Math 201 Mid-term 1 (Fall 2020)_Answer

Model Answer

1- Mark each statement as True or False. **JUSTIFY YOUR ANSWER**.

a. If the **reduced** row echelon form of $A_{n\times n}$ can be obtained using only row replacement operations, then $|A|$ must be 1.

b. If the matrix A has a pivot for each row, then the equation $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ is always consistent.

c. If $A_{n\times n}$ is an idempotent matrix (i.e. $A^2 = A$), then $|A|$ must be 1.

d. A homogeneous system of three linear equations in two variables has infinitely many solutions.

e. If the vectors $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}$ belong to the span of S, then so is $\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}$.

f. The columns of a 3×4 matrix are always linearly dependent.

(6 Marks)

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a. False. Since the number of pivots might be less than n and hence the determinant is 0.

b. True. As no contradiction will occur.

c. False. $|A^2| = |A|^2 = |A| \Rightarrow |A|(|A| - 1) = 0 \Rightarrow |A| = 0 \text{ or } |A| = 1$.

d. False. It might has the trivial solution only (e.g. $x = 0, y = 0, x - y = 0$).

e. True. Since $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}$ belong to all the possible linear combination of a set of vectors, then any linear combination of $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}$ will be too.

f. True. Since any 4 vectors in $\mathbb{R}^3$ will be linearly dependent as the row reduction of the system: $c_1\vec{\boldsymbol{v}}_1 + c_2\vec{\boldsymbol{v}}_2 + c_3\vec{\boldsymbol{v}}_3 + c_4\vec{\boldsymbol{v}}_4 = \vec{\boldsymbol{0}}$ will result in a maximum of 3 pivots, thus there will be at least one free variable and hence the vectors are linearly dependent.

2- Let t be a fixed real number, and let

$$A = \begin{pmatrix} t-1 & t & t \\ t & t-1 & t \\ t & t & t-1 \end{pmatrix}$$

a. Find the value(s) of t for which A is invertible.
b. For $t = 1$, find, if possible, A^{-1}.
c. For $t = 1$, are columns of A linearly independent? **Justify your answer**.
d. For $t = 1$, does the set containing the columns of A span $\mathbb{R}^3$? **Justify your answer**.

(8 Marks)

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a. $|A| = \begin{vmatrix} t-1 & t & t \\ t & t-1 & t \\ t & t & t-1 \end{vmatrix} = (t-1)[(t-1)^2 - t^2] - t[t(t-1) - t^2] +$
$t[t^2 - t(t-1)] = 3t - 1$
So A is invertible if $t \neq 1/3$.

b. For $t = 1$,

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

The inverse can be obtained using the adjugate method,

$$A^{-1} = \frac{1}{|A|} C^T = \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}^T = \begin{pmatrix} -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{pmatrix}$$

or Gauss-Jordan method,

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right]$$

c. Yes, columns of A are linearly independent, since $A\vec{\boldsymbol{x}} = \vec{\mathbf{0}}$ has only the trivial solution or $|A| \neq 0$.

d. Yes, the set containing the columns of A spans $\mathbb{R}^3$ as it contains 3 linearly indepdent vectors in $\mathbb{R}^3$.

3- For

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 3 & 9 \end{pmatrix}, \vec{\boldsymbol{b}} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

a. Find a row echelon form of A.

b. Are columns of A linearly independent? **Justify your answer**.

c. Find a parametric vector form for the solution set of $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{0}}$?

d. Describe, if any, the relation between the solution sets of $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ and $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{0}}$?

(8 Marks)

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a. $A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 3 & 9 \end{pmatrix} \underset{\substack{R_2 - 2R_1 \to R_2 \\ R_3 - 3R_1 \to R_3}}{\sim} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 6 \end{pmatrix} \underset{R_3 - 3R_2 \to R_3}{\sim} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

b. No; as the second and fourth columns are not pivot columns, so they are not linearly independent.

c. $x_1 + 2x_2 + x_4 = 0$

$x_3 + 2x_4 = 0$

$$\vec{\boldsymbol{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 - x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Solution is a hyperplane in $\mathbb{R}^4$ spanned by the two vectors $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

d. For $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$

$$[A \quad \vec{\boldsymbol{b}}] = \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 4 & 2 \\ 3 & 6 & 3 & 9 & 1 \end{pmatrix} \underset{\substack{R_2 - 2R_1 \to R_2 \\ R_3 - 3R_1 \to R_3}}{\sim} \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 3 & 6 & -2 \end{pmatrix} \underset{R_3 - 3R_2 \to R_3}{\sim} \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

In this case, there is no solution because of a contradiction and hence there is no relation between the solution sets of $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ and $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{0}}$.

Hint: you can do the reduction once for the whole problem by reducing $[A \quad \vec{\boldsymbol{b}}]$

4- a. Describe all the possible **reduced** row echelon forms of a nonzero 2×2 matrix. Let '*' denotes any value.

b. Let A and B be two 3×3 matrices such that $det(A) = 3$ and $det(B) = 4$. Evaluate $det(\frac{1}{3}A^T B^{-1} A^2 B^2)$.

c. Let $\vec{\boldsymbol{u}} = \begin{bmatrix} a \\ b \end{bmatrix}, \vec{\boldsymbol{v}} = \begin{bmatrix} c \\ d \end{bmatrix}$ be vectors in $\mathbb{R}^2$. Find, if possible, $(\vec{\boldsymbol{u}}^{\boldsymbol{T}}\vec{\boldsymbol{v}})^{-\mathbf{1}}, (\vec{\boldsymbol{u}}\vec{\boldsymbol{v}}^{\boldsymbol{T}})^{-\mathbf{1}}$.

(8 Marks)

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a. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}$; $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

b. $\left|\frac{1}{3}A^T B^{-1} A^2 B^2\right| = \frac{1}{3^3}\frac{|A|^3|B|^2}{|B|} = \frac{1}{27}\,|A|^3|B| = 4.$

c. $\vec{u}^T\vec{v} = ac + bd \rightarrow \ (\vec{u}^T\vec{v})^{-1} = \frac{1}{ac+bd}$, if $ac + bd \neq 0$

$\vec{u}\vec{v}^T = \begin{bmatrix} ac & ad \\ bc & bd \end{bmatrix} \rightarrow (\vec{u}\vec{v}^T)^{-1}$ doesnot exist as $\begin{vmatrix} ac & ad \\ bc & bd \end{vmatrix} = 0$