Mathematics for Machine Learning

Homework Problems 1

Problem 1. The following vectors are given:

$$\mathbf{a} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ -5 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} 5 \\ 6.5 \\ -7 \end{bmatrix}$$

Find the values of:

a) $\mathbf{a} + \mathbf{b}$,

b) $\mathbf{a} + \mathbf{b} - \mathbf{c}$,

c) $\mathbf{a}^T + \mathbf{b}^T - \mathbf{c}^T$,

d) $3\mathbf{a}^T + 4\mathbf{b}^T - 2\mathbf{c}^T$,

e) $6\mathbf{a}^T + 8\mathbf{b}^T - 4\mathbf{c}^T$,

f) $\mathbf{a} \cdot \mathbf{b}$,

g) $(3\mathbf{a} - 6\mathbf{b}) \cdot \mathbf{c}$,

h) $\mathbf{a} \cdot (4\mathbf{a}^T - \mathbf{b}^T)^T$.

Problem 2. A translation office translated $\mathbf{a} = [24, 17, 9, 13]$ documents from English, French, German and Russian, respectively. For each of those languages, it takes about $\mathbf{b} = [5, 10, 11, 7]$ minutes to translate one page.

How much time did they spend translating in total? How much did each of the translators spend on average if there are 4 translators in the office? Write an expression for this amount in terms of the vectors $\mathbf{a}$ and $\mathbf{b}$.

Problem 3. Calculate the Manhattan (L1) and Euclidean (L2) norms of the following vectors:

a) $\mathbf{a} = \begin{bmatrix} 2 \\ -9 \\ 3 \end{bmatrix}$,

b) $\mathbf{a} - 2\mathbf{b}$, where $\mathbf{a} = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ -2 \\ -1 \end{bmatrix}$,

c) $-3\mathbf{c}$, where $\mathbf{c} = \begin{bmatrix} 2 \\ -5 \\ 6 \end{bmatrix}$.

Problem 4. Find the angles between the following vectors:

a) $\mathbf{a} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$,

b) The vectors $\mathbf{a}$ and $\mathbf{c}$ in the previous problem.

Problem 5. What vectors do you get by applying the matrix

$$A = \begin{bmatrix} 3 & -3 \\ 3 & 3 \end{bmatrix}$$

on the vectors

a) $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,

b) $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

c) $\mathbf{c} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

(Additional:) Draw the vectors before and after multiplying with A. What can you say visually about the matrix? Can you guess how it will act on the vector $[2 \quad -2]$?

Problem 6. Compute the following products:

a) AB, where $A = \begin{bmatrix} 6 & 5 \\ -2 & 7 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 3 \\ 1 & 4 \end{bmatrix}$,

b) $(A-B)(A+B)$, where $A = \begin{bmatrix} 2 & 2 & 4 \\ -3 & -2 & 4 \\ -2 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 2 \\ 1 & 4 & -1 \end{bmatrix}$,

c) $A^2 - B^2$, with the same A and B as in *b)*.

Problem 7 (additional). Consider the following matrix (it is called the *shear matrix*):

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

a) What would you get if you apply S on the vector $[0 \quad 1]$?

b) What would you get if you apply S again on the result of the previous point?

c) What if you apply S one more time?

d) What do you think happens when we apply S 100 times on that vector?

e) Can you compute S^{100}?

Problem 8 (additional). Show that the following set is a vector space:

a) $A = \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} \mid \text{for all numbers } a \in \mathbb{R} \right\}$,

b) $B = \left\{ \begin{bmatrix} a \\ -a \end{bmatrix} \mid \text{for all numbers } a \in \mathbb{R} \right\}$.

Problem 9 (additional). Show that the following set is *not* a vector space:

a) $A = \mathbb{N}$,

b) $B = \left\{ \begin{bmatrix} a \\ 1 \end{bmatrix} \mid \text{for all numbers } a \in \mathbb{R} \right\}$.

Hint: Show that there are some "bad" elements, such that if we add them or multiply with some number (not necessarily positive), the result would not belong to the set.