Homework Problems 3 (Derivative)

Problem 1. Does the following sequence have a limit (as $n \to \infty$)? If so, find it.

a) $\dfrac{3-n}{2}$,

b) $\dfrac{4\sqrt{n}}{n^3}$,

c) $\dfrac{n-1}{n^2-1}$,

d) 1^n,

e) 0.4^n,

f) $(-4)^n$,

g) Choose any number as the first term; divide it by 1.5 to get the next term; then repeat the same step again and again,

h) $a_1 = 1$,
$a_2 =$ the first digit of $a_1/7$ after the decimal point,
$a_3 =$ the first digit of $a_2/7$ after the decimal point,
$\dots$

Hint: It is always useful to write down the first few terms of a sequence to see the dynamics. In the last problem you may want to use your calculator.

Problem 2. The following functions consist of two "parts" which are glued together at $x = 2$. Is there a value of c such that the functions are continuous at 2?

a) $f(x) = \begin{cases} 3x-5, & \text{if } x < 2 \\ x^2 + c, & \text{if } x \geq 2 \end{cases}$

b) $f(x) = \begin{cases} x^3 + 1, & \text{if } x < 2 \\ cx^2, & \text{if } x \geq 2 \end{cases}$

c) $f(x) = \begin{cases} -7, & \text{if } x < 2 \\ c, & \text{if } x = 2 \\ 4 + 3\sin(\pi x), & \text{if } x > 2 \end{cases}$

d) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x-2}, & \text{if } x \geq 2 \\ c, & \text{if } x < 2 \end{cases}$

Problem 3. Zeus, Prometheus and Aramazd invest 1000 gold coins each, in one of the following banks respectively:

1. In OlympusBank, where the amount of money after x years will be $f(x) = 100x + x^3$ gold coins,
2. In TaurusBank, where the amount of money after x years will be $f(x) = 0.18x^3 \cdot \sqrt{x}$ gold coins,
3. In AraratBank, where the amount of money after x years will be $f(x) = 30x^2$ gold coins.

Considering that all three are immortal, who will be the richest of them after infinitely long time?

Problem 4. Find the derivatives of the following functions:

a) $f(x) = 10000$

b) $f(x) = 2x^2 - 7x + 1$

c) $f(x) = 2 \sin x \cdot e^x$

d) $f(x) = e^{x^2}$

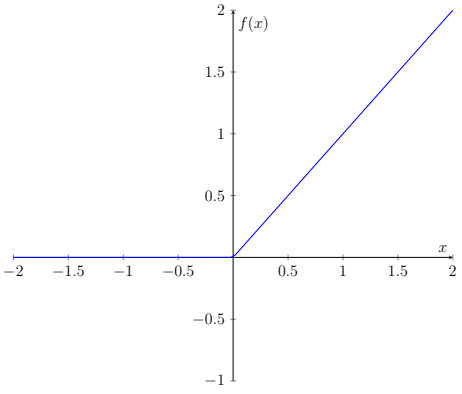
e) $f(x) = x^2 \ln x$

f) $f(x) = \dfrac{x^2}{x^3}$

(Additional:) The following two are called *activation functions* in Machine Learning and have special names:

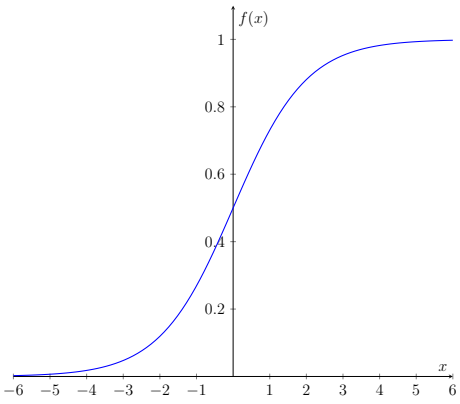
g) ReLU (rectified linear unit):

$$f(x) = \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$



h) Sigmoid:

$$f(x) = \frac{1}{1 + e^{-x}}$$



Problem 5 (additional). In the lecture we claimed that the function $f(x) = |x|$ is not differentiable at $x = 0$. Can you prove it?

Problem 6 (additional). We are interested in what happens if you take a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and change it by a small unit, like this: $A + 0.001 \cdot I$. More specifically, how does it determinant change as we add something small $\times$ the identity matrix I to it?

To this end, we define the following function:

$$f(x) = \det(A + x \cdot I)$$

a) What is $f(0)$?

b) Using the definition of derivative, find $f'(0)$.

c) Do you recognize it?

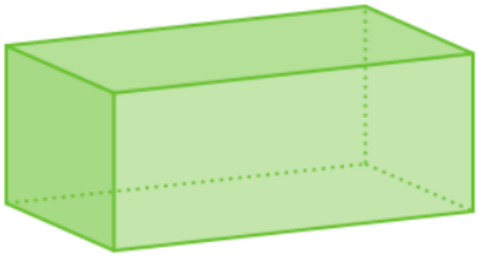
Problem 7. Do the following functions have local maxima and minima? If so, find them:

a) $f(x) = 5x - x^2$

b) $f(x) = 3x + 1$

c) $f(x) = \dfrac{x^3}{e^x}$

Problem 8 (additional). You have 10 square meters of cardboard (as well as scissors, and glue) and you want to make a box like this:



where the left and right sides are squares. What is the maximum volume your box can have?

Hint: Be prepared to do some geometry first. Take one of the non-square sides and denote its length and width by x and y. Can you express y by x? Can you also express the volume by x?

Problem 9. Calculate $\int f(x)\,dx$:

a) $f(x) = 3x^2$

b) $f(x) = x + 6\cos x$

c) $f(x) = \dfrac{3}{x} - 1$

d) $f(x) = \dfrac{4x}{1 - 2x^2}$

e) $f(x) = \operatorname{tg} x$

Problem 10. The speed of a spaceship at time x, as it flies from Earth to the Moon, is given by

$$f(x) = x^5 - x^2$$

thousand kilometers per hour.

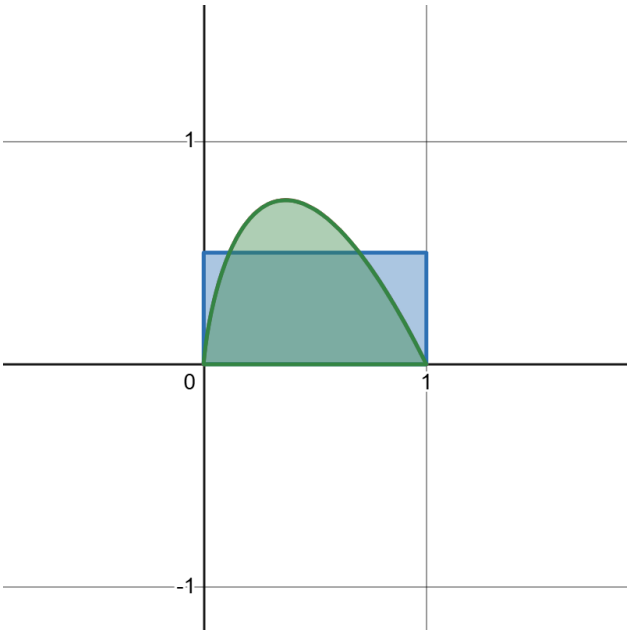
a) How much distance does it pass in the first 3 hours?

b) Consequently, how much was its average speed?

Problem 11 (additional). Here the graph of the function

$$f(x) = -x \cdot \ln(x^2)$$

on the interval $0 < x < 1$ is plotted:



It is known that the area of the green and blue regions are the same. What is the height of the blue rectangle?