Assignment#5

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Question 1 (2 Marks)

Given that,

$$\mathbf{v}_1 = \begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 3 \\ -4 \\ 7 \end{bmatrix}$$

a) Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for $\mathbb{R}^3$.

b) Express $\mathbf{y}$ as a linear combination of the $\mathbf{v}$'s.

a) $V_1 \cdot V_2 = 8 - 8 + 0 = 0$

$V_1 \cdot V_3 = 4 - 4 + 0 = 0$

$V_2 \cdot V_3 = 2 + 2 - 4 = 0$

Then $\{V_1, V_2, V_3\}$ are orthognal basis for R^3

b)

$$c_1 = \frac{y \cdot V_1}{V_1 \cdot V_1} = \frac{12 + 16}{16 + 16} = \frac{28}{32} = \frac{7}{8}$$

$$c_2 = \frac{y \cdot V_2}{V_2 \cdot V_2} = \frac{6 - 8 - 7}{4 + 4 + 1} = \frac{-9}{9} = -1$$

$$c_3 = \frac{y \cdot V_3}{V_3 \cdot V_3} = \frac{3 - 4 + 28}{1 + 1 + 16} = \frac{3}{2}$$

$$y = \frac{7}{8} V_1 - V_2 + \frac{3}{2} V_3$$

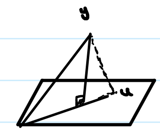
Question 2 (4 Marks)

Given that,

$$\mathbf{u} = \begin{bmatrix}1\\2\\2\end{bmatrix}, \mathbf{y} = \begin{bmatrix}2\\5\\3\end{bmatrix}$$

a) Write $\mathbf{y}$ as the sum of two orthogonal vectors, one in Span$\{\mathbf{u}\}$ and one orthogonal to $\mathbf{u}$.
b) Find the distance between $\mathbf{y}$ and $\mathbf{u}$.
c) Find the distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin.

$d = \sqrt{y^2 + u^2 - 2uy}$



$\cos\theta =$

a) $\hat{y} = \dfrac{y \cdot u}{u \cdot u} \cdot \vec{u} = \dfrac{18}{9}\begin{pmatrix}1\\2\\2\end{pmatrix} = 2u$ $\rightarrow$ $y = \hat{y} + z = \begin{pmatrix}2\\4\\4\end{pmatrix} + \begin{pmatrix}0\\1\\-1\end{pmatrix}$

$z = y - \hat{y} = \begin{pmatrix}2\\5\\3\end{pmatrix} - \begin{pmatrix}2\\4\\4\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix}$

b) $|z| = \sqrt{0+1+1} = \sqrt{2}$

c) $\cos\theta = \dfrac{y \cdot u}{|u||y|} = \dfrac{2+10+6}{3 \cdot \sqrt{38}} = \dfrac{3\sqrt{38}}{19}$

$d = \sqrt{y^2 + u^2 - 2yu\cos\theta} = \sqrt{38 + 9 - 2 \cdot 3 \cdot \sqrt{38} \cdot \dfrac{3\sqrt{38}}{19}} = \sqrt{11}$

Question 3 (3 Marks)

Given that,

$$\mathbf{u}_1 = \begin{bmatrix}4\\-4\\0\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}2\\2\\-1\end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix}4\\4\\7\end{bmatrix}, \mathbf{y} = \begin{bmatrix}3\\-4\\7\end{bmatrix}$$

a) Find the orthogonal projection of $\mathbf{y}$ onto Span$\{\mathbf{u}_1, \mathbf{u}_2\}$.
b) Find the orthogonal projection of $\mathbf{y}$ onto Span$\{\mathbf{u}_2, \mathbf{u}_3\}$.

a) $\hat{y} = \dfrac{y \cdot u_1}{u_1 \cdot u_1}\vec{u}_1 + \dfrac{y \cdot u_2}{u_2 \cdot u_2}\vec{u}_2$

$\hat{y} = \dfrac{7}{8}u_1 - u_2 = \begin{pmatrix}7/2\\-7/2\\0\end{pmatrix} + \begin{pmatrix}-2\\-2\\1\end{pmatrix} = \begin{pmatrix}3/2\\-11/2\\1\end{pmatrix}$

b) $V_2 = u_2$

b) $V_2 = U_2$

$$V_3 = U_3 - \frac{U_3 \cdot V_2}{V_2 \cdot V_2}\vec{V_2} = \begin{pmatrix}4\\4\\7\end{pmatrix} - \frac{9}{9}\begin{pmatrix}2\\2\\-1\end{pmatrix} = \begin{pmatrix}2\\2\\8\end{pmatrix}$$

$$\hat{y} = \frac{y \cdot V_2}{V_2 \cdot V_2}\vec{V_2} + \frac{y \cdot V_3}{V_3 \cdot V_3}\vec{V_3} = -1\begin{pmatrix}2\\2\\-1\end{pmatrix} + \frac{3}{4}\begin{pmatrix}2\\2\\8\end{pmatrix} = \begin{pmatrix}-0.5\\-0.5\\7\end{pmatrix}$$

Question 4 (5 Marks)

Given that,

$$A = \begin{bmatrix}1 & 1\\2 & 3\\2 & 4\end{bmatrix}, b = \begin{bmatrix}1\\7\\3\end{bmatrix}$$

a) Find a QR factorization of A.
b) Find the least squares solutions to $A\mathbf{x} = \mathbf{b}$.
c) Find the projection of $\mathbf{b}$ onto Col A.

a) $$V_1 = a_1 = \begin{pmatrix}1\\2\\2\end{pmatrix}$$

$$V_2 = a_2 - \frac{a_2 \cdot V_1}{V_1 \cdot V_1}\vec{V_1} = \begin{pmatrix}1\\3\\4\end{pmatrix} - \frac{15}{9}\begin{pmatrix}1\\2\\2\end{pmatrix} = \frac{1}{3}\begin{pmatrix}-2\\-1\\2\end{pmatrix}$$

$$U_1 = \frac{V_1}{\|V_1\|} = \begin{pmatrix}1/3\\2/3\\2/3\end{pmatrix}$$

$U_2 = V_2$

$$Q = \frac{1}{3}\begin{pmatrix}1 & -2\\2 & -1\\2 & 2\end{pmatrix}$$

$QR = A \Rightarrow \quad R = Q^T A$

$$R = \frac{1}{3}\begin{pmatrix}1 & 2 & 2\\-2 & -1 & 2\end{pmatrix} \cdot \begin{pmatrix}1 & 1\\2 & 3\\2 & 4\end{pmatrix} = \frac{1}{3}\begin{pmatrix}9 & 15\\0 & 3\end{pmatrix} = \begin{pmatrix}3 & 5\\0 & 1\end{pmatrix}$$

b)

$$A^T A \hat{x} = A^T b$$

$$\begin{pmatrix} 1 & 2 & 2 \\ 1 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 4 \end{pmatrix}\hat{x} = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 9 & 15 \\ 15 & 26 \end{pmatrix}\hat{x} = \begin{pmatrix} 21 \\ 34 \end{pmatrix}$$

$$\begin{pmatrix} 9 & 15 & 21 \\ 15 & 26 & 34 \end{pmatrix} \sim \begin{pmatrix} 9 & 15 & 21 \\ 0 & 1 & -1 \end{pmatrix}$$

$\hat{x}_2 = -1$, $9\hat{x}_1 + 15\hat{x}_2 = 21$

$9\hat{x}_1 - 15 = 21 \rightarrow \hat{x}_1 = \frac{10}{3}$

$$\hat{x} = \begin{pmatrix} \frac{10}{3} \\ -1 \end{pmatrix}$$

c)

$$A\hat{x} = \hat{b}$$

$$\hat{b} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} \frac{10}{3} \\ -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 7 \\ 11 \\ 8 \end{pmatrix}$$

Question 5 (4 Marks)

Find the least squares solutions to $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}$$

$$A^T A \hat{x} = A^T b$$

$$\underset{3\times 4}{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}}\underset{4\times 3}{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}}\hat{x} = \underset{3\times 4}{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}}\underset{4\times 1}{\begin{pmatrix} 1 \\ 3 \\ 8 \\ 2 \end{pmatrix}}$$

$$\begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \hat{x} = \begin{pmatrix} 14 \\ 4 \\ 10 \end{pmatrix}$$

$$\left(\begin{array}{cccc} 2 & 2 & 0 & 4 \\ 2 & 0 & 2 & 10 \\ 4 & 2 & 2 & 14 \end{array}\right) \sim \left(\begin{array}{cccc} 2 & 2 & 0 & 4 \\ 0 & -2 & 2 & 6 \\ 0 & -2 & 2 & 6 \end{array}\right) \sim \left(\begin{array}{cccc} 2 & 2 & 0 & 4 \\ 0 & -2 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$-2\hat{x}_2 + 2\hat{x}_3 = 6 \Rightarrow \hat{x}_2 = \frac{2x_3 - 6}{2} = \hat{x}_3 - 3$$

$$2\hat{x}_1 + 2\hat{x}_2 = 4 \Rightarrow 2\hat{x}_1 + 2\hat{x}_3 - 6 = 4 \Rightarrow \hat{x}_1 = \frac{10 - 2\hat{x}_3}{2} = 5 - \hat{x}_3$$

$$\hat{x} = \begin{pmatrix} 5 \\ -3 \\ 0 \end{pmatrix} + \hat{x}_3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Question 6 (6 Marks)

It is required to find a least-squares fit of the following points by a function of the form $y = \beta_1 x + \beta_2 x^2$. Such a function might arise, for example, as the revenue from the sale of x units of a product.

$$(1, 1), (2, 3), (4, 6)$$

a) Write down the design matrix, the observation vector and the parameters vector.
b) Find the model parameters, β_1, β_2. Use 2 decimal places.
c) Use the model obtained in (b) to predict the value of y at $x = 5$.

$$X\beta = y$$

a)

$$\begin{pmatrix} x_1 & x_1^2 \\ x_2 & x_2^2 \\ x_3 & x_3^2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 4 & 16 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$$

$$X^T X \hat{\beta} = X^T y$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 1 & 4 & 16 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 4 & 16 \end{pmatrix} \hat{\beta} = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 4 & 16 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 21 & 73 \\ 73 & 273 \end{pmatrix} \hat{\beta} = \begin{pmatrix} 31 \\ 109 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 21 & 73 & 31 \\ 73 & 273 & 109 \end{array}\right) \sim \begin{pmatrix} 21 & 73 & 31 \\ 0 & \frac{404}{21} & \frac{26}{21} \end{pmatrix}$$

b) $404 \beta_2 = 26 \rightarrow \boxed{\beta_2 = 0.06}$

$21 \beta_1 + 73 \beta_2 = 31 \rightarrow \boxed{\beta_1 = 1.25}$

$\boxed{y = 1.25 x + 0.06 x^2}$

c) at $x = 5$,

$y = 1.25(5) + 0.06(5)^2 = 7.75$

Question 7 (6 Marks)

Let P_3 have the inner product given by evaluating the polynomials at $-3, -1, 1, 3$.
Let $\mathbf{p}_0(t) = 1, \mathbf{p}_1(t) = t, \mathbf{p}_2(t) = t^2$ and $T: P_3 \to \mathbb{R}^4$ such that:

$$T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(-3) \\ \mathbf{p}(-1) \\ \mathbf{p}(1) \\ \mathbf{p}(3) \end{bmatrix}$$

a) Show that T is an isomorphism.
b) Produce an orthogonal basis for Span$\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$.
c) Find the best approximation to $\mathbf{p}(t) = t^3$ by polynomials in Span$\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$.

a) $T(p) = p(-3)\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + p(-1)\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + p(1)\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + p(3)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

4 variables $\Rightarrow$ range $R^4 \rightarrow$ it is onto

no free variables $\Rightarrow$ one to one

Then it is isomorphism

b) $P_0(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $P_1(t) = \begin{pmatrix} -3 \\ -1 \\ 1 \\ 3 \end{pmatrix}$, $P_2(t) = \begin{pmatrix} 9 \\ 1 \\ 1 \\ 9 \end{pmatrix}$

b)

$$P_0(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad P_1(t) = \begin{pmatrix} -3 \\ -1 \\ 1 \\ 3 \end{pmatrix}, \quad P_2(t) = \begin{pmatrix} 9 \\ 1 \\ 1 \\ 9 \end{pmatrix}$$

$v_1 = 1$

$v_2 = t$

$$v_3 = t^2 - \frac{\langle t^2, 1 \rangle}{\langle 1, 1 \rangle} 1 = t^2 - \frac{20}{4} = t^2 - 5$$

$$\text{basis} = \{ 1, t, t^2 - 5 \}$$

c)

$$\hat{P}_3(t) = \frac{\langle t^3, 1 \rangle}{\langle 1, 1 \rangle} 1 + \frac{\langle t^3, t \rangle}{\langle t, t \rangle} t + \frac{\langle t^3, t^2-5 \rangle}{\langle t^2-5, t^2-5 \rangle} (t^2-5)$$

$$P_3(t) = \begin{pmatrix} -27 \\ -1 \\ 1 \\ 27 \end{pmatrix}$$

$$= \frac{164}{20} t = 8.2\, t$$

$$t^2 - 5 = \begin{pmatrix} 4 \\ -4 \\ -4 \\ 4 \end{pmatrix}$$

Question 8 (4 Marks)

Find Fourier series of the function $f(x) = 2 - x, x \in (0,2)$

$$a_0 = \int_0^2 2 - x \, dx = 2x - \tfrac{1}{2}x^2 \Big|_0^2 = 4 - 2 = 2$$

$$a_n = \int_0^2 (2-x) \cos(n\pi x) \, dx$$

$2 - x \qquad \cos(n\pi x)$

$+$

$-1 \qquad \frac{1}{n\pi} \sin(n\pi x)$

$-$

$0 \qquad \frac{-1}{n^2\pi^2} \cos(n\pi x)$

$$a_n = \frac{1}{n\pi} (2-x) \sin(n\pi x) - \frac{1}{n^2\pi^2} \cos(n\pi x) \Big|_0^2$$

$$a_n = \frac{1}{n\pi}(2-x)\sin(n\pi x) - \frac{1}{n^2\pi^2}\cos(n\pi x)\Big|_0^2 \qquad 0 \qquad \frac{-1}{n^2\pi^2}\cos(n\pi x)$$

$$a_n = -\frac{1}{n^2\pi^2}\cos(2\pi n) + \frac{1}{n^2\pi^2} = 0$$

$$b_n = \int_0^2 (2-x)\sin(n\pi x)\,dx \qquad 2-x \quad \sin(n\pi x)$$

$$-1 \quad \frac{-1}{n\pi}\cos(n\pi x)$$

$$0 \quad \frac{-1}{n^2\pi^2}\sin(n\pi x)$$

$$b_n = -(2-x)\cdot\frac{1}{n\pi}\cos(n\pi x) - \frac{1}{n^2\pi^2}\sin(n\pi x)\Big|_0^2$$

$$b_n = \frac{2}{n\pi}$$

$$\boxed{f(x) = 1 + \sum_{n=1}^{\infty} \frac{2}{\pi n}\sin(n\pi x)}$$

Question 9 (4 Marks)

Given that,

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

a) Is 2 an eigenvalue of A? If so, find its corresponding eigenspace.
b) Find, if possible, an orthogonal matrix P and a digonal matrix D such that $A = PDP^T$.

a) $$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$-2+\lambda$$

$$(3-\lambda)[(3-\lambda)^2 - 1] + [-(3-\lambda)+1] + [1-(3-\lambda)] = 0$$

$$(3-\lambda)^3 - 3 + \cancel{\lambda} + \cancel{\lambda} - 2 \quad -2 + \cancel{\lambda} = 0$$

$$27 - \cancel{\lambda^3} - \cancel{27\lambda} + \cancel{9\lambda^2} + 3\lambda - 7 = 0$$

$$-\lambda^3 + 9\lambda^2 - 24\lambda + 20 = 0$$

$27 - \lambda^3 - 27\lambda + 9\lambda^2 + 3\lambda - 7 = 0$

$-\lambda^3 + 9\lambda^2 - 24\lambda + 20 = 0$

$\lambda = 5, 2, 2$

yes, 2 is eigen value.

$A - 2I =$

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 - x_2 + x_3 = 0 \rightarrow x_1 = x_2 - x_3$

$$X = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

eigen vectors = span $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

at $\lambda = 5$:

$A - 5I =$

$$\begin{pmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{pmatrix} \sim \begin{pmatrix} -1 & -2 & -1 \\ -2 & -1 & 1 \\ 1 & -1 & -2 \end{pmatrix} \sim \begin{pmatrix} -1 & -2 & -1 \\ 0 & 3 & 3 \\ 0 & -3 & -3 \end{pmatrix}$$

$$\sim \begin{pmatrix} -1 & -2 & -1 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$3x_2 + 3x_3 = 0 \rightarrow x_2 = -x_3$

$-x_1 - 2x_2 - x_3 = 0 \rightarrow x_1 = 2x_3 - x_3 = x_3$

$$X = x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$V_1 = a_1$

$V_2 = a_2$

$v_2 = a_2$

$$v_3 = a_3 - \frac{\langle v_3, v_2 \rangle}{\langle v_2, v_2 \rangle} \underline{v_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{-1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} -0.5 \\ 0.5 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -0.5 \\ -1 & 1 & 0.5 \\ 1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{6}}{6} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \sqrt{6}/6 \\ \frac{1}{\sqrt{3}} & 0 & \sqrt{6}/3 \end{pmatrix}$$

$$A = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{6}}{6} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{\sqrt{6}}{6} \\ \frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{6}}{3} \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} \end{pmatrix}$$

Question 10 (2 Marks)

Mark each statement True or False (T/F). Justify your answer.

a) There is no vector that is orthogonal to both the subspaces W and $W^{\perp}$.
b) If the columns of an $m \times n$ matrix A are orthogonal, then the linear mapping $\mathbf{x} \rightarrow A\mathbf{x}$ preserves lengths.
c) If A, B are $n \times n$ orthogonal matrices, then AB is an orthogonal matrix.
d) In a QR factorization, say $A = QR$, if columns of A are linearly independent, then the pseudo-inverse of A can be written as, $A^{+} = R^T Q^T$.

a) True

b) False, if they are orthonormal → then the length is preserved

c) $A^T = A^{-1}$, $B^T = B^{-1}$

$(AB)^T = B^T A^T = B^{-1} A^{-1} = (AB)^{-1}$

Since $(AB)^T = (AB)^{-1}$ → Then AB is orthognal (True)

d) False, $A^{+} = (A^T A)^{-1} A^T$, $R^T Q^T = A^T$

$A^T \neq A^{+}$