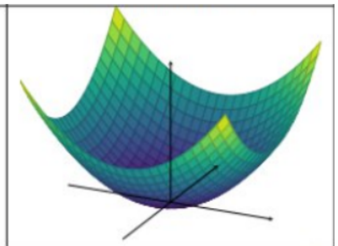
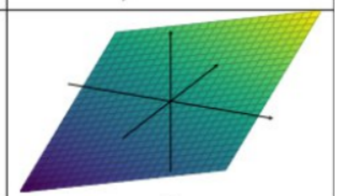
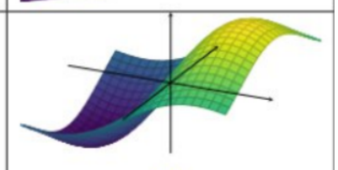
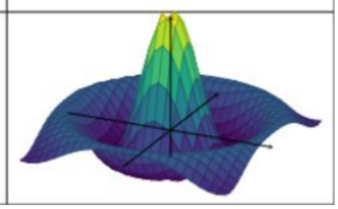
Homework Problems 4 (Functions of Several Variables; Probability)

Problem 1. Match the graphs 1-4 to functions i-iv:

i)	$f(x,y) = 2x + 3y$	1)	
ii)	$f(x,y) = x^2 + y^2$	2)	
iii)	$f(x,y) = \frac{sin\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}}$	3)	
iv)	$f(x,y) = xe^{(-x^2-y^2)}$	4)	

Problem 2. Find the partial derivatives:

a) $f(x,y) = 3x - 2y^2$

b) $f(x,y) = y^7 - 2x^3 + x^2$

c) $f(x,y) = \sin xy$

d) $f(x,y) = x \cdot \ln y + \dfrac{x}{y}$

Problem 3. Compute the directional derivative at the point $(-1,-1)$ along the vector $\mathbf{v} = [0.6,\ 0.8]$:

a) $f(x,y) = 3xy$

b) $f(x,y) = e^{x-y}$

Problem 4. In Lake Sevan, the depth of water at the point with coordinates (x, y) is

$$xy^2 - 6x^2 - 3y^2$$

meters. As the captain of the ship "Noratus" (which is currently at the point $(5, 3)$) wants to get to a deeper part of the lake, his first mate suggests to sail north, while the second mate recommends sailing south. Which mate should the captain listen to?

Problem 5. Does the following function have local extrema? If so, find them:

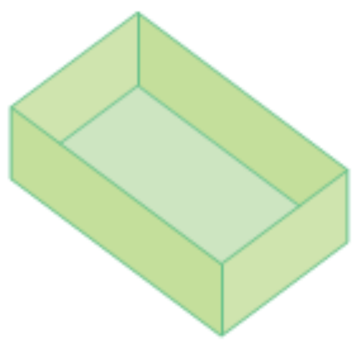
a) $f(x, y) = 3xy$

b) $f(x, y) = x^2 - xy$

c) $f(x, y) = 2x^2 - x^3 - y^2$

You can plot the graph or use the D on the last slide.

Problem 6 (additional). You have 12 square meters of cardboard (as well as scissors, and glue) and this time you want to make a topless box (i.e. no upper side) like this:



What is the maximum volume your box can have?

Hint: This time we have no squares, so you can denote its height, length and width by x, y and z. Can you express z by x and y? Can you then express the volume by x and y? How do we find the maximum value of a function of two variables?

Problem 7 (additional). There are a couple of ways to make one new function from two given functions. One of them is *convolution* which is an important technique in ML.

For any two functions f and g, their convolution is a new function $f * g$, which is defined by the formula:

$$(f * g)(x) = \int_{-1}^{1} f(y)g(x - y)\, dy \qquad \text{(where } x \text{ is fixed)}$$

Given $f(x) = x^2$ and

$$g(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

find the value of $f * g$ at the point $x = 0$.

Hint: Plug in the formula for g, then divide $[-1, 1]$ into two parts where $g = 0$ and where $g = 1$.

Problem 8. Suppose we roll two fair dice. What is the probability of getting

a) 2 on each of them,

b) at least one 1,

c) exactly one 1,

d) one 1 and one 4,

e) 1 on the first die and 4 on the second die?

Problem 9. There are 2 red, 5 blue and 6 yellow pencils in the box. We take two of them out randomly. What is the probability that both of them are

a) red,

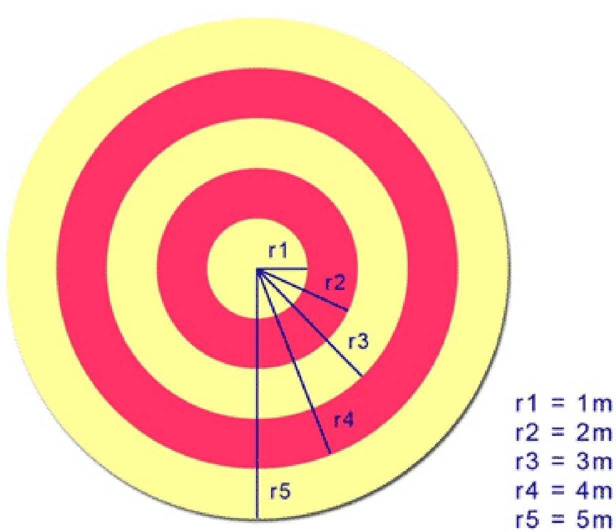
b) of the same color,

c) of different colors,

d) not yellow,

e) not green.

Problem 10.



Assume that any thrown dart will land somewhere within the circular area. The radius of circle 1 (the inner-most yellow circle) is 1 meter. Each radius thereafter increases by 1 m, as shown. We throw a dart randomly. Find the probability that the circle it lands on

a) is the circle 1,

b) is a red circle,

c) is a yellow circle.

Problem 11. A fair coin is tossed 5 times. What is the probability of getting an odd number of heads?

Problem 12. Two fair dice are rolled. What is the probability of getting 1 on at least one of them, given that we know their sum is even?

Problem 13 (additional). 3 cards are drawn from a deck of 52 cards. What is the probability that the first two cards are queens, and the third one is diamonds ♦?

Problem 14 (additional). There are 15 books on a bookshelf, 5 in Armenian, 10 in French. Ruben cannot read French. If he randomly takes 3 books, what is the probability that he can read at least one of them?