Functions of Several Variables

Hayk Aprikyan, Hayk Tarkhanyan

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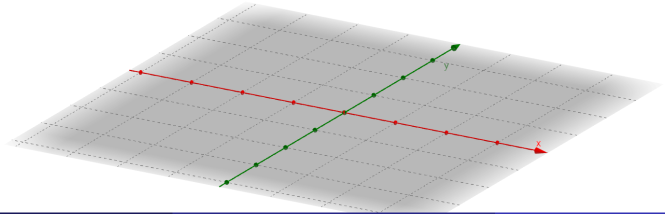

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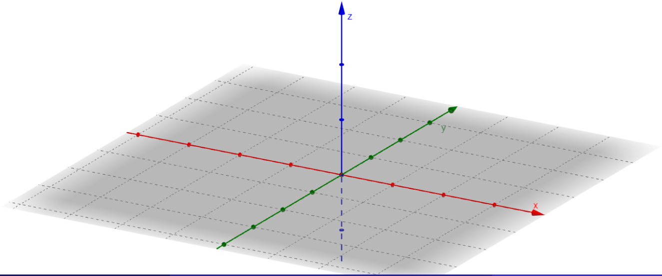

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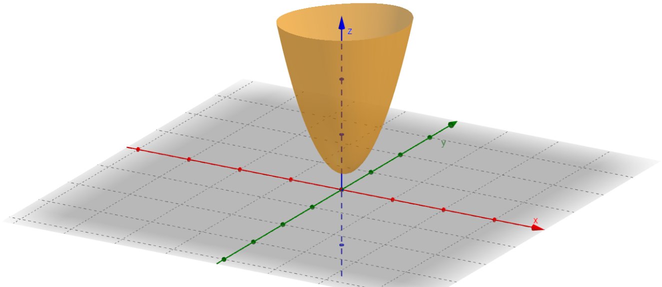

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Partial Derivative

Again, suppose x and y are the costs of apples and peaches, and your profit is given by:

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How can you measure the effect of increasing the cost of apple by a little (i.e. how quickly will f change if x changes)?

By fixing y and then doing the usual derivative stuff with x!

Partial Derivative

Definition

If there exists a finite limit

$$f_x = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

then it is called the **partial derivative** of $f(x, y)$ with respect to x, and denoted by f_x or $\frac{\partial f}{\partial x}$.

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Example

If $f(x, y) = x^2 + y^2$, then:

$$f_x = 2x \qquad \text{and} \qquad f_y = 2y$$

Partial Derivative

The partial derivative is the rate of the function change with respect to only one of the variables, while the others are being kept unchanged (constant).

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Definition

The vector consisting of the partial derivatives of $f(x, y)$:

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$$

is called the **gradient** of $f(x, y)$.

In the previous example, $\nabla f = [2x \quad 2y]$.

Partial Derivative

Similarly, for a function of n variables, $f(x_1, \ldots, x_n) = f(\mathbf{x})$ we define partial derivatives as:

$$f_{x_1}(\mathbf{x}) = \frac{\partial f}{\partial x_1}(\mathbf{x}) = \lim_{h \to 0} \frac{f(x_1 + h, x_2, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{h},$$

$$f_{x_2}(\mathbf{x}) = \frac{\partial f}{\partial x_2}(\mathbf{x}) = \lim_{h \to 0} \frac{f(x_1, x_2 + h, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{h},$$

$$\vdots$$

$$f_{x_n}(\mathbf{x}) = \frac{\partial f}{\partial x_n}(\mathbf{x}) = \lim_{h \to 0} \frac{f(x_1, x_2, \ldots, x_n + h) - f(x_1, x_2, \ldots, x_n)}{h}.$$

And the gradient of $f(\mathbf{x})$ as:

$$\nabla f = \frac{df}{d\mathbf{x}} = \begin{bmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} & \frac{\partial f(\mathbf{x})}{\partial x_2} & \ldots & \frac{\partial f(\mathbf{x})}{\partial x_n} \end{bmatrix} \in \mathbb{R}^{1 \times n}$$

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3. $$\frac{\partial}{\partial x_i}(f(\mathbf{x}) \cdot g(\mathbf{x})) = \frac{\partial f(\mathbf{x})}{\partial x_i} \cdot g(\mathbf{x}) + f(\mathbf{x}) \cdot \frac{\partial g(\mathbf{x})}{\partial x_i}$$

Partial Derivative

Example

Let $f(x,y) = 2x^2$ and $g(x,y) = 4x + 6y$.

$$(f \cdot g)_x = 2x^2(4) + (4x + 6y)(4x) = 24x^2 + 24xy$$
$$(f \cdot g)_y = 2x^2(6) + (4x + 6y)(0) = 12x^2$$

Chain Rule

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How does a change of temperature affect your profit?

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In other words,

- if f depends on x and y
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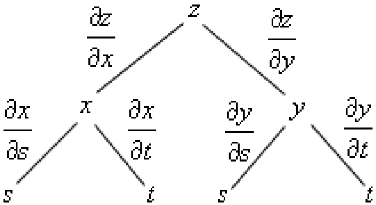

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Let $z = \sin(x^2 + y^2),\ x = t^2 + 3,\ y = t^3$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2x\cos(x^2 + y^2)) \cdot (2t) + (2y\cos(x^2 + y^2)) \cdot (3t^2)$$
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Let $z(x) = x^2 + 4x$, $x(t) = 5t^3 + 2t$. We can again use the chain rule:

$$\frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt} = (2x + 4) \cdot (15t^2 + 2) = (2 \cdot (5t^3 + 2t) + 4) \cdot (15t^2 + 2)$$

$$= 150t^5 + 80t^3 + 60t^2 + 8t + 8$$

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$$\nabla_{\mathbf{v}} f(\mathbf{x}) = \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{v}) - f(\mathbf{x})}{h}$$

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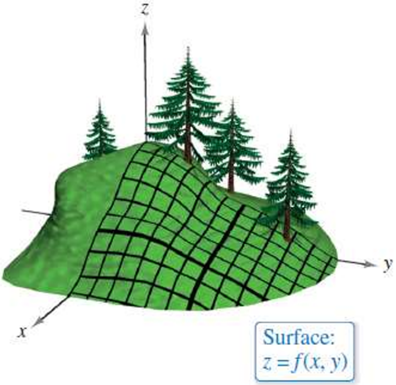
Note that the directional derivative is a *number* (like partial derivatives), not a vector.

Directional Derivative

The directional derivative shows how much our function changes if we "walk" not only along the x or y-axis, but by an arbitrary direction of our choice.



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A particularly important question you might ask is:

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By which direction should I move, so the function increases the most?

In other words, along which direction does $\nabla_{\mathbf{v}} f$ take its highest value?

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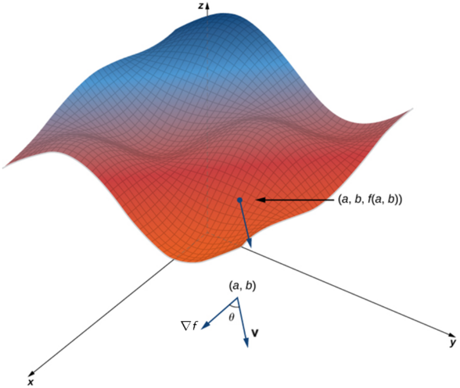

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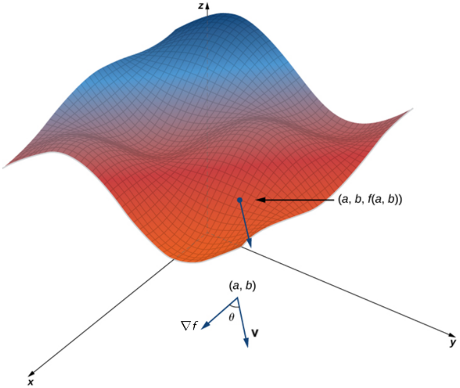

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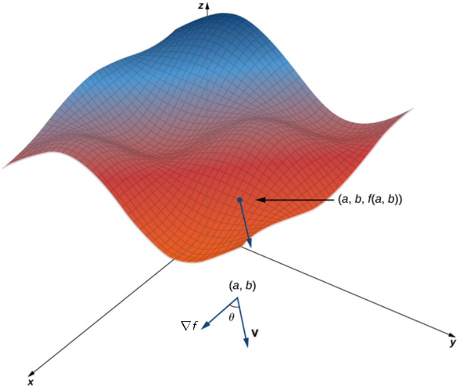

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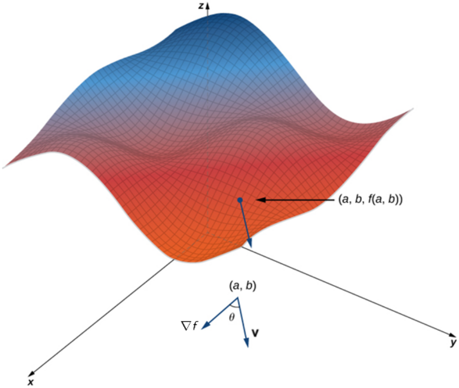

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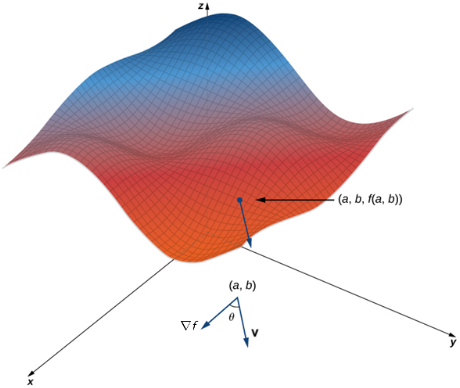

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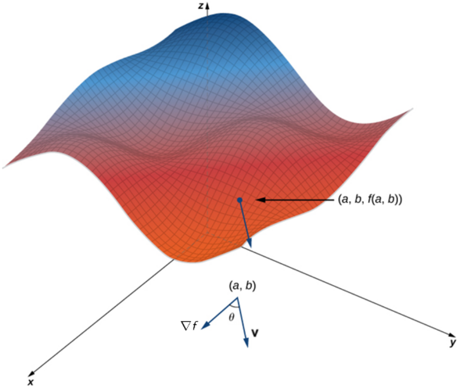

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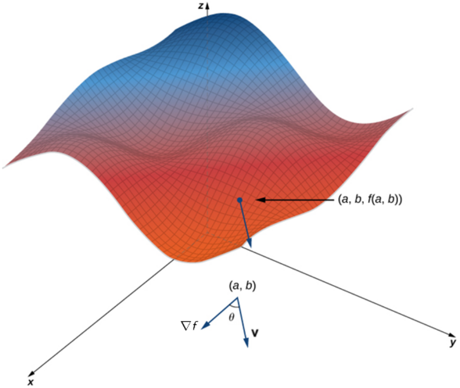

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Similarly, $-\nabla f$ is the fastest decreasing direction of the function.

Extrema of a Function

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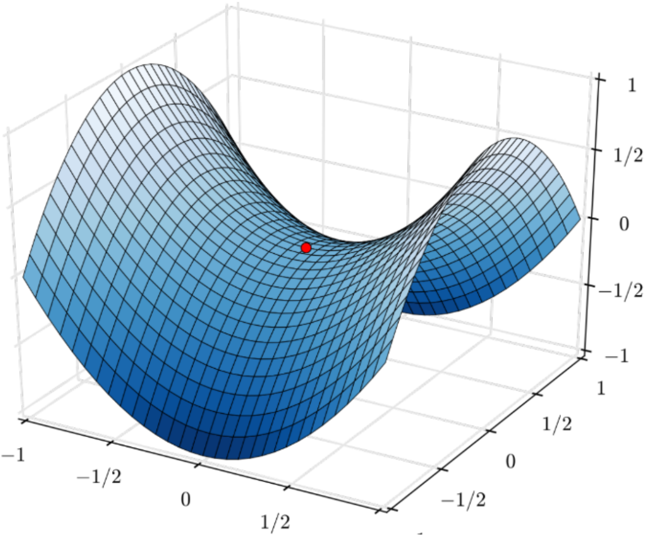
Definition

$\mathbf{x}_0$ is called a **saddle point** of f if $\nabla f(\mathbf{x}_0) = \mathbf{0}$ but it's not an extremum point.

Extrema of a Function



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Theorem

If $\nabla f(a, b) = \mathbf{0}$ at some point (a, b), and

- $D > 0$ and $f_{xx} > 0$ $\Rightarrow$ local minimum
- $D > 0$ and $f_{xx} < 0$ $\Rightarrow$ local maximum
- $D < 0$ $\Rightarrow$ saddle point