

Expected Value, Variance, Distributions

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Expected Value



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Would you play this game?

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Would you play this game? What if instead of \$36, you won \$150 if it fell on 8?

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Since the chance of winning is only $\frac{1}{38}$, if you play it a couple of thousands times (say 38000), then you can expect to win about ~ 1000 times and lose ~ 37000 times. Your net revenue would then be:

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These examples motivate the notion of the **mean** or **expected value** of a random variable.

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In words, the expected value is the weighted average of all its possible values, each of the values being weighted by its probability.

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If X is a random variable and $g(x)$ is a continuous function, then $g(X)$ is a random variable too.

Theorem

If X is a continuous random variable, then for any continuous function g ,

$$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f_X(x) dx$$

If X is a discrete random variable, then for any continuous function g ,

$$\mathbb{E}(g(X)) = \sum_{x_i} g(x_i) \cdot \mathbb{P}(X = x_i)$$

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Theorem

If X and Y are independent random variables, then

$$\mathbb{E}(XY) = \mathbb{E}(X) \cdot \mathbb{E}(Y)$$

The converse is not necessarily true.

Now assume you are offered to play one of these two games:

- You toss a coin and win \$1 if it is Heads, otherwise you lose \$1,
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In this case, we say that the winnings of the second game have a **higher variance** than those of the first one.

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The standard deviation shows how much, in average, do the values of the random variable deviate from their average ($\mathbb{E}(X)$).

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To calculate $\text{Var}(X)$, we need $\mathbb{E}(X^2)$:

$$\mathbb{E}(X^2) = 1 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 9 \cdot \frac{1}{6} + 16 \cdot \frac{1}{6} + 25 \cdot \frac{1}{6} + 36 \cdot \frac{1}{6} = \frac{91}{6}$$

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$$\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} \approx 2.92$$

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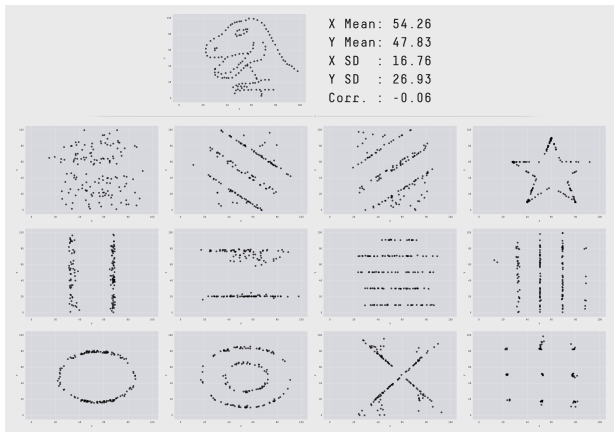
$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$$

Why do you think the 4th point makes sense?

Variance

Warning

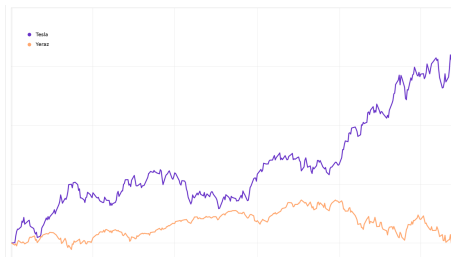
Expected value and variance are very useful to describe random variables, **but they are not everything!** They do not replace CDF/PDF/PMF!



[\[source\]](#)

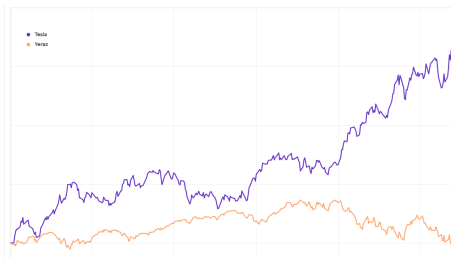
Covariance

Suppose X is the stock price of Tesla, Y is the stock price of Yeraz, and you have some Yeraz stocks.



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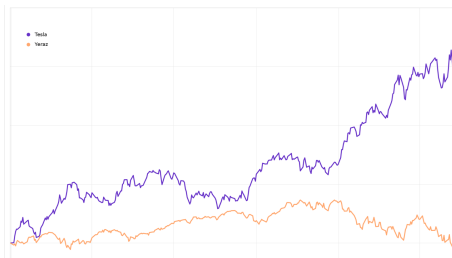
Due to some reasons, the stock of Tesla starts to decrease. Naturally, you are interested in how could that affect your Yeraz stocks, i.e.

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How would a change of X affect Y ?

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More specifically, if X goes up by 1 unit, how much would Y change?

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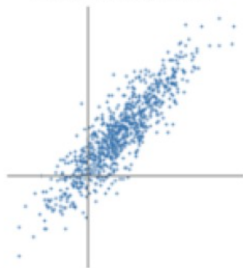
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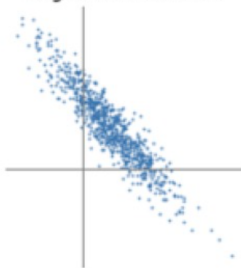
Covariance shows how much the linear growth of one RV is related to the linear growth of the other RV. It is very similar to the concept of dot product of the two vectors.

Covariance

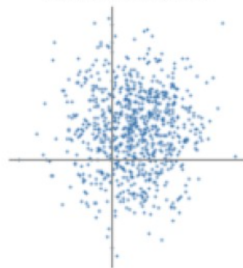
Positive covariance



Negative covariance



Weak covariance



Properties

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$\text{Cov}(X, Y)$ can be *any* number (positive/negative, large/small, zero, etc). What if we want a normalized, universal method to measure the relatedness level of two random variables?

Definition

For two non-constant ($\text{Var}(X), \text{Var}(Y) \neq 0$) random variables X and Y , the **correlation** (or **Pearson correlation coefficient**) between them is defined as:

$$\rho(X, Y) = \text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \cdot \text{Var}(Y)}}$$

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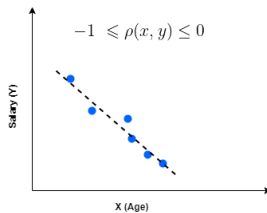
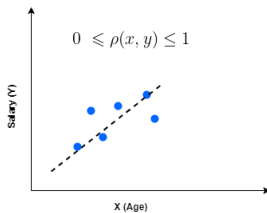
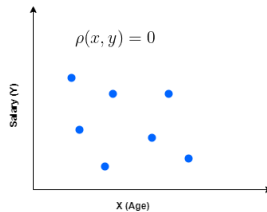
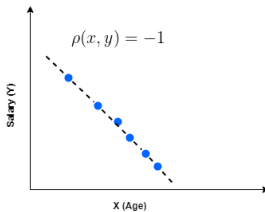
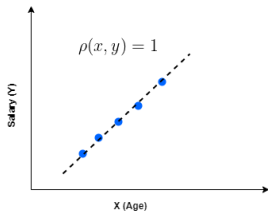
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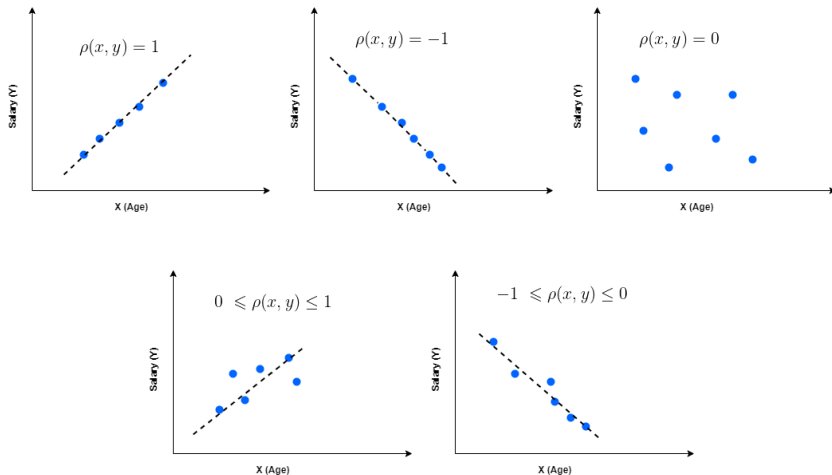
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- 2 $-1 \leq \rho(X, Y) \leq 1$,
- 3 $Y = aX + b$ for some constants a, b if and only if $\rho(X, Y) = \pm 1$.

Correlation



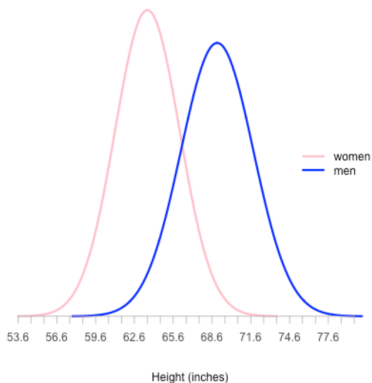
Correlation



- Play with this correlation visualization!

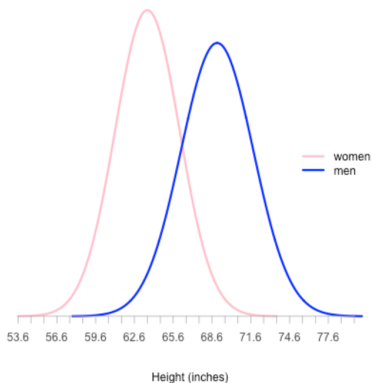
Distributions

Very often in practice, many random variables share similar properties. In particular, the probabilities of their values seem to follow a common pattern, i.e. their CDFs (or PMFs/PDFs) are similar to each other:



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The way the values of an RV are distributed is called a *distribution*.

Bernoulli Distribution

Consider these situations:

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Definition

A random variable X is said to be a **Bernoulli** random variable with parameter p , denoted by $X \sim \text{Bernoulli}(p)$, if it only takes two values and its PMF is given by:

$$\mathbb{P}(X = x) = \begin{cases} p & \text{for } x = 1 \\ 1 - p & \text{for } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

where $0 < p < 1$.

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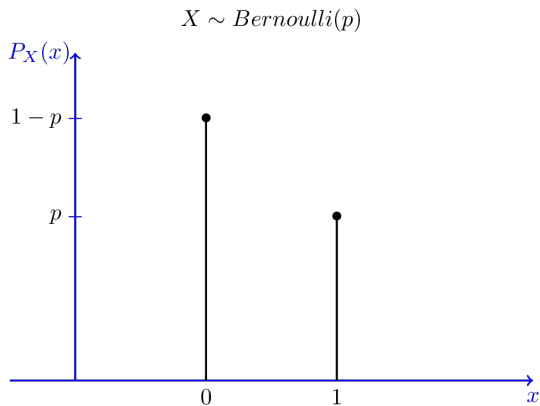
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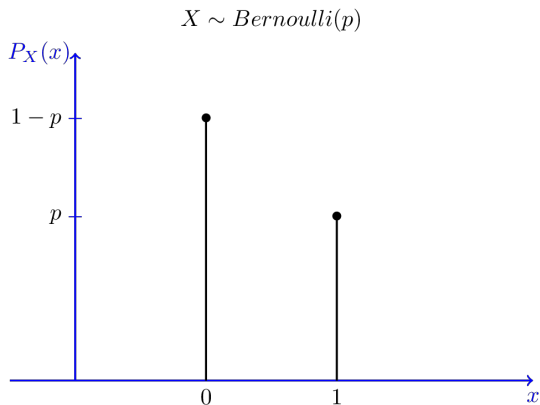
where $0 < p < 1$.

A series of n independent experiments all following Bernoulli distribution $\text{Bernoulli}(p)$, is called **Bernoulli trials**.

Bernoulli Distribution



Bernoulli Distribution



If $X \sim \text{Bernoulli}(p)$,

$$\mathbb{E}(X) = p, \quad \text{Var}(X) = p(1 - p)$$

Geometric Distribution

Suppose we have a coin with $\mathbb{P}(\text{Heads}) = p$ and we toss it until we observe the first Heads, after which we stop. We define X as the total number of coin tosses in this experiment. Then X is said to have *geometric distribution* with parameter p .

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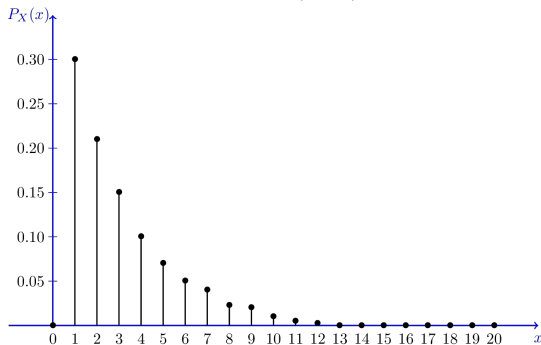
A random variable X is said to be a **geometric** random variable with parameter p , denoted by $X \sim \text{Geo}(p)$, if its PMF is given by:

$$\mathbb{P}(X = k) = \begin{cases} (1 - p)^{k-1}p & \text{for } k = 1, 2, 3, \dots \\ 0 & \text{otherwise} \end{cases}$$

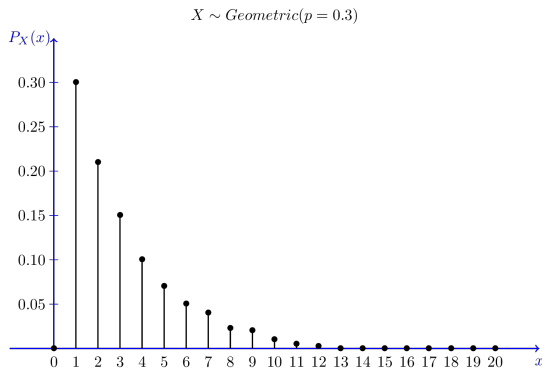
where $0 < p < 1$.

Geometric Distribution

$$X \sim \text{Geometric}(p = 0.3)$$



Geometric Distribution



If $X \sim \text{Geo}(p)$,

$$\mathbb{E}(X) = \frac{1}{p}, \quad \text{Var}(X) = \frac{1-p}{p^2}$$

Binomial Distribution

Suppose we have a coin with $\mathbb{P}(\text{Heads}) = p$ and we toss it n times. We define X to be the total number of Heads observed. Then X is said to have *binomial distribution* with parameter n and p .

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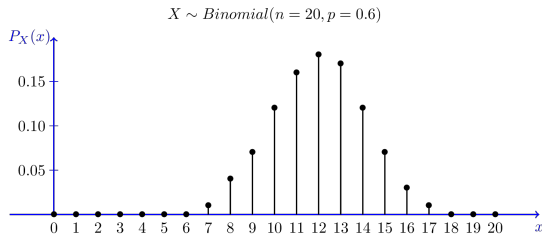
Definition

A random variable X is said to be a **binomial** random variable with parameters n and p , denoted by $X \sim B(n, p)$, if its PMF is given by:

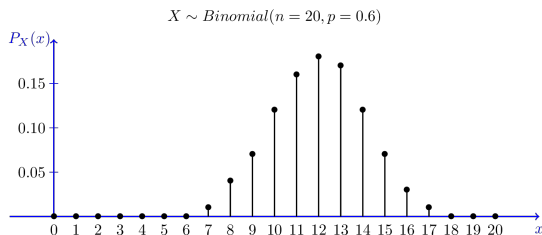
$$\mathbb{P}(X = k) = \begin{cases} C_n^k p^k (1 - p)^{n-k} & \text{for } k = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

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Binomial Distribution



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If $X \sim B(n, p)$,

$$\mathbb{E}(X) = np, \quad \text{Var}(X) = np(1 - p)$$

Poisson Distribution

The Poisson distribution is one of the most widely used probability distributions. It is usually used in scenarios where we are counting the occurrences of certain events in an interval of time or space. In practice, it is often an approximation of a real-life random variable.

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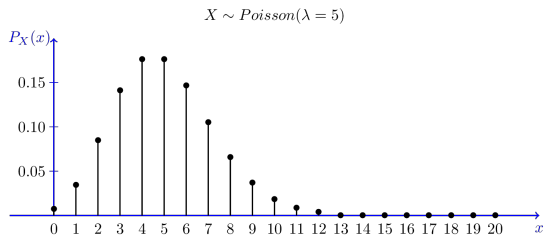
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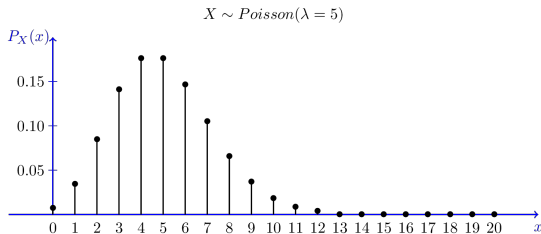
A random variable X is said to be a **Poisson** random variable with parameter λ , denoted by $X \sim \text{Poisson}(\lambda)$, if its PMF is given by:

$$\mathbb{P}(X = k) = \begin{cases} \frac{\lambda^k}{k!} e^{-\lambda} & \text{for } k = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

Poisson Distribution



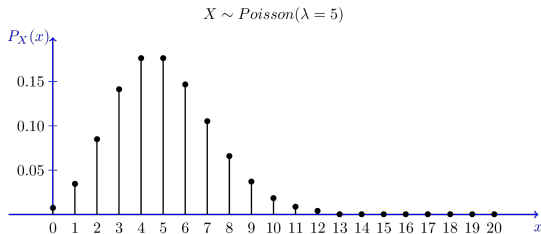
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Remark

If $X \sim B(n, p)$, then

$$\mathbb{P}(X = k) \approx \frac{\lambda^k}{k!} e^{-\lambda}, \quad \text{where } \lambda = np$$

Uniform Distribution

So far, we have considered only discrete RVs. Let's observe some common distributions for continuous RVs.

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A random variable X is said to be a **uniform** random variable over the interval $[a, b]$, denoted by $X \sim U(a, b)$, if its PDF is given by:

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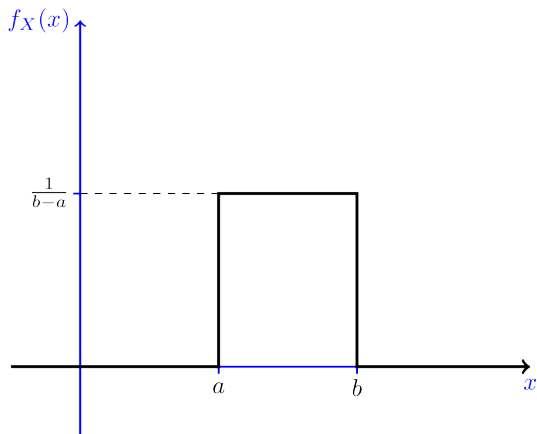
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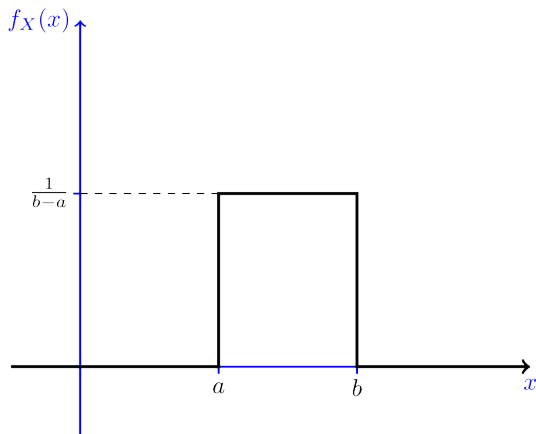
$$f(x) = \begin{cases} \frac{1}{b-a} & x \in (a, b) \\ 0 & x \notin (a, b) \end{cases}$$

$$F(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x \geq b \end{cases}$$

Uniform Distribution



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If $X \sim U(a, b)$,

$$\mathbb{E}(X) = \frac{a+b}{2}, \quad \text{Var}(X) = \frac{(b-a)^2}{12}$$

Exponential Distribution

The *exponential* distribution is the continuous analog of the geometric distribution. It is one of the widely used continuous distributions, and is often used to model the time elapsed between events.

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A random variable X is said to be a **exponential** random variable with parameter $\lambda > 0$, denoted by $X \sim \text{Exp}(\lambda)$, if its PDF is given by:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

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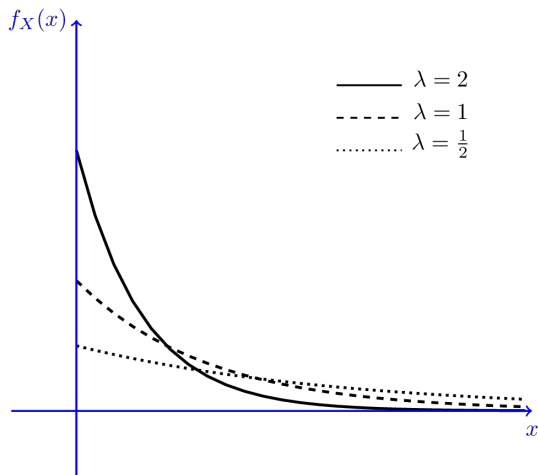
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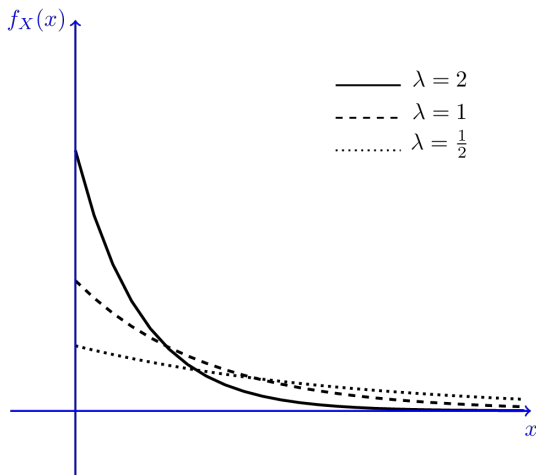
If $X \sim \text{Exp}(\lambda)$, then X is a **memoryless** random variable, that is

$$\mathbb{P}(X > x + a \mid X > a) = \mathbb{P}(X > x), \quad \text{for } a, x \geq 0.$$

Exponential Distribution



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If $X \sim \text{Exp}(\lambda)$,

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Normal Distribution

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Definition

A random variable X is said to be a **normal** random variable with mean μ and variance σ^2 , denoted by $X \sim N(\mu, \sigma^2)$, if its PDF is given by:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}$$

Normal Distribution

