

Problem 3. Find the gradient and the Hessian of the following functions:

- a) $f(x, y) = 6x - y^2$,
 b) $f(x, y) = x^2y^2 - 4xy + 1$,
 c) $f(x, y) = e^{\pi x} - \sin(\pi y) - \pi xy$

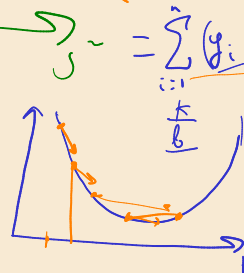
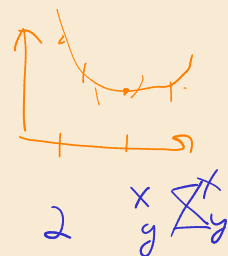
$$f(x, y, z) \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix}$$

xy

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xy^2 - 4y \\ 2x^2y - 4x \end{bmatrix}$$

$$\nabla f(x, y) = \begin{bmatrix} f'_{xx} & f'_{xy} \\ f'_{yx} & f'_{yy} \end{bmatrix} =$$

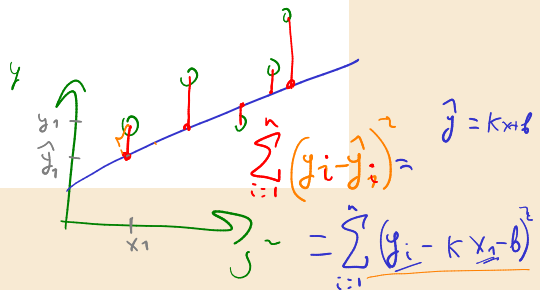
∇ -matrix $n \times n$



$$10^2 \text{ } 10 \times 10 \text{ } \text{Taylor 2-}$$

$$f''(x) > 0$$

$$f''(x) < 0$$



Problem 5. Does the following function have local extrema? If so, find them:

- a) $f(x, y) = 3xy$
 b) $f(x, y) = x^2 - xy$
 c) $f(x, y) = 2x^2 - x^3 - y^2$

You can plot the graph or use the D on the last slide.

For the functions of one variable we looked at the sign of f'' .
 In case of two variables, we look at:

$$D = f_{xx}f_{yy} - f_{xy}^2$$

Theorem		
If at some point (a, b)		
$D > 0$ and $f_{xx} > 0$	\Rightarrow	local minimum
$D > 0$ and $f_{xx} < 0$	\Rightarrow	local maximum
$D \leq 0$	\Rightarrow	saddle point

$$\nabla f(x, y) = \begin{bmatrix} 2x - y \\ -x \end{bmatrix}$$

$$\nabla^2 f(x, y) = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$$

Problem 4. Consider the bivariate function $f : \mathbb{R}^2 \rightarrow \mathbb{R}, (x, y) \mapsto x^2 + 0.5y^2 + xy$.

- a) Find the direction of greatest increase of f at $(x, y) = (1, 1)$.
 b) Find the direction of greatest decrease of f at $(x, y) = (1, 1)$.
 c) Calculate the directional derivative at the point $(x, y) = (1, 1)$ along the vector $\mathbf{v} = [0.6, 0.8]^T$.
 d) Find a direction in which f does not instantly change at $(x, y) = (1, 1)$.

$$x^2 + \frac{y^2}{2} + xy$$

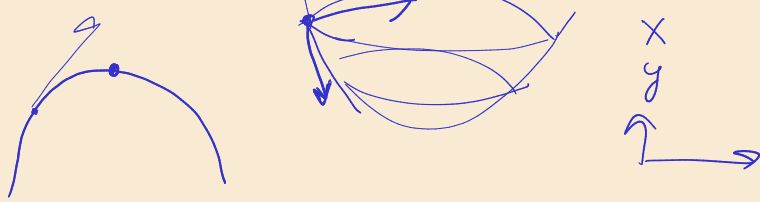
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{v_x}{\|v\|_2} \\ \frac{v_y}{\|v\|_2} \end{bmatrix}$$

$$\nabla f \cdot \mathbf{v} = 0$$

$$\frac{f_x}{\sqrt{5}} + 2 \frac{f_y}{\sqrt{5}}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} / \sqrt{2^2 + 1^2}$$



Problem 5. What is the probability that a randomly generated point within the square will lay in the circle:

$$1 - \frac{\pi}{4} = \frac{3135}{4000} \approx \frac{\pi}{4}$$

$x, y \rightarrow \approx \pi$
 $\sqrt{4} \approx 2$
 $2r$

$$\frac{\pi r^2}{4r^2} = \frac{\pi}{4}$$



0 ≠ null

Problem 6. Bertrand's paradox

Problem 7. A fair coin is tossed 3 times. What is the probability of getting at least one tail?

Problem 8. An urn contains 3 red balls and 5 blue balls. Two balls are drawn at random without replacement. What is the probability that both balls are red?

Problem 9. A standard deck of 52 playing cards is shuffled. What is the probability that the top card is a heart or a queen?



0.5 H, T

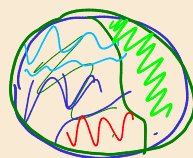
H T H

T H H

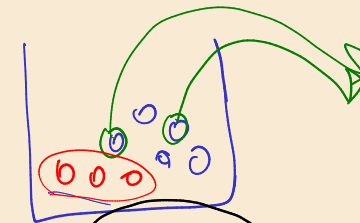
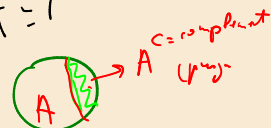
$$1 - \frac{1}{8} = \frac{7}{8}$$

$$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \text{ func. q.}$$

1, 2, 3



$$P(X_1=H, X_2=H, X_3=H) = \prod_{i=1}^3 P(X_i=H)$$



$$\frac{3}{8} \cdot \frac{2}{7}$$

$$\frac{\# \text{ hearts} - 1}{\# \text{ cards}} = \frac{3}{8}$$

$$P(1 \cup 2) = P(1) \cdot P(2)$$

$$C_8^2 = \frac{8 \cdot 7}{2} = \frac{8!}{2! \cdot (8-2)!}$$

$$C_3^2 = \frac{3 \cdot 2}{2} = \frac{3 \cdot 2}{2 \cdot 1}$$

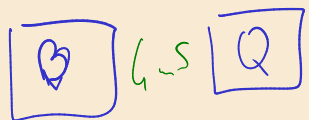
$$\rightarrow P(1 - 1/8, 1/4, 1/8)$$

$$\rightarrow P(2 - 1/4, 1/8, 1/4)$$

52

1, 2, 3, ..., 13, Q, K, A

4



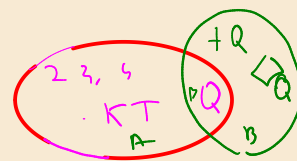
$$\heartsuit \frac{13}{52} = \frac{1}{4}$$

$$Q \frac{4}{52} = \frac{1}{13}$$

$$\heartsuit Q = \frac{1}{4} \cdot \frac{1}{13} = \frac{1}{52}$$

$$\heartsuit \sim Q = \frac{1}{4} + \frac{1}{13} = \frac{4+13}{52} = \frac{17}{52}$$

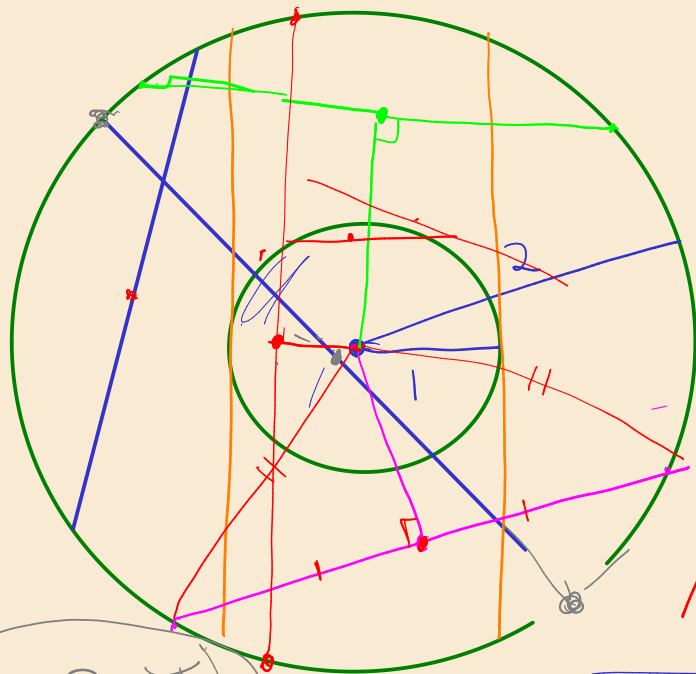
♥Q, Q, ♥Q+Q
♥2, ♥3, ♥4, ♥5, ♥6, ♥7, ♥8, ♥9, ♥10, ♥J, ♥K, ♥A



$$P(\heartsuit \cup Q) = P(\heartsuit) + P(Q) - P(\heartsuit \cap Q)$$

$$= \frac{1}{4} + \frac{1}{13} - \frac{1}{52}$$

$$P(A \cup B) = \frac{P(A) \cup P(B) - P(A \cap B)}{4+13-1}$$



$$\frac{1}{2}$$

$$\frac{P(0)}{\phi(0)} = \frac{\pi 4^2}{\pi 2^2} = \frac{1}{4}$$

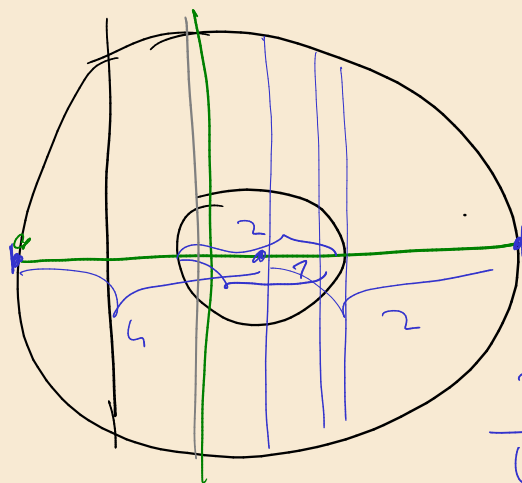
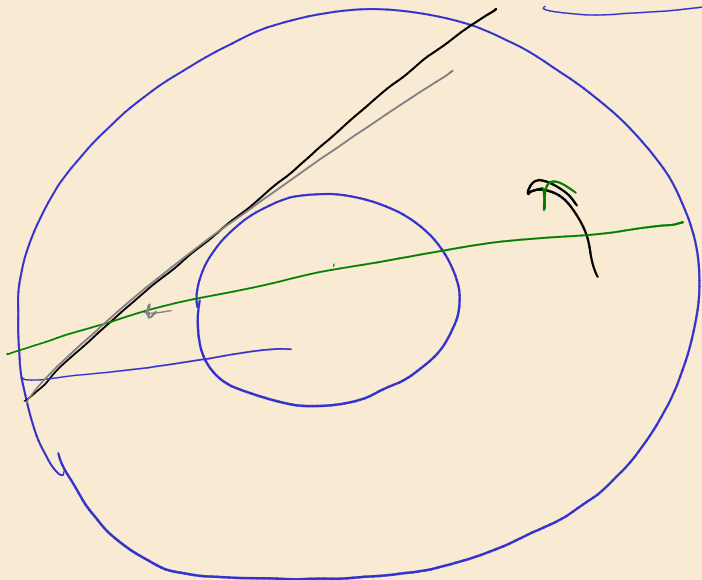
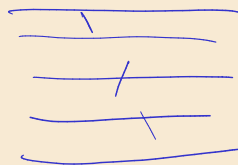
BezL



$$\frac{1}{3}$$

Bu f fano's aralle

(P, Ω, F)



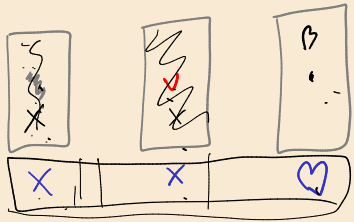
$$\frac{2}{4} = \frac{1}{2}$$



I - 7m2
 II - 4m2

updating belief
 X 80%

$$P(A|B) = P(2m^2 | 1m^2) = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{2}} = \frac{1}{2}$$

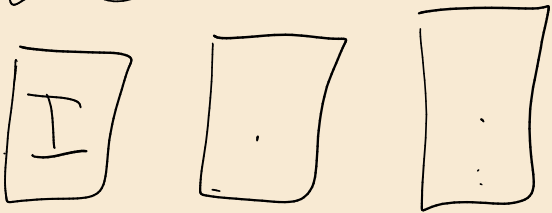


$$\frac{1}{3} \rightarrow \frac{1}{2}$$



I - 4m2
 II - 4m2

$\frac{1}{3} \cdot \text{believe}(X \rightarrow \text{long pants})$



21

X
 ya
 (2)
 (X)

$$\frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

$$(B) \quad 1 - \frac{1}{3} = \frac{2}{3}$$

$$\frac{2}{3}$$