Limit, Derivative, Extrema of a Function

Hayk Aprikyan, Hayk Tarkhanyan

Motivation

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with real-time values $x = 4$, $y = 0.4$, $z = 0.8$, $t = 55$, and you should decide whether to increase or decrease each of x, y, z, t (and how much).

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In machine learning you often have 1.000.000+ such parameters.

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Example

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- $1, -1, 1, -1, 1, \ldots$
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We usually fix a letter, say a, and denote the first term by a_1, the second term by a_2, and so on. In general, for the n^{th} term we write a_n, and to denote the whole sequence we use $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

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Sometimes it also comes in handy to give the formula of the general n^{th} term, e.g. $a_n = n^2$ or $\{a_n\} = \{n^2\}$, which means:

$$a_1 = 1, \qquad a_2 = 4, \qquad a_3 = 9, \qquad \ldots$$

Limit of a Sequence

There are many interesting examples of sequences. Take, for example, the sequence $a_n = \dfrac{1}{n}$:

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Does the sequence become equal to 0 at some point?

Interestingly, it does not: The numbers come arbitrarily close to 0 but they never actually become 0. This shows that the sequence may or may not eventually equal to its limit.

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Definition

We say that $\{a_n\}$ **converges** to the number L (or that the number L is its **limit**), denoted as

$$\lim_{n\to\infty} a_n = L \qquad (\text{or } a_n \to L)$$

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and then whatever number you say (e.g. "not further than 0.002"), we can point out some number N (say, $N = 1000$) such that after the N^{th} term, all others are close to L by 0.002, i.e.

$$|a_N - L| < 0.002, \quad |a_{N+1} - L| < 0.002, \quad |a_{N+2} - L| < 0.002, \quad \ldots$$

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So more technically, $\lim_{n \to \infty} a_n = L$ means that

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If $\{a_n\}$ has a *finite* limit, we say that it is **convergent**, otherwise it is **divergent**.

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- $c^n \to +\infty$ if $c > 1$, but $c^n \to 0$ if $|c| < 1$
- If a sequence consists of the same number (or if it becomes constant starting from some point), the limit is that number.

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More examples (we will not go further into details):

Example

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Consider the sequence $\{0.3n\}$. $\lim_{n\to\infty} 0.3n = \infty$ (it is divergent).

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Properties

1. If $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = M$, then

$$\lim_{n\to\infty}(a_n + b_n) = L + M$$

$$\lim_{n\to\infty}(a_n - b_n) = L - M$$

$$\lim_{n\to\infty}(a_n \cdot b_n) = L \cdot M$$

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3. If $a_n \leq b_n \leq c_n$ for all n and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Limit of a Function

Now that we have the notion of

$$\lim_{n\to\infty}\left(2+\frac{1}{n}\right)^3$$

what do you think the expression

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Similarly to sequences, we can define the limit of the above expression, i.e. of the function

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as x approaches 0.

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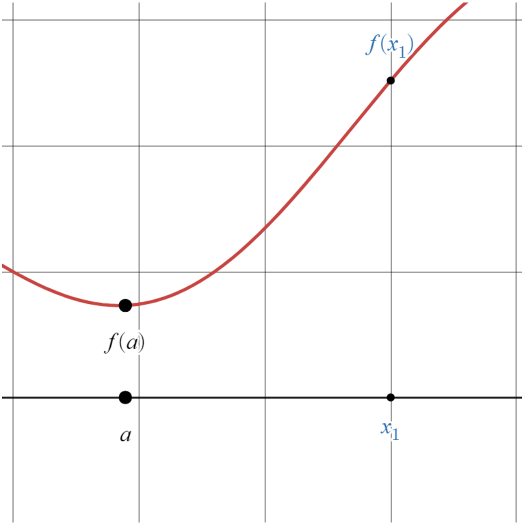
How do we do that? We can say: take any sequence x_n that converges to a, calculate the values of $f(x)$ at $x_1, x_2, \ldots$, and see what happens.

Limit of a Function



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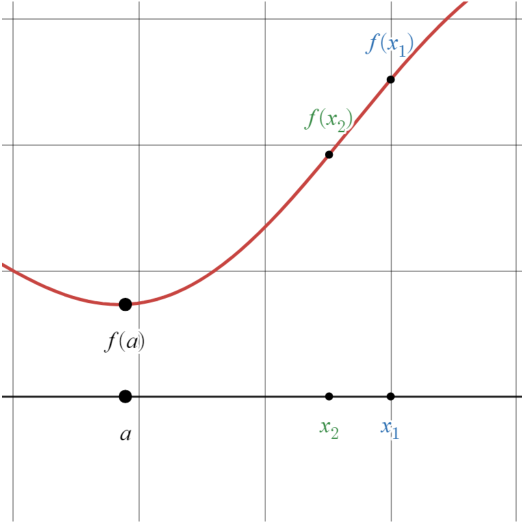

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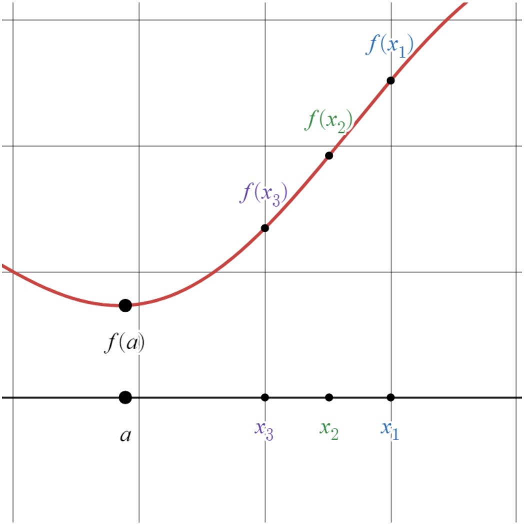

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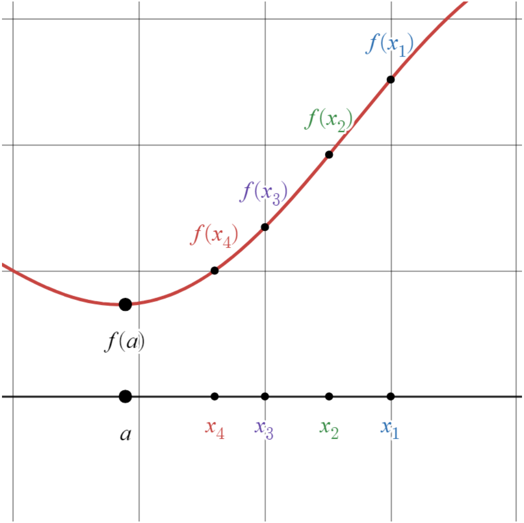

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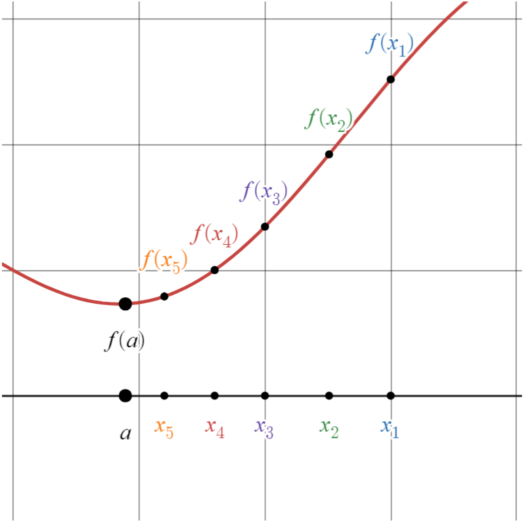

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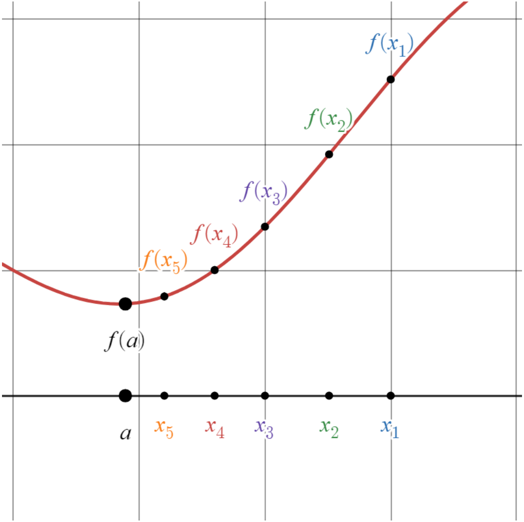

If numbers $f(x_1)$, $f(x_2)$, ... approach some limit L, then $\lim_{x \to a} f(x) = L$

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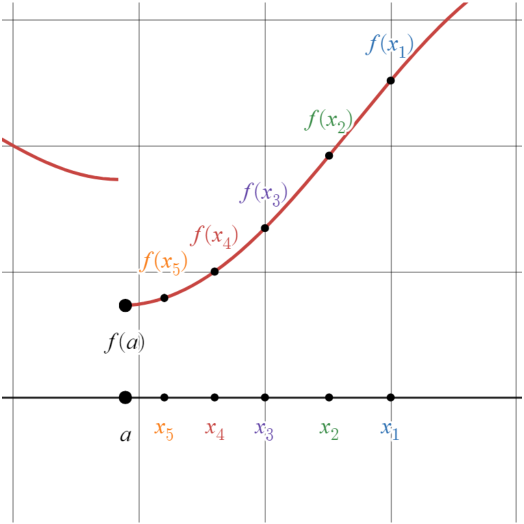

It may happen that if $x_n \to a$ from the other side, we get another "limit".

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converges to a certain number L, then we say

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In other words, if the value of $f(x)$ always approaches L, as its input approaches a.

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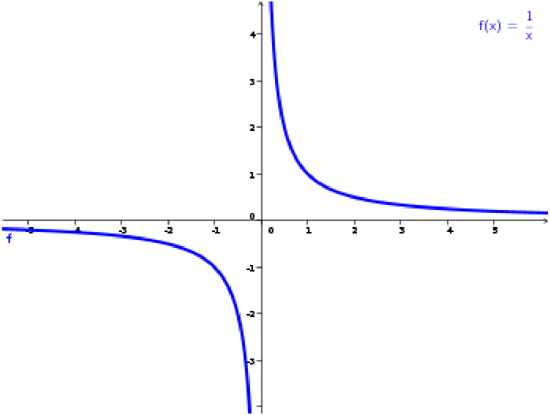

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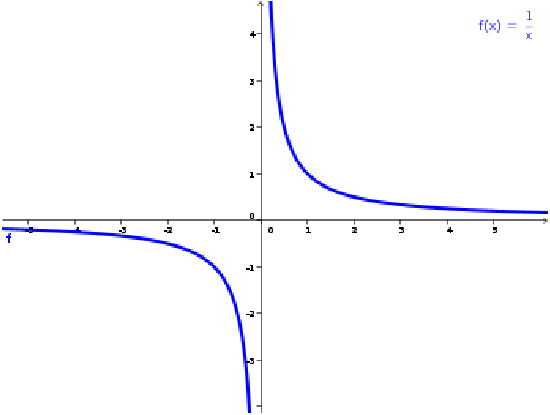

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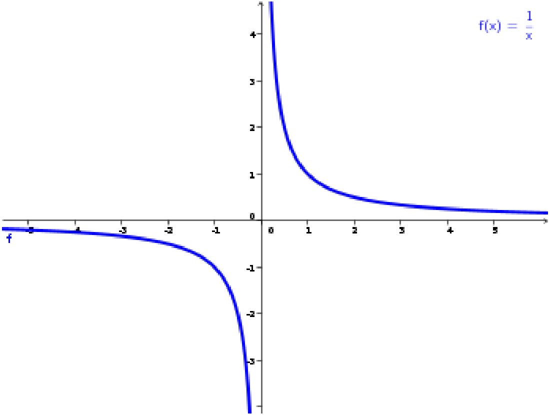

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A function $f(x)$ is said to be **continuous** at the point c if it is defined at c and

1. $\lim\limits_{x \to c} f(x)$ exists,
2. $\lim\limits_{x \to c} f(x) = f(c)$ (i.e. limit = value).

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You can draw its graph without lifting pencil on $(-\infty, 0)$ and $(0, +\infty)$ but not at 0. That's why we say that it is **continuous** all points except for 0. We say that it is **discontinuous** at 0.

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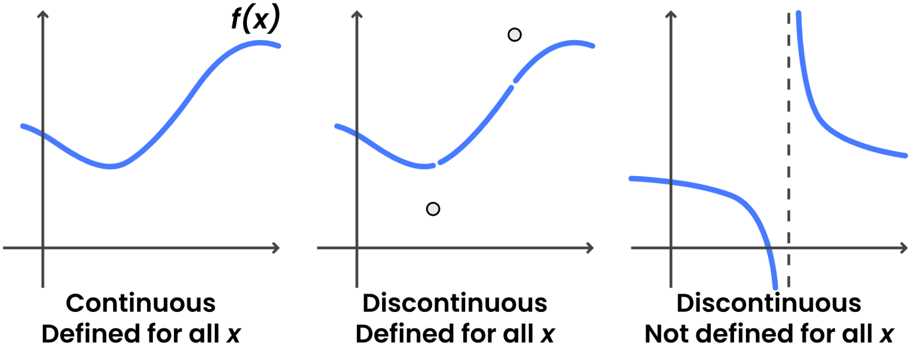
A function $f(x)$ is said to be **continuous** at the point c if it is defined at c and

1. $\lim\limits_{x \to c} f(x)$ exists,
2. $\lim\limits_{x \to c} f(x) = f(c)$ (i.e. limit = value).

If a function is continuous at all points, it is called a **continuous function**.

Continuity



f(x)
Continuous
Defined for all x
Discontinuous
Defined for all x
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Continuity

Properties

If f and g are continuous at some point a, then

- $f + g$

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are also continuous at a.

In fact, most "good" functions are continuous (in their domains!):

- Polynomials (e.g. $x^2 + 7x - 1$, $xy - y^4 + z$)
- Root functions (e.g. $\sqrt{x}, \sqrt[5]{x}$)
- Exponential and logarithmic functions (e.g. $2^x, e^{3x}$, $\ln x$)
- Trigonometric functions and their inverses (e.g. $\cos(3x), \arcsin x$)

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Continuous functions are better and more preferable to work with (continuous $= \heartsuit$), and many processes in the world are actually described by continuous functions.

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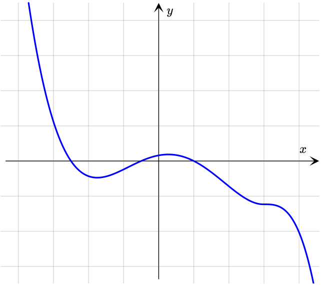

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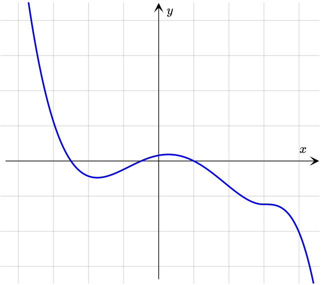

Now assume we want to maximize or minimize this function.
Notice how at some points it changes "faster" than at the others.
How can we measure that?

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Definition

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or, equivalently,

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exists and is finite. We denote it by $f'(a)$ or $\frac{df}{dx}(a)$ and call the **derivative** of function $f(x)$ at a.

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Note that $f'(x)$ is a **function** itself and not a fixed number!

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Example

Consider the function $f(x) = x^2$. The derivative of $f(x)$ is:

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Consider the function $f(x) = x^2$. The derivative of $f(x)$ is:

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The function $f(x) = |x|$ is continuous but it is not differentiable at point $x = 0$.

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Similarly, we can compute the derivative of $f'(x)$ itself (it will show the speed of the speed of $f(x)$, i.e. its *acceleration*).

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Derivatives tell about amazingly many interesting properties of the function.

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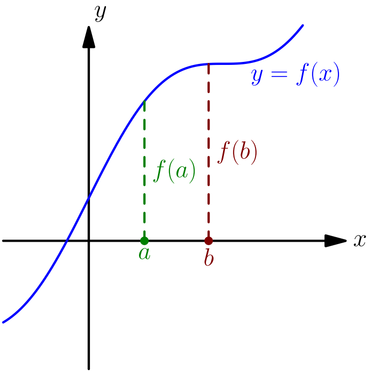

We say that a function $f(x)$ is **increasing** at the point a if

$$f(c) < f(a) < f(b) \qquad \text{when} \qquad c < a < b$$

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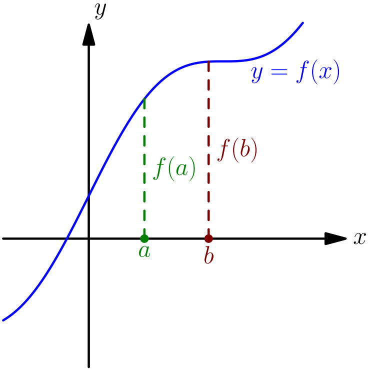

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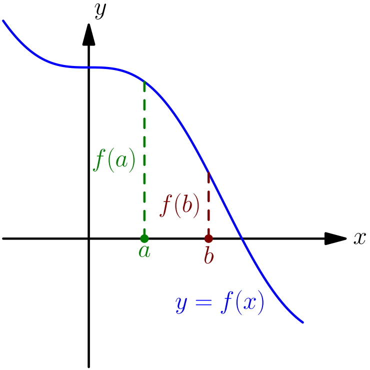

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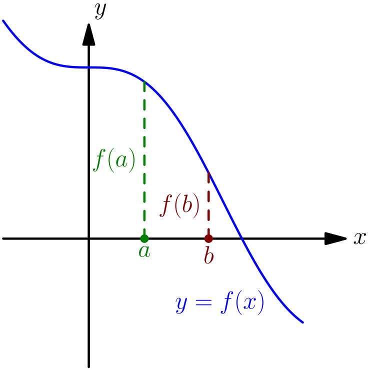

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Theorem

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Question

What if $f'(a) = 0$?

Finally, what about the largest/smallest values of the function?

Extrema of a Function

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Definition

We say x_0 is a **local maximum (minimum) point** of f, if the largest (smallest) value of f on a small enough interval $(x_0 - \delta, x_0 + \delta)$ is the value $f(x_0)$ itself.

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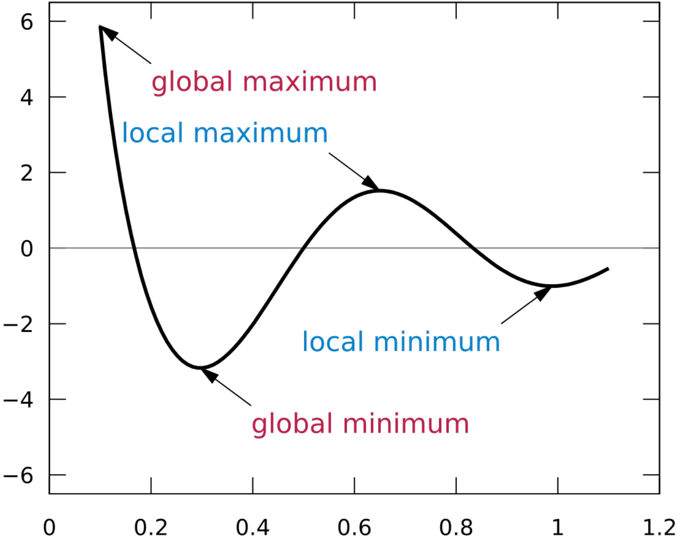
Theorem

Every continuous function f has both a global maximum and a global minimum on any **closed** interval $[a, b]$.

Extrema of a Function



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Example

Consider the function $f_1(x) = x^2$ which has a minimum at $x = 0$.

$$f_1'(x) = 2x, \qquad f_1'(0) = 0$$

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If x_0 is an extremum point of f and there exists $f'(x_0)$, then $f'(x_0) = 0$.

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Consider the function $f_1(x) = x^2$ which has a minimum at $x = 0$.

$$f_1'(x) = 2x, \qquad f_1'(0) = 0$$

Moreover, the function $f_2(x) = -x^2$ has a maximum at $x = 0$ and again:

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Hence, the condition $f'(x) = 0$ is necessary but *not sufficient*.

Extrema of a Function

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How can we tell if a critical point is a local minimum/maximum point?

Extrema of a Function

Theorem 1 (f'' at one point)

If $f'(x_0) = 0$ and there exists finite $f''(x_0)$, then

1. If $f''(x_0) > 0$, then x_0 is a local minimum point,
2. If $f''(x_0) < 0$, then x_0 is a local maximum point.

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Theorem 2 (f' at multiple points)

If for some $\delta > 0$, f is differentiable in the intervals $(x_0 - \delta, x_0)$ and $(x_0, x_0 + \delta)$ and continuous at x_0, then

1. If $f'(x) > 0$ for $x \in (x_0 - \delta, x_0)$ and $f'(x) < 0$ for $x \in (x_0, x_0 + \delta)$, then x_0 is a local maximum point.
2. If $f'(x) < 0$ for $x \in (x_0 - \delta, x_0)$ and $f'(x) > 0$ for $x \in (x_0, x_0 + \delta)$, then x_0 is a local minimum point.
3. If $f'(x)$ doesn't change its sign, then x_0 is not an extremum point.

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Step 2: If $f(x)$ is given on a closed interval $[a, b]$, also check its values on endpoints a and b.
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Step 3: a) If there exists finite $f''(x_0) \neq 0$, use Theorem 1.
b) If you find the sign of $f'(x)$ on left and right "sides" of x_0, use Theorem 2.