}\int_0^{t_c} t\, dt$$

$$= \frac{1}{T_0} * \frac{t^2}{2} = \frac{1}{2T_0} * t^2$$

$$= \frac{1}{2T_0} * \left(\frac{{T_0}^2}{4} - 0\right) = \frac{T_0}{8}$$

Another solution

Average DC level $=\frac{area\ undr\ one\ of\ the\ triangle}{T_0} = \frac{0.5*\frac{T_0}{2}*\frac{T_0}{2}}{T_0} = \frac{T_0}{8}$

3–69 Compute the Fourier Series Coefficients for a Periodic Pulse Signal

Solution

PROBLEM:

Use the signal $x(t)$ defined by the equation

$$x(t) = \begin{cases} t & 0 \le t \le t_c \\ 0 & t_c < t \le T_0 \end{cases}$$

where $t_c = \frac{1}{2}T_0$.

(a) Use the Fourier $\boxed{analysis}$ integral (for $k \neq 0$)

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}\,dt$$

to determine a general formula for the Fourier Series coefficients a_k. Your final result for a_k should depend on k.

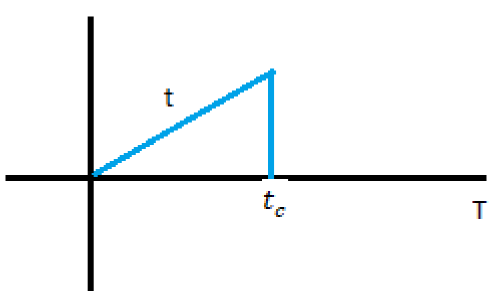
Notes: This Fourier integral requires integration by parts; in addition, the Fourier integral can be done over any period of the signal; in this case, the most convenient choice is from 0 to T_0.

Note: the frequency ω_0 would be given in rads/sec, but it does not have a specific value. However, you can simplify your formulas by using the identity $\omega_0 T_0 = 2\pi$.

(b) Use the Fourier Series coefficients to sketch the spectrum of $x(t)$ for the case $\omega_0 = 2\pi(\frac{1}{4})$ rad/sec and $t_c = \frac{1}{2}T_0$. Include *only* those frequency components corresponding to $k = 0, \pm1, \pm2, \pm3$. Label each component with its frequency and its complex amplitude (i.e., numerical values of magnitude and phase).

Fourier Coefficients | Pulse Signal

Solution



a) $\mathbf{a_k}$ **?**

$$a_k = \frac{1}{T_0}\int x(t)\, e^{-j\, k w_0 t}\, dt$$

$$t_c = \frac{T_0}{2}$$

$$w_0 = 2\pi f = \frac{2\pi}{T_0}$$

$$a_{k\neq 0} = \frac{1}{T_0}\int_0^{\frac{T_0}{2}} t\, e^{-j\, k w_0 t}\, dt$$

By using integration by parts

t	$e^{-j\, k w_0 t}$
1	$\frac{e^{-j\, k w_0 t}}{-j\, k w_0 t}$
0	$\frac{e^{-j\, k w_0 t}}{(-j\, k w_0 t)^2}$

$$a_{k\neq 0} = \frac{1}{T_0}\left[\frac{t e^{-j\, k w_0 t}}{-j\, k w_0 t} - \frac{e^{-j\, k w_0 t}}{(-j\, k w_0 t)^2}\right]$$

$$= \frac{1}{T_0}\left[\left(\frac{T_0}{2} * \frac{e^{-j\, k \frac{2\pi}{T_0}\frac{T_0}{2}}}{-j\, k \frac{2\pi}{T_0}} - 0\,\cancel{\frac{\frac{T_0}{2}\, e^0}{-j\, k \frac{2\pi}{T_0}}}\right) - \left(\frac{e^{-j\, k \frac{2\pi}{T_0}\frac{T_0}{2}}}{\left(-j\, k \frac{2\pi}{T_0}\right)^2} - \frac{e^0}{\left(-j\, k \frac{2\pi}{T_0}\right)^2}\right)\right]$$

$$= \frac{1}{T_0}\left[\frac{T_0}{2} * \frac{e^{-j\, k\pi}.T_0}{-j\, 2\pi\, k} + \frac{e^{-j\, k\pi}.{T_0}^2}{4\pi^2 k^2} - \frac{{T_0}^2}{4\pi^2 k^2}\right]$$

$$= \frac{1}{T_0}\, {T_0}^2 \left[\frac{0.5\, e^{-j\, k\pi}}{-j\, 2\pi\, k} + \frac{e^{-j\, k\pi}}{4\pi^2 k^2} - \frac{1}{4\pi^2 k^2}\right] * \frac{2}{2}$$

$$= \frac{T_0}{2}\left[\frac{-\,\mathrm{e}^{-\mathrm{j}\,\mathrm{k}\,\pi}}{\mathrm{j}\;2\pi\;\mathrm{k}} + \frac{2\mathrm{e}^{-\mathrm{j}\,\mathrm{k}\,\pi}}{4\pi^2 k^2} - \frac{2}{4\pi^2 k^2}\right]$$

b)

$$w = 2\pi\left(\frac{1}{4}\right)$$

$$f = \frac{1}{4}\ Hz$$

$$t = 4\ sec$$

$$\mathrm{a_k} = \ 0, \pm1, \pm2, \pm3$$

k	$\mathrm{a_k} = \frac{T_0}{2}\left[\frac{-\,\mathrm{e}^{-\mathrm{j}\,\mathrm{k}\,\pi}}{\mathrm{j}\;2\pi\;\mathrm{k}} + \frac{2\mathrm{e}^{-\mathrm{j}\,\mathrm{k}\,\pi}}{4\pi^2 k^2} - \frac{2}{4\pi^2 k^2}\right]$
1	
-1	
2	
-2	
3	
-3	

3–90 Evaluate Fourier Series Integral for a Specific Signal

Solution

PROBLEM:

A signal $x(t)$ is periodic with period $T_0 = 10$. Therefore it can be represented as a Fourier series of the form

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/10)kt}.$$

It is known that the Fourier series coefficients for this representation of a particular signal $x(t)$ are given by the integral

$$a_k = \frac{1}{10}\int_0^5 (t)e^{-j(2\pi/10)kt}\,dt. \qquad (1)$$

NOTE: Parts (c) and (d) can be worked independently of parts (a) and (b).

(a) In the expression for a_k in Equation (1) above, the integral and its limits define the signal $x(t)$. Determine an equation for $x(t)$ that is valid over one period.

Solution

a)

$$\mathbf{a_k = \frac{1}{T_0}\int x(t)\, e^{-j\,k w_0 t}\, dt\ ,,,,\ FS\ eq}$$

$$a_k = \frac{1}{10}\int_0^t t\, e^{-j\frac{2\pi}{10}kt}\, dt$$

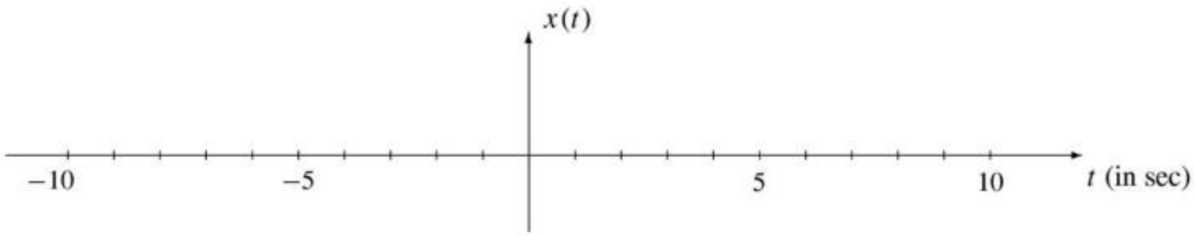
one period $T \ldots\ldots 0 \le t \le 10$

$$x(t) = t$$

$$w_0 = \frac{2\pi}{10}$$

$$\therefore x(t) = \begin{cases} t & 0 \le t \le 5 \\ 0 & 5 \le t \le 10 \end{cases}$$

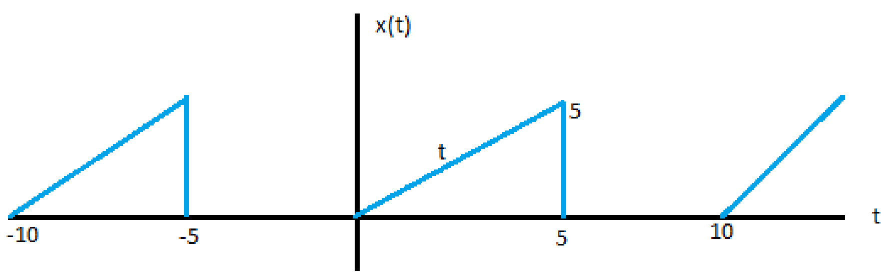
(b) Using your result from part (a), draw a plot of $x(t)$ over the range $-10 \leq t \leq 10$ seconds. Label it carefully.



(c) Which value of k in Equation (1) gives the DC (or average) value of $x(t)$? $k =$

(d) Determine the DC value of $x(t)$.

b)



c)

$$k = 0$$

d)

$$a_k = \frac{1}{10}\int_0^t t\,e^{-j\frac{2\pi}{10}kt}\,dt$$

$$a_0 = \frac{1}{10}\int_0^5 t\,e^{-0t}\,dt$$

$$= \frac{1}{10}\int_0^5 t\;dt$$

$$= \frac{1}{10}\,\frac{t^2}{2}$$

$$= \frac{1}{10} \frac{1}{2} (5^2 - 0) = 10.25$$

Another solution

$$DC = \frac{1}{T_0} * Area$$

$$= \frac{1}{10} * 2.5 * 5 = 1.25$$

Notes

$$x(t) = \sum_{-\infty}^{\infty} a_k \, e^{jw_0 t} \;\; \textit{Synthis eq}$$

$$\mathbf{a_k} = \frac{\mathbf{1}}{\boldsymbol{T_0}} \int \mathbf{x(t)\, e^{-j\,kw_0 t}\, dt} \;\; \textit{Analysis eq}$$

$$\cos x = \frac{e^{jx} + e^{-jx}}{2}$$

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

$$\sin x = \cos(x - \frac{\pi}{2})$$

$$-\sin x = \cos(x + \frac{\pi}{2})$$

Integration by parts:

$$\int u\, dv \; = uv - \int v\, du$$

How to use or apply the Tabular integration by parts method and its formulas?

Sign	F(x) Differentiate	F(y) Integration
+ $\Rightarrow$	F(x)	F(y)
- $\Rightarrow$	First derivative of F(x)	First Integrate of F(y)
+ $\Rightarrow$	Second derivative of F(x)	Second Integrate of F(y)
- $\Rightarrow$	Third derivative of F(x)	Third Integrate of F(y)