Linear Algebra

Chapter 5: Eigenvalues & Eigenvectors

Solution of highlighted problems

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12. Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}.$$

(12)

a) $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \rightsquigarrow \det(A - \lambda I) = 0$

$\begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0 \rightarrow -(9-\lambda^2) - 16 = 0$

$\rightarrow \lambda^2 = 25 \rightarrow \lambda = \mp 5$

* $\lambda = -5 \Rightarrow \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow N(A+5I) = c \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

* $\lambda = 5 \Rightarrow \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow N(A-5I) = c \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Eigen value $= \{-5, 5\}$ Eigen vector $= \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

18. Suppose A has eigenvalues 0, 3, 5 with independent eigenvectors u, v, w.

(a) Give a basis for the nullspace and a basis for the column space.

(b) Find a particular solution to $Ax = v + w$. Find all solutions.

(c) Show that $Ax = u$ has no solution. (If it had a solution, then ____ would be in the column space.)

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$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix} \Rightarrow \det(A - \lambda I) = 0 \Rightarrow (0-\lambda)(3-\lambda)(5-\lambda) = 0$$

$\lambda = 0$, $\lambda = 3$, $\lambda = 5$

$$* \ \lambda = 0 \Rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow N(A - 0I) = c_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$* \ \lambda = 3 \Rightarrow \begin{bmatrix} -3 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow N(A - 3I) = c_2\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

$$* \ \lambda = 5 \Rightarrow \begin{bmatrix} -5 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow N(A - 5I) = c_3\begin{bmatrix} 3/5 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{cases} -5x_1 + x_2 + x_3 = 0 \rightarrow -5x_1 = -3x_2 \\ -2x_2 + x_3 = 0 \end{cases} \Rightarrow x_1 = \frac{3}{5}x_2 \ ; \ x_3 = 2x_2$$

$$\text{Eigen vector} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3/5 \\ 1 \\ 2 \end{bmatrix} \right\}$$

a) Nullspace of $A = c_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ Column space of A: $c_1\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$

b) $$Ax = v + w \longrightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8/5 \\ 4 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow{r_3 = \frac{1}{5}r_3} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 = r_1 - r_3} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 = r_2 - r_3} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 = r_2 - 3r_1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{rref}$$

x_1 is free var ; x_2, x_3 are pivot vars

$$\text{Set } x_1 \xrightarrow{} 0 \quad \begin{bmatrix} 0 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8/5 \\ 4 \\ 2 \end{bmatrix} \xrightarrow[\text{Back substitution}]{} x_3 = 2/5 \rightarrow \begin{matrix} 3x_2 = 18/5 \\ x_2 = 6/5 \end{matrix} \Rightarrow x_2 + x_3 = 8/5 \ \checkmark$$

$$\text{Particular solution} = \begin{bmatrix} 0 \\ 6/5 \\ 2/5 \end{bmatrix}$$

Nullspace of A

$x_1 = 0$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = 1$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow Ax = v + w$

Complete solution $= x_p + x_N$

$$= \begin{bmatrix} 0 \\ 6/5 \\ 2/5 \end{bmatrix} + c\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

c) $\underline{u}$ is in column space

15. Factor these two matrices into $A = S\Lambda S^{-1}$:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

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a) $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \Rightarrow \det(A - \lambda I) = 0 \Rightarrow (1-\lambda)(3-\lambda) = 0 \Rightarrow \lambda = \{1, 3\}$

$$\left.\begin{aligned} &* \ (A - 1I)X = 0 \rightarrow \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} X = 0 \rightarrow X = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &* \ (A - 3I)X = 0 \rightarrow \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} X = 0 \rightarrow X = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}\right\} \rightarrow S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$S^{-1} = \frac{1}{\det(S)} C_S^T = \frac{1}{1}\left(\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\right)^T = \frac{1}{1}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

19. True or false: If the n columns of S (eigenvectors of A) are independent, then

(a) A is invertible.

(b) A is diagonalizable.

(c) S is invertible.

(d) S is diagonalizable.

(19)

a) A is invertible $\rightarrow$ the factor of non-singularity may not ensure that the matrix has $\underline{\underline{n}}$ independent eigen vectors. False

b) A is diagonalizable

$\rightarrow$ If the matrix S is Full-rank or non-singular, the matrix A is diagonalizable True

c) S is invertible $\rightarrow$ because $\underline{\underline{S}}$ has $\underline{\underline{n}}$ independent Column vector and respectively non-singular, It is invertible True

d) S is diagonalizable $\rightarrow$ the non-singularity of S may not be resulted in diagonalizability of $\underline{\underline{S}}$.

32. Diagonalize A and compute $S\Lambda^k S^{-1}$ to prove this formula for A^k:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{has} \quad A^k = \frac{1}{2}\begin{bmatrix} 3^k+1 & 3^k-1 \\ 3^k-1 & 3^k+1 \end{bmatrix}.$$

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$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow \det(A - I\lambda) = 0$$

$$\Rightarrow (2-\lambda)^2 - 1 = 0 \rightarrow \lambda = 1, 3$$

$$\left.\begin{aligned} * \ (A-I)X &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} X = 0 \rightarrow X = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ * \ (A-3I)X &= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} X = 0 \rightarrow X = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}\right\} \Rightarrow S = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$S^{-1} = \frac{1}{\det(S)} C_S^T = \frac{1}{-2}\left(\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}\right)^T = \frac{1}{-2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$\Lambda^k = \begin{bmatrix} 1^k & 0 \\ 0 & 3^k \end{bmatrix}$$

$$A^k = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1^k & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} \times 3^k & \frac{1}{2} \times 3^k \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1+3^k) & \frac{1}{2}(3^k-1) \\ \frac{1}{2}(3^k-1) & \frac{1}{2}(1+3^k) \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1+3^k & 3^k-1 \\ 3^k-1 & 1+3^k \end{bmatrix}$$

2. Bernadelli studied a beetle "which lives three years only. and propagates in as third year." They survive the first year with probability $\frac{1}{2}$, and the second with probability $\frac{1}{3}$, and then produce six females on the way out:

$$\textbf{Beetle matrix} \qquad A = \begin{bmatrix} 0 & 0 & 6 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}.$$

Show that $A^3 = I$, and follow the distribution of 3000 beetles for six years.

(2)

$$\text{Beetle Matrix} = \begin{bmatrix} 0 & 0 & 6 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix} = A$$

$$A^3 = \begin{bmatrix} 0 & 0 & 6 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 6 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 6 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 6 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1/6 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$u_0 = \begin{bmatrix} 3000 \\ 0 \\ 0 \end{bmatrix}$$

$$u(6) = A^6 u(0) = (A^3)^2 u(0) = I^2 u(0) = u(0) = \begin{bmatrix} 3000 \\ 0 \\ 0 \end{bmatrix}$$

10. Find the limiting values of y_k and $_k$ $(k \to \infty)$ if

$$y_{k+1} = .8y_k + .3z_k \qquad y_0 = 0$$
$$z_{k+1} = .2y_k + .7z_k \qquad z_0 = 5.$$

Also find formulas for y_k and z_k from $A^k = S\Lambda^k S^{-1}$.

(10) $u_k = \begin{bmatrix} y_k \\ z_k \end{bmatrix}$

$u_{k+1} = M u_k \longrightarrow \begin{bmatrix} y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} y_k \\ z_k \end{bmatrix}$

$\det(M - \lambda I) = \begin{vmatrix} 0.8-\lambda & 0.3 \\ 0.2 & 0.7-\lambda \end{vmatrix} = (0.8-\lambda)(0.7-\lambda) - 0.06 = 0$

$\Rightarrow 0.56 - \lambda(0.8) - \lambda(0.7) + \lambda^2 - 0.06 = 0.5 - 1.5\lambda + \lambda^2 = (0.5-\lambda)(1-\lambda) = 0$

$\Rightarrow \lambda_1 = 1, \ \lambda_2 = 0.5$

$\lambda_1 = 1$

$\begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{cases} -0.2x_1 + 0.3x_2 = 0 \\ 0.2x_1 - 0.3x_2 = 0 \end{cases} \qquad x_1 = 3/2\, x_2$

$N(M-I) = c\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$

$\lambda_2 = 0.5$

$\begin{bmatrix} 0.3 & 0.3 \\ 0.2 & 0.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow N(M - 0.5I) = c\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$u_0 = c_1x_1 + c_2x_2 = \begin{bmatrix} 0 \\ 5 \end{bmatrix} = c_1\begin{bmatrix} 3/2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 3/2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} \longrightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

$u_k = A^k u_0 = 2\lambda_1^k \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} + (-3)\lambda_2^k \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2(1)^k \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} + (-3)(0.5)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lim_{k\to\infty} \left(\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} -3(0.5)^k \\ 3(0.5)^k \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$A^k = S\Lambda^k S^{-1}$

$S = \begin{bmatrix} 3/2 & 1 \\ 1 & -1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \quad S^{-1} = \frac{1}{|S|} C_S^T = \frac{1}{-5/2} \left(\begin{bmatrix} -1 & -1 \\ -1 & 3/2 \end{bmatrix} \right)^T$

$S^{-1} = \begin{bmatrix} 2/5 & 2/5 \\ 2/5 & -3/5 \end{bmatrix} \Rightarrow A^k = \begin{bmatrix} 3/2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & (\frac{1}{2})^k \end{bmatrix} \begin{bmatrix} 2/5 & 2/5 \\ 2/5 & -3/5 \end{bmatrix}$

$= \begin{bmatrix} 3/2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2/5 & 2/5 \\ (2/5)(\frac{1}{2})^k & (-3/5)(\frac{1}{2})^k \end{bmatrix} = \begin{bmatrix} (3/2)(2/5) + (2/5)(\frac{1}{2})^k & (3/2)(2/5) + (-3/5)(\frac{1}{2})^k \\ (2/5) - (2/5)(\frac{1}{2})^k & (2/5) + (3/5)(\frac{1}{2})^k \end{bmatrix}$

$= \begin{bmatrix} 3/5 + \frac{1}{5}(\frac{1}{2})^{k-1} & 3/5 - 3/5(\frac{1}{2})^k \\ 2/5 - \frac{1}{5}(\frac{1}{2})^{k-1} & 2/5 + 3/5(\frac{1}{2})^k \end{bmatrix}$

$u_k = A^k u_0 \Rightarrow \lim_{k\to\infty} u_k = \lim_{k\to\infty} A^k u_0 = \begin{bmatrix} 3/5 & 3/5 \\ 2/5 & 2/5 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

1. Following the first example in this section, find the eigenvalues and eigenvectors, and the exponential e^{At}, for

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

(1)

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \det(A-\lambda I)=0 \rightarrow (-1-\lambda)^2 - 1 = 0 \rightarrow 1 + 2\lambda + \lambda^2 - 1 = 0$$

$$\lambda = \{0, -2\}$$

$$* \ (A - 0I)X = 0 \rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \rightarrow x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$* \ (A + 2I)X = 0 \rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \rightarrow x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\rightarrow S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$e^{At} = S e^{\Lambda t} S^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{0t} & 0 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ -1/2\, e^{-2t} & 1/2\, e^{-2t} \end{bmatrix}$$

$$\boxed{S^{-1} = \frac{1}{\det(S)} \left(\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \right)^T = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}}$$

$$= \begin{bmatrix} 1/2 + \frac{1}{2}e^{-2t} & \frac{1}{2} - \frac{1}{2}e^{-2t} \\ \frac{1}{2} - \frac{1}{2}e^{-2t} & 1/2 + 1/2\, e^{-2t} \end{bmatrix}$$

5. A diagonal matrix like $\Lambda = \left[\begin{smallmatrix}1 & 0\\ 0 & 2\end{smallmatrix}\right]$ satisfies the usual rule $e^{\Lambda(t+T)} = e^{\Lambda t}e^{\Lambda T}$, because the rule holds for each diagonal entry.

(a) Explain why $e^{A(t+T)} = e^{At}e^{AT}$, using the formula $e^{At} = Se^{\Lambda t}S^{-1}$.

(b) Show that $e^{A+B} = e^Ae^B$ is *not true* for matrices, from the example

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \qquad \text{(use series for } e^A \text{ and } e^B\text{).}$$

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a) $e^{A(t+T)} = Se^{\Lambda(t+T)}S^{-1} = Se^{\Lambda t}e^{\Lambda T}S^{-1} = Se^{\Lambda t}S^{-1}Se^{\Lambda T}S^{-1}$

$= e^{At}e^{AT}$

b) e^{A+B}, e^Ae^B

$e^A \approx I + \frac{A}{1!}$, $e^B \approx I + \frac{B}{1!}$ $\Rightarrow$ $e^A \approx \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$e^B \approx \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$e^{A+B} \approx I + \frac{(A+B)}{1!} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$e^{A+B} \approx \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ $\Rightarrow$ $e^{A+B} \neq e^Ae^B$

$e^Ae^B \approx \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$

8. Suppose the rabbit population r and the wolf population w are governed by

$$\frac{dr}{dt} = 4r - 2w$$
$$\frac{dw}{dt} = r + w.$$

(a) Is this system stable, neutrally stable, or unstable?

(b) If initially $r = 300$ and $w = 200$, what are the populations at time t?

(c) After a long time, what is the proportion of rabbits to wolves?

(8)

$$\frac{du}{dt} = \begin{bmatrix} dr/dt \\ dw/dt \end{bmatrix} = \underbrace{\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} r \\ w \end{bmatrix}}_{u}$$

a)

$$\det(A - \lambda I) = 0 \rightarrow (4-\lambda)(1-\lambda) + 2 = 6 - 5\lambda + \lambda^2 = 0 \rightarrow \lambda = \{2, 3\} \Rightarrow \lambda_i > 1$$

$\Downarrow$ unstable

$$* \; (A - 2I)X = 0 \rightarrow \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} X = 0 \rightarrow X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$* \; (A - 3I)X = 0 \rightarrow \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} X = 0 \rightarrow X = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

b)

$$u(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$t = 0$

$$u(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 300 \\ 200 \end{bmatrix} \rightarrow c_1 = 100, \; c_2 = 100$$

$$\hookrightarrow u(t) = 100\, e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 100\, e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 100\, e^{2t} + 200\, e^{3t} \\ 100\, e^{2t} + 100\, e^{3t} \end{bmatrix}$$

c)

$$\lim_{t \to \infty} \frac{100\, e^{2t} + 200\, e^{3t}}{100\, e^{2t} + 100\, e^{3t}} = \lim_{t \to \infty} \frac{100\, e^{3t} (e^{-t} + 2)}{100\, e^{3t} (e^{-t} + 1)} = \frac{0 + 2}{0 + 1} = 2$$

Extra problem

Solve the differential equation $y''' + 2y'' - y' - 2y = 0$ for the general.

What is the matrix A?

Find the first column of $\exp(At)$?

$$y''' = -2y'' + y' + 2y$$

$$\underbrace{\begin{bmatrix} y''' \\ y'' \\ y' \end{bmatrix}}_{u'(t)} = \underbrace{\begin{bmatrix} -2 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y'' \\ y' \\ y \end{bmatrix}}_{u(t)}$$

$$\det(A - \lambda I) = 0 \rightarrow \begin{vmatrix} -2-\lambda & 1 & 2 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$(-2-\lambda)(\lambda^2) + (-1)(-\lambda) + (2)(1) = 0$$

$$-2\lambda^2 - \lambda^3 + \lambda + 2 = 0 \rightarrow \lambda^2(-2-\lambda) + \lambda + 2 = 0$$

$$\rightarrow (\lambda+2)(1-\lambda^2) = (\lambda+2)(1-\lambda)(1+\lambda) = 0 \rightarrow \lambda = \{-2, 1, -1\}$$

$$* \ \lambda_1 = 1 \quad (A - I)X = 0 \rightarrow \begin{bmatrix} -3 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$* \ \lambda_2 = -1 \quad (A + I)X = 0 \rightarrow \begin{bmatrix} -1 & 1 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow X = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$* \ \lambda_3 = -2 \quad (A - 2I)X = 0 \rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow X = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

$$u(t) = c_1 e^{t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} y''(t) \\ y'(t) \\ y(t) \end{bmatrix}$$

$$y(t) = c_1 e^{t} + c_2 e^{-t} + c_3 e^{-2t}$$

$$\exp(At) = S e^{\Lambda t} S^{-1} = \begin{bmatrix} 1 & 1 & 4 \\ 1 & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1/6 & \cdots & \cdots \\ -1/2 & \cdots & \cdots \\ 1/3 & \cdots & \cdots \end{bmatrix} =$$

$$\boxed{S^{-1} = \frac{1}{|S|} C_S^T = \frac{1}{6} \begin{bmatrix} 1 & \cdots & \cdots \\ -3 & \cdots & \cdots \\ 2 & \cdots & \cdots \end{bmatrix}}$$

$$\boxed{\frac{1}{6} e^{t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \frac{1}{3} e^{-2t} \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}}$$

5. (a) If $x = re^{i\theta}$ what are x^2, x^{-1}, and $\overline{x}$ in polar coordinates? Where are the complex numbers that have $x^{-1} = \overline{x}$?

(b) At $t = 0$, the complex number $e^{(-1+i)t}$ equals one. Sketch its path in the complex plane as t increases from 0 to 2π.

5

a) $x = re^{i\theta}$

$x^2 = r^2 e^{2i\theta}$; $x^{-1} = \frac{1}{r} e^{-i\theta}$; $\bar{x} = \overline{re^{i\theta}} = re^{-i\theta}$

$(*)$ $x^{-1} = \bar{x} \rightsquigarrow \frac{1}{r} e^{-i\theta} = r e^{-i\theta} \to \frac{1}{r} = r \to r^2 = 1$

The equation $(*)$ is satisfied if the value of $\underline{r}$ is equal to $\underline{1}$.

we can consider any θ for the equation.

b)

$e^{(-1+i)t} = e^{-t} \cdot e^{it}$

In this term, its value is reduced with the increase of $\underline{t}$.

This term represent polar form of complex number with radius of $\underline{1}$.

This term represent polar form of complex number with radius of $\underline{1}$ that circles around origin of complex plane with the increase of $\underline{t}$.

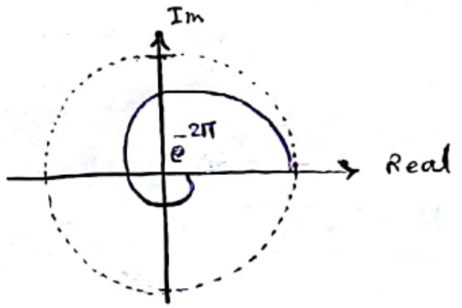
At $t=0$

$e^{-t} \cdot e^{it} = 1$

At $t = 2\pi$

$e^{-2\pi} e^{it}$

by the variation of $\underline{t}$ from $\underline{0}$ to $\underline{2\pi}$, the circular movement of the function around the origin is in conjuction with a considerable decrease in the value of radius. $\to$ spiral movement



7. Write out the matrix A^{H} and compute $C = A^{\mathrm{H}}A$ if

$$A = \begin{bmatrix} 1 & i & 0 \\ i & 0 & 1 \end{bmatrix}.$$

What is the relation between C and C^{H}? Does it hold whenever C is constructed from some $A^{\mathrm{H}}A$?

⑦

$$A = \begin{bmatrix} 1 & i & 0 \\ i & 0 & 1 \end{bmatrix} \rightarrow \bar{A} = \begin{bmatrix} 1 & -i & 0 \\ -i & 0 & 1 \end{bmatrix} \rightarrow \bar{A}^{T} = A^{H} = \begin{bmatrix} 1 & -i \\ -i & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = A^{H}A = \begin{bmatrix} 1 & -i \\ -i & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & i & 0 \\ i & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & i & -i \\ -i & 1 & 0 \\ i & 0 & 1 \end{bmatrix} \rightarrow C^{H} = \begin{bmatrix} 2 & i & -i \\ -i & 1 & 0 \\ i & 0 & 1 \end{bmatrix}$$

$\rightarrow C^{H} = C$

* If we define Matrix A as a non-Complex one, It satisfies the relation between C^{H} and C in question a.

15. What is the dimension of the space S of all n by n real symmetric matrices? The spectral theorem says that every symmetric matrix is a combination of n projection matrices. Since the dimension exceeds n, how is this difference explained?

(15)

A is 3 by 3 symmetric matrix

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

it is represented in form below:

$$A = a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + e\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Its dimension is equal to $\frac{n(n+1)}{2}$ because it has 6 different matrix space in its basis.

* n entries on the main diagonal.

* $\frac{n^2-n}{2}$ entries on side of the diagonal that must be equal to the other side.

$\Rightarrow$ to sumup we have $\frac{n^2-n+2n}{2} = \frac{n(n+1)}{2}$ matrix space in symmeric matrix A's basis.

Another way of decomposing symmetric matrix is to find it's e.values and e.vectors to represent it in the form below:

$$A = \lambda_1 (q_1 \cdot q_1^T) + \lambda_2 (q_2 \cdot q_2^T) + \lambda_3 (q_3 \cdot q_3^T)$$

with any changes in the value of A, respectively all the e.values and e.vectors change too.

* There is no relation between the value of dimension and the linear combination of projection matrices.

29. When you multiply a Hermitian matrix by a real number c, is cA still Hermitian? If $c = i$, show that iA is skew-Hermitian. The 3 by 3 Hermitian matrices are a subspace, provided that the "scalars" are real numbers.

If A is Hermitian then $K = iA$ is skew-Hermitian.

$$K = iA = \begin{bmatrix} 2i & 3+3i \\ -3+3i & 5i \end{bmatrix} = -K^{\mathrm{H}}.$$

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a) $A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix} \rightarrow \bar{A} = \begin{bmatrix} 2 & 3+3i \\ 3-3i & 5 \end{bmatrix} \rightarrow \bar{A}^T = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix} = A^H \Rightarrow A = A^H$

A is Hermitian

$cA = \begin{bmatrix} 2c & (3-3i)c \\ (3+3i)c & 5c \end{bmatrix} \rightarrow c\bar{A} = \begin{bmatrix} 2c & (3+3i)c \\ (3-3i)c & 5c \end{bmatrix} \rightarrow c\bar{A}^T = \begin{bmatrix} 2c & (3-3i)c \\ (3+3i)c & 5c \end{bmatrix} \Rightarrow cA = cA^H$

cA is Hermitian

$K = iA = \begin{bmatrix} 2i & 3i-3i^2 \\ 3i+3i^2 & 5i \end{bmatrix} \rightarrow K = \begin{bmatrix} 2i & 3+3i \\ -3+3i & 5i \end{bmatrix} \rightarrow K^H = \begin{bmatrix} -2i & -3-3i \\ 3-3i & -5i \end{bmatrix} \Rightarrow K = -K^H$

K is skew-Hermitian

Thanks for your attention