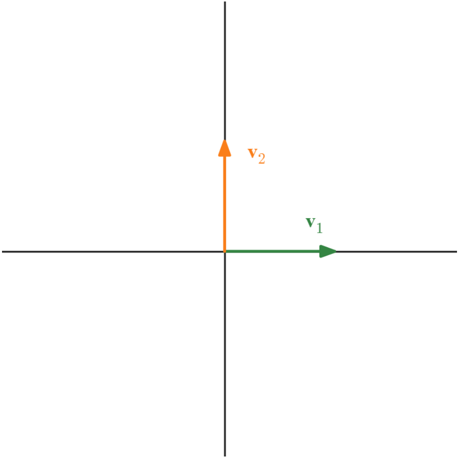
Basis, Eigenvalues and Eigenvectors

Hayk Aprikyan, Hayk Tarkhanyan

March 28, 2025

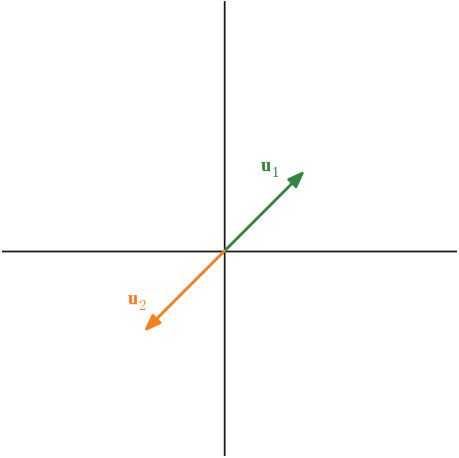
Motivation

When talking about vectors/matrices, why do we focus on these vectors?



Motivation

And not on these:

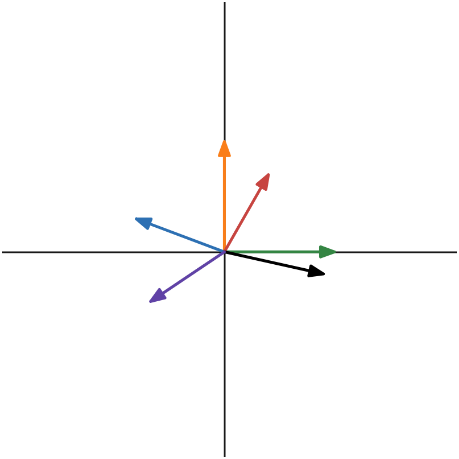


Motivation

Because you can get any vector with $\mathbf{v}_1$ and $\mathbf{v}_2$!

Motivation

Because you can get any vector with $\mathbf{v}_1$ and $\mathbf{v}_2$! Then why not these?



Motivation

Because now they are too much: 2 vectors are enough for $\mathbb{R}^2$.

Motivation

Because now they are too much: 2 vectors are enough for $\mathbb{R}^2$.

Now, let's go step-by-step. For the first two vectors $\mathbf{v}_1 = [1 \quad 0]$ and $\mathbf{v}_2 = [0 \quad 1]$, we can express any vector using only those two.

Motivation

Because now they are too much: 2 vectors are enough for $\mathbb{R}^2$.

Now, let's go step-by-step. For the first two vectors $\mathbf{v}_1 = [1 \quad 0]$ and $\mathbf{v}_2 = [0 \quad 1]$, we can express any vector using only those two.

For example, the vector $[4 \quad 7]$ can be written as:

$$\begin{bmatrix} 4 \\ 7 \end{bmatrix} =$$

Motivation

Because now they are too much: 2 vectors are enough for $\mathbb{R}^2$.

Now, let's go step-by-step. For the first two vectors $\mathbf{v}_1 = [1 \ \ 0]$ and $\mathbf{v}_2 = [0 \ \ 1]$, we can express any vector using only those two.

For example, the vector $[4 \ \ 7]$ can be written as:

$$\begin{bmatrix} 4 \\ 7 \end{bmatrix} = 4 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} +$$

Motivation

Because now they are too much: 2 vectors are enough for $\mathbb{R}^2$.

Now, let's go step-by-step. For the first two vectors $\mathbf{v}_1 = [1 \quad 0]$ and $\mathbf{v}_2 = [0 \quad 1]$, we can express any vector using only those two.

For example, the vector $[4 \quad 7]$ can be written as:

$$\begin{bmatrix} 4 \\ 7 \end{bmatrix} = 4 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 7 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$

Motivation

Because now they are too much: 2 vectors are enough for $\mathbb{R}^2$.

Now, let's go step-by-step. For the first two vectors $\mathbf{v}_1 = [1 \quad 0]$ and $\mathbf{v}_2 = [0 \quad 1]$, we can express any vector using only those two.

For example, the vector $[4 \quad 7]$ can be written as:

$$\begin{bmatrix} 4 \\ 7 \end{bmatrix} = 4 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 7 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 4\mathbf{v}_1 + 7\mathbf{v}_2$$

Motivation

Because now they are too much: 2 vectors are enough for $\mathbb{R}^2$.

Now, let's go step-by-step. For the first two vectors $\mathbf{v}_1 = [1 \ \ 0]$ and $\mathbf{v}_2 = [0 \ \ 1]$, we can express any vector using only those two.

For example, the vector $[4 \ \ 7]$ can be written as:

$$\begin{bmatrix} 4 \\ 7 \end{bmatrix} = 4 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 7 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 4\mathbf{v}_1 + 7\mathbf{v}_2$$

We call expressions like these:

$$\text{something} \cdot \mathbf{v}_1 + \text{something} \cdot \mathbf{v}_2$$

the **linear combinations** of $\mathbf{v}_1$ and $\mathbf{v}_2$.

In our case, the vector $[4 \ \ 7]$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Span

More generally,

Definition

For any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and for any scalars $c_1, c_2, \ldots, c_k$, the expression

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$$

is called a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

Span

More generally,

Definition

For any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and for any scalars $c_1, c_2, \ldots, c_k$, the expression

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$$

is called a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

So in this sense, all vectors of $\mathbb{R}^2$ can be written as linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$! In this case, we say that $\mathbb{R}^2$ is the **span** of $\mathbf{v}_1$ and $\mathbf{v}_2$:

Definition

The set of all possible linear combinations of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is called their **span**, i.e.

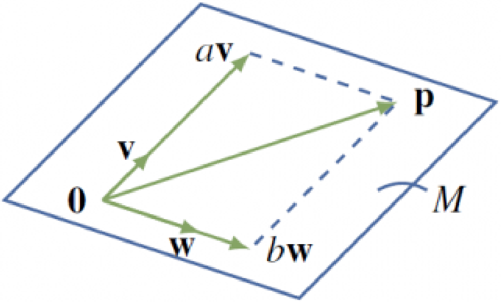
$$\text{span}\{\mathbf{v}_1,\mathbf{v}_2, \ldots, \mathbf{v}_n\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n \mid c_1, c_2, \ldots, c_n \in \mathbb{R}\}.$$

Span

Geometrically, the span represents all the vectors that we can get by adding multiples of the given vectors.

Span

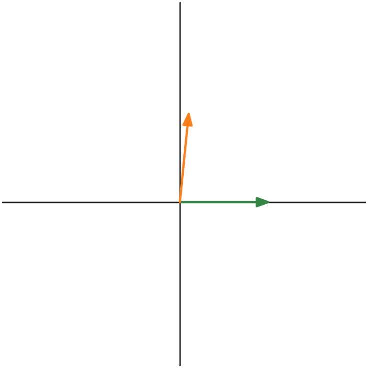
Geometrically, the span represents all the vectors that we can get by adding multiples of the given vectors.



Span

Question

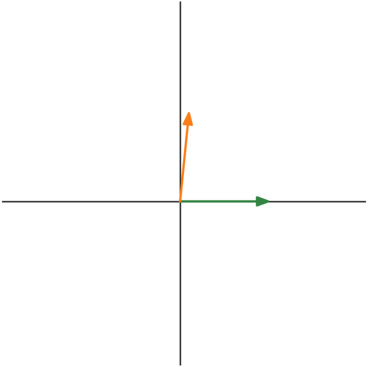
What is the span of vectors $[1 \quad 0]$ and $[0.1 \quad 1]$?



Span

Question

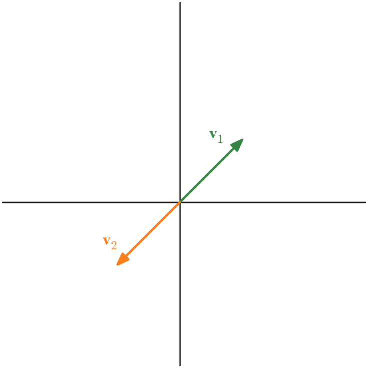
What is the span of vectors $[1 \quad 0]$ and $[0.1 \quad 1]$?



Again, it is the whole $\mathbb{R}^2$: We can express any vector using these two.

Question

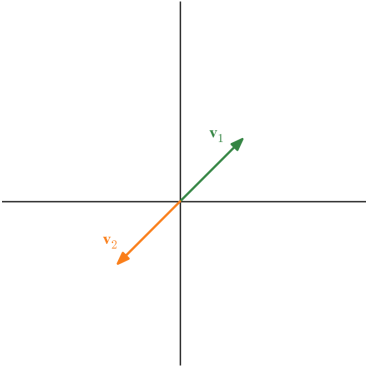
What is the span of vectors $[1 \quad 1]$ and $[-1 \quad -1]$?



Span

Question

What is the span of vectors $[1 \quad 1]$ and $[-1 \quad -1]$?

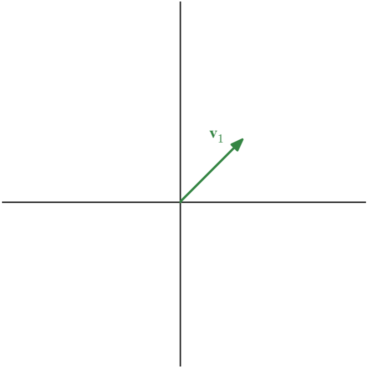


Since they both lie on the line $y = x$, their span is the line $y = x$ itself.

Span

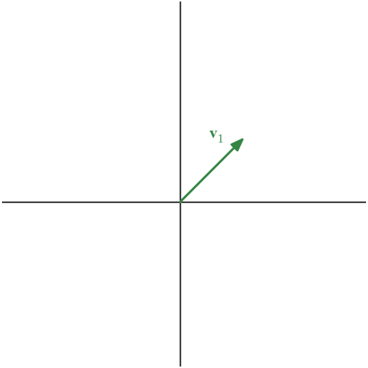
Question

What about the span of the single vector $[1 \quad 1]$?



Question

What about the span of the single vector $[1 \quad 1]$?



Again, the span of $[1 \quad 1]$ is the line $y = x$.

Span

Notice that in all cases so far, the span was either $\mathbb{R}^2$ or some subspace of $\mathbb{R}^2$. Indeed,

Span

Notice that in all cases so far, the span was either $\mathbb{R}^2$ or some subspace of $\mathbb{R}^2$. Indeed,

Theorem

The span of vectors **is a vector space** itself.

Span

Notice that in all cases so far, the span was either $\mathbb{R}^2$ or some subspace of $\mathbb{R}^2$. Indeed,

Theorem

The span of vectors **is a vector space** itself.

But what is the reason that

- in one case (e.g. $\mathbf{v}_1 = [1 \;\; 0]$ and $\mathbf{v}_2 = [0 \;\; 1]$) the span is the whole $\mathbb{R}^2$,
- but in another case (e.g. $\mathbf{u}_1 = [1 \;\; 1]$ and $\mathbf{u}_2 = [-1 \;\; -1]$) the span is only a line?

Span

Notice that in all cases so far, the span was either $\mathbb{R}^2$ or some subspace of $\mathbb{R}^2$. Indeed,

Theorem

The span of vectors **is a vector space** itself.

But what is the reason that

- in one case (e.g. $\mathbf{v}_1 = [1 \;\; 0]$ and $\mathbf{v}_2 = [0 \;\; 1]$) the span is the whole $\mathbb{R}^2$,
- but in another case (e.g. $\mathbf{u}_1 = [1 \;\; 1]$ and $\mathbf{u}_2 = [-1 \;\; -1]$) the span is only a line?

Because in the second case, one of the vectors can be expressed by another!

Linear Independence

Indeed, you can express $\mathbf{u}_2$ with $\mathbf{u}_1$:

$$\mathbf{u}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (-1) \cdot \mathbf{u}_1,$$

but you **cannot express** $\mathbf{v}_2$ with $\mathbf{v}_1$ (or vice versa).

Linear Independence

Indeed, you can express $\mathbf{u}_2$ with $\mathbf{u}_1$:

$$\mathbf{u}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (-1) \cdot \mathbf{u}_1,$$

but you **cannot express** $\mathbf{v}_2$ with $\mathbf{v}_1$ (or vice versa).

In this case, we say that

- the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are **linearly dependent,**
- while the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are **linearly independent**.

Linear Independence

Indeed, you can express $\mathbf{u}_2$ with $\mathbf{u}_1$:

$$\mathbf{u}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (-1) \cdot \mathbf{u}_1,$$

but you **cannot express** $\mathbf{v}_2$ with $\mathbf{v}_1$ (or vice versa).

In this case, we say that

- the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are **linearly dependent,**
- while the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are **linearly independent**.

More generally,

Definition

The vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ are called **linearly independent** if none of them can be written as a linear combination of the others.

Indeed, you can express $\mathbf{u}_2$ with $\mathbf{u}_1$:

$$\mathbf{u}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (-1) \cdot \mathbf{u}_1,$$

but you **cannot express** $\mathbf{v}_2$ with $\mathbf{v}_1$ (or vice versa).

In this case, we say that

- the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are **linearly dependent,**
- while the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are **linearly independent**.

More generally,

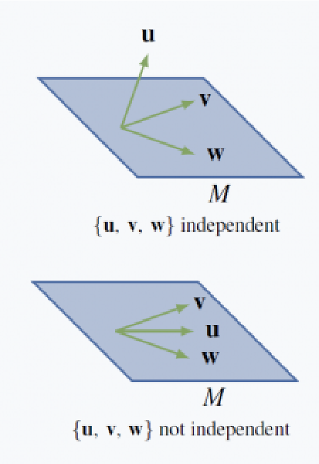
Definition

The vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ are called **linearly independent** if none of them can be written as a linear combination of the others.

And we say that they are linearly dependent if one of them, say $\mathbf{v}_n$, can be written as

$$\mathbf{v}_n = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{n-1}\mathbf{v}_{n-1}$$

Linear Independence



Check these animations:

- www.desmos.com/calculator/9rnbn0ycdd
- www.desmos.com/calculator/aje8cboe0j

Linear Independence

Geometrically,

- Two vectors are linearly dependent if they lie on the same line,
- Three vectors are linearly dependent if they lie on the same plane,

and so on.

Linear Independence

Geometrically,

- Two vectors are linearly dependent if they lie on the same line,
- Three vectors are linearly dependent if they lie on the same plane,

and so on.

There is also another characterization of linear independence (try to prove it by yourself):

Theorem

Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent if and only if the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$$

is *only* true if $c_1 = c_2 = \ldots = c_n = 0$.

(i.e. if you plug in any numbers other than 0, the sum will not be $\mathbf{0}$).

Basis

So now we can say that the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

are linearly independent and their span is $\mathbb{R}^2$.

We call the pairs of vectors like $\mathbf{v}_1$ and $\mathbf{v}_2$ the **basis** of the space $\mathbb{R}^2$.

Definition

The vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are called a **basis** of the vector space V if:

1. $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent,
2. V is equal to the span of $\mathbf{v}_1, \dots, \mathbf{v}_n$.

Basis

So now we can say that the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

are linearly independent and their span is $\mathbb{R}^2$.

We call the pairs of vectors like $\mathbf{v}_1$ and $\mathbf{v}_2$ the **basis** of the space $\mathbb{R}^2$.

Definition

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are called a **basis** of the vector space V if:

1. $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent,
2. V is equal to the span of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

In other words, there are no "irrelevant", "redundant" vectors, and any vector of V can be expressed with $\mathbf{v}_1, \ldots, \mathbf{v}_n$.
(In fact, such representation is always unique, i.e. there is only one way to express $[3 \;\; 4]$ with $\mathbf{v}_1 = [1 \;\; 0]$ and $\mathbf{v}_2 = [0 \;\; 1]$: $\quad 3 \cdot \mathbf{v}_1 + 4 \cdot \mathbf{v}_2$)

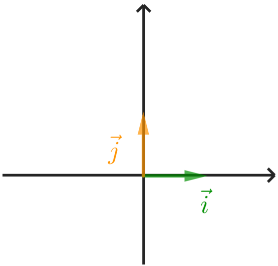
Basis

Example

The vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

form a basis for $\mathbb{R}^2$ and are called the **standard basis**. They are often denoted $\hat{i}, \hat{j}$.

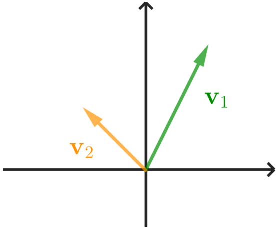


Example

The linearly independent vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

form a basis for $\mathbb{R}^2$ as these vectors are linearly independent and their span is $\mathbb{R}^2$.



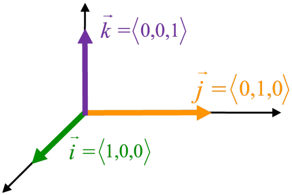
Basis

Example

The vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for $\mathbb{R}^3$ and are called the **standard basis**. They are often denoted $\hat{i}, \hat{j}, \hat{k}$.



Basis

Example (too many vectors)

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Basis

Example (too many vectors)

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

are not a basis for $\mathbb{R}^2$ because they are linearly dependent.

Basis

Example (too many vectors)

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

are not a basis for $\mathbb{R}^2$ because they are linearly dependent.

Example (not enough vectors)

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Basis

Example (too many vectors)

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

are not a basis for $\mathbb{R}^2$ because they are linearly dependent.

Example (not enough vectors)

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

are not a basis for $\mathbb{R}^3$ because their span is not the whole $\mathbb{R}^3$.

Basis

Example (too many vectors)

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

are not a basis for $\mathbb{R}^2$ because they are linearly dependent.

Example (not enough vectors)

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

are not a basis for $\mathbb{R}^3$ because their span is not the whole $\mathbb{R}^3$.

Example (horrible vectors)

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Basis

Example (too many vectors)

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

are not a basis for $\mathbb{R}^2$ because they are linearly dependent.

Example (not enough vectors)

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

are not a basis for $\mathbb{R}^3$ because their span is not the whole $\mathbb{R}^3$.

Example (horrible vectors)

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

are not a basis for $\mathbb{R}^2$ because they are dependent, their span is not $\mathbb{R}^2$.

Basis

So far we have noticed that

- There are many different bases for the same space,

Basis

So far we have noticed that

- There are many different bases for the same space,
- All bases have the same number of vectors,

Basis

So far we have noticed that

- There are many different bases for the same space,
- All bases have the same number of vectors,
- Any n independent vectors are a basis for $\mathbb{R}^n$.

Basis

So far we have noticed that

- There are many different bases for the same space,
- All bases have the same number of vectors,
- Any n independent vectors are a basis for $\mathbb{R}^n$.

The same is true for any vector space.

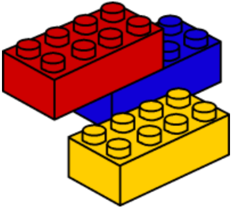
Basis

So far we have noticed that

- There are many different bases for the same space,
- All bases have the same number of vectors,
- Any n independent vectors are a basis for $\mathbb{R}^n$.

The same is true for any vector space.

The basis is our Lego set of "building blocks" out of which we build our castle (i.e. the vector space). Since they all have the same number of vectors, we call that number the **dimension** of the space.



Basis

Definition

The number of vectors in any basis of a vector space V is called the **dimension** of V and is denoted by $\dim V$.

Basis

Definition

The number of vectors in any basis of a vector space V is called the **dimension** of V and is denoted by $\dim V$.

For example,

- The dimension of $\mathbb{R}^1$ is 1,

Basis

Definition

The number of vectors in any basis of a vector space V is called the **dimension** of V and is denoted by $\dim V$.

For example,

- The dimension of $\mathbb{R}^1$ is 1,
- The dimension of $\mathbb{R}^2$ is 2,

Basis

Definition

The number of vectors in any basis of a vector space V is called the **dimension** of V and is denoted by $\dim V$.

For example,

- The dimension of $\mathbb{R}^1$ is 1,
- The dimension of $\mathbb{R}^2$ is 2,
- $\ldots$

Basis

Definition

The number of vectors in any basis of a vector space V is called the **dimension** of V and is denoted by $\dim V$.

For example,

- The dimension of $\mathbb{R}^1$ is 1,
- The dimension of $\mathbb{R}^2$ is 2,
- $\ldots$
- The dimension of $\mathbb{R}^n$ is n,

Basis

Definition

The number of vectors in any basis of a vector space V is called the **dimension** of V and is denoted by $\dim V$.

For example,

- The dimension of $\mathbb{R}^1$ is 1,
- The dimension of $\mathbb{R}^2$ is 2,
- ...
- The dimension of $\mathbb{R}^n$ is n,
- The dimension of a line is 1,

Basis

Definition

The number of vectors in any basis of a vector space V is called the **dimension** of V and is denoted by $\dim V$.

For example,

- The dimension of $\mathbb{R}^1$ is 1,
- The dimension of $\mathbb{R}^2$ is 2,
- ...
- The dimension of $\mathbb{R}^n$ is n,
- The dimension of a line is 1,
- The dimension of a plane is 2,

Basis

Definition

The number of vectors in any basis of a vector space V is called the **dimension** of V and is denoted by $\dim V$.

For example,

- The dimension of $\mathbb{R}^1$ is 1,
- The dimension of $\mathbb{R}^2$ is 2,
- ...
- The dimension of $\mathbb{R}^n$ is n,
- The dimension of a line is 1,
- The dimension of a plane is 2,
- etc.

Basis

Definition

The number of vectors in any basis of a vector space V is called the **dimension** of V and is denoted by $\dim V$.

For example,

- The dimension of $\mathbb{R}^1$ is 1,
- The dimension of $\mathbb{R}^2$ is 2,
- $\ldots$
- The dimension of $\mathbb{R}^n$ is n,
- The dimension of a line is 1,
- The dimension of a plane is 2,
- etc.

The dimension describes how **big** our vector space is. It shows how many linearly independent vectors are there in that vector space at most.

Geometric Interpretation of Matrices (last time)

Getting back to our matrices, our main question remains:

Question

How can we tell which transformation a matrix represents by just looking at it?

Geometric Interpretation of Matrices (last time)

Getting back to our matrices, our main question remains:

Question

How can we tell which transformation a matrix represents by just looking at it?

Turns out, the answer is hidden in the concept of basis.

Consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Geometric Interpretation of Matrices (last time)

Consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Where do you think it takes the basis vectors $\mathbf{e}_1 = [1 \;\; 0]$ and $\mathbf{e}_2 = [0 \;\; 1]$?

Geometric Interpretation of Matrices (last time)

Consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Where do you think it takes the basis vectors $\mathbf{e}_1 = [1 \quad 0]$ and $\mathbf{e}_2 = [0 \quad 1]$?

$$A\mathbf{e}_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Geometric Interpretation of Matrices (last time)

Consider the matrix

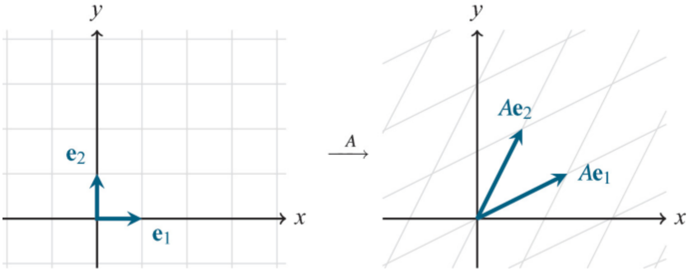
$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Where do you think it takes the basis vectors $\mathbf{e}_1 = [1 \;\; 0]$ and $\mathbf{e}_2 = [0 \;\; 1]$?

$$A\mathbf{e}_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A\mathbf{e}_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Geometric Interpretation of Matrices (last time)



As we see, applying a matrix **transforms the basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$** into the **columns of the matrix**:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \text{1st column of } A$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \text{2nd column of } A$$

Geometric Interpretation of Matrices (last time)

Therefore any vector with coordinates $\begin{bmatrix} a \\ b \end{bmatrix}$ is transformed into

$$a \begin{bmatrix} 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

i.e. the linear transformation sends our basis vectors to its columns, which become a new basis for our space.

Geometric Interpretation of Matrices (last time)

Therefore any vector with coordinates $\begin{bmatrix} a \\ b \end{bmatrix}$ is transformed into

$$a \begin{bmatrix} 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

i.e. the linear transformation sends our basis vectors to its columns, which become a new basis for our space.

More precisely, if the columns of A are linearly independent, then they form a basis.

Definition

The number of linearly independent columns of the matrix A is called the **rank** of A.

Geometric Interpretation of Matrices (last time)

Therefore any vector with coordinates $\begin{bmatrix} a \\ b \end{bmatrix}$ is transformed into

$$a \begin{bmatrix} 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

i.e. the linear transformation sends our basis vectors to its columns, which become a new basis for our space.

More precisely, if the columns of A are linearly independent, then they form a basis.

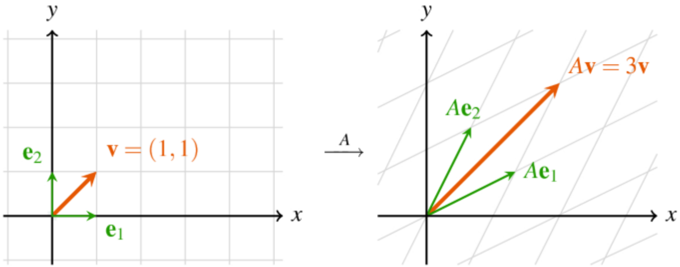
Definition

The number of linearly independent columns of the matrix A is called the **rank** of A.

Finally, peace.

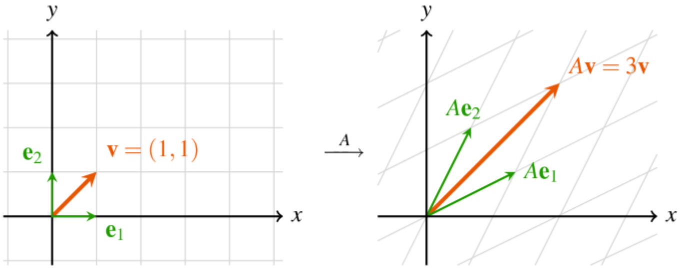
Geometric Interpretation of Matrices (last time)

Sometimes the matrix also has some vectors which **do not change their direction** when being multiplied (transformed) by that matrix, rather they get scaled by some number:



Geometric Interpretation of Matrices (last time)

Sometimes the matrix also has some vectors which **do not change their direction** when being multiplied (transformed) by that matrix, rather they get scaled by some number:



Vectors like these are of special interest to us, and we call them *eigenvectors*.

Eigenvalues and Eigenvectors

Definition

If for some number λ and some non-zero vector $\mathbf{v}$

$$A\mathbf{v} = \lambda\mathbf{v}$$

then we say

- λ is an **eigenvalue** of A,
- $\mathbf{v}$ is an **eigenvector** of A corresponding to the eigenvalue λ.

Eigenvalues and Eigenvectors

Definition

If for some number λ and some non-zero vector $\mathbf{v}$

$$A\mathbf{v} = \lambda\mathbf{v}$$

then we say

- λ is an **eigenvalue** of A,
- $\mathbf{v}$ is an **eigenvector** of A corresponding to the eigenvalue λ.

Example

For the matrix $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ and vector $\mathbf{v} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$,

$$A\mathbf{v} = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 4 \end{bmatrix} = 4\mathbf{v}$$

so $\lambda = 4$ is an eigenvalue of A with eigenvector $\mathbf{v}$.

Eigenvalues and Eigenvectors

Definition

If for some number λ and some non-zero vector $\mathbf{v}$

$$A\mathbf{v} = \lambda\mathbf{v}$$

then we say

- λ is an **eigenvalue** of A,
- $\mathbf{v}$ is an **eigenvector** of A corresponding to the eigenvalue λ.

Example

For the matrix $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ and vector $\mathbf{v} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$,

$$A\mathbf{v} = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 4 \end{bmatrix} = 4\mathbf{v}$$

so $\lambda = 4$ is an eigenvalue of A with eigenvector $\mathbf{v}$. What about $-3\mathbf{v}$?

Eigenvalues and Eigenvectors

Remark

If $\mathbf{v}$ is an eigenvector of A, then for any scalar $c \neq 0$, $c\mathbf{v}$ is also an eigenvector for A.

Eigenvalues and Eigenvectors

Remark

If $\mathbf{v}$ is an eigenvector of A, then for any scalar $c \neq 0$, $c\mathbf{v}$ is also an eigenvector for A.

Definition

For $A \in \mathbb{R}^{n \times n}$, the set of all eigenvectors of A associated with an eigenvalue λ, together with the zero vector, is called the **eigenspace** of A with respect to λ and is denoted by E_λ.

Eigenvalues and Eigenvectors

Remark

If $\mathbf{v}$ is an eigenvector of A, then for any scalar $c \neq 0$, $c\mathbf{v}$ is also an eigenvector for A.

Definition

For $A \in \mathbb{R}^{n \times n}$, the set of all eigenvectors of A associated with an eigenvalue λ, together with the zero vector, is called the **eigenspace** of A with respect to λ and is denoted by E_λ.

Definition

The set of all eigenvalues of A is called the **spectrum** of A.

Eigenvalues and Eigenvectors

Remark

If $\mathbf{v}$ is an eigenvector of A, then for any scalar $c \neq 0$, $c\mathbf{v}$ is also an eigenvector for A.

Definition

For $A \in \mathbb{R}^{n \times n}$, the set of all eigenvectors of A associated with an eigenvalue λ, together with the zero vector, is called the **eigenspace** of A with respect to λ and is denoted by E_λ.

Definition

The set of all eigenvalues of A is called the **spectrum** of A.

How can we find the eigenvalues and eigenvectors of a given matrix?

Eigenvalues and Eigenvectors

Suppose $A \in \mathbb{R}^{n \times n}$ is a matrix, and we want to find $x \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n$ such that:

$$
\begin{aligned}
A\mathbf{v} = x\mathbf{v} = I(x\mathbf{v}) &= xI\mathbf{v} \\
A\mathbf{v} - xI\mathbf{v} &= \mathbf{0} \\
(A - xI)\mathbf{v} &= \mathbf{0}
\end{aligned}
$$

Eigenvalues and Eigenvectors

Suppose $A \in \mathbb{R}^{n \times n}$ is a matrix, and we want to find $x \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n$ such that:

$$A\mathbf{v} = x\mathbf{v} = I(x\mathbf{v}) = xI\mathbf{v}$$
$$A\mathbf{v} - xI\mathbf{v} = \mathbf{0}$$
$$(A - xI)\mathbf{v} = \mathbf{0}$$

Since $\mathbf{v} \neq \mathbf{0}$,

$$\det(A - xI) = 0$$

Eigenvalues and Eigenvectors

Suppose $A \in \mathbb{R}^{n\times n}$ is a matrix, and we want to find $x \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n$ such that:

$$
\begin{aligned}
A\mathbf{v} = x\mathbf{v} = I(x\mathbf{v}) = xI\mathbf{v} \\
A\mathbf{v} - xI\mathbf{v} = \mathbf{0} \\
(A - xI)\mathbf{v} = \mathbf{0}
\end{aligned}
$$

Since $\mathbf{v} \neq \mathbf{0}$,

$$
\det(A - xI) = 0
$$

The polynomial above is called the **characteristic polynomial** of A. Its roots are the eigenvalues of A.

Eigenvalues and Eigenvectors

Example

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$:

Example

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$:

$$A - xI =$$

Example

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$:

$$A - xI = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} =$$

Example

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$:

$$A - xI = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} = \begin{bmatrix} 3-x & 5 \\ 1 & -1-x \end{bmatrix}$$

Eigenvalues and Eigenvectors

Example

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$:

$$A - xI = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} = \begin{bmatrix} 3 - x & 5 \\ 1 & -1 - x \end{bmatrix}$$

$$p_A(x) = \det(A - xI) = (3 - x)(-1 - x) - 5 = x^2 - 2x - 8$$

Example

Hence, the roots of $p_A(x)$ are $\lambda = 4$ and $\lambda = -2$, so these are the eigenvalues of A. The spectrum of A is $\{4, -2\}$.

Example

Hence, the roots of $p_A(x)$ are $\lambda = 4$ and $\lambda = -2$, so these are the eigenvalues of A. The spectrum of A is $\{4, -2\}$. For $\lambda = 4$,

$$A\mathbf{v} = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3v_1 + 5v_2 \\ v_1 - v_2 \end{bmatrix},$$

Example

Hence, the roots of $p_A(x)$ are $\lambda = 4$ and $\lambda = -2$, so these are the eigenvalues of A. The spectrum of A is $\{4, -2\}$. For $\lambda = 4$,

$$A\mathbf{v} = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3v_1 + 5v_2 \\ v_1 - v_2 \end{bmatrix}, \quad 4\mathbf{v} = \begin{bmatrix} 4v_1 \\ 4v_2 \end{bmatrix}$$

$$A\mathbf{v} - 4\mathbf{v} = \begin{bmatrix} -v_1 + 5v_2 \\ v_1 - 5v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Eigenvalues and Eigenvectors

Example

Hence, the roots of $p_A(x)$ are $\lambda = 4$ and $\lambda = -2$, so these are the eigenvalues of A. The spectrum of A is $\{4, -2\}$. For $\lambda = 4$,

$$A\mathbf{v} = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3v_1 + 5v_2 \\ v_1 - v_2 \end{bmatrix}, \quad 4\mathbf{v} = \begin{bmatrix} 4v_1 \\ 4v_2 \end{bmatrix}$$

$$A\mathbf{v} - 4\mathbf{v} = \begin{bmatrix} -v_1 + 5v_2 \\ v_1 - 5v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad v_1 = 5v_2$$

Example

Hence, the roots of $p_A(x)$ are $\lambda = 4$ and $\lambda = -2$, so these are the eigenvalues of A. The spectrum of A is $\{4, -2\}$. For $\lambda = 4$,

$$A\mathbf{v} = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3v_1 + 5v_2 \\ v_1 - v_2 \end{bmatrix}, \quad 4\mathbf{v} = \begin{bmatrix} 4v_1 \\ 4v_2 \end{bmatrix}$$

$$A\mathbf{v} - 4\mathbf{v} = \begin{bmatrix} -v_1 + 5v_2 \\ v_1 - 5v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad v_1 = 5v_2$$

Setting $v_2 = a$ and $v_1 = 5a$ for any scalar $a \in \mathbb{R}$, we will get the solution. There are infinite solutions which are all multiplies of each other:

$$\mathbf{v} = a \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Eigenvalues and Eigenvectors

Example

Hence, the roots of $p_A(x)$ are $\lambda = 4$ and $\lambda = -2$, so these are the eigenvalues of A. The spectrum of A is $\{4, -2\}$. For $\lambda = 4$,

$$A\mathbf{v} = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3v_1 + 5v_2 \\ v_1 - v_2 \end{bmatrix}, \quad 4\mathbf{v} = \begin{bmatrix} 4v_1 \\ 4v_2 \end{bmatrix}$$

$$A\mathbf{v} - 4\mathbf{v} = \begin{bmatrix} -v_1 + 5v_2 \\ v_1 - 5v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad v_1 = 5v_2$$

Setting $v_2 = a$ and $v_1 = 5a$ for any scalar $a \in \mathbb{R}$, we will get the solution. There are infinite solutions which are all multiplies of each other:

$$\mathbf{v} = a \begin{bmatrix} 5 \\ 1 \end{bmatrix} \qquad E_4 = \left\{ a \begin{bmatrix} 5 \\ 1 \end{bmatrix} \mid \text{for any } a \in \mathbb{R} \right\}$$

Eigenvalues and Eigenvectors

Example

Hence, the roots of $p_A(x)$ are $\lambda = 4$ and $\lambda = -2$, so these are the eigenvalues of A. The spectrum of A is $\{4, -2\}$. For $\lambda = 4$,

$$A\mathbf{v} = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3v_1 + 5v_2 \\ v_1 - v_2 \end{bmatrix}, \quad 4\mathbf{v} = \begin{bmatrix} 4v_1 \\ 4v_2 \end{bmatrix}$$

$$A\mathbf{v} - 4\mathbf{v} = \begin{bmatrix} -v_1 + 5v_2 \\ v_1 - 5v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad v_1 = 5v_2$$

Setting $v_2 = a$ and $v_1 = 5a$ for any scalar $a \in \mathbb{R}$, we will get the solution. There are infinite solutions which are all multiplies of each other:

$$\mathbf{v} = a \begin{bmatrix} 5 \\ 1 \end{bmatrix} \qquad E_4 = \left\{ a \begin{bmatrix} 5 \\ 1 \end{bmatrix} \mid \text{for any } a \in \mathbb{R} \right\}$$

Similarly we get $E_{-2} = \{a \cdot [-1 \quad 1]^T \mid \text{for any } a \in \mathbb{R}\}$.

Eigenvalues and Eigenvectors

Again, what we are concerned with, is the *concept* and not the computation.

Eigenvalues and Eigenvectors

Again, what we are concerned with, is the *concept* and not the computation.

As a bonus, we have a surprising theorem:

Theorem

The determinant of a matrix is equal to the product of its eigenvalues:

$$\det(A) = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n$$

and the trace of a matrix is equal to the sum of its eigenvalues:

$$\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_n$$

Systems of Linear Equations

Let us consider one last application of the matrices.

Question

Imagine scrolling Facebook, when you suddenly see the following problem: You have 2 types of fruits, apples and oranges. You buy 2 apples and 3 oranges for a total cost of 11 dollars. Additionally, you buy 1 apple and 4 oranges for a total cost of 7 dollars.
Only people with 140 IQ can find the prices of apples and oranges.

Systems of Linear Equations

Let us consider one last application of the matrices.

Question

Imagine scrolling Facebook, when you suddenly see the following problem: You have 2 types of fruits, apples and oranges. You buy 2 apples and 3 oranges for a total cost of 11 dollars. Additionally, you buy 1 apple and 4 oranges for a total cost of 7 dollars.
Only people with 140 IQ can find the prices of apples and oranges.

Let x be the cost of one apple and y be the cost of one orange. The problem can be represented as a 2×2 system of linear equations:

$$\begin{cases} 2x + 3y & = 11 \\ x + 4y & = 7 \end{cases}$$

Solving this system will give us the prices of apples (x) and oranges (y).

Systems of Linear Equations

Definition

A **system of linear equations** is a collection of two or more linear equations involving the same set of variables.

Systems of Linear Equations

Definition

A **system of linear equations** is a collection of two or more linear equations involving the same set of variables.

A system of m linear equations with n variables can be written as:

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & \dots & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m
\end{array}
$$

Systems of Linear Equations

Definition

A **system of linear equations** is a collection of two or more linear equations involving the same set of variables.

A system of m linear equations with n variables can be written as:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ldots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m \end{array}$$

Definition

A **particular solution** to the system is a set of values for the variables $(x_1, x_2, \ldots, x_n)$ that satisfies all equations simultaneously. The collection of all particular solutions is called the **general solution**.

Systems of Linear Equations

Going back to our example,

$$\begin{cases} 2x + 3y & = 11 \\ x + 4y & = 7 \end{cases}$$

we may notice that

Going back to our example,

$$\begin{cases} 2x + 3y & = 11 \\ x + 4y & = 7 \end{cases}$$

we may notice that

- the first row is the dot product of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$,

Systems of Linear Equations

Going back to our example,

$$\begin{cases} 2x + 3y & = 11 \\ x + 4y & = 7 \end{cases}$$

we may notice that

- the first row is the dot product of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$,
- the second row is the dot product of $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$,

Going back to our example,

$$\begin{cases} 2x + 3y & = 11 \\ x + 4y & = 7 \end{cases}$$

we may notice that

- the first row is the dot product of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$,
- the second row is the dot product of $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$,

implicating

Going back to our example,

$$\begin{cases} 2x + 3y & = 11 \\ x + 4y & = 7 \end{cases}$$

we may notice that

- the first row is the dot product of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$,
- the second row is the dot product of $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$,

implicating

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

Systems of Linear Equations

Going back to our example,

$$\begin{cases} 2x + 3y &= 11 \\ x + 4y &= 7 \end{cases}$$

we may notice that

- the first row is the dot product of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$,
- the second row is the dot product of $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$,

implicating

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} =$$

Systems of Linear Equations

Going back to our example,

$$\begin{cases} 2x + 3y & = 11 \\ x + 4y & = 7 \end{cases}$$

we may notice that

- the first row is the dot product of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$,
- the second row is the dot product of $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$,

implicating

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 7 \end{bmatrix}$$

Systems of Linear Equations

Going back to our example,

$$\begin{cases} 2x + 3y & = 11 \\ x + 4y & = 7 \end{cases}$$

we may notice that

- the first row is the dot product of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$,
- the second row is the dot product of $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$,

implicating

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 7 \end{bmatrix}$$

So Facebook is just asking: On which vector should you apply this matrix to get $[11 \quad 7]$?

Systems of Linear Equations

Let's consider three systems of linear equations:

a)

$$\begin{cases} 2x + 3y & = 7 \\ 4x - y & = 5 \end{cases}$$

b)

$$\begin{cases} 2x + 3y & = 7 \\ 4x + 6y & = 14 \end{cases}$$

c)

$$\begin{cases} 2x + 3y & = 7 \\ 4x + 6y & = 15 \end{cases}$$

a)

$$\begin{cases} 2x + 3y &= 7 \\ 4x - y &= 5 \end{cases}$$

$$2x + 3y = 7 \Rightarrow 2x = 7 - 3y \Rightarrow x = \frac{7 - 3y}{2}$$

a)

$$\begin{cases} 2x + 3y & = 7 \\ 4x - y & = 5 \end{cases}$$

$$2x + 3y = 7 \Rightarrow 2x = 7 - 3y \Rightarrow x = \frac{7 - 3y}{2}$$

$$4\left(\frac{7 - 3y}{2}\right) - y = 5 \Rightarrow 7 - 3y - y = 5 \Rightarrow -4y = -2 \Rightarrow y = \frac{1}{2}$$

Systems of Linear Equations

a)

$$\begin{cases} 2x + 3y &= 7 \\ 4x - y &= 5 \end{cases}$$

$$2x + 3y = 7 \Rightarrow 2x = 7 - 3y \Rightarrow x = \frac{7 - 3y}{2}$$

$$4\left(\frac{7 - 3y}{2}\right) - y = 5 \Rightarrow 7 - 3y - y = 5 \Rightarrow -4y = -2 \Rightarrow y = \frac{1}{2}$$

$$x = \frac{7 - 3(\frac{1}{2})}{2} = 2$$

a)

$$\begin{cases} 2x + 3y & = 7 \\ 4x - y & = 5 \end{cases}$$

$$2x + 3y = 7 \Rightarrow 2x = 7 - 3y \Rightarrow x = \frac{7 - 3y}{2}$$

$$4\left(\frac{7 - 3y}{2}\right) - y = 5 \Rightarrow 7 - 3y - y = 5 \Rightarrow -4y = -2 \Rightarrow y = \frac{1}{2}$$

$$x = \frac{7 - 3(\frac{1}{2})}{2} = 2$$

Solution: $x = 2, \quad y = \frac{1}{2}$

b)

$$\begin{cases} 2x + 3y &= 7 \\ 4x + 6y &= 14 \end{cases}$$

$$2x + 3y = 7 \Rightarrow 2x = 7 - 3y \Rightarrow x = \frac{7 - 3y}{2}$$

b)

$$\begin{cases} 2x + 3y &= 7 \\ 4x + 6y &= 14 \end{cases}$$

$$2x + 3y = 7 \Rightarrow 2x = 7 - 3y \Rightarrow x = \frac{7 - 3y}{2}$$

$$4\left(\frac{7 - 3y}{2}\right) + 6y = 14 \Rightarrow 14 - 6y + 6y = 14 \Rightarrow 14 = 14$$

b)

$$\begin{cases} 2x + 3y &= 7 \\ 4x + 6y &= 14 \end{cases}$$

$$2x + 3y = 7 \Rightarrow 2x = 7 - 3y \Rightarrow x = \frac{7 - 3y}{2}$$

$$4\left(\frac{7 - 3y}{2}\right) + 6y = 14 \Rightarrow 14 - 6y + 6y = 14 \Rightarrow 14 = 14$$

y can be any number

b)

$$\begin{cases} 2x + 3y & = 7 \\ 4x + 6y & = 14 \end{cases}$$

$$2x + 3y = 7 \Rightarrow 2x = 7 - 3y \Rightarrow x = \frac{7 - 3y}{2}$$

$$4\left(\frac{7-3y}{2}\right) + 6y = 14 \Rightarrow 14 - 6y + 6y = 14 \Rightarrow 14 = 14$$

y can be any number

Infinite solutions: $x = \frac{7-3y}{2}, \quad \text{for any } y \in \mathbb{R}$

c)

$$\begin{cases} 2x + 3y &= 7 \\ 4x + 6y &= 15 \end{cases}$$

Multiplying the first equation by 2 gives:

$$4x + 6y = 14$$

which contradicts the second equation.

c)

$$\begin{cases} 2x + 3y &= 7 \\ 4x + 6y &= 15 \end{cases}$$

Multiplying the first equation by 2 gives:

$$4x + 6y = 14$$

which contradicts the second equation.
No solution.

Systems of Linear Equations

So as we saw, in general a system of linear equations can have a *unique solution*, *no solution*, or *infinitely many solutions*.

Definition

A system of linear equations is **consistent** if it has at least one solution. A system is *inconsistent* if it has no solutions.

Systems of Linear Equations

Consider the system of three linear equations:

$$\begin{cases} 2x + y - z &= 5 \\ -3x - 2y + 2z &= -8 \\ x + 4y - 3z &= 1 \end{cases}$$

Systems of Linear Equations

Consider the system of three linear equations:

$$\begin{cases} 2x + y - z & = 5 \\ -3x - 2y + 2z & = -8 \\ x + 4y - 3z & = 1 \end{cases}$$

We can write it in the form:

$$A\mathbf{x} = \mathbf{b},$$

where

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -3 & -2 & 2 \\ 1 & 4 & -3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ -8 \\ 1 \end{bmatrix}$$

Systems of Linear Equations

Consider the system of three linear equations:

$$\begin{cases} 2x + y - z & = 5 \\ -3x - 2y + 2z & = -8 \\ x + 4y - 3z & = 1 \end{cases}$$

We can write it in the form:

$$A\mathbf{x} = \mathbf{b},$$

where

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -3 & -2 & 2 \\ 1 & 4 & -3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ -8 \\ 1 \end{bmatrix}$$

Theorem (very fundamental)

The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for any vector $\mathbf{b} \in \mathbb{R}^n$, if and only if $\det A \neq 0$ (i.e. A is invertible).