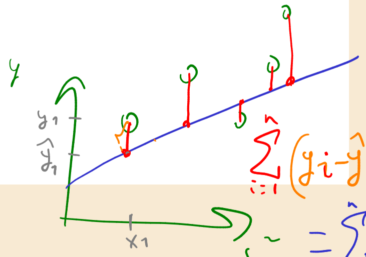
Problem 3. Find the gradient and the Hessian of the following functions:

a) $f(x,y) = 6x - y^2$,

b) $f(x,y) = x^2y^2 - 4xy + 1$,

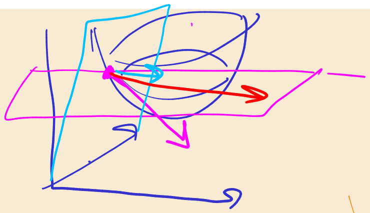
c) $f(x,y) = e^{\pi x} - \sin(\pi y) - \pi xy$



$$\sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} (y_i - kx_i - b)^2$$

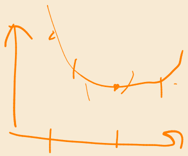
$\hat{y} = kx + b$

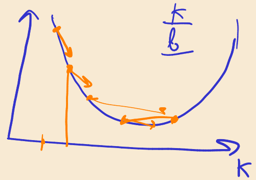
$$f(x,y,z) \quad \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix}$$



xy

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \boxed{2xy^2 - 4y} \\ 2x^2y - 4x \end{bmatrix}$$





$k = 3$

$$\theta^{[t]} = \theta^{[t-1]} - \alpha \frac{\partial f}{\partial \theta}$$

Gradient Descent

$10^2 \quad 10 \times 10$

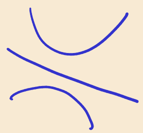
Taylor 2-

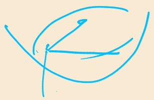
$\nabla^2 f(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} =$

∇ - nabla

$n \times n$

$f''(x) > 0$, < 0





Problem 5. Does the following function have local extrema? If so, find them:

a) $f(x,y) = 3xy$

b) $f(x,y) = x^2 - xy$

c) $f(x,y) = 2x^2 - x^3 - y^2$

You can plot the graph or use the D on the last slide.

For the functions of one variable we looked at the sign of f''.
In case of two variables, we look at:

- f_{xx}
- and $D = f_{xx}f_{yy} - f_{xy}^2$

$$\begin{matrix} \boxed{f_{xx}} & f_{xy} \\ f_{yx} & f_{yy} \end{matrix}$$

Theorem

If at some point (a,b)

- $D > 0$ and $f_{xx} > 0$ $\Rightarrow$ local minimum
- $D > 0$ and $f_{xx} < 0$ $\Rightarrow$ local maximum
- $D < 0$ $\Rightarrow$ saddle point

$x^2 - xy$

$$\nabla f(x,y) = \begin{bmatrix} 2x - y \\ -x \end{bmatrix}$$

$$\begin{bmatrix} 2x - y \\ -x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad x = 0 \Rightarrow y = 0$$

$$\nabla^2 f(x,y) = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$$

$|\nabla^2 f(x,y)| = 2 \cdot 0 - (-1)\cdot(-1) = -1$

$f_{xx} = 2$

Problem 4. Consider the bivariate function $f : \mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto x^2 + 0.5y^2 + xy$.

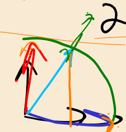
a) Find the direction of greatest increase of f at $(x,y) = (1,1)$.

b) Find the direction of greatest decrease of f at $(x,y) = (1,1)$.

c) Calculate the directional derivative at the point $(x,y) = (1,1)$ along the vector $\mathbf{v} = [0.6, 0.8]^T$

d) Find a direction in which f does not instantly change at $(x,y) = (1,1)$.

$x^2 + \frac{y^2}{2} + xy$



$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

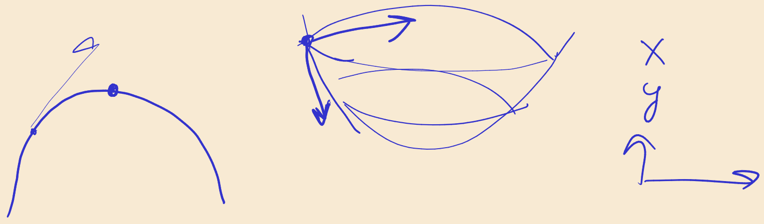
$$\begin{bmatrix} f_x \\ f_y \end{bmatrix} \cdot \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

$$\begin{bmatrix} \frac{v_x}{\|v\|_2} \\ \frac{v_y}{\|v\|_2} \end{bmatrix}$$

$\nabla f \cdot v = 0$

$$\frac{f_x}{\sqrt{5}} + \frac{2f_y}{\sqrt{5}}$$

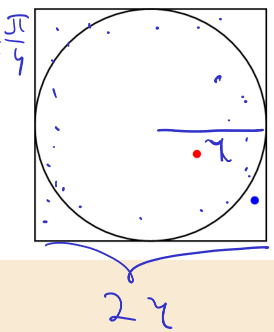
$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} / \sqrt{2^2 + 1^2}$$



Problem 5. What is the probability that a randomly generated point withing the square will lay in the circle:

1 $\quad \frac{\bigcirc}{\square} = \frac{3135}{4000} \approx \frac{\pi}{4}$

$\times 4 \to \approx \pi$



$$\frac{\pi r^2}{4r^2} = \frac{\pi}{4}$$

Problem 6. Bertrand's paradox

Problem 7. A fair coin is tossed 3 times. What is the probability of getting at least one tail?

Problem 8. An urn contains 3 red balls and 5 blue balls. Two balls are drawn at random without replacement. What is the probability that both balls are red?

Problem 9. A standard deck of 52 playing cards is shuffled. What is the probability that the top card is a heart or a queen?

H H H

$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$

$\boxed{0.5}$ H, T

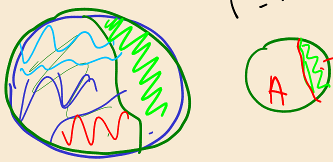
$P(X_1 = H, X_2 = H, X_3 = H) = \prod_{i=1}^{3} P(X_i = H)$

H T H

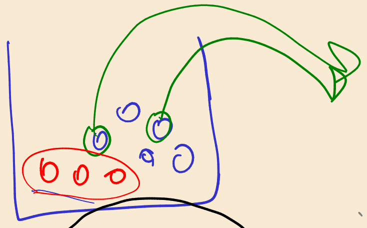
T H H

$1 - \frac{1}{8} = \frac{7}{8}$

1, 2, 3



A^{c} = complement



$\frac{3}{8}\cdot\frac{2}{7}$

$C_8^2 = \frac{8\cdot 7}{2} = \frac{8!}{2!\cdot(8-2)!}$

$C_3^2 = \frac{3\cdot 2}{2}$

$\frac{3\cdot 2}{\frac{8\cdot 7}{2}}$

$\frac{\#}{\#} = \frac{3}{8}$

$P(1\cup 2) \stackrel{?}{=} P(1)\cdot P(2)$

$$52 \qquad \overbrace{1, 2, 3, \dots, Q, K, A}^{13} \qquad 4$$

$\boxed{♥}$ یا $\boxed{Q}$

$$♥ \quad \frac{13}{52} = \frac{1}{4}$$

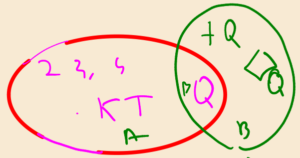
$$Q \quad \frac{4}{52} = \frac{1}{13}$$

$$♥Q = \frac{1}{4} \cdot \frac{1}{13} = \frac{1}{52}$$

$$♥ \text{ یا } Q = \frac{1}{4} + \frac{1}{13} = \frac{4+13}{52} = \frac{17}{52}$$

♥Q, Q, ♦Q + Q

♥2, ♥3, ..., ♥Q

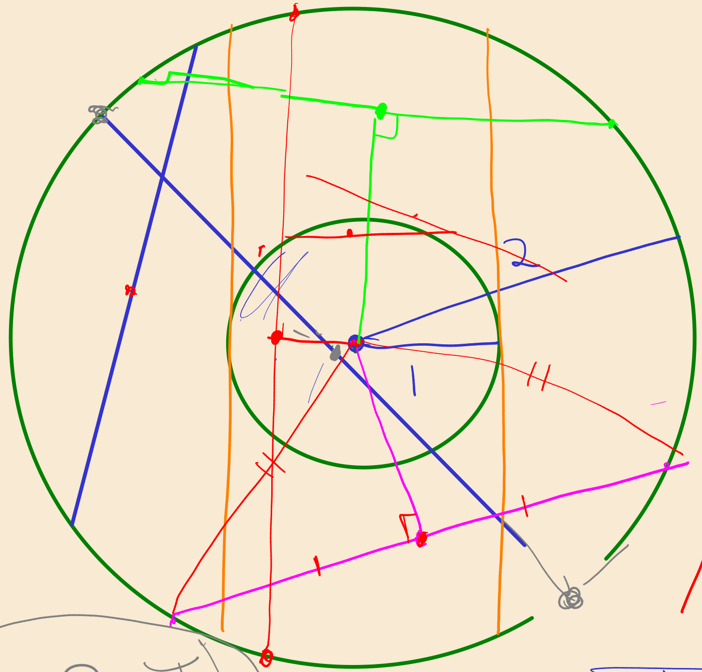


$$P(A \cup B) = \underbrace{P(A) \cup P(B)}_{\text{یا}} - P(A \cap B)$$

$$P(♥ \cup Q) =$$

$$= P(♥) + P(Q) - \underbrace{P(♥ \cup Q)}$$

$$\frac{1}{4} + \frac{1}{13} - \frac{1}{52}$$



$\frac{1}{2}$

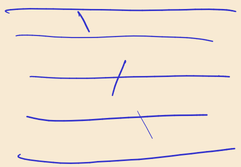
1

$$\frac{P(o)}{P(O)} = \frac{\pi r^2}{\pi 2^2} = \frac{1}{4}$$

Bezh

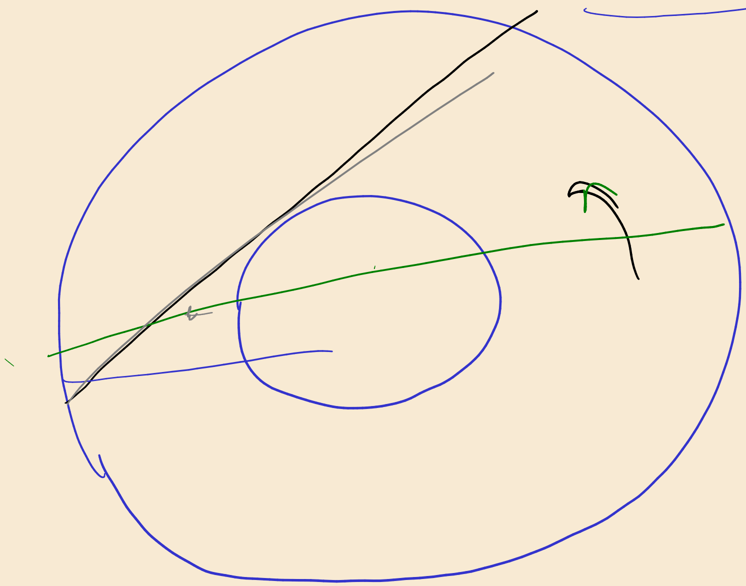
$1 \rightarrow$

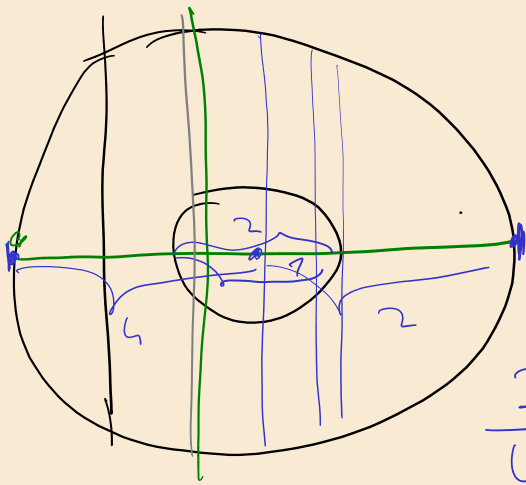
$(P, \Omega, \mathcal{F})$



$\frac{1}{3}$

Buffon's needle





$\frac{2}{4} = \frac{1}{2}$

I - [illegible]

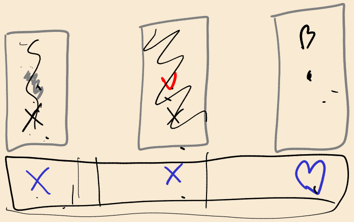
II - [illegible]

updating belief

X 80%

$P(A|B) =$ $P($[illegible]$) =$

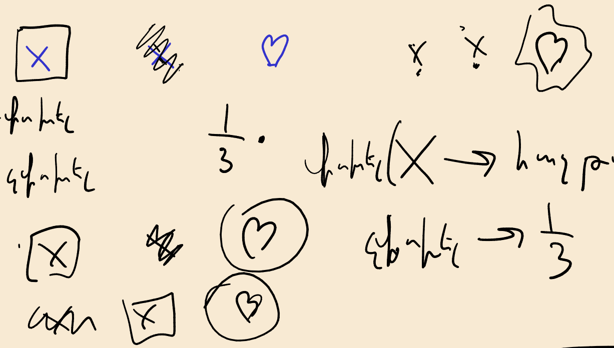
II [illegible] $\frac{1}{2}$ $\frac{1}{2}$ $= P($[illegible]$|$[illegible]



$\frac{1}{3} \rightarrow \frac{1}{2}$

$\frac{1}{2}$

X



I - [illegible]

II - [illegible]

$\frac{1}{3} \cdot$ [illegible] (X → [illegible])

[illegible] $\rightarrow \frac{1}{3}$

2 - [illegible]

(B)

1 - [illegible]

$\frac{2}{3}$

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