Practice 1: Vectors

Exercise: Finding a Perpendicular Vector

Context:
In linear algebra, two vectors are perpendicular (or orthogonal) if their dot product is zero. In this exercise, you will find a vector in $\mathbb{R}^2$ that is perpendicular to a given vector.

Given:
Let $\mathbf{v} = [2, 3]$.

Tasks:

1. **Find a Perpendicular Vector:**
   - Find a non-zero vector $\mathbf{w} = [x, y]$ such that $\mathbf{v}$ and $\mathbf{w}$ are perpendicular.
2. **Verification:**
   - Show that your chosen vector $\mathbf{w}$ indeed satisfies the condition $\mathbf{v} \cdot \mathbf{w} = 0$.
3. **Unit Perpendicular Vector:**
   - Find a unit vector in the direction of $\mathbf{w}$ by computing $\frac{\mathbf{w}}{\|\mathbf{w}\|}$, where $\|\mathbf{w}\|$ is the Euclidean norm of $\mathbf{w}$.
4. **Bonus Discussion:**
   - Explain why there are infinitely many vectors perpendicular to $\mathbf{v}$ and describe the general form of all such vectors.

Exercise: Finding the Closest Word with 2D Embeddings

Context:
In NLP, words can be represented as vectors. Here, each word is represented by a 2-dimensional vector. By comparing these vectors using Euclidean distance and cosine similarity, you can determine which word is "closer" in meaning.

Given Word Embeddings:

- **cheese:** `[1, 2]`
- **mushroom:** `[3, 1]`
- **tasty:** `[2, 2]`

Tasks:

1. **Euclidean Distance:**
    - **a.** Compute the Euclidean distance between **tasty** and **cheese**.
    - **b.** Compute the Euclidean distance between **tasty** and **mushroom**.
    - **c.** Which word is closer to **tasty** based on the Euclidean distance?
2. **Cosine Similarity:**

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$$

    - **a.** Compute the cosine similarity between **tasty** and **cheese** using the formula above.
    - **b.** Compute the cosine similarity between **tasty** and **mushroom**.
    - **c.** Based on cosine similarity, which word is closer to **tasty**?
3. **Discussion:**
    - Compare the outcomes from the Euclidean distance and cosine similarity calculations.
    - Discuss why one metric might be preferred over the other in different NLP applications.

> **Note**
>
> Cool video by 3blue1brown discussing word vectors (embeddings)

Exercise: Linear transformation matrix power

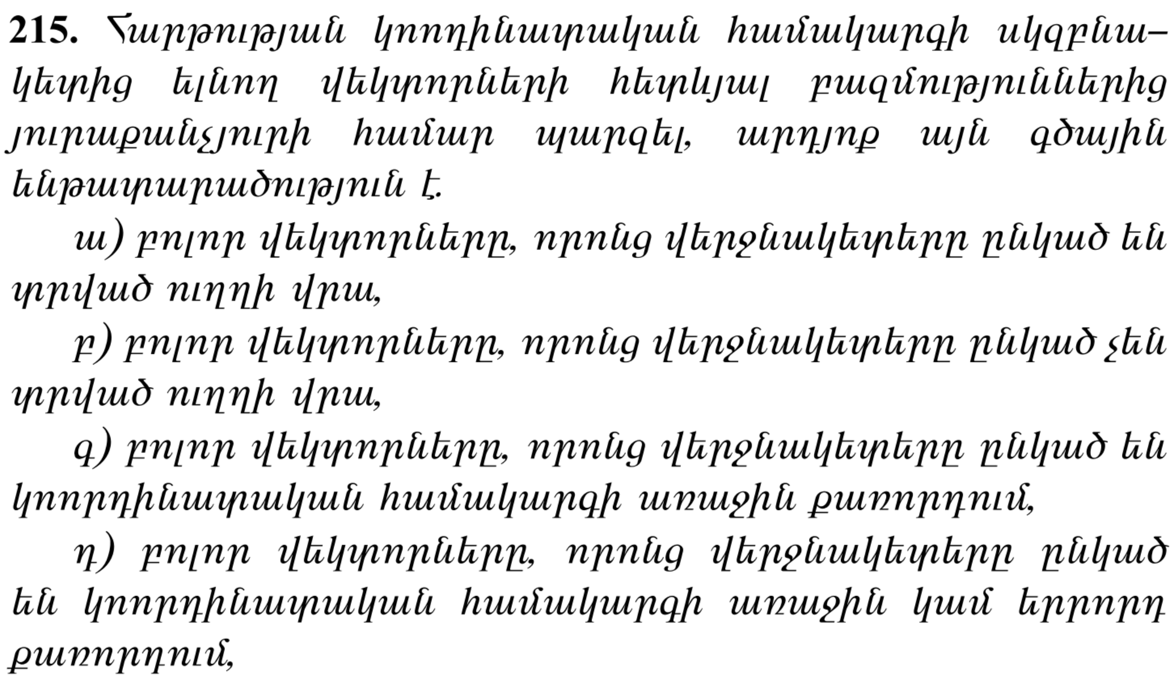
Tasks: 1. **Matrix Power:**
- Compute the matrix power of the following matrix A to the power of n:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

- What does the result represent in terms of linear transformations?

Exercise: Subspace

Tasks:



215. *Հարթության կոորդինատական համակարգի սկզբնակետից ելնող վեկտորների հետևյալ բազմություններից յուրաքանչյուրի համար պարզել, արդյոք այն գծային ենթատարածություն է.*

ա) բոլոր վեկտորները, որոնց վերջնակետերը ընկած են տրված ուղղի վրա,

բ) բոլոր վեկտորները, որոնց վերջնակետերը ընկած չեն տրված ուղղի վրա,

գ) բոլոր վեկտորները, որոնց վերջնակետերը ընկած են կոորդինատական համակարգի առաջին քառորդում,

դ) բոլոր վեկտորները, որոնց վերջնակետերը ընկած են կոորդինատական համակարգի առաջին կամ երրորդ քառորդում,

Figure 1: subspace_exercise

Exercise: Vector Space

Definition Let V be a set on which two operations, called *addition* and *scalar multiplication*, have been defined. If $\mathbf{u}$ and $\mathbf{v}$ are in V, the *sum* of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} + \mathbf{v}$, and if c is a scalar, the *scalar multiple* of $\mathbf{u}$ by c is denoted by $c\mathbf{u}$. If the following axioms hold for all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d, then V is called a ***vector space*** and its elements are called ***vectors***.

1. $\mathbf{u} + \mathbf{v}$ is in V. — Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ — Commutativity
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ — Associativity
4. There exists an element $\mathbf{0}$ in V, called a ***zero vector,*** such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is an element $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. $c\mathbf{u}$ is in V. — Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ — Distributivity
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ — Distributivity
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

Figure 2: Poole_vec_space

Problem 3. Check if the following set is a vector space:

a) $A = \mathbb{Z}$, with the usual operations $+$ and $\cdot$,

b) $B = \left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \mid \text{ for all real numbers } a \in \mathbb{R} \right\}$ with the usual operations $+$ and $\cdot$,

c) $C = \mathbb{R}^2 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \mid \text{ for all numbers } a, b \in \mathbb{R} \right\}$, with the usual operation $\cdot$ and the addition defined as:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 + 1 \end{bmatrix},$$

d) The set of all polynomials of degree ≤ 2, with the usual operations $+$ and $\cdot$.

Figure 3: vec_space_exercise