Linear Algebra

Chapter 3: Orthogonality

Solution of highlighted problems

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3. Two lines in the plane are perpendicular when the product of their slopes is -1. Apply this to the vectors $x = (x_1, x_2)$ and $y = (y_1, y_2)$, whose slopes are x_2/x_1 and y_2/y_1, to derive again the orthogonality condition $x^T y = 0$.

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$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \text{slope} = \frac{x_2}{x_1} \qquad \text{slope} = \frac{y_2}{y_1}$$

$$\text{Product of slope} = \frac{x_2 \cdot y_2}{x_1 \cdot y_1} = -1 \longrightarrow x_2 y_2 = -x_1 y_1 \quad (*)$$

$$x^T y = [x_1 \quad x_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1 y_1 + x_2 y_2 = x_1 y_1 - x_1 y_1 = 0$$

orthogonality condition checked ✓

7. Find a vector x orthogonal to the row space of A, and a vector y orthogonal to the column space, and a vector z orthogonal to the nullspace:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}.$$

(7) $$AX = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{\text{Elimination}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ x_3 &= 0 \\ x_3 &= 0 \end{aligned}\right\} \longrightarrow x_1 = -2x_2 \;;\; x_3 = 0 \longrightarrow x = c\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$A^T y = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{\text{Elimination}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} y_1 + 2y_2 + 3y_3 &= 0 \\ y_2 + y_3 &= 0 \end{aligned}\right\} \longrightarrow y_1 = -y_3 \;;\; y_2 = -y_3 \longrightarrow y = c\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$z^T x = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = 0 \longrightarrow -2z_1 + z_2 = 0 \longrightarrow z_2 = 2z_1$$

$$\rightsquigarrow z = c_1\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

27. (a) If $Ax = b$ has a solution and $A^T y = 0$, then y is perpendicular to ____.

(b) If $A^T y = c$ has a solution and $Ax = 0$, then x is perpendicular to ____.

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(a) $Ax = b$ $\rightarrow$ it has a solution $\rightarrow$ $\underline{\underline{b}}$ is in the column space of $\underline{\underline{A}}$

$A^T y = 0$ $\rightarrow$ Since y is perpendicular to column space of $\underline{\underline{A}}$, we can conclude that y is also perpendicular to $\underline{\underline{b}}$.

(b) $A^T y = c$ $\rightarrow$ it has a solution $\rightarrow$ $\underline{\underline{c}}$ is in row space of $\underline{\underline{A}}$.

$Ax = 0$ $\rightarrow$ since x is perpendicular to row space of A, we can conclude that x is also perpendicular to $\underline{\underline{c}}$.

35. The floor and the wall are not orthogonal subspaces because they share a nonzero vector (along the line where they meet). Two planes in $\mathbf{R}^3$ cannot be orthogonal! Find a vector in both column spaces $C(A)$ and $C(B)$:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 4 \\ 6 & 3 \\ 5 & 1 \end{bmatrix}.$$

This will be a vector Ax and also $B\widehat{x}$. Think 3 by 4 with the matrix $[A \ \ B]$.

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$$C = [A \,\vdots\, B] = \begin{bmatrix} 1 & 2 & 5 & 4 \\ 1 & 3 & 6 & 3 \\ 1 & 2 & 5 & 1 \end{bmatrix}$$

a vector, as an assumption, by the name of $\underline{\underline{v}}$ which lie both on the column spaces of $\underline{\underline{A}}$ and $\underline{\underline{B}}$.

It can be considered as feasible right hand side of the below 2 equation. As it lies on the $C(A)$ and $C(B)$.

$$AX = v \;,\; B\hat{X} = v \longrightarrow AX - B\hat{X} = 0 \longrightarrow \begin{bmatrix} 1 & 2 & 5 & 4 \\ 1 & 3 & 6 & 3 \\ 1 & 2 & 5 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ -\hat{X}_1 \\ -\hat{X}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{\text{Elimination}} \begin{bmatrix} 1 & 2 & 5 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ -\hat{X}_1 \\ -\hat{X}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{cases} X_1 + 2X_2 - 5\hat{X}_1 - 4\hat{X}_2 = 0 \\ X_2 - \hat{X}_1 + \hat{X}_2 = 0 \\ +3\hat{X}_2 = 0 \end{cases}$$

$$\longrightarrow \hat{X}_2 = 0 \;;\; X_2 = \hat{X}_1 \;;\; X_1 = 3\hat{X}_1. \qquad X = \begin{bmatrix} 3 \\ 1 \end{bmatrix} ,\; \hat{X} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

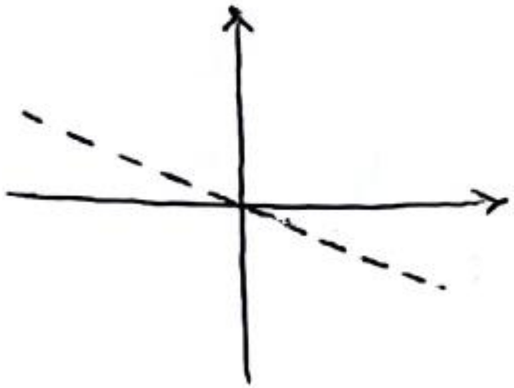
$$AX = B\hat{X} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 5 \end{bmatrix} = v$$

it lies both on the $C(A)$ and $C(B)$

12. Find the matrix that projects every point in the plane onto the line $x+2y=0$.

(12)

Line $x+2Y=0 \longrightarrow X=-2Y$ $\qquad a=\begin{bmatrix}-2\\1\end{bmatrix}$



$$P=\frac{aa^T}{a^Ta}=\frac{\begin{bmatrix}-2\\1\end{bmatrix}\begin{bmatrix}-2&1\end{bmatrix}}{\begin{bmatrix}-2&1\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix}}=\frac{\begin{bmatrix}4&-2\\-2&1\end{bmatrix}}{5}=\begin{bmatrix}4/5&-2/5\\-2/5&1/5\end{bmatrix}$$

so we end up with a Matrix P that is able to Project all the given point in R^2 onto the line $x+2Y$

17. Project the vector b onto the line through a. Check that e is perpendicular to a:

(a) $b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. (b) $b = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $a = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}$.

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a) $b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$P = \frac{aa^T}{a^Ta} = \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = \frac{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}{3} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$p = Pb = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 5/3 \\ 5/3 \end{bmatrix}$$

$$e = b - p = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 5/3 \\ 5/3 \\ 5/3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

check $a^Te = 0$ (perpendicularity check)

$$a^Te = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix} = 0$$

25. In Problem 24, the projection of b onto the *plane* of a_1 and a_2 will equal b. Find $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ for $A = [a_1 \ a_2] \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

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$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$$

$$P = A(A^TA)^{-1}A^T$$

$$A^TA = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 25 \end{bmatrix} \Rightarrow (A^TA)^{-1} = \frac{1}{\det(A^TA)}\begin{bmatrix} 25 & -3 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 25/16 & -3/16 \\ -3/16 & 1/16 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 25/16 & -3/16 \\ -3/16 & 1/16 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & -12/16 \\ 0 & 4/16 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

6. Find the projection of b onto the column space of A:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}.$$

Split b into $p+q$, with p in the column space and q perpendicular to that space. Which of the four subspaces contains q?

(b) $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}$ $\quad b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -8 \\ -8 & 18 \end{bmatrix}$$

$$\det(A^TA) = 6 \times 18 - 64 = 108 - 64 = 44$$

$$p = Pb \longrightarrow p = A\underbrace{(A^TA)^{-1}A^Tb}_{\hat{x}}$$

$$(A^TA)^{-1} = \frac{1}{44}\begin{bmatrix} 18 & 8 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} 18/44 & 8/44 \\ 8/44 & 6/44 \end{bmatrix}$$

$$\hat{x} = (A^TA)^{-1}A^Tb = \begin{bmatrix} 18/44 & 8/44 \\ 8/44 & 6/44 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 18/44 & 8/44 \\ 8/44 & 6/44 \end{bmatrix} \begin{bmatrix} -11 \\ 27 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-11 \times 18}{44} + \frac{8 \times 27}{44} \\ \frac{-11 \times 8}{44} + \frac{6 \times 27}{44} \end{bmatrix} = \begin{bmatrix} \frac{18}{44} \\ \frac{74}{44} \end{bmatrix}$$

$$p = A\hat{x} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} \frac{18}{44} \\ \frac{74}{44} \end{bmatrix} = \begin{bmatrix} 92/44 \\ -56/44 \\ 260/44 \end{bmatrix}$$

$$b = p + q \longrightarrow q = b - p = \frac{1}{44}\left(\begin{bmatrix} 44 \\ 88 \\ 308 \end{bmatrix} - \begin{bmatrix} 92 \\ -56 \\ 260 \end{bmatrix} \right) = \frac{1}{44}\begin{bmatrix} -48 \\ 144 \\ 48 \end{bmatrix} = \begin{bmatrix} -48/44 \\ 144/44 \\ 48/44 \end{bmatrix}$$

check Perpendicularity

$$A^Tq = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} -48/44 \\ 144/44 \\ 48/44 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \checkmark$$

we can conclude that vector q is in the left Null space of A

$$\Longrightarrow q \in N(A^T)$$

13. Find the best straight-line fit (least squares) to the measurements

$$b=4 \quad \text{at} \quad t=-2, \qquad b=3 \quad \text{at} \quad t=-1,$$
$$b=1 \quad \text{at} \quad t=0, \qquad b=0 \quad \text{at} \quad t=2.$$

Then find the projection of $b=(4,3,1,0)$ onto the column space of

$$A=\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

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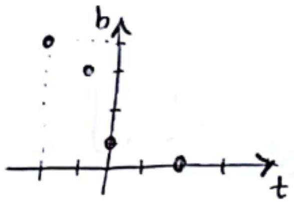
$b=4 \quad t=-2 \; ; \; b=3 \quad t=-1$

$b=1 \quad t=0 \; ; \; b=0 \quad t=2$



$$A=\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \quad x=\begin{bmatrix} C \\ D \end{bmatrix} \quad b=\begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix} \longrightarrow Ax=b$$

As vector b doesnot lie on the column space of A, we shoud Project it on $C(A)$. In this way can Compute the estimation of vector $\hat{x}$.

$$\hat{x}=(A^TA)^{-1}A^Tb \longrightarrow A^TA=\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} 4 & -1 \\ -1 & 9 \end{bmatrix}$$

$$(A^TA)^{-1}=\frac{1}{35}\begin{bmatrix} 9 & 1 \\ 1 & 4 \end{bmatrix}=\begin{bmatrix} 9/35 & 1/35 \\ 1/35 & 4/35 \end{bmatrix}$$

$$\hat{x}=\begin{bmatrix} 9/35 & 1/35 \\ 1/35 & 4/35 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

$$=\begin{bmatrix} 9/35 & 1/35 \\ 1/35 & 4/35 \end{bmatrix}\begin{bmatrix} 8 \\ -11 \end{bmatrix}=\begin{bmatrix} 61/35 \\ -36/35 \end{bmatrix}$$

line: $\hat{b}=61/35-36/35\,t$

$$P=A\hat{x}=\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 61/35 \\ -36/35 \end{bmatrix}=\begin{bmatrix} 133/35 \\ 97/35 \\ 61/35 \\ -11/35 \end{bmatrix}$$

2. Project $b = (0, 3, 0)$ onto each of the orthonormal vectors $a_1 = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$ and $a_2 = (-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$, and then find its projection p onto the plane of a_1 and a_2.

(2)

$$b = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \qquad Q = \begin{array}{cc} a_1 & a_2 \\ \end{array} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{bmatrix}$$

$$P_1 = \left(\frac{a_1 a_1^T}{a_1^T a_1}\right) b = \frac{\begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix} \begin{bmatrix} 2/3 & 2/3 & -1/3 \end{bmatrix}}{1} \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{9} & \frac{4}{9} & -\frac{2}{9} \\ \frac{4}{9} & \frac{4}{9} & -\frac{2}{9} \\ -\frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 4/3 \\ -2/3 \end{bmatrix}$$

$$P_2 = \left(\frac{a_2 a_2^T}{a_2^T a_2}\right) b = \frac{\begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & 2/3 \end{bmatrix}}{1} \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{9} & -\frac{2}{9} & -\frac{2}{9} \\ -\frac{2}{9} & \frac{4}{9} & 4/9 \\ -\frac{2}{9} & 4/9 & 4/9 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 4/3 \\ 4/3 \end{bmatrix}$$

$$\hat{x} = (Q^TQ)^{-1} Q^T b \longrightarrow \hat{x} = (I)^{-1} Q^T b = Q^T b$$

$$P = Q\hat{x} = QQ^Tb = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 8/3 \\ 2/3 \end{bmatrix}$$

6. Find a third column so that the matrix

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & \\ 1/\sqrt{3} & 2/\sqrt{14} & \\ 1/\sqrt{3} & -3/\sqrt{14} & \end{bmatrix}$$

is orthogonal. It must be a unit vector that is orthogonal to the other columns; how much freedom does this leave? Verify that the rows automatically become orthonormal at the same time.

(6)

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & x_1 \\ 1/\sqrt{3} & 2/\sqrt{14} & x_2 \\ 1/\sqrt{3} & -3/\sqrt{14} & x_3 \end{bmatrix} \rightarrow \begin{cases} x_1 + x_2 + x_3 = 0 \quad (*) \\ x_1 + 2x_2 - 3x_3 = 0 \rightarrow x_3 = \dfrac{x_1 + 2x_2}{3} \quad (**) \\ x_1^2 + x_2^2 + x_3^2 = 1 \end{cases}$$

$$\underline{(*),(**)} \rightarrow x_1 + x_2 + \frac{x_1 + x_2}{3} = 0 \rightarrow 3x_1 + 3x_2 + x_1 + 2x_2 = 0$$

$$\rightarrow 4x_1 + 5x_2 = 0 \rightarrow x_1 = -\frac{5}{4}x_2 \quad (o)$$

$$\underline{(o),(**)} \rightarrow x_3 = \frac{-5/4\, x_2 + 8/4\, x_2}{3} = \frac{3/4\, x_2}{3} = \frac{1}{4}x_2 \rightarrow x_2 = 4x_3$$

$$\Rightarrow X = \begin{bmatrix} -5\cancel{x_3} \\ 4\cancel{x_3} \\ \cancel{x_3} \end{bmatrix} \Rightarrow \begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix} \longrightarrow \text{unit vector} \left\{ \begin{bmatrix} -5/\sqrt{42} \\ 4/\sqrt{42} \\ 1/\sqrt{42} \end{bmatrix}, \begin{bmatrix} 5/\sqrt{42} \\ -4/\sqrt{42} \\ -1/\sqrt{42} \end{bmatrix} \right\}$$

$$\rightarrow Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & -5/\sqrt{42} \\ 1/\sqrt{3} & 2/\sqrt{14} & 4/\sqrt{42} \\ 1/\sqrt{3} & -3/\sqrt{14} & 1/\sqrt{42} \end{bmatrix}$$

13. Apply the Gram-Schmidt process to

$$a = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and write the result in the form $A = QR$.

(13)

$$a = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad c = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A = a \qquad B = b - \frac{A^T b}{A^T A} A \longrightarrow B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$q_a = \frac{A}{\|A\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad q_b = \frac{B}{\|B\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad q_c = \frac{C}{\|C\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} q_a & q_b & q_c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A = QR \longrightarrow A = \begin{bmatrix} \vdots & \vdots & \vdots \\ a & b & c \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} q_a & q_b & q_c \end{bmatrix} \begin{bmatrix} q_a^T a & q_a^T b & q_a^T c \\ & q_b^T b & q_b^T c \\ & & q_c^T c \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Thanks for your attention