Linear Algebra

Chapter 2: Vector spaces

Solution of highlighted problems

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3. Describe the column space and the nullspace of the matrices

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

③

a) $A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \rightarrow C(A) = c\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad N(A) = c\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

b) $B = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix} \rightarrow C(B) = c_1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$BX = \vec{0} \Rightarrow \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0} \Rightarrow X_3 = 0, \quad x_1 = -2x_2$

$N(B) = c\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix}$

c) $C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow C(C) = c\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad N(C) = \mathbb{R}^3$

Problem set 2.1

24. For which vectors (b_1, b_2, b_3) do these systems have a solution?

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

㉔

a) Each column vector in matrix "A" is independent from the other ones. So, they are the basis of C(A)

The equation (*) is solvable if and only if the vector "b" can be expressed as a combination of the columns of "A". So, "b" must be in C(A)

$$A\vec{X} = \vec{b} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad (*)$$

$$\searrow x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

b) $$A\vec{X} = \vec{b} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad (**)$$

basis of "A" $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (x_2 + x_3) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

the equation (**) is solvable

" " " " ".

5. Write the complete solutions $x = x_p + x_n$ to these systems, as in equation (4):

$$\begin{bmatrix}1&2&2\\2&4&5\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}=\begin{bmatrix}1\\4\end{bmatrix} \qquad \begin{bmatrix}1&2&2\\2&4&4\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}=\begin{bmatrix}1\\4\end{bmatrix}.$$

⑤

a)

$$\overset{A}{\begin{bmatrix}1&2&2\\2&4&5\end{bmatrix}}\begin{bmatrix}u\\v\\w\end{bmatrix}=\begin{bmatrix}1\\4\end{bmatrix}$$

$$\xrightarrow{R_2=R_2-2R_1}\begin{bmatrix}1&2&2\\0&0&1\end{bmatrix}\xrightarrow{R_1=R_1-2R_2}\begin{bmatrix}1&2&0\\0&0&1\end{bmatrix}$$

u and w are pivot vars.

v free variable.

Null space ?

$R\vec{N}=\vec{0}$

$$v=1 \Rightarrow \begin{bmatrix}1&2&2\\0&0&1\end{bmatrix}\begin{bmatrix}u\\1\\w\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \longrightarrow \begin{cases}w=0\\u+2w=-2\end{cases} \longrightarrow w=0,\ u=-2,\ v=1$$

$$v=0 \Rightarrow \begin{bmatrix}1&2&2\\0&0&1\end{bmatrix}\begin{bmatrix}u\\0\\w\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \longrightarrow \begin{cases}w=0\\u+2w=0\end{cases} \longrightarrow w=0,\ u=0,\ v=0$$

$$N(A)=c\begin{bmatrix}-2\\1\\0\end{bmatrix}$$

Particular solution ?

set "v" to zero

$$\begin{bmatrix}1&2&2\\2&4&5\end{bmatrix}\begin{bmatrix}u\\0\\w\end{bmatrix}=\begin{bmatrix}1\\4\end{bmatrix} \longrightarrow \begin{cases}u+2w=1\\2u+5w=4\end{cases} \longrightarrow \begin{cases}-2u-4w=-2\\2u+5w=4\end{cases}$$

$w=2$
$u=-3$

$$x_p=\begin{bmatrix}-3\\0\\2\end{bmatrix}$$

Complete solution ?

$$x_{complete}=x_p+x_N=\begin{bmatrix}-3\\0\\2\end{bmatrix}+c\begin{bmatrix}-2\\1\\0\end{bmatrix}$$

b)

$$\begin{bmatrix}1&2&2\\2&4&4\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}=\begin{bmatrix}1\\4\end{bmatrix}$$

$$RREF=\begin{bmatrix}1&2&2\\0&0&0\end{bmatrix}$$

u is pivot variable

v, w are free vars

Nullspace ?

$$v=1\quad w=0 \rightarrow \begin{bmatrix}1&2&2\\0&0&0\end{bmatrix}\begin{bmatrix}u\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \rightarrow u=-2,\ v=1,\ w=0$$

$$v=0,\ w=1 \rightarrow \begin{bmatrix}1&2&2\\0&0&0\end{bmatrix}\begin{bmatrix}u\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \rightarrow u=-2,\ v=0,\ w=1$$

$$N(A)=C_1\begin{bmatrix}-2\\1\\0\end{bmatrix}+C_2\begin{bmatrix}-2\\0\\1\end{bmatrix}$$

Particular solution ?

set v, w to zero

$$\begin{bmatrix}1&2&2\\2&4&4\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\4\end{bmatrix} \quad \times$$

No particular solution

Complete solution ?

So, the complete solution will be the set of all vectors in the null space of the system.

Problem set 2.2

13. Find the reduced row echelon forms R and the rank of these matrices:

(a) The 3 by 4 matrix of all 1s.

(b) The 4 by 4 matrix with $a_{ij} = (-1)^{ij}$.

(c) The 3 by 4 matrix with $a_{ij} = (-1)^j$.

⑬ A = 4 by 4 matrix with $a_{ij} = (-1)^{ij}$

b)

$$A = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[P_4 = P_4 + P_1]{\substack{P_2 = P_2 + P_1 \\ P_3 = P_3 - P_1}} \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix} \xrightarrow{P_4 = P_4 - P_2} \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{P_2 = P_2/2} \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{P_1 = P_1 - P_2} \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{P_1 = -P_1} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{rank} = 2$$

20. *If A has rank r, then it has an r by r submatrix S that is invertible.* Find that submatrix S from the pivot rows and pivot columns of each A:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

⑳

a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \longrightarrow \text{rref} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ rank = 2 $\longrightarrow S_{2\times 2} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \longrightarrow \text{rref} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ rank = 1 $\longrightarrow S_{1\times 1} = 1$

35. What conditions on b_1, b_2, b_3, b_4 make each system solvable? *Solve for* x:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

ⓢ35

a) $$\left[\begin{array}{cc:c} 1 & 2 & b_1 \\ 2 & 4 & b_2 \\ 2 & 5 & b_3 \\ 3 & 9 & b_4 \end{array}\right] \longrightarrow \begin{bmatrix} 1 & 2 & b_1 \\ 0 & 0 & b_2-2b_1 \\ 0 & 1 & b_3-2b_1 \\ 0 & 3 & b_4-3b_1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & b_1-2b_3+4b_1 \\ 0 & 0 & b_2-2b_1 \\ 0 & 1 & b_3-2b_1 \\ 0 & 0 & b_4-3b_1-3b_3+6b_1 \end{bmatrix}$$

Solvability Condition

$$\begin{cases} b_2-2b_1=0 \\ b_4+3b_1-3b_3=0 \end{cases}$$

61. Construct a matrix whose nullspace consists of all multiples of $(4, 3, 2, 1)$.

(61)

① $N(A) = c\begin{bmatrix}4\\3\\2\\1\end{bmatrix} \longrightarrow A = \begin{bmatrix}1 & 0 & 0 & -4\\0 & 1 & 0 & -3\\0 & 0 & 1 & -2\end{bmatrix}$

② $N(A) = \begin{bmatrix}11\\8\\3\end{bmatrix} \longrightarrow A = \begin{bmatrix}1 & 0 & -\frac{11}{3}\\0 & 1 & -\frac{8}{3}\end{bmatrix}$

3. Prove that if $a = 0$, $d = 0$, or $f = 0$ (3 cases), the columns of U are dependent:

$$U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}.$$

③

$$U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \xrightarrow{d=0} \begin{bmatrix} a & b & c \\ 0 & 0 & e \\ 0 & 0 & f \end{bmatrix}$$

$$c_1 \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} c \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (*)$$

if $c_3 = 0$, $c_1 = -b$, $c_2 = a$, then the equation $(*)$ is equal to zero.
we can conclude that the columns are dependent.

20. Find a basis for each of these subspaces of $\mathbf{R}^4$:

(a) All vectors whose components are equal.

(b) All vectors whose components add to zero.

(c) All vectors that are perpendicular to $(1,1,0,0)$ and $(1,0,1,1)$.

(d) The column space (in $\mathbf{R}^2$) and nullspace (in $\mathbf{R}^5$) of $U = \begin{bmatrix}1&0&1&0&1\\0&1&0&1&0\end{bmatrix}$.

(20)

d) $U = \begin{bmatrix}1&0&1&0&1\\0&1&0&1&0\end{bmatrix}$

* $C(U) = c_1\begin{bmatrix}1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\end{bmatrix}$ basis $= \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$

* $U\vec{N}=\vec{0}$ $\begin{bmatrix}\underbrace{\begin{matrix}1&0\\0&1\end{matrix}}_{\text{Pivot vars}} & \underbrace{\begin{matrix}1&0&1\\0&1&0\end{matrix}}_{\text{free vars}}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$x_3=1,\ x_4=0,\ x_5=0$

$$\begin{bmatrix}1&0&1&0&1\\0&1&0&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \longrightarrow \begin{cases}x_1+1=0\\x_2=0\end{cases} \longrightarrow X=\begin{bmatrix}-1\\0\\1\\0\\0\end{bmatrix}$$

$x_3=0,\ x_4=1,\ x_5=0$

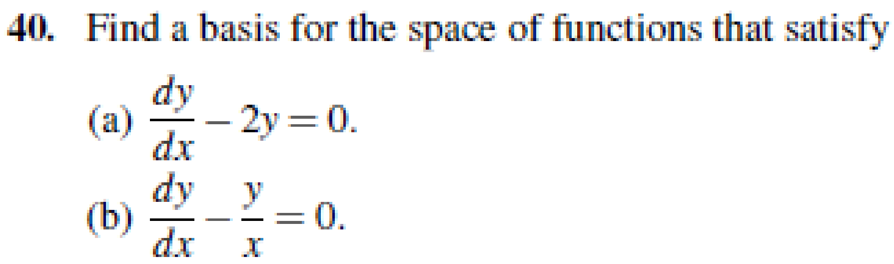
$$\begin{bmatrix}1&0&1&0&1\\0&1&0&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \longrightarrow \begin{cases}x_1=0\\x_2+1=0\end{cases} \longrightarrow X=\begin{bmatrix}0\\-1\\0\\1\\0\end{bmatrix}$$

$x_3=0,\ x_4=0,\ x_5=1$

$$\begin{bmatrix}1&0&1&0&1\\0&1&0&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \longrightarrow \begin{cases}x_1+1=0\\x_2=0\end{cases} \longrightarrow X=\begin{bmatrix}-1\\0\\0\\0\\1\end{bmatrix}$$

$$N(U) = c_1\begin{bmatrix}-1\\0\\1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\-1\\0\\1\\0\end{bmatrix} + c_3\begin{bmatrix}-1\\0\\0\\0\\1\end{bmatrix}$$

Problem set 2.3



40. Find a basis for the space of functions that satisfy

(a) $\frac{dy}{dx} - 2y = 0.$

(b) $\frac{dy}{dx} - \frac{y}{x} = 0.$

(40)

a) $\frac{dy}{dx} - 2y = 0 \longrightarrow \frac{dy}{dx} = 2y \longrightarrow y = c_1 e^{2x}$ basis of "y" = e^{2x}

b) $\frac{dy}{dx} - \frac{y}{x} = 0 \longrightarrow \frac{dy}{dx} = \frac{y}{x} \longrightarrow y = C_1 x$ basis of "y" = x

6. Suppose A is an m by n matrix of rank r. Under what conditions on those numbers does

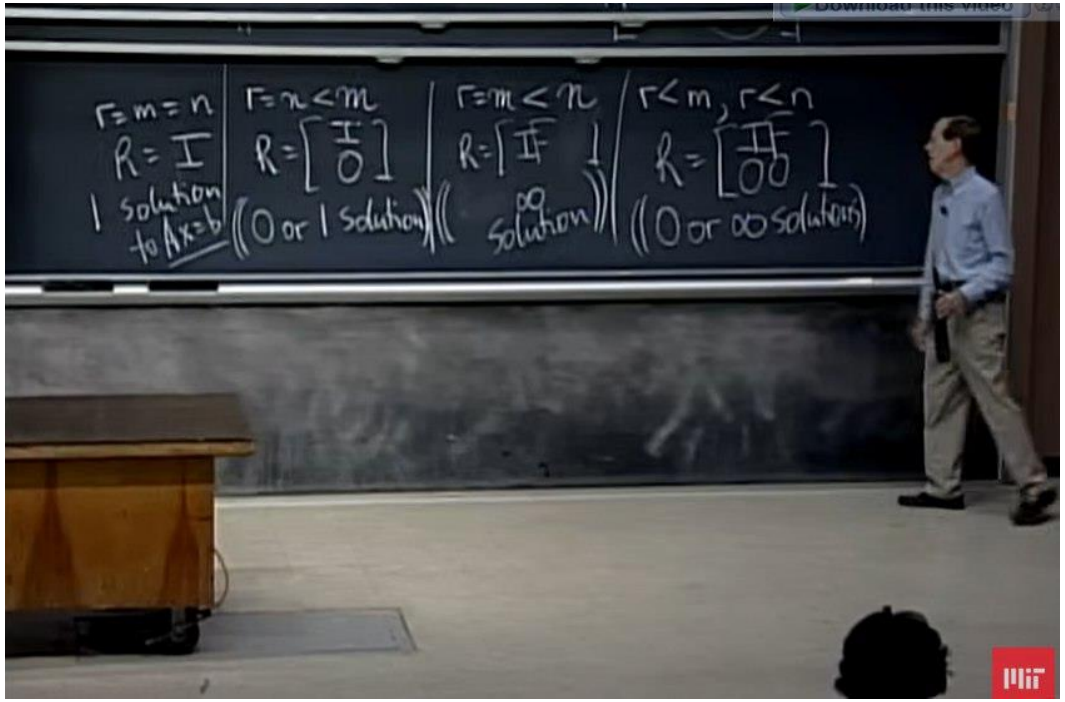
(a) A have a two-sided inverse: $AA^{-1} = A^{-1}A = I$?

(b) $Ax = b$ have *infinitely many solutions* for *every* b?

⑥ a) $AA^{-1} = I$, $A^{-1}A = I$

The above equations are true if and only if $m=n=r$ or we have full rank square matrix.

b) $Ax=b$ has infinitely many solutions for every "b" if and only if
$r<n$



9. Find a 1 by 3 matrix whose nullspace consists of all vectors in $\mathbf{R}^3$ such that $x_1 + 2x_2 + 4x_3 = 0$. Find a 3 by 3 matrix with that same nullspace.

⑨

$$x_1 + 2x_2 + 4x_3 = 0$$

$$A = \begin{bmatrix} \underset{\text{Pivot var}}{\boxed{1}} & \overset{\text{free vars}}{\boxed{2 \quad 4}} \end{bmatrix} \longrightarrow \text{rank} = 1$$

$x_2 = 1,\ x_3 = 0$
$$\begin{bmatrix} 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ 1 \\ 0 \end{bmatrix} = 0 \longrightarrow x_1 + 2 = 0 \longrightarrow x_1 = -2$$

$x_2 = 0,\ x_3 = 1$
$$\begin{bmatrix} 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ 1 \end{bmatrix} = 0 \longrightarrow x_1 + 4 = 0 \longrightarrow x_1 = -4$$

$$\text{basis of } N(A) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Example
$$\longrightarrow A_{3\times3} = \begin{bmatrix} 0.5 & 1 & 2 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$$

12. Find the rank of A and write the matrix as $A = uv^{\mathrm{T}}$:

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 6 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 2 & -2 \\ 6 & -6 \end{bmatrix}.$$

⑫

a) $$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 6 \end{bmatrix} \longrightarrow \text{rank}(A) = 1$$

Every matrix of rank 1 has a simple form of $A = uv^T$ = column times row

$$A = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 3 \end{bmatrix}$$

b) $$A = \begin{bmatrix} 2 & -2 \\ 6 & -6 \end{bmatrix} \longrightarrow \text{rank}(A) = 1$$

$$A = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix}$$

35. Suppose A is the sum of two matrices of rank one: $A = uv^{\mathrm{T}} + wz^{\mathrm{T}}$.

(a) Which vectors span the column space of A?

(b) Which vectors span the row space of A?

(c) The rank is less than 2 if ____ or if ____.

(d) Compute A and its rank if $u = z = (1,0,0)$ and $v = w = (0,0,1)$.

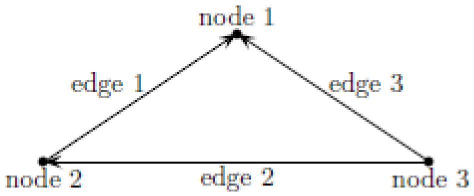
㉟ $A = uv^T + wz^T$

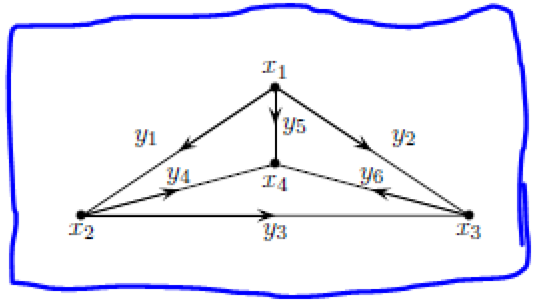
a) vectors span column space of $A \Rightarrow u, w$

b) vector span colum space of A^T or row space of $A \rightarrow v, z$

c) $\underline{u}$ and $\underline{\underline{w}}$ are linearly dependent

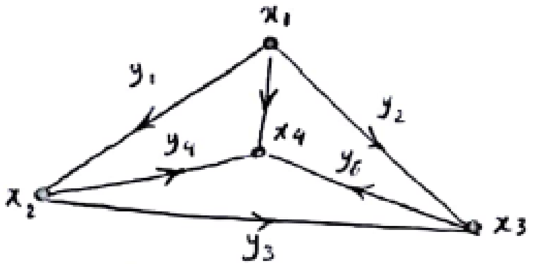
1. For the 3-node triangular graph in the figure following, write the 3 by 3 incidence matrix A. Find a solution to $Ax = 0$ and describe all other vectors in the nullspace of A. Find a solution to $A^{\mathrm{T}}y = 0$ and describe all other vectors in the left nullspace of A.





①

b)



$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \longrightarrow A\vec{x} = \vec{0} \longrightarrow N(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$A^T = \begin{bmatrix} -1 & -1 & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \longrightarrow A^T\vec{y} = \vec{0}$$

vector representation of inner loops that lie on the left null space

$$N(A^T) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

8. Write down the dimensions of the four fundamental subspaces for this 6 by 4 incidence matrix, and a basis for each subspace.

⑧

Column space $C(A) \longrightarrow \dim(C(A)) = 3 = \text{rank}(A)$

basis of $C(A) = \left\{ \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}$

Row space $C(A^T) \longrightarrow \dim(C(A^T)) = 3 = \text{rank}(A^T)$

basis of $C(A^T) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Null space $(N(A)) \longrightarrow \dim(N(A)) = n - r = 4 - 3 = 1$

lef null space $(N(A^T)) \longrightarrow \dim(N(A^T)) = m - r = 6 - 3 = 3$

Thanks for your attention