Random Variables

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April 15, 2025

Recap:

During the previous lecture we had a problem like this:

Example

Both the bus and you get to the bus stop at random times between 12 pm and 1 pm. When the bus arrives, it waits for 5 minutes before leaving. When you arrive, you wait for 20 minutes before leaving if the bus doesn't come. What is the probability that you catch the bus?

Here we did not know the **exact values** of the arrival times, so in order to compare them, we **denoted them** by y and b – and calculated their probabilities.

In this case, we say that y and b are **random variables**.

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You throw a fair dice and win \$1 if it is prime, lose \$1 if it is composite (4 or 6), and stay even otherwise. In other words, if X is your profit, then

$$X = \begin{cases} 1, & \text{if } \omega \in \{2,3,5\} \\ -1, & \text{if } \omega \in \{4,6\} \\ 0, & \text{if } \omega = 1 \end{cases}$$

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What is the probability that $X = 1$?

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What is the probability that $X = 1$? (We'll get to this later.)

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What values can each of the random variables above take?

Random Variables (optional)

Not a random variable

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- who threw the dice,
- the color of the sky,
- what will show up if you had thrown a third die

are not random variables, as their values *do not depend* on the outcome of the dice (even if you knew the outcome, you could not answer these).

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In practice, every unknown thing in the given problem is a random variable.

Technically speaking, although, the value of the random variable should somehow depend on the outcome ω of the experiment. In other words, if you ask "When is $X = 4$?", there should be a way to measure that probability, i.e. $\{X = 4\}$ should be an event.

In your textbook you will find it as $\{\omega \text{ for which } X = 4\} \in \mathcal{F}$, but we skip the technicalities.

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Otherwise,

Definition

If the values of X cannot be represented as a list (e.g. they are an interval), it is called a **continuous random variable.**

PMF

Of course, what we are interested in the most, is the probability that X takes a certain value.

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$$X = \begin{cases} 1, & \text{if } \omega \in \{2, 3, 5\} \\ -1, & \text{if } \omega \in \{4, 6\} \\ 0, & \text{if } \omega = 1 \end{cases}$$

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The thing above, $\mathbb{P}(X = \text{something})$, is called the **probability mass function** or just the **PMF** of X.

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k (right answers)	$\mathbb{P}(X=k)$
0	$11/21$
1	$44/105$
2	$6/105$

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If there were, say, 5 questions and more than 3 points needed to pass, then we would be interested in

$$\mathbb{P}(X \leq 3) \qquad \text{(probability of failing)}$$

or

$$\mathbb{P}(X > 3) \qquad \text{(probability of passing)}$$

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Properties

- $0 \leq F_X(x) \leq 1,$ for any $x \in \mathbb{R}$
- $F_X(x)$ is a non-decreasing function
- $\lim\limits_{x \to -\infty} F_X(x) = 0 \qquad$ and $\qquad \lim\limits_{x \to +\infty} F_X(x) = 1$

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Let X indicate the number on a fair die. Then

$$F_X(6.1) =$$

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$$F_X(6.1) = 1$$
$$F_X(6) = 1$$
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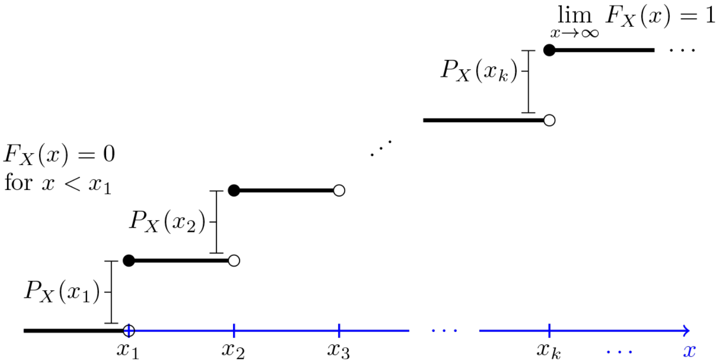
$$F_X(x) = \begin{cases} 0, & \text{if } x < 1, \\ \frac{1}{6}, & \text{if } 1 \leq x < 2, \\ \frac{2}{6}, & \text{if } 2 \leq x < 3, \\ \frac{3}{6}, & \text{if } 3 \leq x < 4, \\ \frac{4}{6}, & \text{if } 4 \leq x < 5, \\ \frac{5}{6}, & \text{if } 5 \leq x < 6, \\ 1, & \text{if } x \geq 6 \end{cases}$$

Graphs of CDFs usually usually look like this:



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- While the students may be asked different questions,
- $\Rightarrow$ it may happen that X and Y are not equal,
- still Y has the same PMF and CDF as X (check it!)

In this case, we say that X and Y are identically distributed:

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They both have the save PMFs:

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PDF

Finally, what if X is a continuous random variable?

Say you randomly put your finger on some number X in $(0, 10)$. What is the probability that $\mathbb{P}(X = 0.5)$?

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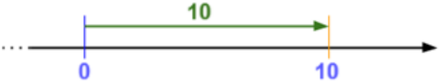

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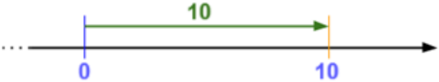

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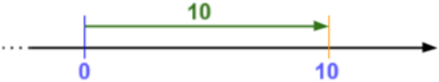

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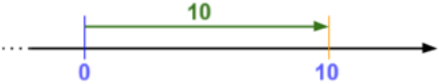

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So another question we might ask is:

Question

What do you think $\mathbb{P}(X \leq 5)$ is?

PDF

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If X is a continuous random variable, then there exists a nonnegative function $f(x)$ such that for any $c \in \mathbb{R}$,

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The function f is called the **probability density function** or **PDF** of X.

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The PDF of a continuous random variable plays essentially the same role as the PMF of a discrete random variable.

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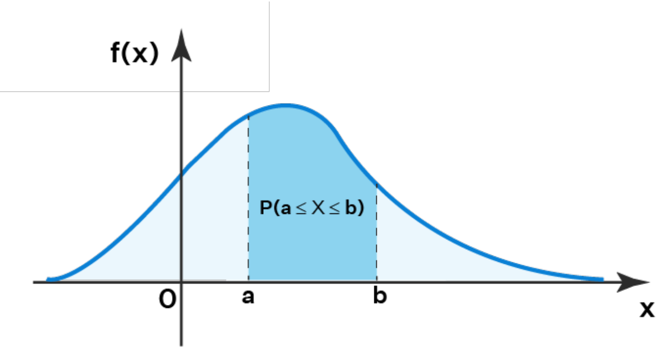

PDF

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Example

Ani chooses a random real number X uniformly from the interval $[a, b]$.

By "uniformly" we mean that for any two intervals of the same length (e.g. $(1.3, 1.5)$ and $(4.7, 4.9)$) X can belong to them with the same probability.

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For $a \leq x \leq b$, we have:

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(X \in [a, x]) = \frac{x - a}{b - a}$$

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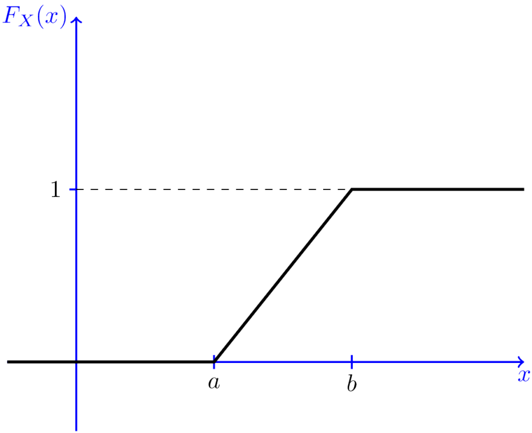

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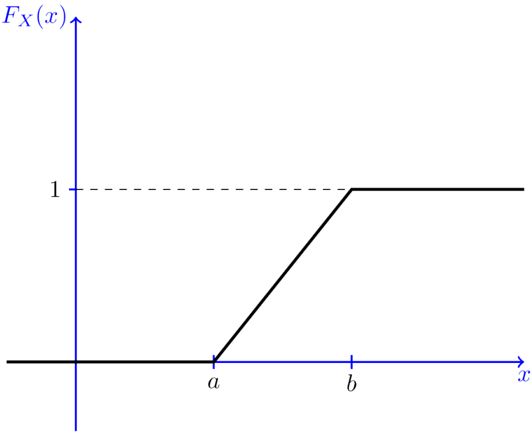

Thus,

$$F_X(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } x > b \end{cases}$$

PDF

To find $f_X(x)$, we take the derivative of $F_X(x)$:

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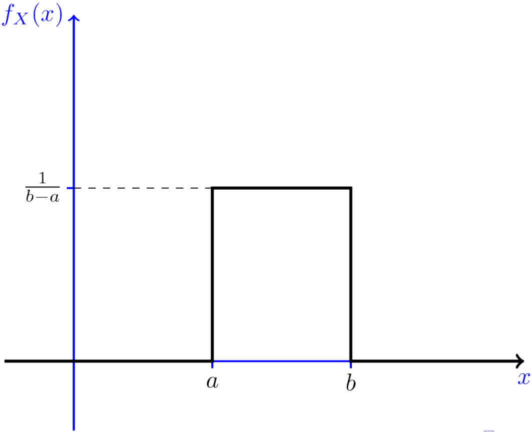

Independence

Recall that two events A and B are called independent if they do not affect each other, i.e.

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

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Definition

X and Y are called **independent** if

$$\mathbb{P}(X \leq a \ \text{ and } \ Y \leq b) = \mathbb{P}(X \leq a) \cdot \mathbb{P}(Y \leq b)$$

for any $a, b \in \mathbb{R}$.

So the probability of both X and Y simultaneously being less than some numbers is just their *separate* probabilities multiplied together.