Probability, Independence, Bayes Rule

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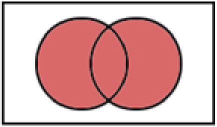
Preliminaries

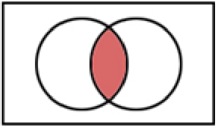
Recall the set operations:

$A \cup B$



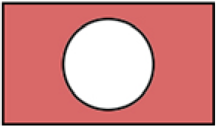
Union $A \cup B$:
All elements that belong to A or B or both

$A \cap B$



Intersection $A \cap B$:
All elements that belong to *both* A and B

A^c



Complement A^c:
All elements that do not belong to A

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- E.g. predicting the score of a football match, or the outcome of a tossed coin.
- While we cannot tell the exact outcome of such an event, we can still speculate about the likely outcomes (e.g. which outcome is more likely than the others).
- The mathematical notion associated with the likeliness of a particular output to happen is called **probability**.

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We denote the sample space with the letter Ω, so

$$\Omega = \{\text{all possible outcomes}\}$$

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Example

A football match is a random experiment, where:

- Outcome is the score of the game.
- For example, one possible outcomes is "Pyunik 2 - 1 Alashkert". One way to denote it is $(2, 1)$.
- The sample space is:

$$\Omega = \{(0,0); (0,1); (1,0); (1,1); (2,0); \dots\}$$

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Informally, we can say that both outcomes are equally likely: 50/50.

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- Outcome is the time until the bus arrives.
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Whenever Ω is an interval (e.g. $(0, 1), [1, 10], [0, +\infty)$), the probability of each outcome is 0.

(of course, this does not mean that they are impossible – watch this!)

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- The set of all events is called the **event space** and denoted by $\mathcal{F}$.

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Can A and B happen at the same time? How would you denote the event of both A and B happening? What about B and C?

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Two events A and B of the same experiment are called **disjoint** or **mutually exclusive** if $A \cap B = \varnothing$.

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Example

When rolling a die, the events $A = \{1, 4\}$ and $B = \{2, 5\}$ are disjoint, while A and $C = \{3, 4, 5\}$ are not.

Example

When waiting for a bus, the events $A = [0, 20]$ and $B = [30, 40]$ are disjoint, but none of them is disjoint with $C = [10, 40]$.

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Similarly, each of the outcomes $\{1, 2, 3, 4, 5, 6\}$ also has the probability

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Experiments like this (where all outcomes have the same probability of occuring) are called **equiprobable**.

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since A has 3 elements and the total number of outcomes is 6.

In equiprobable case, the probability of any event $A \in \mathcal{F}$ is:

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Now we roll two fair dice. Our sample space will be
$\Omega = \{(x, y) \mid 1 \leq x, y \leq 6\}$.
Assuming that all of 36 outcomes are equally likely to show up, the probability of each (x, y) outcome is:

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As it shows, probability can also be not equiprobable.

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So in general, in different problems the *probability measure* $\mathbb{P}$ can be different, and it depends on the specifics of the problem how it is computed.

There are, however, three properties which the probability measure $\mathbb{P}$ always satisfies:

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1. $\mathbb{P}(A) \geq 0$ for any event $A \in \mathcal{F}$,
2. $\mathbb{P}(\Omega) = 1$,
3. For any disjoint events $A_1, A_2, \ldots$,

$$\mathbb{P}\left(A_1 \cup A_2 \cup \ldots\right) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \ldots$$

i.e.

$$\mathbb{P}\left(\bigcup_n A_n\right) = \sum_n \mathbb{P}(A_n)$$

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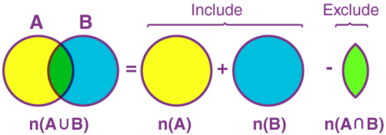

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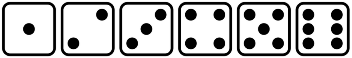

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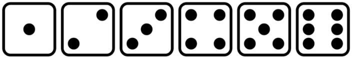

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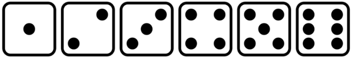

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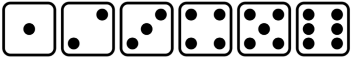

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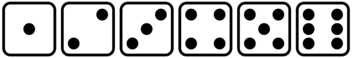

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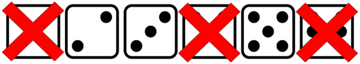

Here we know that the outcome **cannot be** 1, 4 or 6.

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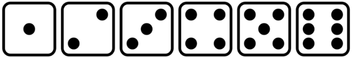
a. What is the probability that the outcome is even?



Since there are 6 possible outcomes, out of which only 3 are even,

$$A = \{2, 4, 6\}, \qquad \mathbb{P}(A) = \frac{1}{2}$$

b. What is the probability that the outcome is even, given that we know that it is prime?

Here we know that the outcome **cannot be** 1, 4 or 6. Therefore we are left with only three possible outcomes $B = \{2, 3, 5\}$, out of which only 2 is even.

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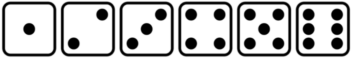

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Here we know that the outcome **cannot be** 1, 4 or 6. Therefore we are left with only three possible outcomes $B = \{2, 3, 5\}$, out of which only 2 is even. So in this case the probability of being even is $\frac{1}{3}$.

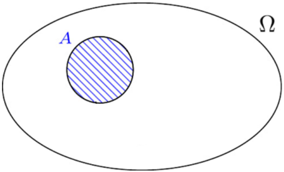
Conditional Probability

Definition

For any events A and B (if $\mathbb{P}(B) \neq 0$), the following number:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

is called the **conditional probability** of A **given** B
(or the probability of A under the condition of B).



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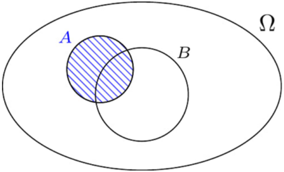

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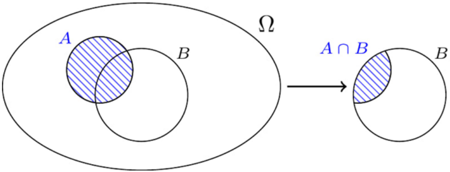

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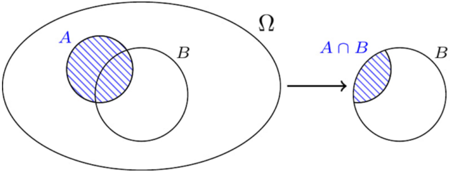

In our problem, if A means being even and B means being prime, we had:

$$\mathbb{P}(\text{even} \mid \text{given that prime}) = \mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{1/6}{3/6} = \frac{1}{3}$$

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Example

Suppose we roll two fair dice. What is the probability that the first one is 2, given that their sum is no greater than 5?

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	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Usually, there would be 36 total outcomes, but since we *know* the sum is 5, they are only 10 possible outcomes left. Out of them only three outcomes ("2-1", "2-2" and "2-3") are desired. So the probability is $\frac{3}{10}$.

Law of Total Probability

Question

In some university, 3/5 of all students are women and the rest are men. It is known that 15% of men are over left-handed, while only 10% of women are. If you choose a random student, what is the probability of the student being left-handed?

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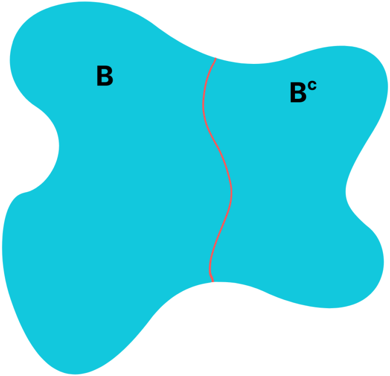
Let B denote the event that the randomly selected student is a woman, and B^c that he is a man.

With another letter, say A, let us denote the event of being left-handed.



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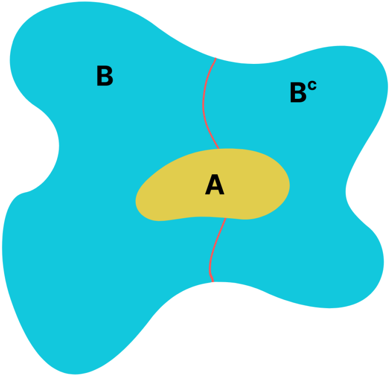

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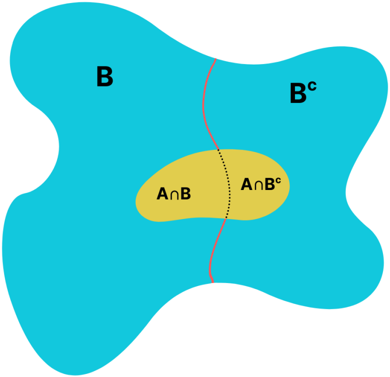

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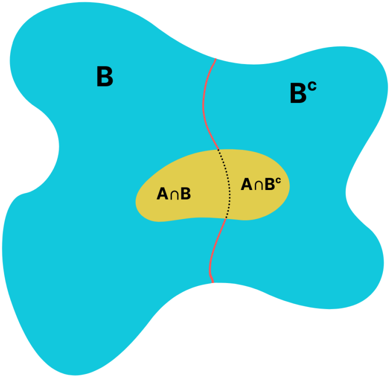

$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c) = \mathbb{P}(B) \cdot \mathbb{P}(A|B) + \mathbb{P}(B^c) \cdot \mathbb{P}(A|B^c)$$

Law of Total Probability

What we get is known as the Law of Total Probability:

Theorem

If A and B are some events such that $\mathbb{P}(B) \neq 0$, then

$$\mathbb{P}(A) = \mathbb{P}(B) \cdot \mathbb{P}(A|B) + \mathbb{P}(B^c) \cdot \mathbb{P}(A|B^c)$$

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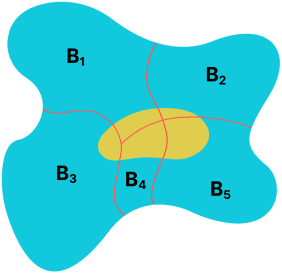

We can also generalize it to the case of three or more subgroups:

Theorem

If $B_1, B_2, \ldots, B_n$ are some disjoint events such that $A \subset \bigcup_{k=1}^{n} B_k$, then

$$\mathbb{P}(A) = \mathbb{P}(B_1) \cdot \mathbb{P}(A|B_1) + \cdots + \mathbb{P}(B_n) \cdot \mathbb{P}(A|B_n)$$

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Example

There are 52 cards in a deck. One of the cards is randomly removed from the deck, 51 are left. What is the probability that if we take a card, it will be a diamond?

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If we denote by A the probability of taking a diamond, and by B the probability that the removed card was a diamond, then either:

- the taken card was a diamond, and there are 12 left,
- the taken card wasn't a diamond, and there are 13 left.

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The total probability will be:

$$\mathbb{P}(A) = \mathbb{P}(B) \cdot \mathbb{P}(A|B) + \mathbb{P}(B^c) \cdot \mathbb{P}(A|B^c)$$

$$\mathbb{P}(A) = \frac{13}{52} \cdot \frac{12}{51} + \frac{39}{52} \cdot \frac{13}{51} = \frac{1}{4}$$

Bayes Rule

One more simple yet powerful tool is the so called Bayes Rule:

Theorem

If A and B are some events (with non-zero probabilities), then

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Let F denote dangerous fire, S smoke. Then:

$$\mathbb{P}(F|S) = \frac{\mathbb{P}(F) \cdot \mathbb{P}(S|F)}{\mathbb{P}(S)} = \frac{1}{100} \cdot \frac{90}{100} : \frac{10}{100} = 0.09$$

Independence

Now let's consider another simple experiment: flipping a fair coin and rolling a six-sided die. The probability of getting **Heads** on the coin flip is independent of the probability of rolling a specific number on the die, as these two events *do not affect* each other.

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This definition makes sense because it means that the probability of both A and B occurring together is simply the product of their individual probabilities: the outcome of one event has no effect on the other.

Independence

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
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Suppose A and B are some events with $\mathbb{P}(A) \neq 0$ and $\mathbb{P}(B) \neq 0$.

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When solving a problem, principles like total probability, Bayes rule, and independence help us a lot. Let's explore one more useful problem solving technique.

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Your bus is coming at a random time between 12 pm and 1 pm. If you show up at 12:30 pm, how likely are you to catch the bus?

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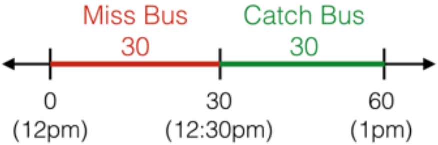


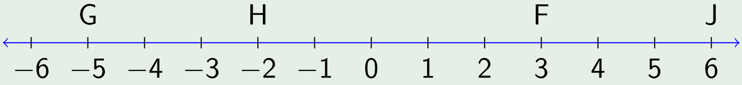
If we draw the hours on a number line and mark the time points for which we will miss or catch the bus, we see that the *length* of the segment for catching the bus is half the total length:

$$\mathbb{P}(\text{catching the bus}) = \frac{30}{30 + 30} = \frac{1}{2}$$

Geometric Probability

Example

A selection is to be made between points G and J as seen below



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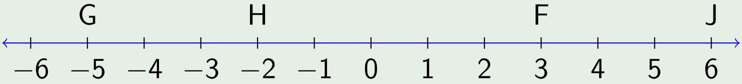

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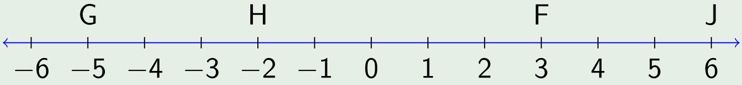

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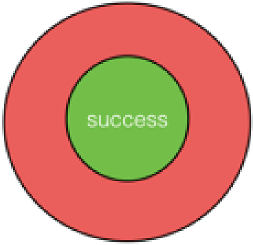
So in general, we draw the sample space and the set of desired outcomes as lines or line segments, and divide the length of the "desired outcomes" by the length of the "sample space" to get the probability.

Geometric Probability

Imagine now that we are playing darts with two circles, and the smaller circle has two times smaller radius than the bigger one.



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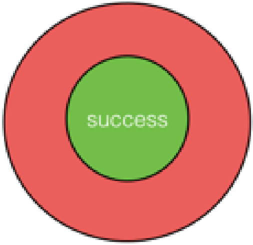

A dart is thrown at a random. What is the probability that it lands in the smaller circle?

In this case, since the sample space and the set of desired outcomes is 2-dimensional, we should divide the *area* of the smaller circle by the *area* of the bigger circle.

The probability that the dart falls in the smaller circle, is $\frac{\pi r^2}{\pi (2r)^2} = \frac{1}{4}$.

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Both the bus and you get to the bus stop at random times between 12 pm and 1 pm. When the bus arrives, it waits for 5 minutes before leaving. When you arrive, you wait for 20 minutes before leaving if the bus doesn't come. What is the probability that you catch the bus?

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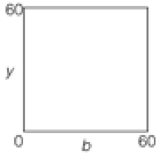

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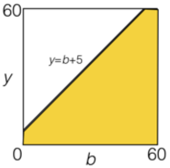
Then, we need to determine the region of "success"; that is, the points where we catch the bus. Since the bus will wait for 5 minutes, you need to arrive within 5 minutes of the bus' arrival, so $y \leq b + 5$.



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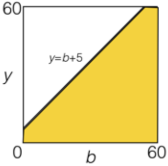
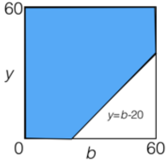
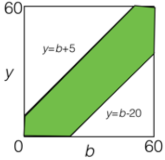


However, you only wait for 20 minutes, so you can't arrive more than 20 minutes before the bus, so $y \geq b - 20$.



Geometric Probability

Combining our two conditions, we can draw the region of success:



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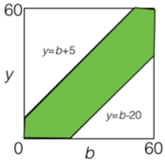

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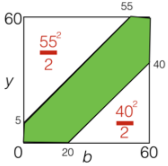

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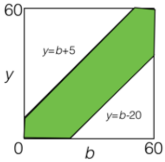

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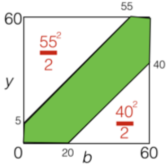

$$\mathbb{P}(\text{catching the bus}) = \frac{60^2 - \frac{55^2}{2} - \frac{40^2}{2}}{60^2} = \frac{103}{288}$$