Linear Algebra

Chapter 6: Positive Definite Matrices

Solution of highlighted problems

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2. Decide for or against the positive definiteness of these matrices, and write out the corresponding $f = x^{\mathrm{T}}Ax$:

(a) $\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. (c) $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$. (d) $\begin{bmatrix} -1 & 2 \\ 2 & -8 \end{bmatrix}$.

The determinant in (b) is zero; along what line is $f(x,y) = 0$?

2 **checking Positive definiteness**

a) $\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix} \to \det(A-\lambda I)=0 \to \begin{vmatrix} 1-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = (1-\lambda)(5-\lambda)-9 = 5-6\lambda+\lambda^2-9$

$= \lambda^2-6\lambda-4=0 \qquad \lambda = \dfrac{6 \mp \sqrt{36-4(1)(-4)}}{2(1)} = \dfrac{6\mp\sqrt{52}}{2} = 3 \mp \sqrt{13}$

* As all the eigen values are not positive, we can conclude that the matrix is not a positive definite matrix.

d) $\begin{bmatrix} -1 & 2 \\ 2 & 8 \end{bmatrix} \to \det(A-\lambda I)=0 \to \begin{vmatrix} -1-\lambda & 2 \\ 2 & 8-\lambda \end{vmatrix} = -(1+\lambda)(8-\lambda)-4 = 0$

$-(8-\lambda+8\lambda-\lambda^2)-4=0 \to -8+\lambda-8\lambda+\lambda^2-4=0 \to -8-7\lambda+\lambda^2-4=0$

$\to \lambda^2-7\lambda-12=0 \to \lambda = \dfrac{7\mp\sqrt{97}}{2}$ not positive definite

$\begin{bmatrix} \boxed{-1} & 2 \\ 2 & 8 \end{bmatrix} \to \det([-1]) = -1 \ ; \ \det\left(\begin{bmatrix} -1 & 2 \\ 2 & 8 \end{bmatrix}\right) = -8-4 = -12$

* As the 2 sub determinants are not positive, the matrix is not Positive definite.

b) $f(x_1,x_2) = X^TBX = [x_1 \ x_2]\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1 \ x_2]\begin{bmatrix} x_1-x_2 \\ -x_1+x_2 \end{bmatrix} = x_1(x_1-x_2)+x_2(-x_1+x_2)$

$= x_1^2 - x_1x_2 - x_2x_1 + x_2^2 = x_1^2 - 2x_1x_2 + x_2^2 = (x_1-x_2)^2 = 0 \longrightarrow x_1 = x_2$

This means that the quadratic form $f(x_1,x_2)$ is zero along the line where $x_1 = x_2$

4. Decide between a minimum, maximum, or saddle point for the following functions.

(a) $F = -1 + 4(e^x - x) - 5x\sin y + 6y^2$ at the point $x = y = 0$.

(b) $F = (x^2 - 2x)\cos y$, with stationary point at $x = 1$, $y = \pi$.

(4)

a) $F = -1 + 4(e^x - x) - 5x\sin y + 6y^2 \qquad (x,y) = (0,0)$

$\dfrac{\partial F}{\partial x} = 4e^x - 4 - 5\sin y \rightarrow \dfrac{\partial F}{\partial x}(0,0) = 0$

$\dfrac{\partial F}{\partial y} = -5x\cos y + 12y \rightarrow \dfrac{\partial F}{\partial y}(0,0) = 0$

In both first-order partial derivatives, we have the values of zero at the point $(0,0)$, so it's the critical point.

$\dfrac{\partial^2 F}{\partial x^2} = 4e^x \qquad \dfrac{\partial^2 F}{\partial y^2} = 12 + 5x\sin y \qquad \dfrac{\partial^2 F}{\partial x \partial y} = -5\cos y$

At point $(0,0)$, there second-order partial derivatives are equal to:

$\dfrac{\partial^2 F}{\partial x^2}(0,0) = 4 \qquad \dfrac{\partial^2 F}{\partial y^2}(0,0) = 12 \qquad \dfrac{\partial^2 F}{\partial x \partial y}(0,0) = -5$

at $(0,0)$

$\rightarrow f_{xx} f_{yy} = 4 \times 12 = 48$

$f_{xy} = -5 \rightarrow f_{xy}^2 = 25$

$\rightarrow f_{xx} f_{yy} - (f_{xy})^2 = 48 - 25 = 23 \quad (*)$

$f_{xx} = 4 \quad (**)$

by considering $(*)$, $(**)$, we can conclude that $F(x,y)$ has a minimum at $(0,0)$.

4. Decide between a minimum, maximum, or saddle point for the following functions.

(a) $F = -1 + 4(e^x - x) - 5x\sin y + 6y^2$ at the point $x = y = 0$.

(b) $F = (x^2 - 2x)\cos y$, with stationary point at $x = 1$, $y = \pi$.

b)

$F = (x^2 - 2x)\cos y$ stationary Point = (0, 0)

$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}(x^2\cos y - 2x\cos y) = 2x\cos y - 2\cos y = 2(x-1)\cos y$

$\frac{\partial F}{\partial y} = (-\sin y)(x^2 - 2x) = (2x - x^2)\sin y$

at $(1, \pi)$ both derivatives are equal to zero, so $(1, \pi)$ is stationary or critical point.

$\frac{\partial^2 F}{\partial x^2} = 2\cos y$ $\frac{\partial^2 F}{\partial y^2} = (2x - x^2)\cos y$ $\frac{\partial^2 F}{\partial x \partial y} = 2(1-x)\sin y$

Alternatively, we can consider another approach to figure out whether the critical point is minimum, maximum or saddle point.

Hessian Matrix $= \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ $\longrightarrow$ the matrix is symmetric.

- $\det(H) = f_{xx} f_{yy} - f_{xy}^2 \xrightarrow{\text{at } (1,\pi)} (-2)(-1) - (0) = 2$
- $f_{xx} = -2$

$\left. \begin{array}{l} \det(H) \\ f_{xx} < 0 \end{array} \right\} \rightarrow F(x,y)$ has a Maximum.

20. For $F_1(x,y) = \frac{1}{4}x^4 + x^2y + y^2$ and $F_2(x,y) = x^3 + xy - x$, find the second derivative matrices A_1 and A_2:

$$A = \begin{bmatrix} \partial^2 F/\partial x^2 & \partial^2 F/\partial x\partial y \\ \partial^2 F/\partial y\partial x & \partial^2 F/\partial y^2 \end{bmatrix}.$$

A_1 is positive definite, so F_1 is concave up (= convex). Find the minimum point of F_1 and the saddle point of F_2 (look where first derivatives are zero).

(20)

$F_1(x,y) = \frac{1}{4}x^4 + x^2y + y^2 \longrightarrow F_1(x,y) = (\frac{1}{2}x^2+y)^2 \geq 0$

$\frac{\partial^2 F_1}{\partial x^2} = \frac{\partial}{\partial x}(x^3 + 2xy) = 3x^2 + 2y$

$\frac{\partial^2 F_1}{\partial y^2} = \frac{\partial}{\partial y}(x^2+2y) = 2$ $\quad \frac{\partial^2 F_1}{\partial x \partial y} = \frac{\partial}{\partial x}(x^2+2y) = 2x$

Hessian Matrix $= \begin{bmatrix} 3x^2+2y & 2x \\ 2x & 2 \end{bmatrix}$

Principal minors

- $\det(A_1) = 2(3x^2+2y) - 4x^2 = 6x^2+4y-4x^2 = 2x^2+4y = x^2+2y$
- $3x^2+2y$

$f_x = x^3+2xy = 0$

$\Rightarrow (x^2+2y)(x) = 0$

$x=0$

$y = -\frac{x^2}{2}$

$f_{1y} = x^2+2y = 0$

$\Rightarrow x^2 = -2y \rightarrow y = -\frac{x^2}{2}$

So the critical point is $(x, -\frac{x^2}{2})$

by considering the critical point, we have positive semi-definite matrix.

$\xrightarrow{y=-\frac{x^2}{2}} H = \begin{bmatrix} 2x^2 & 2x \\ 2x & 2 \end{bmatrix} \rightarrow$ ① $\det(H) = 2(2x^2) - 4x^2 = 0$

② $2x^2 \geq 0$

As the matrix's all principal minors are non-negative our Hessian matrix is positive semi-definite.

$x^T A x \geq 0$

being a positive semi definite indicates that the quadratic function has a minimum or a plateau at the critical point.

due to having minimum with respect to any value of $x\ (f(x, -\frac{x^2}{2}) = 0)$, the quadratic function has Plateau on the critical points.

20. For $F_1(x,y) = \frac{1}{4}x^4 + x^2y + y^2$ and $F_2(x,y) = x^3 + xy - x$, find the second derivative matrices A_1 and A_2:

$$A = \begin{bmatrix} \partial^2 F/\partial x^2 & \partial^2 F/\partial x \partial y \\ \partial^2 F/\partial y \partial x & \partial^2 F/\partial y^2 \end{bmatrix}.$$

A_1 is positive definite, so F_1 is concave up (= convex). Find the minimum point of F_1 and the saddle point of F_2 (look where first derivatives are zero).

20

$$F_2(x,y) = x^3 + xy - x$$

$$\frac{\partial F_2}{\partial x} = 3x^2 + y - 1 = 0 \xrightarrow{(*)} y = 1$$

$$\frac{\partial F_2}{\partial y} = x = 0 \quad (*)$$

so the critical point is $(0,1)$

$$\frac{\partial^2 F_2}{\partial x^2} = 6x \qquad \frac{\partial^2 F_2}{\partial x \partial y} = \frac{\partial}{\partial x}(x) = 1 \qquad \frac{\partial^2 F_2}{\partial y^2} = \frac{\partial}{\partial y}(x) = 0$$

$$H = \begin{bmatrix} 6x & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Principal minors

- $|0| = 0$
- $\det(H) = (0)(0) - 1 = -1$

eigen values

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \rightarrow \lambda = \{-1, 1\}$$

our function has a saddle point, as the eigen values take both signs at the critical point. and our matrix is indefinite.

1. For what range of numbers a and b are the matrices A and B positive definite?

$$A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & b & 8 \\ 4 & 8 & 7 \end{bmatrix}.$$

(1)

a) $A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$ A is a positive definite matrix if all of its sub determinants are greater than zero.

$(*)\ |a| = a > 0$

$(**)\ \begin{vmatrix} a & 2 \\ 2 & a \end{vmatrix} = a^2 - 4 > 0 \rightarrow a > 2 \ , \ a < -2$

$(***)\ \begin{vmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{vmatrix} = (a)\begin{vmatrix} a & 2 \\ 2 & a \end{vmatrix} - (2)\begin{vmatrix} 2 & 2 \\ 2 & a \end{vmatrix} + 2\begin{vmatrix} 2 & a \\ 2 & 2 \end{vmatrix}$

$= a(a^2-4) - 2(2a-4) + 2(4-2a)$

$= a(a+2)(a-2) - 4(a-2) - 4(a-2) = a(a+2)(a-2) - 8(a-2)$

$= (a-2)(a^2+2a-8) = (a-2)(a-2)(a+4) = (a-2)^2(a+4) > 0$

$\Rightarrow a > -4$

$\rightarrow (*) \cap (**) \cap (***) \Rightarrow a > 2$

3. Construct an indefinite matrix with its largest entries on the main diagonal:

$$A = \begin{bmatrix} 1 & b & -b \\ b & 1 & b \\ -b & b & 1 \end{bmatrix} \quad \text{with } |b| < 1 \text{ can have } \det A < 0.$$

③ $A = \begin{bmatrix} 1 & b & -b \\ b & 1 & b \\ -b & b & 1 \end{bmatrix}$ $|b|<1$ $\det A < 0$

Principal minors

$|A_1| = 1 > 0$

$|A_3| < 0$

$\hookrightarrow (1)\begin{vmatrix} 1 & b \\ b & 1 \end{vmatrix} - b\begin{vmatrix} b & b \\ -b & 1 \end{vmatrix} - b\begin{vmatrix} b & 1 \\ -b & b \end{vmatrix}$

$= (1-b^2) - b(b+b^2) - b(b^2+b) = 0$

$= (1-b)(1+b) - b^2(1+b) - b^2(b+1) = 0$

$= (1+b)(1-b-2b^2) = 0 \longrightarrow b = \{-1, \frac{1 \mp \sqrt{9}}{4}\}$

$b = \{-1, -\frac{1}{2}, 1\}$

		-1	$-\frac{1}{2}$	1
$(1+b)$	$-$	$+$	$+$	$+$
$(1-b-2b^2)$	$-$	$-$	$+$	$-$
	$+$	$-$	$+$	$-$

$\longrightarrow -1 < b < -\frac{1}{2}$

14. Decide whether the following matrices are positive definite, negative definite, semidefinite, or indefinite:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 6 & -2 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 3 \end{bmatrix}, \qquad C = -B, \qquad D = A^{-1}.$$

Is there a real solution to $-x^2 - 5y^2 - 9z^2 - 4xy - 6xz - 8yz = 1$?

(14)

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{bmatrix}$

$(*)\ |1| > 0 \qquad (**)\ \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 5 - 4 > 0$

$(***)\ (1)\begin{vmatrix} 5 & 4 \\ 4 & 9 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} + 3\begin{vmatrix} 2 & 5 \\ 3 & 4 \end{vmatrix} = 29 - 2(6) + 3(-7) = 29 - 33 = -4 < 0$

Since the principal minors didn't change the sign in order, the matrix isn't Negative definite and Negative semi-definite. Regarding det(A) is equal to -4 it isn't Positive definite and Positive semi-definite. it's indefinite.

$D = A^{-1} = \dfrac{1}{|A|}\left(\begin{bmatrix} 29 & -6 & -7 \\ -6 & 0 & 2 \\ -7 & 2 & 1 \end{bmatrix}\right)^T = \dfrac{-1}{4}\begin{bmatrix} 29 & -6 & -7 \\ -6 & 0 & 2 \\ -7 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -29/4 & 6/4 & 7/4 \\ 6/4 & 0 & -2/4 \\ 7/4 & -2/4 & -1/4 \end{bmatrix}$

$(*)\ |-29/4| = -\frac{29}{4} < 0 \qquad (**)\ \begin{vmatrix} -29/4 & 6/4 \\ 6/4 & 0 \end{vmatrix} = -\frac{36}{16} < 0$

$(***)\ |D| = -\frac{6}{4}\begin{vmatrix} 6/4 & 7/4 \\ -2/4 & -1/4 \end{vmatrix} + \frac{2}{4}\begin{vmatrix} -29/4 & 6/4 \\ 7/4 & -2/4 \end{vmatrix}$

$= -\frac{6}{4}\left(-\frac{6}{16} + \frac{14}{16}\right) + \frac{2}{4}\left(\frac{58}{16} - \frac{42}{16}\right) = \frac{-6}{4}\cdot\frac{8}{16} + \frac{2}{4}\left(\frac{16}{16}\right)$

$= \frac{-3}{2}\cdot\frac{1}{2} + \frac{1}{2} = \frac{-3}{4} + \frac{2}{4} = \frac{-1}{4} < 0.$

As the principal minors are all negative, they don't satisfy the conditions of being positive definite, positive semi-definite, negative definite, and negative semi-definite. therefore, our matrix is indefinite.

14. Decide whether the following matrices are positive definite, negative definite, semidefinite, or indefinite:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 6 & -2 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 3 \end{bmatrix}, \qquad C = -B, \qquad D = A^{-1}.$$

Is there a real solution to $-x^2 - 5y^2 - 9z^2 - 4xy - 6xz - 8yz = 1$?

$$B = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 6 & -2 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 3 \end{bmatrix}$$

(*) $|1| > 0$ (**) $\begin{vmatrix} 1 & 2 \\ 2 & 6 \end{vmatrix} = 6 - 4 = 2 > 0$

(***) $\begin{vmatrix} 1 & 2 & 0 \\ 2 & 6 & -2 \\ 0 & -2 & 5 \end{vmatrix} = (1)\begin{vmatrix} 6 & -2 \\ -2 & 5 \end{vmatrix} - 2\begin{vmatrix} 2 & -2 \\ 0 & 5 \end{vmatrix} = (1)(30-4) + (-2)(10) = 26 - 20 = 6 > 0$

(***) $|B| = (1)\begin{vmatrix} 6 & -2 & 0 \\ -2 & 5 & -2 \\ 0 & -2 & 3 \end{vmatrix} + (2)\begin{vmatrix} 2 & -2 & 0 \\ 0 & 5 & -2 \\ 0 & -2 & 3 \end{vmatrix}$

$= (1)(6(11) + (2)(-6)) + (2)(2)(11) = 66 - 12 + 44 = 98 > 0$

all principal minors are greater than zero, so B is positive definite.

$$C = -B = \begin{bmatrix} -1 & -2 & 0 & 0 \\ -2 & -6 & 2 & 0 \\ 0 & 2 & -5 & 2 \\ 0 & 0 & 2 & -3 \end{bmatrix}$$

(*) $|-1| < 0$ (**) $\begin{vmatrix} -1 & -2 \\ -2 & -6 \end{vmatrix} = (-1)^2(2) = 2 > 0$

(***) $\begin{vmatrix} -1 & -2 & 0 \\ -2 & -6 & 2 \\ 0 & 2 & -5 \end{vmatrix} = (-1)^3(6) = -6 < 0$

(****) $C = |-B| = (-1)^4|B| = (-1)^4(98) = 98 > 0$

Its principal minors are alternating in sign and start with negative value, so we can conclude that our matrix is negative definite.

14. Decide whether the following matrices are positive definite, negative definite, semidefinite, or indefinite:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 6 & -2 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 3 \end{bmatrix}, \qquad C = -B, \qquad D = A^{-1}.$$

Is there a real solution to $-x^2 - 5y^2 - 9z^2 - 4xy - 6xz - 8yz = 1$?

$F(x,y,z) = -x^2 - 5y^2 - 9z^2 - 4xy - 6xz - 8yz$

$\frac{\partial^2 F}{\partial x^2} = -2 \quad \frac{\partial^2 F}{\partial z^2} = -18 \quad \frac{\partial^2 F}{\partial y^2} = -10$

$\frac{\partial^2 F}{\partial x \partial y} = -4 \quad \frac{\partial^2 F}{\partial x \partial z} = -6 \quad \frac{\partial^2 F}{\partial z \partial y} = -8$

$H = \begin{bmatrix} -2 & -4 & -6 \\ -4 & -10 & -8 \\ -6 & -8 & -18 \end{bmatrix}$

$(*)\ |-2| < 0$

$(**)\ \begin{vmatrix} -2 & -4 \\ -4 & -10 \end{vmatrix} = 20 - 16 = 4 > 0$

$(***)\ \det(H) = (-2)\begin{vmatrix} -10 & -8 \\ -8 & -18 \end{vmatrix} + (4)\begin{vmatrix} -4 & -8 \\ -6 & -18 \end{vmatrix} + (-6)\begin{vmatrix} -4 & -10 \\ -6 & -8 \end{vmatrix}$

$= (-2)(180 - 64) + (4)(72 - 48) + (-6)(32 - 60)$

$= -232 + 96 + 168 = -232 + 264 = 32$

As the subdeterminants have both signs, the hessian matrix is indefinite.

Hence we can say that $F(x,y,z) = 1$ has a real solution.

Example

1. Compute $A^{\mathrm{T}}A$ and its eigenvalues σ_1^2, 0 and unit eigenvectors v_1, v_2:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}.$$

(1)

$$A^T A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 20 \\ 20 & 80 \end{bmatrix}$$

$$\rightarrow \begin{vmatrix} 5-\lambda & 20 \\ 20 & 80-\lambda \end{vmatrix} = 0 \rightarrow (5-\lambda)(80-\lambda) - 400 = 0$$

$$\rightarrow 400 - 85\lambda + \lambda^2 - 400 = 0 \rightarrow \lambda(\lambda - 85) = 0 \rightarrow \lambda_1 = 0 \quad \lambda_2 = 85$$

$$* \begin{bmatrix} 5 & 20 \\ 20 & 80 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

$$* \begin{bmatrix} -80 & 20 \\ 20 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\rightarrow V = \begin{bmatrix} -4/\sqrt{17} & 1/\sqrt{17} \\ 1/\sqrt{17} & 4/\sqrt{17} \end{bmatrix}$$

$\lambda_1 = \sigma_1^2$

$\lambda_2 = \sigma_2^2$

5. Compute $A^{\mathrm{T}}A$ and AA^{T}, and their eigenvalues and unit eigenvectors, for

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Multiply the three matrices $U\Sigma V^{\mathrm{T}}$ to recover A.

(5)

$$A^TA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)\left((2-\lambda)(1-\lambda)-1\right) - 1(1-\lambda)$$

$$= (1-\lambda)(2-3\lambda+\lambda^2-1) - 1(1-\lambda) = (1-\lambda)(\lambda^2-3\lambda) = (1-\lambda)\lambda(\lambda-3) = 0$$

$$\rightarrow \lambda \in \{1, 0, 3\}$$

$$\xrightarrow{\lambda_1=1} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow x = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \xrightarrow[\text{unit}]{\text{make}} \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$\xrightarrow{\lambda_2=0} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{\text{Elimination}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow x = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \xrightarrow[\text{unit}]{\text{make}} \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$$

$$\xrightarrow{\lambda_3=3} \begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{\text{Elimination}} \begin{bmatrix} -1 & 0.5 & 0 \\ 0 & -0.5 & 1 \\ 0 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 0.5 & 0 \\ 0 & 0 & 0 \\ 0 & 0.5 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\longrightarrow x = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \xrightarrow[\text{unit}]{\text{make}} \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 3/\sqrt{6} \\ 3/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A u_2 = \frac{1}{\sqrt{1}}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \rightarrow U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$A = U\Sigma V^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix}$$

10. Suppose A is a 2 by 2 symmetric matrix with unit eigenvectors u_1 and u_2. If its eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$, what are U, Σ, and V^{T}?

(10)

1. A is symmetric

2. Its eigenvalues are 3, -2.

$$A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \qquad A^T A = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\det(A^T A - \lambda I) = 0 \longrightarrow \lambda = \{9, 4\} \longrightarrow \sigma = \sqrt{\lambda} = \{3, 2\}$$

$$\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$(AA^T - 9I)X = 0 \longrightarrow \begin{bmatrix} 0 & 0 \\ 0 & -5 \end{bmatrix} X = 0 \longrightarrow X = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = v_1$$

$$(AA^T - 4I)X = 0 \longrightarrow \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} X = 0 \longrightarrow X = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = v_2$$

$$\longrightarrow V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = u_1$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{2}\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = u_2$$

$$\longrightarrow U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A = U \Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thanks for your attention