Linear Algebra

Chapter 4: Determinants

Solution of highlighted problems

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1. If a 4 by 4 matrix has $\det A = \frac{1}{2}$, find $\det(2A)$, $\det(-A)$, $\det(A^2)$, and $\det(A^{-1})$.

(1)

(a) $\det A = \frac{1}{2} \longrightarrow \det(2A) = \begin{vmatrix} 2a & 2b & 2c & 2d \\ 2e & 2f & 2g & 2H \\ 2i & 2j & 2k & 2l \\ 2m & 2n & 2o & 2p \end{vmatrix} = 2(2(2(2(\det A)))) = 2^4 \det A$

$= 2^4\left(\frac{1}{2}\right) = 2^3$

(b)

$\det(-A) = -1(-1(-1(-1(\det A)))) = (-1)^4 \det A = 1 \times \frac{1}{2} = \frac{1}{2}$

(c)

$\det(A^2) = |A^2| = |A||A| = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$

(d) $\det(A^{-1}) = ?$

$\det(A^{-1}A) = \det(I) \longrightarrow \det(A^{-1}) \cdot \det(A) = \det(I) \longrightarrow \det(A^{-1}) = \dfrac{1}{\det(A)}$

$\longrightarrow \det(A^{-1}) = \dfrac{1}{1/2} = 2$

4. By applying row operations to produce an upper triangular U, compute

$$\det\begin{bmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix} \quad \text{and} \quad \det\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix}.$$

Exchange rows 3 and 4 of the second matrix and recompute the pivots and determinant.

(4)

b) $A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix} \xrightarrow{\text{Gaussian Elimination}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix} = A'$

$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2/3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A' = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix} = A''$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3/4 & 1 \end{bmatrix} A'' = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & -11/4 \end{bmatrix} = A'''$

$\det(A''') = 2\left(\tfrac{3}{2}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{-11}{4}\right) = -11$

$\det(A) = \det(A''')$

new $A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & -1 & 2 & -1 \end{bmatrix} \longrightarrow A' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & -1 & 2 & -1 \end{bmatrix}$

$A'' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{2}{3} & 0 & 1 \end{bmatrix} A' = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 4/3 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{4}{3} & 1 \end{bmatrix} A'' = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -\frac{11}{3} \end{bmatrix} = A'''$

$\longrightarrow \det(A''') = 2 \times \frac{3}{2} \times -1 \times -\frac{11}{3} = \underline{\underline{11}} = \det(\overset{\text{new}}{A}) = \det(\underset{\text{old}}{PA})$

$\det(\underset{\text{old}}{PA}) = \det(P) \cdot \det(\underset{\text{old}}{A}) = \det(\underset{\substack{\text{Permutation matrix} \\ \text{for 1 exchange.}}}{P}) \cdot (-11) = (-1)(-11) = \underline{11}$

7. Find the determinants of:

(a) a rank one matrix

$$A = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}$$

⑦

a) $A = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 \\ 8 & -4 & 8 \\ 4 & -2 & 4 \end{bmatrix} \xrightarrow[r_3 = r_3 - 2r_1]{r_2 = r_2 - 4r_1} \begin{bmatrix} 2 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ⑤

$\det(A) = \det\left(\begin{bmatrix} 2 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \overset{⑥}{\Longrightarrow} \det(A) = 0$

(b) the upper triangular matrix

$$U = \begin{bmatrix} 4 & 4 & 8 & 8 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

(c) the lower triangular matrix U^{T}.

(d) the inverse matrix U^{-1}.

(e) the "reverse-triangular" matrix that results from row exchanges,

$$M = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 1 & 2 & 2 \\ 4 & 4 & 8 & 8 \end{bmatrix}.$$

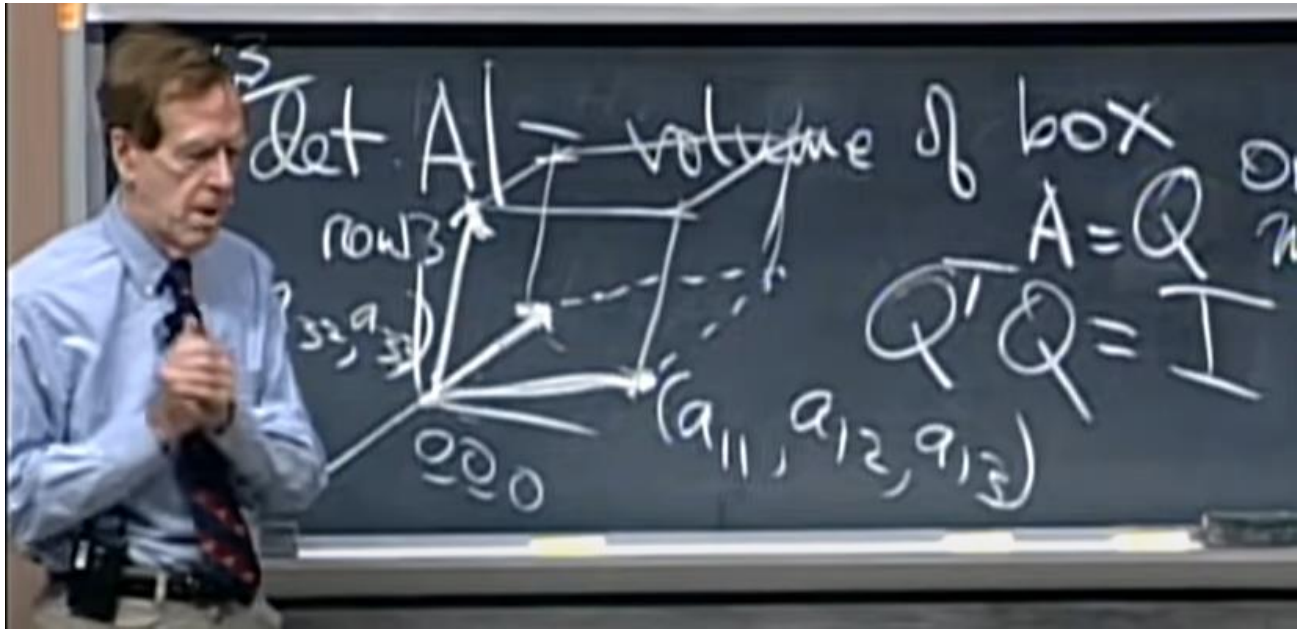
10. If Q is an orthogonal matrix, so that $Q^{\mathrm{T}}Q = I$, prove that $\det Q$ equals $+1$ or -1. What kind of box is formed from the rows (or columns) of Q?

(10)

$$Q^TQ = I \rightarrow \det(Q^TQ) = \det(I) = \det(Q^T)\det(Q) = \det(I)$$

$$\rightarrow \det(Q^T) = \frac{\det(I)}{\det(Q)} = \frac{1}{\det(Q)} \xrightarrow{\det(Q^T) = \det(Q)} \det(Q)^2 = 1$$

$$\rightarrow \det(Q) = \mp 1$$



14. True or false, with reason if true and counterexample if false:

(a) If A and B are identical except that $b_{11} = 2a_{11}$, then $\det B = 2\det A$.

(b) The determinant is the product of the pivots.

(c) If A is invertible and B is singular, then $A+B$ is invertible.

(d) If A is invertible and B is singular, then AB is singular.

(e) The determinant of $AB - BA$ is zero.

(14)

(a) True

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(1) → $\det(A) = (1)(1)(1)$, $\det B = (2)(1)(1) \Rightarrow \det(B) = 2\det(A)$

(3) → $|B| = \begin{vmatrix} 2(1) & 2(0) & 2(0) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \Rightarrow \det(B) = 2\det(A)$

(b) False

If the matrix is not square, it doesn't have determinant by itself.

(c) False

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow A+B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 = r_1 - r_2 - r_3} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) → $\det(A+B) = 0 \longrightarrow$ ∴ $(A+B)$ matrix is not invertible

(d) True

$\det(A) \neq 0$

$\det(B) = 0 \leftarrow B$ is singular

$AB \rightsquigarrow \det(AB) = \det(A)\det(B) = 0 \longrightarrow$ we can conclude that $\underline{AB}$ will be singular, if B is defined as singular matrix.

(e) False

$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$AB - BA = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -5 & -3 \\ -3 & 5 \end{bmatrix}$

→ $\det\left(\begin{bmatrix} -5 & -3 \\ -3 & 5 \end{bmatrix}\right) = (-25) - (9) = \boxed{-34}$

24. Use row operations to simplify and compute these determinants:

$$\det\begin{bmatrix}101 & 201 & 301\\ 102 & 202 & 302\\ 103 & 203 & 303\end{bmatrix} \quad \text{and} \quad \det\begin{bmatrix}1 & t & t^2\\ t & 1 & t\\ t^2 & t & 1\end{bmatrix}.$$

(24)

a) $\det\begin{bmatrix}101 & 201 & 301\\ 102 & 202 & 302\\ 103 & 203 & 303\end{bmatrix} \overset{(5)}{=} \det\begin{bmatrix}101 & 201 & 301\\ 1 & 1 & 1\\ 2 & 2 & 2\end{bmatrix} \overset{(5)}{=} \det\begin{bmatrix}101 & 201 & 301\\ 1 & 1 & 1\\ 0 & 0 & 0\end{bmatrix} \overset{(6)}{=} 0$

b) $\det\begin{bmatrix}1 & t & t^2\\ t & 1 & t\\ t^2 & t & 1\end{bmatrix} = ?$

$$\underbrace{\begin{bmatrix}1 & 0 & 0\\ -t & 1 & 0\\ 0 & 0 & 1\end{bmatrix}}_{E_{21}}\begin{bmatrix}1 & t & t^2\\ t & 1 & t\\ t^2 & t & 1\end{bmatrix} = \begin{bmatrix}1 & t & t^2\\ 0 & 1-t^2 & t-t^3\\ t^2 & t & 1\end{bmatrix} \rightarrow \underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -t^2 & 0 & 1\end{bmatrix}}_{E_{31}}\begin{bmatrix}1 & t & t^2\\ 0 & 1-t^2 & t-t^3\\ t^2 & t & 1\end{bmatrix} = \begin{bmatrix}1 & t & t^2\\ 0 & 1-t^2 & t-t^3\\ 0 & t-t^3 & 1-t^4\end{bmatrix}$$

$$\underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -t & 1\end{bmatrix}}_{E_{32}} \rightsquigarrow \begin{bmatrix}1 & t & t^2\\ 0 & 1-t^2 & t-t^3\\ 0 & 0 & (1-t^4)-t(t-t^3)\end{bmatrix} \Rightarrow \text{DET} = (1)(1-t)(1+t)\left[(1-t^2)(1+t^2) - t^2(1-t)(1+t)\right]$$

$$\Rightarrow \text{DET} = (1-t^2)(1-t^2)\left[(1+t^2)-t^2\right] = (1-t^2)^2$$

25. Elimination reduces A to U. Then $A = LU$:

$$A = \begin{bmatrix} 3 & 3 & 4 \\ 6 & 8 & 7 \\ -3 & 5 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{bmatrix} = LU.$$

Find the determinants of L, U, A, $U^{-1}L^{-1}$, and $U^{-1}L^{-1}A$.

25 $A = LU \rightarrow$ U is obtained from elimination procedure that will lead to no changes in the value of initial matrix A

$\star$ $\det(U) = (3)(2)(-1) = -6 = \det(A)$

$\star$ $A = LU \rightarrow \det(A) = \det(LU) \rightarrow \det(A) = \det(L)\det(U) \rightarrow \det(L) = 1$

$\star$ $U^{-1}L^{-1} = (LU)^{-1} \xrightarrow{LU = A} (LU)^{-1}(LU) = I \rightarrow \det((LU)^{-1}) = \dfrac{1}{\det(LU)} = \dfrac{1}{\det(A)} = -\dfrac{1}{6}$

$\star$ $U^{-1}L^{-1}A = (LU)^{-1}A = \dfrac{1}{\det(LU)} \cdot \det(A) = \dfrac{1}{\det(A)} \cdot \det(A) = 1$

Proof of Big Formula

determinant → square matrix → $n = m$ → n^n

2x2 (1)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix}$$

$\boxed{2^2 = 4}$

$\boxed{3^3 = 27}$

3x3 (2)

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ d & e & f \\ g & h & i \end{vmatrix} = \left(\begin{vmatrix} a & 0 & 0 \\ d & 0 & 0 \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & 0 & 0 \\ 0 & e & 0 \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & 0 & 0 \\ 0 & 0 & f \\ g & h & i \end{vmatrix} \right)$$

$$+ \left(\begin{vmatrix} 0 & b & 0 \\ d & 0 & 0 \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ 0 & e & 0 \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ 0 & 0 & f \\ g & h & i \end{vmatrix} \right) + \left(\begin{vmatrix} 0 & 0 & c \\ d & 0 & 0 \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ 0 & e & 0 \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ 0 & 0 & f \\ g & h & i \end{vmatrix} \right)$$

$$= \left(\begin{vmatrix} a & 0 & 0 \\ d & 0 & 0 \\ g & 0 & 0 \end{vmatrix} + \begin{vmatrix} a & 0 & 0 \\ d & 0 & 0 \\ 0 & h & 0 \end{vmatrix} + \begin{vmatrix} a & 0 & 0 \\ d & 0 & 0 \\ 0 & 0 & i \end{vmatrix} \right) + \left(\begin{vmatrix} a & 0 & 0 \\ 0 & e & 0 \\ g & 0 & 0 \end{vmatrix} + \begin{vmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & h & 0 \end{vmatrix} + \begin{vmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{vmatrix} \right) + \left(\begin{vmatrix} a & 0 & 0 \\ 0 & 0 & f \\ g & 0 & 0 \end{vmatrix} + \begin{vmatrix} a & 0 & 0 \\ 0 & 0 & f \\ 0 & h & 0 \end{vmatrix} + \begin{vmatrix} a & 0 & 0 \\ 0 & 0 & f \\ 0 & 0 & i \end{vmatrix} \right)$$

$$+ \left(\begin{vmatrix} 0 & b & 0 \\ d & 0 & 0 \\ g & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ d & 0 & 0 \\ 0 & h & 0 \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ d & 0 & 0 \\ 0 & 0 & i \end{vmatrix} \right) + \left(\begin{vmatrix} 0 & b & 0 \\ 0 & e & 0 \\ g & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ 0 & e & 0 \\ 0 & h & 0 \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{vmatrix} \right) + \left(\begin{vmatrix} 0 & b & 0 \\ 0 & 0 & f \\ g & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ 0 & 0 & f \\ 0 & h & 0 \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ 0 & 0 & f \\ 0 & 0 & i \end{vmatrix} \right)$$

$$+ \left(\begin{vmatrix} 0 & 0 & c \\ d & 0 & 0 \\ g & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ d & 0 & 0 \\ 0 & h & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ d & 0 & 0 \\ 0 & 0 & i \end{vmatrix} \right) + \left(\begin{vmatrix} 0 & 0 & c \\ 0 & e & 0 \\ g & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ 0 & e & 0 \\ 0 & h & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ 0 & e & 0 \\ 0 & 0 & i \end{vmatrix} \right) + \left(\begin{vmatrix} 0 & 0 & c \\ 0 & 0 & f \\ g & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ 0 & 0 & f \\ 0 & h & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ 0 & 0 & f \\ 0 & 0 & i \end{vmatrix} \right)$$

$$= \sum_{\text{all perm}} \left(a_{1\alpha}\, a_{2\beta}\, a_{3\gamma} \right) \det(P)$$

1. For these matrices, find the only nonzero term in the big formula (6):

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

①

a)

Exchange row 1 & row 2
row 3 & row 4

$$A = \begin{bmatrix} 0 & \underline{1} & 0 & 0 \\ \underline{1} & 0 & 1 & 0 \\ 0 & 1 & 0 & \underline{1} \\ 0 & 0 & \underline{1} & 0 \end{bmatrix} \longrightarrow \det A = (+1)[a_{12}\, a_{21}\, a_{34}\, a_{43}] = \boxed{1}$$

Exchange row 1 & row 3

b) $$B = \begin{bmatrix} 0 & 0 & \underline{1} & 2 \\ 0 & \underline{3} & 4 & 5 \\ \underline{6} & 7 & 8 & 9 \\ 0 & 0 & 0 & \underline{1} \end{bmatrix} \longrightarrow \det B = (-1)[b_{13}\, b_{22}\, b_{31}\, b_{44}] = (-1)[1 \times 3 \times 6 \times 1] = -18$$

3. *True or false?*

(a) The determinant of $S^{-1}AS$ equals the determinant of A.

(b) If $\det A = 0$ then at least one of the cofactors must be zero.

(c) A matrix whose entries are 0s and 1s has determinant 1, 0, or -1.

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a) True

$$\det(S^{-1}AS) = |S^{-1}||A||S| = \frac{1}{|S|} \cdot |A| \cdot |S| = |A|$$

b)

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{Cofactor matrix of } A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\det(A) = 0 \longrightarrow \text{False}$$

c)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \longrightarrow \det A = (1)(1) + (-1)(-1) = 2 \longrightarrow \text{False}$$

11. If A is m by n and B is n by m, explain why

$$\det\begin{bmatrix} 0 & A \\ -B & I \end{bmatrix} = \det AB. \qquad \left(\textit{Hint:} \text{ Postmultiply by } \begin{bmatrix} I & 0 \\ B & I \end{bmatrix}.\right)$$

Do an example with $m < n$ and an example with $m > n$. Why does your second example automatically have $\det AB = 0$?

(11)

Prove $\det\begin{bmatrix} 0 & A \\ -B & I \end{bmatrix}$; $\underline{\underline{A}}$ is m by n $\underline{\underline{B}}$ is n by m

$$\underbrace{\begin{bmatrix} 0_{m\times m} & A_{m\times n} \\ -B_{n\times m} & I_{n\times n} \end{bmatrix}}_{Z} \underbrace{\begin{bmatrix} I_{m\times m} & 0_{m\times n} \\ B_{n\times m} & I_{n\times n} \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} AB_{m\times m} & A_{m\times n} \\ 0_{n\times m} & I_{n\times n} \end{bmatrix}}_{Y}$$

$\det(Z).\det(X) = \det(Y)$

$\det(X) = 1$

$$\det(Y) = \sum_{\text{all permutations}} (\underbrace{a_{1\alpha}\, a_{2\beta} \cdots a_{m\gamma}}_{m}\ \underbrace{1\times 1\times - \times 1}_{n})\ \det(P_{m\times m}) =$$

$$= \sum (a_{1\alpha}\, a_{2\beta} - a_{m\gamma}) \det(P_{m\times m}) = \sum \mp (a_{1\alpha}\, a_{2\beta} - a_{m\gamma}) = \det(AB)$$

example :

(1) $m < n \Rightarrow m = 1,\ n = 2$

$A = \begin{bmatrix} 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$

$$\det\begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 6 & 0 & 1 \end{bmatrix} = (6)(0-2) + (1)(0-2) = -14 = \det(AB)$$

(2) $m > n \Rightarrow m = 2,\ n = 1$

$A = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $B = \begin{bmatrix} 5 & -2 \end{bmatrix}$

$$\det\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 3 \\ -5 & 2 & 1 \end{bmatrix} = (1)(0-0) = 0 = \det AB$$

$$\det AB = \begin{vmatrix} 5 & -2 \\ 15 & -6 \end{vmatrix} = -30 + 30 = 0$$

24. Find cofactors and then transpose. Multiply C_A^{T} and C_B^{T} by A and B!

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix}.$$

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$$A = \begin{bmatrix} 2 & 1 \\ 3 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix}$$

a) $$C_A = \begin{bmatrix} 6 & -3 \\ -1 & 2 \end{bmatrix} \qquad C_A^T A = \begin{bmatrix} 6 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} = \det(A) I$$

b) $$C_B = \begin{bmatrix} 0 & 42 & -35 \\ 0 & -21 & 14 \\ -3 & 6 & -3 \end{bmatrix} \qquad C_B^T B = \begin{bmatrix} 0 & 0 & -3 \\ 42 & -21 & 6 \\ -35 & 14 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -21 & 0 & 0 \\ 0 & -21 & 0 \\ 0 & 0 & -21 \end{bmatrix} = \det(B) I$$

28. The n by n determinant C_n has 1s above and below the main diagonal:

$$C_1 = \begin{vmatrix} 0 \end{vmatrix} \quad C_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \quad C_3 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \quad C_4 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}.$$

(a) What are the determinants of C_1, C_2, C_3, C_4?

(b) By cofactors find the relation between C_n and C_{n-1} and C_{n-2}. Find C_{10}.

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$$C_1 = \begin{bmatrix} 0 \end{bmatrix} \quad C_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad C_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad C_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\det(C_1) = 0 \qquad \det(C_2) = 0 - 1 = -1$

$\det(C_3) = (1)(-1)(0) = 0$

$\det(C_4) = (1)(-1)\begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = (-1)(1)(-1)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$|C_4| = -|C_2| \qquad |C_3| = -|C_1| \qquad \Rightarrow |C_n| = -|C_{n-2}|$

$|C_{10}| = -|C_8| \rightarrow |C_{10}| = -(-|C_6|) = -(-(-|C_4|)) = -1$

1. Find the determinant and all nine cofactors C_{ij} of this triangular matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Form C^{T} and verify that $AC^{\mathrm{T}} = (\det A)I$. What is A^{-1}?

1

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} \rightarrow C_A = \begin{bmatrix} 20 & 0 & 0 \\ -10 & 5 & 0 \\ -12 & 0 & 4 \end{bmatrix}$$

$$\det(A) = (4)(+1)(5) = 20$$

$$A C_A^{T} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 20 & -10 & -12 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix} = \det(A)\, I \quad \checkmark$$

$$A^{-1} = \frac{1}{\det(A)} C^{T} = \frac{1}{20} \begin{bmatrix} 20 & -10 & -12 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 & -3/5 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$$

Problem set 4.4

3. Find x, y, and z by Cramer's Rule in equation (4):

$$\begin{aligned} ax + by &= 1 \\ cx + dy &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x + 4y - z &= 1 \\ x + y + z &= 0 \\ 2x \qquad + 3z &= 0. \end{aligned}$$

3

b) $$\underbrace{\begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{b}$$

$$x_1 = \frac{\begin{vmatrix} 1 & 4 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 4 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{vmatrix}} = \frac{(3)(+1)(1-0)}{(2)(+1)(4+1) + (3)(+1)(1-4)} = \frac{3}{10-9} = 3$$

$$x_2 = \frac{\begin{vmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 0 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 4 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{vmatrix}} = \frac{(1)(-1)(3-2)}{10-9} = \frac{-1}{1} = -1$$

$$x_3 = \frac{\begin{vmatrix} 1 & 4 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 4 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{vmatrix}} = \frac{(1)(+1)(0-2)}{10-9} = -2$$

$$A^{-1} = \frac{1}{\det(A)} C_A^T \longrightarrow x = A^{-1} b = \frac{1}{\det A} C_A^T b$$

$$\longrightarrow x_i = \frac{\det B_i}{\det A} \rightsquigarrow B_i = \underline{\underline{A}} \text{ with column } \underline{\underline{i}} \text{ replaced by } \underline{\underline{b}}$$

14. Use Cramer's Rule to solve for y (only). Call the 3 by 3 determinant D:

(a) $\begin{aligned} ax + by &= 1 \\ cx + dy &= 0. \end{aligned}$ (b) $\begin{aligned} ax + by + cz &= 1 \\ dx + ey - fz &= 0 \\ gx + hy + iz &= 0. \end{aligned}$

(14)

b) $\begin{bmatrix} a & b & c \\ d & e & -f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$y = \frac{\begin{vmatrix} a & 1 & c \\ d & 0 & -f \\ g & 0 & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & -f \\ g & h & i \end{vmatrix}} = \frac{(1)(-1)(di+gf)}{a(ei+hf) - b(di+gf) + c(dh-eg)}$$

29. (a) The corners of a triangle are $(2,1)$, $(3,4)$, and $(0,5)$. What is the area?

(b) A new corner at $(-1,0)$ makes it lopsided (four sides). Find the area.

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a) $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 4 & 1 \\ 0 & 5 & 1 \end{bmatrix} \to \det A = -5(2-3) + 1(8-3) = 10$

area of triangle $= \frac{1}{2} \det A = 5$

NOTE: we can change the coordinates with respect to the origin.

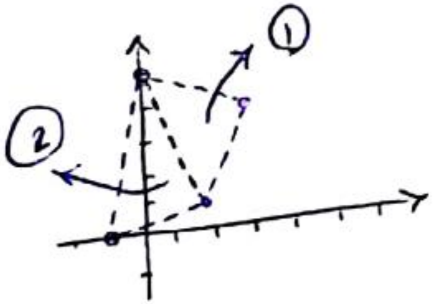
we should subtract all the point from $(2,1)$.

$(0,0)$; $(1,3)$; $(-2,4)$

area of triangle $= \frac{1}{2} \det \left(\begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix} \right) = \frac{1}{2}(4+6) = 5$

b)

$$\text{area} = \frac{1}{2}\left(\det\left(\begin{bmatrix} 2 & 1 & 1 \\ 3 & 4 & 1 \\ 0 & 5 & 1 \end{bmatrix} \right) + \det\left(\begin{bmatrix} 2 & 1 & 1 \\ 0 & 5 & 1 \\ -1 & 0 & 1 \end{bmatrix} \right) \right)$$



$= \frac{1}{2}(10 + (-1)(1-5) + (1)(10))$

$= \frac{1}{2}(10 + 4 + 10) = 12$

Thanks for your attention