Mathematics for Machine Learning

Homework Problems 2 (Matrices)

Problem 1. Match the descriptions a)-f) with matrices 1)-6): *Descriptions:*

a) Rotation by $70°$

b) Not changing anything

c) Flipping around the x-axis

d) Stretching out 5 times in y-direction

e) Sending all vectors onto the line $y = 6x$

f) Sending all vectors onto the line $y = 6x + 1$

Matrices:

1) $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$

2) Such matrix does not exist

3) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4) $\begin{bmatrix} 0.342 & -0.939 \\ 0.939 & 0.342 \end{bmatrix}$

5) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

6) $\begin{bmatrix} 3 & -1 \\ 18 & -6 \end{bmatrix}$

Hint: Compute the determinants.

Problem 2. What is the determinant of the matrix

a) which rotates everything by $14°$ and stretches 1.5 times in the x-direction,

b) which does the inverse of a),

c) which sends $[1 \ \ 0]$ to $[2 \ \ 4]$ and $[0 \ \ 1]$ to $[-1 \ \ 0]$,

d) which has the form $\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$,

e) which has the form $= \begin{bmatrix} 8 & 3 & 5 \\ 1 & 4 & 2 \\ -4 & 0 & 4 \end{bmatrix}$,

f) which is the matrix of e) raised to the power 3.

Problem 3. Fill in the missing entries of

$$\begin{bmatrix} 2 & * \\ -1 & * \end{bmatrix}$$

if it

a) stretches $[0 \ \ 1]$ by 2 times,

b) has trace 2 and determinant 3,

c) has eigenvalues 1 and 5.

Problem 4. A subspace of which dimension (at most) can you construct with the vectors:

a) $\mathbf{v}_1 = \begin{bmatrix} 6 \\ 1 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$,

b) $\mathbf{v}_1 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$,

c) $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 8 \\ -1 \\ 7 \end{bmatrix}$,

d) $\mathbf{v}_1 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ -4 \\ 4 \end{bmatrix}$.

Problem 5. Find the eigenvalues and eigenvectors of the following matrices:

a) $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$,

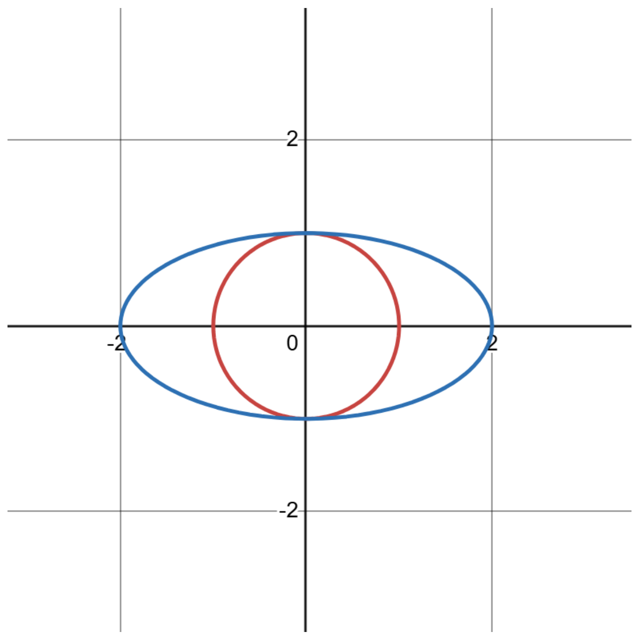
b) $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$,

c) $C = \begin{bmatrix} 8 & 1 \\ 8 & 1 \end{bmatrix}$,

d) $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

You may use any of the techniques (characteristic polynomial / trace and determinant / columns-basis relationship) but try to also think of the problem visually.

Problem 6. If you take the unit circle $x^2 + y^2 = 1$ (i.e. the circle with center at $(0, 0)$ and radius 1) and stretch it out 2 times along the x-axis, you will get an *ellipse*. What is its area equal to?



(Additional:) Can you generalize the result to derive a formula for the area of ellipse?

Problem 7. It is known that the matrix

$$A = \begin{bmatrix} 4 & -8 \\ 1 & 2 \end{bmatrix}$$

sends a certain vector $\mathbf{v}$ to

a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$,

b) $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

Can you find the vector $\mathbf{v}$?

Problem 8 (additional). Consider the 5×5 matrix

$$A = \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & 0 & a_5 \end{bmatrix}$$

where $a_1, \ldots, a_5$ are some fixed positive numbers.

a) What is the standard basis of $\mathbb{R}^5$?

b) Where does A send the standard basis vectors of $\mathbb{R}^5$?

c) What do you think is the determinant of A?

d) What do you think are the eigenvalues and eigenvectors of A?