

# Limit, Derivative, Extrema of a Function

Hayk Aprikyan, Hayk Tarkhanyan

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# Motivation

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$$f(x, y, z, t) = 3xy^2 - y \log t - (1 - y) \log(1 - t) + \frac{z^3}{t}$$

with real-time values  $x = 4$ ,  $y = 0.4$ ,  $z = 0.8$ ,  $t = 55$ , and you should decide whether to increase or decrease each of  $x, y, z, t$  (and how much).

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We usually fix a letter, say  $a$ , and denote the first term by  $a_1$ , the second term by  $a_2$ , and so on. In general, for the  $n^{\text{th}}$  term we write  $a_n$ , and to denote the whole sequence we use  $\{a_n\}$  or  $\{a_n\}_{n=1}^{\infty}$ .

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Sometimes it also comes in handy to give the formula of the general  $n^{\text{th}}$  term, e.g.  $a_n = n^2$  or  $\{a_n\} = \{n^2\}$ , which means:

$$a_1 = 1, \quad a_2 = 4, \quad a_3 = 9, \quad \dots$$

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There are many interesting examples of sequences. Take, for example, the sequence  $a_n = \frac{1}{n}$ :

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Interestingly, it does not: The numbers come arbitrarily close to 0 but they never actually become 0. This shows that the sequence may or may not eventually equal to its limit.

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## Definition

We say that  $\{a_n\}$  **converges** to the number  $L$  (or that the number  $L$  is its **limit**), denoted as

$$\lim_{n \rightarrow \infty} a_n = L \quad (\text{or } a_n \rightarrow L)$$

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and then whatever number you say (e.g. "not further than 0.002"), we can point out some number  $N$  (say,  $N = 1000$ ) such that after the  $N^{\text{th}}$  term, all others are close to  $L$  by 0.002, i.e.

$$|a_N - L| < 0.002, \quad |a_{N+1} - L| < 0.002, \quad |a_{N+2} - L| < 0.002, \quad \dots$$

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If  $\{a_n\}$  has a *finite* limit, we say that it is **convergent**, otherwise it is **divergent**.

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- $c^n \rightarrow +\infty$  if  $c > 1$ , but  $c^n \rightarrow 0$  if  $|c| < 1$
- If a sequence consists of the same number (or if it becomes constant starting from some point), the limit is that number.

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More examples (we will not go further into details):

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Consider the sequence  $\{0.3n\}$ .  $\lim_{n \rightarrow \infty} 0.3n = \infty$  (it is divergent).

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## Properties

① If  $\lim_{n \rightarrow \infty} a_n = L$  and  $\lim_{n \rightarrow \infty} b_n = M$ , then

$$\lim_{n \rightarrow \infty} (a_n + b_n) = L + M$$

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- ③ If  $a_n \leq b_n \leq c_n$  for all  $n$  and  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$ , then  $\lim_{n \rightarrow \infty} b_n = L$ .

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Now that we have the notion of

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what do you think the expression

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as  $x$  approaches 0.

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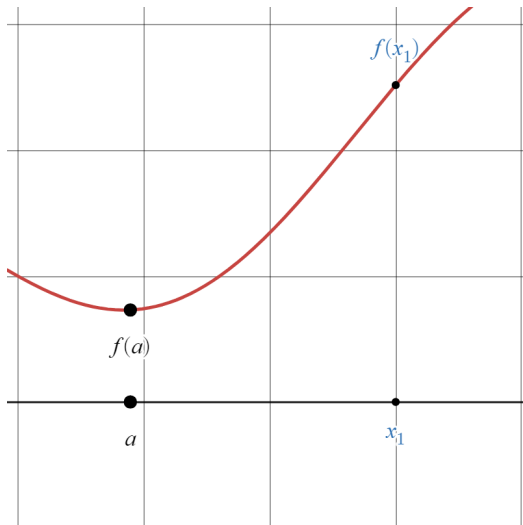
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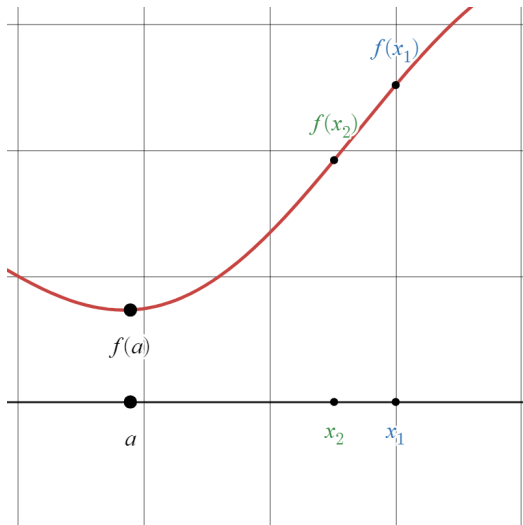
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How do we do that? We can say: take any sequence  $x_n$  that converges to  $a$ , calculate the values of  $f(x)$  at  $x_1, x_2, \dots$ , and see what happens.

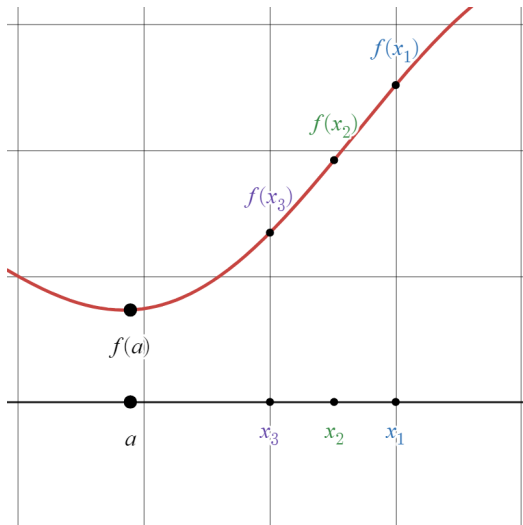
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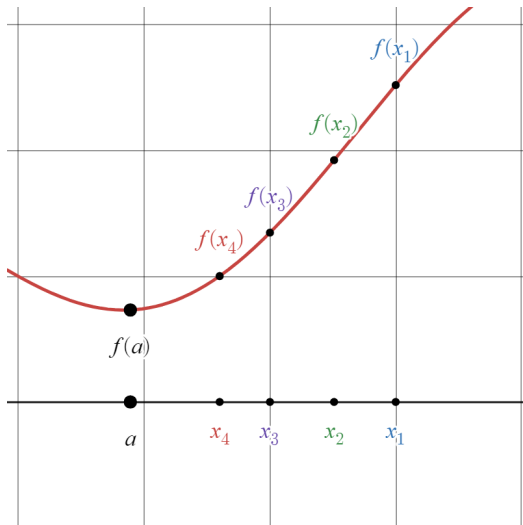


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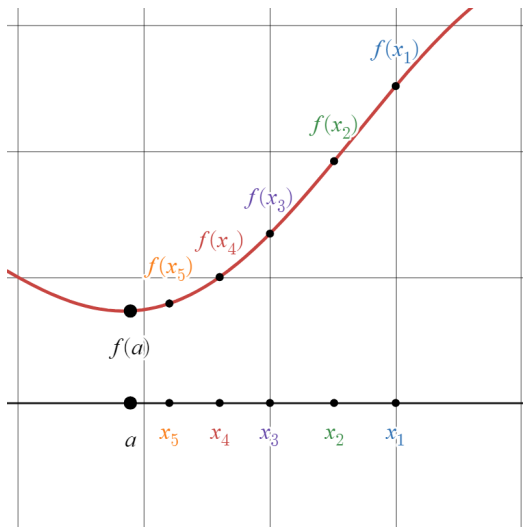




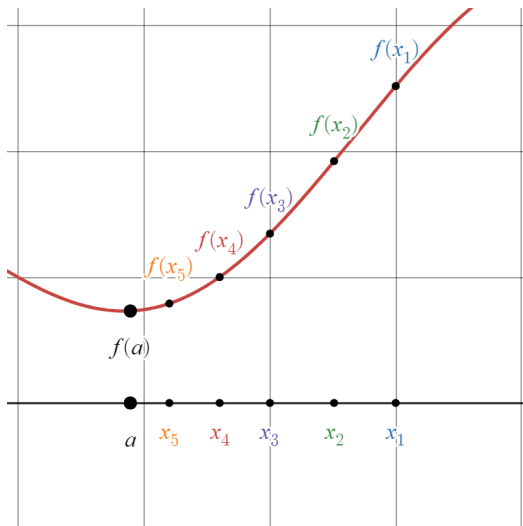
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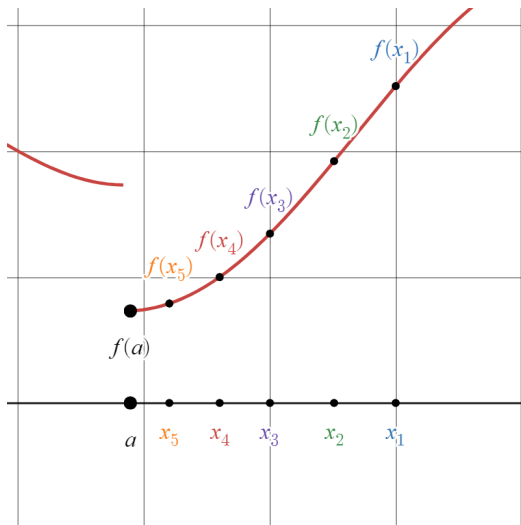


# Limit of a Function



If numbers  $f(x_1), f(x_2), \dots$  approach some limit  $L$ , then  $\lim_{x \rightarrow a} f(x) = L$

# Limit of a Function



It may happen that if  $x_n \rightarrow a$  from the other side, we get another "limit".

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## Definition

If for **all** sequences  $x_n \rightarrow a$  (no matter from left or right), the sequence

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converges to a certain number  $L$ , then we say

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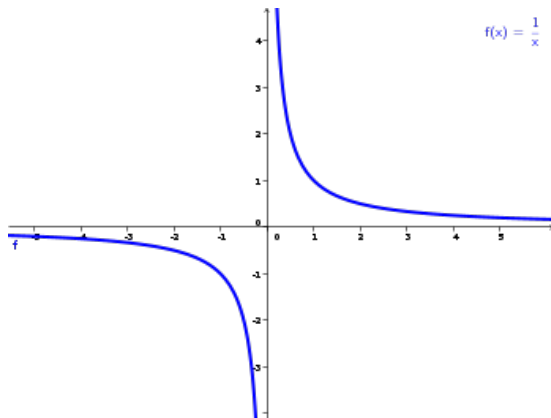
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In other words, if the value of  $f(x)$  always approaches  $L$ , as its input approaches  $a$ .

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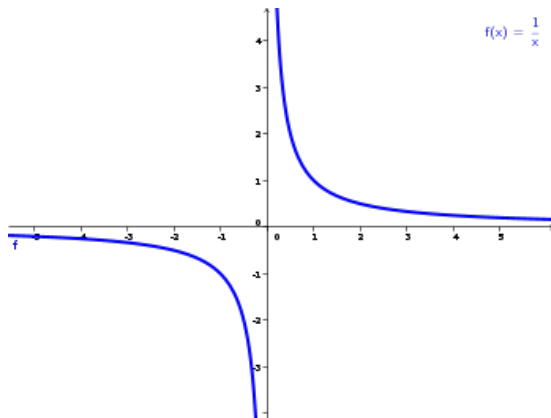


## Example

$$\lim_{x \rightarrow 3} \frac{1}{x} = \frac{1}{3}$$



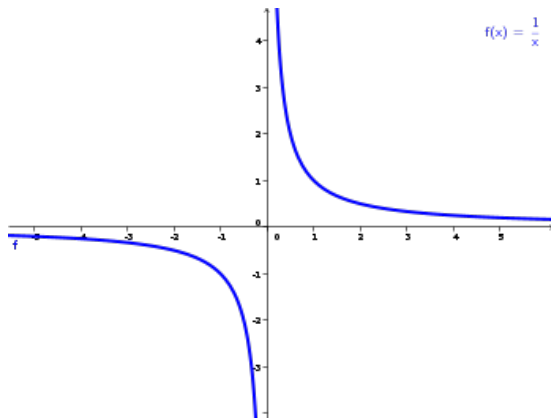
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$$\lim_{x \rightarrow 3} \frac{1}{x} = \frac{1}{3} \quad \text{but} \quad \lim_{x \rightarrow 0} \frac{1}{x}$$

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## Example

$$\lim_{x \rightarrow 3} \frac{1}{x} = \frac{1}{3} \quad \text{but} \quad \lim_{x \rightarrow 0} \frac{1}{x} \text{ does not exist.}$$

# Continuity

Notice how the graph of  $f(x) = \frac{1}{x}$  looks like it consists of two separate graphs put together at  $x = 0$ .

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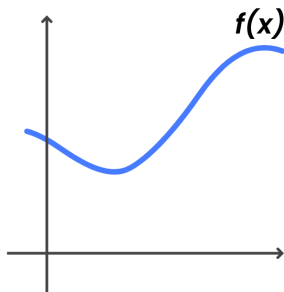
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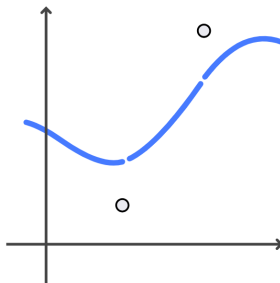
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If a function is continuous at all points, it is called a **continuous function**.

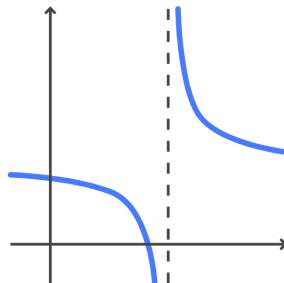
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In fact, most "good" functions are continuous (in their domains!):

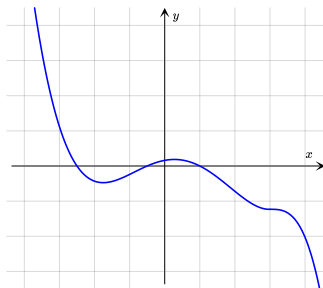
- Polynomials (e.g.  $x^2 + 7x - 1$ ,  $xy - y^4 + z$ )
- Root functions (e.g.  $\sqrt{x}$ ,  $\sqrt[5]{x}$ )
- Exponential and logarithmic functions (e.g.  $2^x$ ,  $e^{3x}$ ,  $\ln x$ )
- Trigonometric functions and their inverses (e.g.  $\cos(3x)$ ,  $\arcsin x$ )

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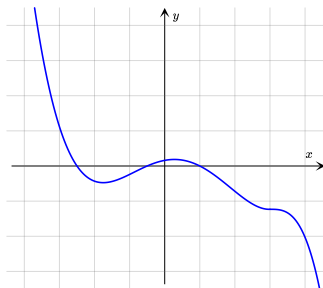


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Now assume we want to maximize or minimize this function. Notice how at some points it changes "faster" than at the others. How can we measure that?

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We say  $f(x)$  is **differentiable at a point**  $a$ , if the following limit

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Note that  $f'(x)$  is a **function** itself and not a fixed number!

## Example

Consider the function  $f(x) = x^2$ . The derivative of  $f(x)$  is:

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The function  $f(x) = |x|$  is continuous but it is not differentiable at point  $x = 0$ .



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Similarly, we can compute the derivative of  $f'(x)$  itself (it will show the speed of the speed of  $f(x)$ , i.e. its *acceleration*).

We denote the derivative of  $f'(x)$  by  $f''(x)$ , that of  $f''(x)$  by  $f'''(x)$ , and so on. The derivative taken of  $f(x)$   $n$  times is also denoted by  $f^{(n)}(x)$ .

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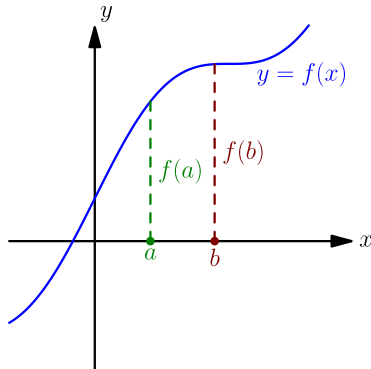
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Derivatives tell about amazingly many interesting properties of the function.

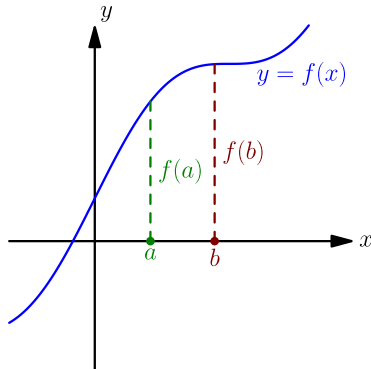
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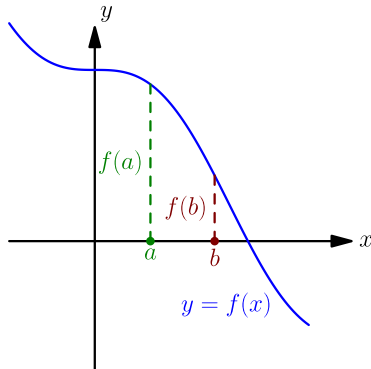
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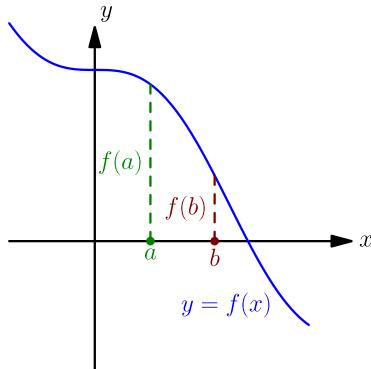
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## Question

What if  $f'(a) = 0$ ?

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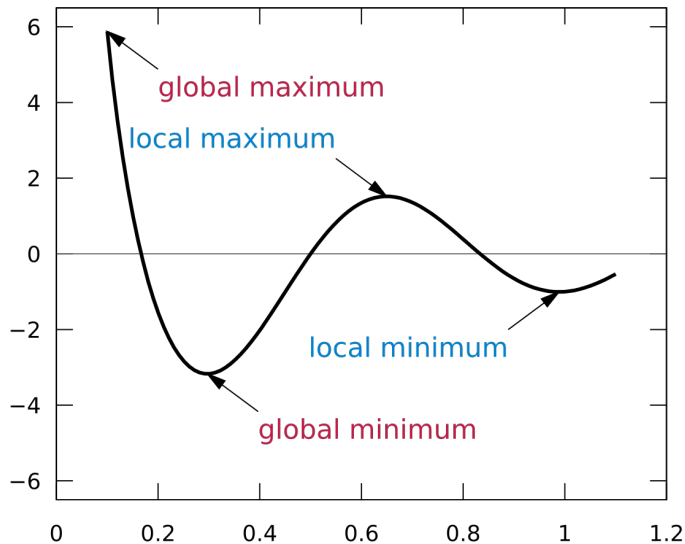
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Together, global minima and maxima are called global extrema of  $f$ . Apparently, every global extrema is also a local extrema (and the converse is not true).

## Theorem

Every continuous function  $f$  has both a global maximum and a global minimum on any **closed** interval  $[a, b]$ .

# Extrema of a Function



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If  $x_0$  is an extremum point of  $f$  and there exists  $f'(x_0)$ , then  $f'(x_0) = 0$ .

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Hence, the condition  $f'(x) = 0$  is necessary but *not sufficient*.

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How can we tell if a critical point is a local minimum/maximum point?

# Extrema of a Function

## Theorem 1 ( $f''$ at one point)

If  $f'(x_0) = 0$  and there exists finite  $f''(x_0)$ , then

- 1 If  $f''(x_0) > 0$ , then  $x_0$  is a local minimum point,
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## Theorem 2 ( $f'$ at multiple points)

If for some  $\delta > 0$ ,  $f$  is differentiable in the intervals  $(x_0 - \delta, x_0)$  and  $(x_0, x_0 + \delta)$  and continuous at  $x_0$ , then

- 1 If  $f'(x) > 0$  for  $x \in (x_0 - \delta, x_0)$  and  $f'(x) < 0$  for  $x \in (x_0, x_0 + \delta)$ , then  $x_0$  is a local maximum point.
- 2 If  $f'(x) < 0$  for  $x \in (x_0 - \delta, x_0)$  and  $f'(x) > 0$  for  $x \in (x_0, x_0 + \delta)$ , then  $x_0$  is a local minimum point.
- 3 If  $f'(x)$  doesn't change its sign, then  $x_0$  is not an extremum point.



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- Step 3:** a) If there exists finite  $f''(x_0) \neq 0$ , use Theorem 1.  
b) If you find the sign of  $f'(x)$  on left and right "sides" of  $x_0$ , use Theorem 2.