Inverse, Determinant

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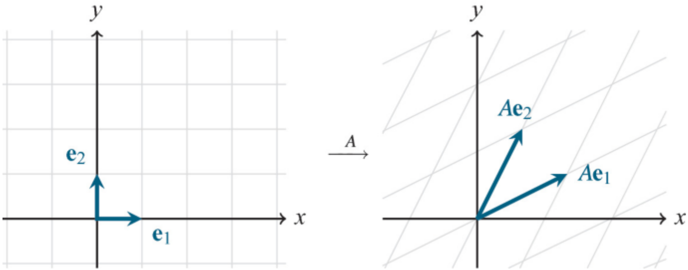
Recap:
When you multiply, say, a 2×2 matrix A by a vector $\mathbf{v} \in \mathbb{R}^2$, what you get is another vector $\mathbf{u} = A\mathbf{v} \in \mathbb{R}^2$. We call this $\mathbf{u}$ the **transformed version** of $\mathbf{v}$ (and we say that A is a linear transformation).



Geometric Interpretation

As we will see later, the resulting "transformed version" $\mathbf{u}$ is just the same old $\mathbf{v}$ except it is **rotated** and **scaled** to become longer or shorter (and possibly, flipped).

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In this sense, all matrices are either just rotating vectors by some degree, or flipping them horizontally/vertically, or scale them, or do all three.

The key thing is: whatever a matrix "does" to one vector, it does the same to all other vectors too (when being multiplied with them).

Check different matrices yourself:

- visualize-it.github.io/linear_transformations/simulation.html
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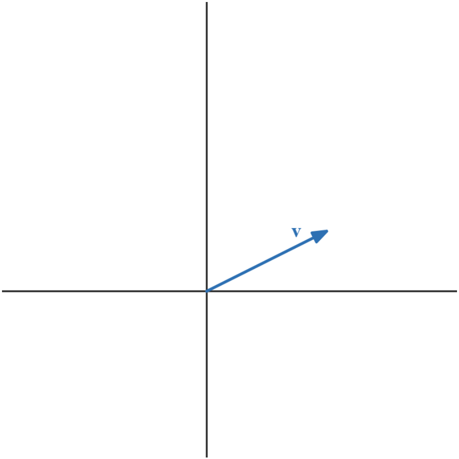
Now that we know what matrix $\times$ vector multiplication means, what about matrix $\times$ matrix multiplication? Why is it defined the way it is?

Geometric Interpretation

Suppose $\mathbf{v} \in \mathbb{R}^2$, $A \in \mathbb{R}^{2\times 2}$, $B \in \mathbb{R}^{2\times 2}$:



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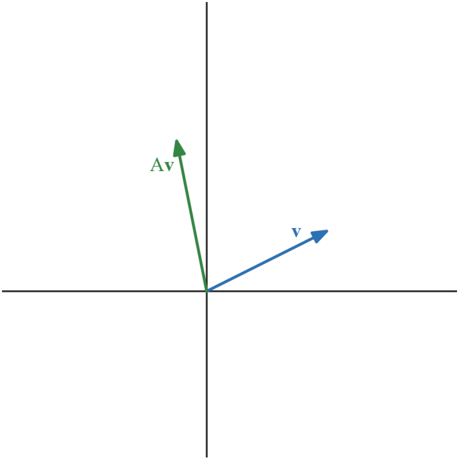

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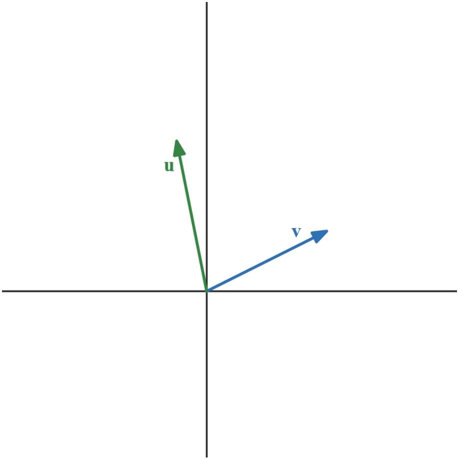

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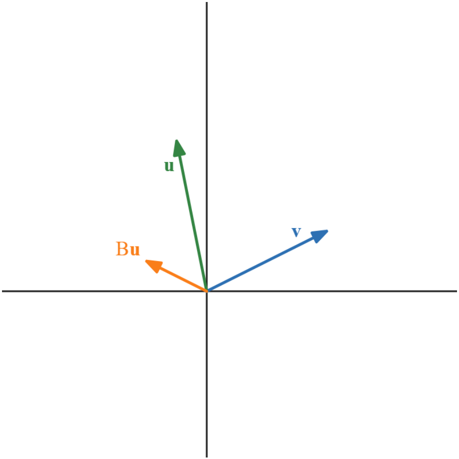
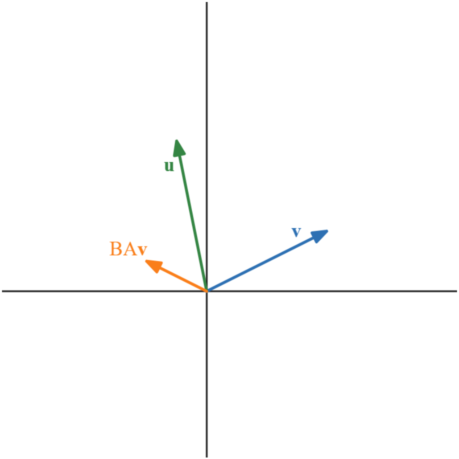
Now applying B on $\mathbf{u}$, we get a transformed version of $\mathbf{u}$, i.e. $B\mathbf{u}$



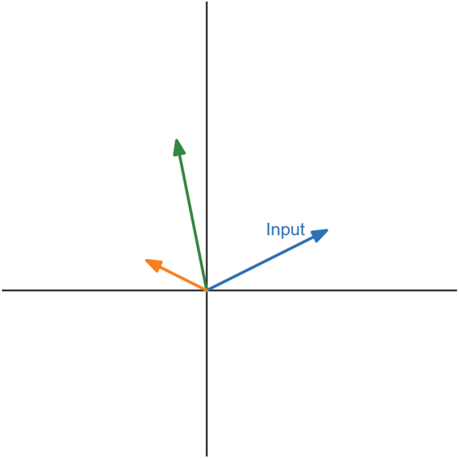
Geometric Interpretation

Now applying B on $\mathbf{u}$, we get a transformed version of $\mathbf{u}$, i.e. $B\mathbf{u} = BA\mathbf{v}$



Geometric Interpretation

So what is the product BA? To get $(BA)(\mathbf{v})$, we do:



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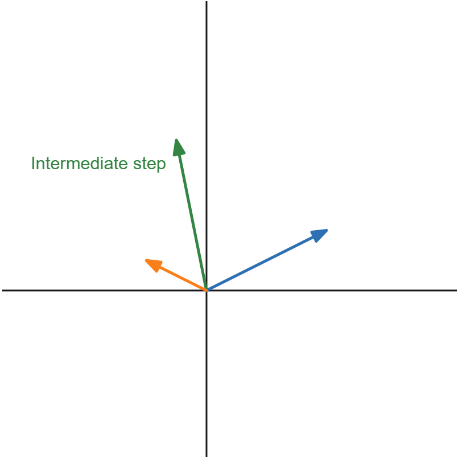

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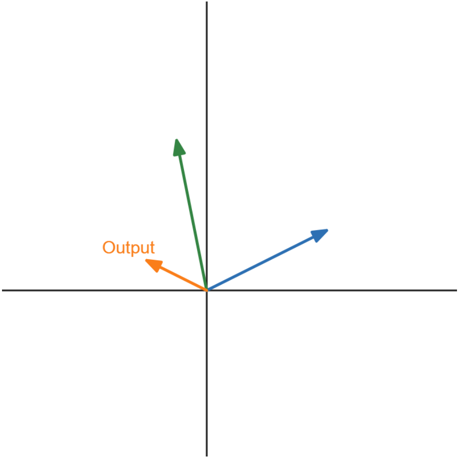

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Hence BA is the linear transformation that we get by **first applying** A, and **then applying** B!

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Question

Suppose A is the matrix that rotates the vectors by 30°, B the one that rotates by 50°, and C by 260°.

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Suppose A is the matrix that rotates the vectors by 30°, B the one that rotates by 50°, and C by 260°.
What would the product matrix BA be?

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What would the product matrix BA be? What about CBA?

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Suppose A is the matrix that rotates the vectors by 30°, B the one that rotates by 50°, and C by 260°.
What would the product matrix BA be? What about CBA?

Which matrix leaves everything in its place (does not touch anything)?

Identity Matrix

Definition

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Example

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -3 & 4 \\ 1 & 4 & 6 \end{bmatrix}$$

This matrix is both symmetric and (of course) square.

Identity Matrix

Definition

The **main diagonal** (or just the **diagonal**) of a matrix A are the terms a_{ii} for which the row and column indices are the same $(a_{11}, a_{22}, \ldots)$, so from the upper left element to the lower right.

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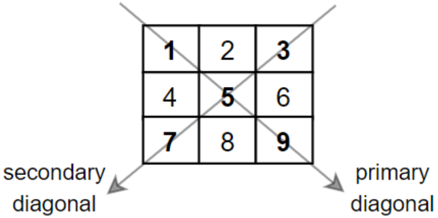

Identity Matrix

For example, here the main diagonal is marked with red:

$$\begin{bmatrix} \color{red}{1} & 0 & 0 \\ 0 & \color{red}{1} & 0 \\ 0 & 0 & \color{red}{1} \end{bmatrix} \quad \begin{bmatrix} \color{red}{1} & 0 & 0 & 0 \\ 0 & \color{red}{1} & 0 & 0 \\ 0 & 0 & \color{red}{1} & 0 \end{bmatrix} \quad \begin{bmatrix} \color{red}{1} & 0 & 0 \\ 0 & \color{red}{1} & 0 \\ 0 & 0 & \color{red}{1} \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \color{red}{1} & 0 & 0 & 0 \\ 0 & \color{red}{1} & 0 & 0 \\ 0 & 0 & \color{red}{1} & 0 \\ 0 & 0 & 0 & \color{red}{1} \end{bmatrix}$$

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For example, here the main diagonal is marked with red:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The secondary diagonal is marked with red:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

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Therefore, we can say:

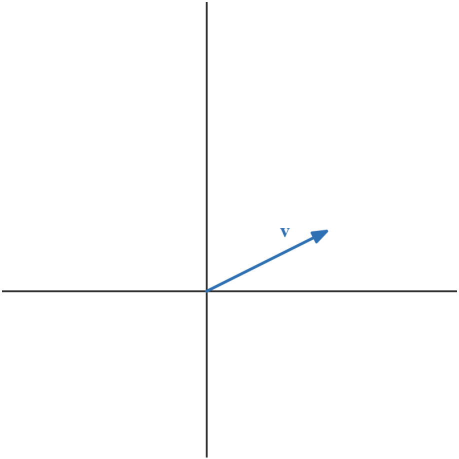
Property

For any matrix $A \in \mathbb{R}^{m\times n}$,

$$I_m A = A I_n = A$$

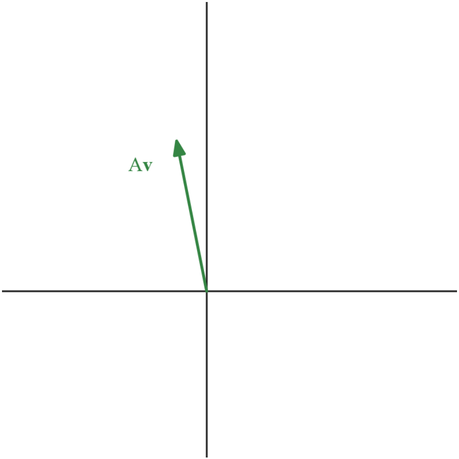
Inverse Matrix

Finally, what if we have a vector in $\mathbb{R}^n$,



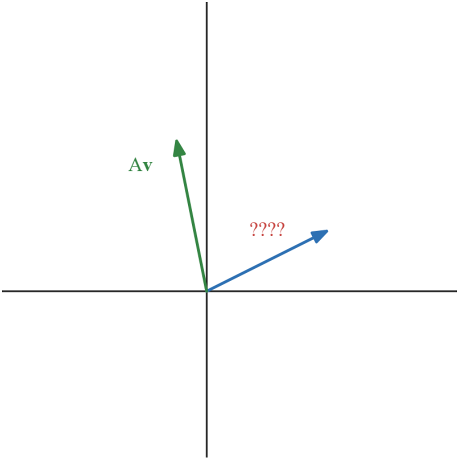
Inverse Matrix

Finally, what if we have a vector in $\mathbb{R}^n$, and we accidentally transform it?



Inverse Matrix

How to get back to the original vector?



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In other words, in terms of what we learned about matrix multiplication,

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Question

Assume the matrix $A \in \mathbb{R}^{n \times n}$ does the following when applied on a vector:

1. scales the vector up 2 times in the horizontal direction,
2. then rotates it by 30° clockwise,
3. then squishes it down 3 times in the vertical direction,
4. and then flips it horizontally (around the x-axis) $\sim$

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Given $\mathbf{v} = A\mathbf{u}$, could we recover the original $\mathbf{u}$?

The answer is yes, i.e. the matrix A has an inverse. As we will see soon, only some square matrices actually have an inverse.

Trace

Definition

The **trace** of a square matrix A, denoted as $\mathrm{tr}(A)$, is the sum of the elements on its main diagonal.

$$\mathrm{tr}(A) = a_{11} + a_{22} + \ldots + a_{nn}$$

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$$A = \begin{bmatrix} 2 & 5 & 1 \\ 0 & -3 & 4 \\ 7 & 2 & 6 \end{bmatrix}$$

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Note that only square matrices have a trace.

Trace

Trace Properties

For any matrices A and B, and any scalar c, the trace of a matrix satisfies the following properties:

- $\mathrm{tr}(cA) = c \cdot \mathrm{tr}(A)$
- $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$
- $\mathrm{tr}(AB) = \mathrm{tr}(BA)$
- $\mathrm{tr}(A^T) = \mathrm{tr}(A)$

Determinant of a 2×2 Matrix

Determinant Formula

For a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the determinant is given by

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For the matrix

$$A = \begin{bmatrix} 2 & 5 \\ -3 & 4 \end{bmatrix}$$

the determinant is $\det(A) = (2)(4) - (5)(-3) = 8 + 15 = 23.$

Determinant of a 3×3 Matrix

Determinant Formula

For a 3×3 matrix

$$C = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

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Forget that formula–remember the algorithm!

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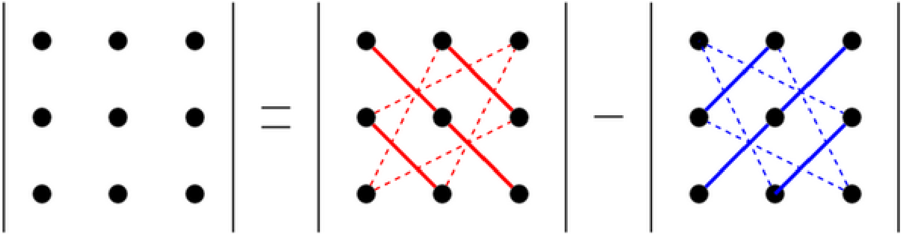
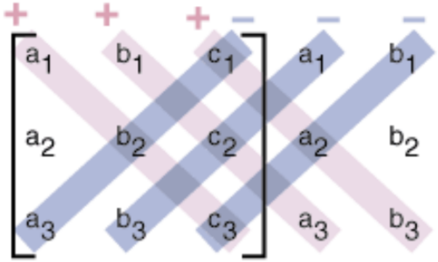


Alternatively,



$$\det A = (a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3) - (a_3 b_2 c_1 + b_3 c_2 a_1 + c_3 a_2 b_1)$$

Determinant of a 3 × 3 Matrix

Example

For the matrix

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\det(C) = 1 \cdot 5 \cdot 9 + 2 \cdot 6 \cdot 7 + 3 \cdot 4 \cdot 8 - 3 \cdot 5 \cdot 7 - 2 \cdot 4 \cdot 9 - 1 \cdot 6 \cdot 8 = 0$$

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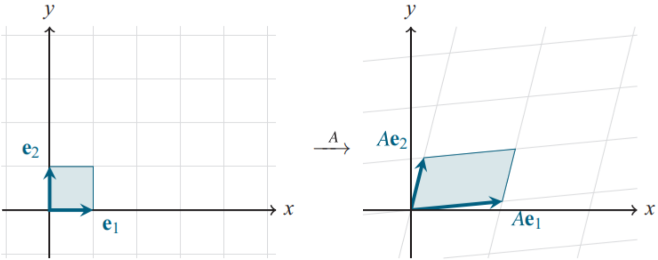
But what does the determinant show, and how do we need it?

Determinant

If we take, for example, the so-called "unit square" formed by the vectors $\mathbf{e}_1 = [1 \;\; 0]$ and $\mathbf{e}_2 = [0 \;\; 1]$, we can see that their transformed versions, $A\mathbf{e}_1$ and $A\mathbf{e}_2$, form a parallelogram:



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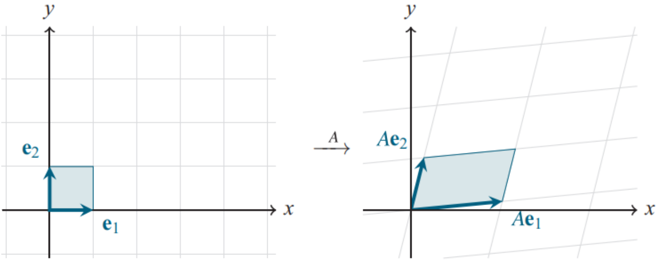
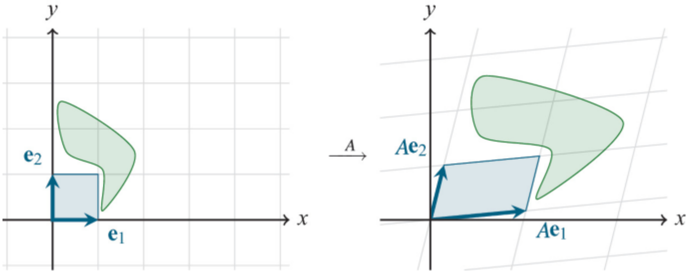


Then $\det(A)$ is the area of that parallelogram.

Determinant

More generally, after we apply the transformation A (play that animation in your head), the area of *any shape* gets scaled by the factor of $\det(A)$:



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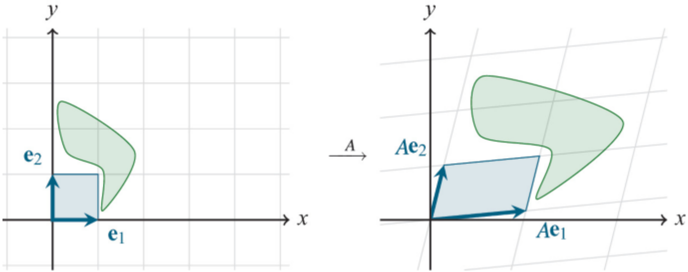

So the determinant shows how much the matrix scales up everything in average. Note that it is defined **only** for square matrices.

Determinant

Determinant Properties

Let $A, B \in \mathbb{R}^{n \times n}$ be square matrices of the same size, and let $c \in \mathbb{R}$ be any scalar. Then:

- $\det(cA) = c^n \cdot \det(A)$ *(where n is the size of the matrix)*
- $\det(AB) = \det(A) \cdot \det(B)$ *(multiplicativity)*
- $\det(I) = 1$
- If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$
- $\det(A^T) = \det(A)$ *(invariance under transpose)*
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It would be an exercise of huge importance to attempt proving these properties (except the last three) by playing the matrices in your head.

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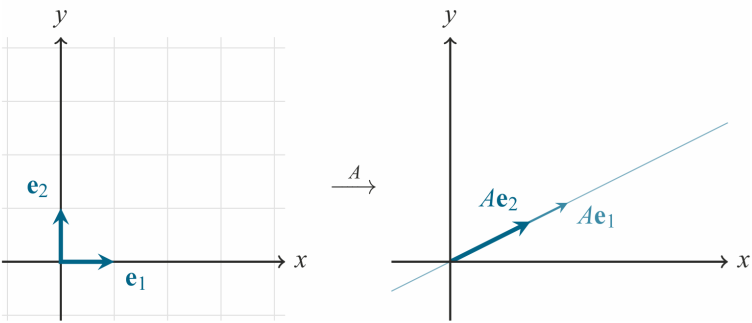

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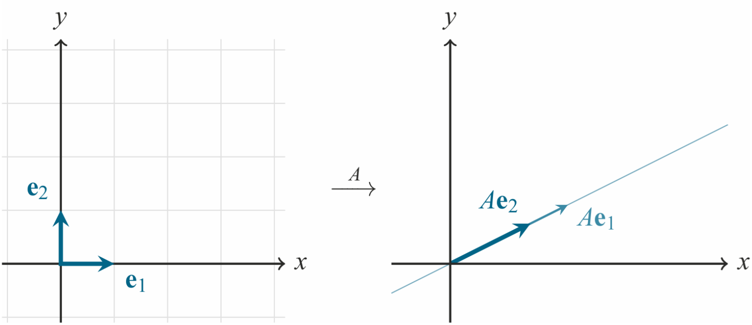

Theorem

A square matrix A has an inverse if and only if its determinant is not zero.

Inverse Matrix

Formula for 2x2

For a 2×2 invertible matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse A^{-1} can be calculated using the formula:

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Example

Given $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ with $\det A = (2 \times 4) - (3 \times 1) = 5$, we can calculate the inverse as follows:

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$