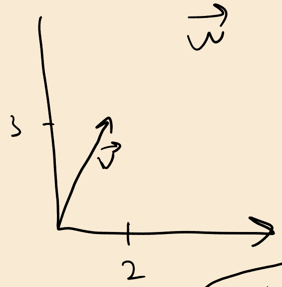
Exercise: Finding a Perpendicular Vector

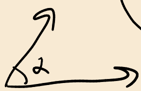
Context:
In linear algebra, two vectors are perpendicular (or orthogonal) if their dot product is zero. In this exercise, you will find a vector in $\mathbb{R}^2$ that is perpendicular to a given vector.

Given:
Let $\mathbf{v} = [2, 3]$.

Tasks:

1. **Find a Perpendicular Vector:**
   - Find a non-zero vector $\mathbf{w} = [x, y]$ such that $\mathbf{v}$ and $\mathbf{w}$ are perpendicular.
2. **Verification:**
   - Show that your chosen vector $\mathbf{w}$ indeed satisfies the condition $\mathbf{v} \cdot \mathbf{w} = 0$.
3. **Unit Perpendicular Vector:**
   - Find a unit vector in the direction of $\mathbf{w}$ by computing $\frac{\mathbf{w}}{\|\mathbf{w}\|}$, where $\|\mathbf{w}\|$ is the Euclidean norm of $\mathbf{w}$.
4. **Bonus Discussion:**
   - Explain why there are infinitely many vectors perpendicular to $\mathbf{v}$ and describe the general form of all such vectors.





$\cos\alpha = \frac{\vec{v}\cdot\vec{w}}{\|v\|\cdot\|w\|}$

$\vec{v}\cdot\vec{w} = \cos\alpha \cdot \|w\| \|w\|$

$\alpha = 90$

$\vec{v}\cdot\vec{w} = 0$

dot product

$\vec{w} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

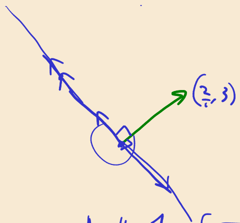
$2x + 3y = 0$

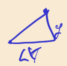
$x, y = 0$

$2x = -3y$

$x = -\frac{3}{2}y$

$1 \quad -\frac{2}{3}$; $x = 2 \quad y = -\frac{4}{3}$





$\|w\| = 1$

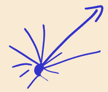
$\begin{cases} \sqrt{x^2 + y^2} = 1 \\ x = -1.5y \end{cases}$

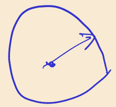
$w_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

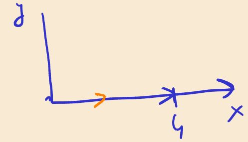
$\|w^*\|_2 = 1$

$\frac{w_1}{\|w_1\|} \rightarrow \sqrt{3^2 + (-2)^2}$

$\begin{bmatrix} \frac{3}{\sqrt{13}} \\ \frac{-2}{\sqrt{13}} \end{bmatrix}$

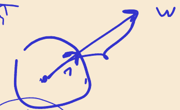






$a = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

$\|a^*\|_2 = 1 \rightarrow \frac{a}{\|a\|}$



Exercise: Finding the Closest Word with 2D Embeddings

Context:
In NLP, words can be represented as vectors. Here, each word is represented by a 2-dimensional vector. By comparing these vectors using Euclidean distance and cosine similarity, you can determine which word is "closer" in meaning.

Given Word Embeddings:

- **cheese:** `[1, 2]`
- **mushroom:** `[3, 1]`
- **tasty:** `[2, 2]`

Tasks:

1. **Euclidean Distance:**
   - **a.** Compute the Euclidean distance between **tasty** and **cheese**.
   - **b.** Compute the Euclidean distance between **tasty** and **mushroom**.
   - **c.** Which word is closer to **tasty** based on the Euclidean distance?
2. **Cosine Similarity:**

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

   - **a.** Compute the cosine similarity between **tasty** and **cheese** using the formula above.
   - **b.** Compute the cosine similarity between **tasty** and **mushroom**.
   - **c.** Based on cosine similarity, which word is closer to **tasty**?
3. **Discussion:**
   - Compare the outcomes from the Euclidean distance and cosine similarity calculations.
   - Discuss why one metric might be preferred over the other in different NLP applications.

> **Note**
> Cool video by 3blue1brown discussing word vectors (embeddings)

$[1, 2, 15, 30, 0, 0, 0, 0]$

$[3, 7, 10, 22, 4, 0, 0, 0]$

token

GPT

$\|v - w\|_2$

$\dfrac{v \cdot w}{\|v\|\|w\|}$

$w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$v_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\left\| \begin{matrix} 1-3 \\ 2-1 \end{matrix} \right\|_2 = \dfrac{x_1}{x_2}$

$\cos\alpha_1 = \dfrac{w \cdot v_1}{\|v_1\|\|w\|}$

460-500

30°

Exercise: Linear transformation matrix power

Tasks: 1. **Matrix Power:**

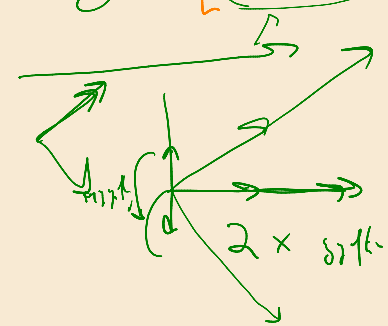
- Compute the matrix power of the following matrix A to the power of n:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

- What does the result represent in terms of linear transformations?

$$\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\cdot 0 + 0\cdot 1 \\ 0\cdot 0 + -1\cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$



$$\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}^3 = \begin{bmatrix} 2^3 & 0 \\ 0 & (-1)^3 \end{bmatrix}$$

215. *Հարթության կոորդինատական համակարգի սկզբնակետից ելնող վեկտորների հետևյալ բազմություններից յուրաքանչյուրի համար պարզել, արդյոք այն գծային ենթատարածություն է.*

ա) բոլոր վեկտորները, որոնց վերջնակետերը ընկած են տրված ուղղի վրա,

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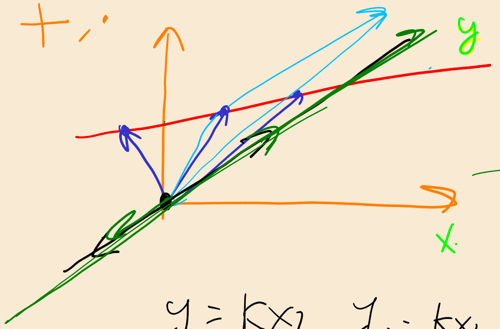
$(\Omega = \{x_1, x_2, x_3\}, \oplus, \odot)$

1) $\forall x, y \in \Omega$ $\quad x + y \in \Omega$

2) $\forall c \in \mathbb{R}$ $\quad \forall x \in \Omega$ $\quad c \cdot x \in \Omega$



$y = kx + c$

$c = 0$

0. $\theta \in \Omega$

$y_1 = kx_1 \quad y_2 = kx_2 \quad y_1 + y_2 = k(x_1 + x_2)$

$\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$

$\forall x, y \in \mathbb{R}$

$3 + 5 = 8$

$4(3+5) = 4 \cdot (8)$

Definition Let V be a set on which two operations, called *addition* and *scalar multiplication*, have been defined. If $\mathbf{u}$ and $\mathbf{v}$ are in V, the *sum* of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} + \mathbf{v}$, and if c is a scalar, the *scalar multiple* of $\mathbf{u}$ by c is denoted by $c\mathbf{u}$. If the following axioms hold for all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d, then V is called a ***vector space*** and its elements are called ***vectors***.

1. $\mathbf{u} + \mathbf{v}$ is in V. — Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ — Commutativity
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ — Associativity
4. There exists an element $\mathbf{0}$ in V, called a ***zero vector***, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is an element $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. $c\mathbf{u}$ is in V. — Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ — Distributivity
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ — Distributivity
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

Problem 3. Check if the following set is a vector space:

a) $A = \mathbb{Z}$, with the usual operations $+$ and $\cdot$,

b) $B = \left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \mid \text{for all real numbers } a \in \mathbb{R} \right\}$ with the usual operations $+$ and $\cdot$,

c) $C = \mathbb{R}^2 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \mid \text{for all numbers } a, b \in \mathbb{R} \right\}$, with the usual operation $\cdot$ and the addition defined as:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \doteq \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 + 1 \end{bmatrix},$$

d) The set of all polynomials of degree ≤ 2, with the usual operations $+$ and $\cdot$.

$8 + 0 = 8$

$\begin{bmatrix} a \\ b \end{bmatrix}$; $a, b \in \mathbb{R}$