Linear Algebra

Chapter 1: Matrices and Gaussian Elimination

Solution of highlighted problems

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2. Solve to find a combination of the columns that equals b:

Triangular system

$$\begin{aligned} u - v - w &= b_1 \\ v + w &= b_2 \\ w &= b_3. \end{aligned}$$

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$$\begin{aligned} u - v - w &= b_1 \\ v + w &= b_2 \\ w &= b_3 \end{aligned} \longrightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \rightarrow$$

$$u \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

w = b3
v = b2 – b3
u = b1+b2

14. For two linear equations in three unknowns x, y, z, the row picture will show (2 or 3) (lines or planes) in (two or three)-dimensional space. The column picture is in (two or three)-dimensional space. The solutions normally lie on a ______.

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$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{21} & a_{23} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = B$$

Row picture = 2 plane 3 dim

column picture = 2 dim space

if there are solution, they may lie on a strenght line.

22. If (a,b) is a multiple of (c,d) with $abcd \neq 0$, *show that* (a,c) *is a multiple of* (b,d). This is surprisingly important: call it a challenge question. You could use numbers first to see how a, b, c, and d are related. The question will lead to:

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has dependent rows then it has dependent columns.

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$c = c_1 a \qquad d = c_1 b$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow E_{21} A = \begin{bmatrix} 1 & 0 \\ -c_1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

$$x_1 \begin{bmatrix} a \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow x_1 = -b, \ x_2 = -a$$

Hence the column are linearly dependent

1. What multiple ℓ of equation 1 should be subtracted from equation 2?

$$\begin{aligned} 2x + 3y &= 1 \\ 10x + 9y &= 11. \end{aligned}$$

After this elimination step, write down the upper triangular system and circle the two pivots. The numbers 1 and 11 have no influence on those pivots.

2. Solve the triangular system of Problem 1 by back-substitution, y before x. Verify that x times $(2,10)$ plus y times $(3,9)$ equals $(1,11)$. If the right-hand side changes to $(4,44)$, what is the new solution?

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$$\begin{bmatrix} 2 & 3 \\ 10 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

Augmented form

$$[A \vdots B] = \left[\begin{array}{cc:c} 2 & 3 & 1 \\ 10 & 9 & 11 \end{array}\right] \longrightarrow E_{21}[A \vdots B] = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 10 & 9 & 11 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -6 & 6 \end{bmatrix}$$

Pivots

Back-substitution

$$2x + 3y = 1$$
$$-6y = 6 \longrightarrow y = -1 \ , \ x = 2$$

12. Which number d forces a row exchange, and what is the triangular system (not singular) for that d? Which d makes this system singular (no third pivot)?

$$\begin{aligned} 2x + 5y + z &= 0 \\ 4x + dy + z &= 2 \\ y - z &= 3. \end{aligned}$$

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Augmented form $\rightarrow A \left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 4 & d & 1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right] = [A:B]$

Row exchange?
make singular?

$$E_{21}[A:B] = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & 1 & 0 \\ 4 & d & 1 & 2 \\ 0 & 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 1 & 0 \\ 0 & d-10 & -1 & 2 \\ 0 & 1 & -1 & 3 \end{bmatrix}$$

$\rightarrow d = 10$

$$P_{23} E_{21} [A|B] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \left[\boxed{\begin{matrix} 2 & 5 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{matrix}} \begin{matrix} 0 \\ 3 \\ 2 \end{matrix} \right]$$

to make the matrix singular

$$A = \begin{bmatrix} 2 & 5 & 1 & 0 \\ 0 & d-10 & -1 & 2 \\ 0 & 1 & -1 & 3 \end{bmatrix} \rightarrow d = 11 \qquad E_{32} A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} A = \left[\boxed{\begin{matrix} 2 & 5 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{matrix}} \begin{matrix} 0 \\ 2 \\ 1 \end{matrix} \right]$$

Singular matrix

11. The first row of AB is a linear combination of all the rows of B. What are the coefficients in this combination, and what is the first row of AB, if

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}?$$

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$$A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & -1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Linear Combination of Rows

1st Row of $AB = 2\begin{bmatrix} 1 & 1 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \end{bmatrix}$

20. The matrix that rotates the x-y plane by an angle θ is

$$A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Verify that $A(\theta_1)A(\theta_2) = A(\theta_1 + \theta_2)$ from the identities for $\cos(\theta_1 + \theta_2)$ and $\sin(\theta_1 + \theta_2)$. What is $A(\theta)$ times $A(-\theta)$?

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$$A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$A(\theta_1)\,A(\theta_2) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix} = A(\theta_1+\theta_2)$$

$$A(\theta)\,A(-\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \overset{\cos\theta}{\cancel{\cos(-\theta)}} & \overset{\sin\theta}{\cancel{-\sin(-\theta)}} \\ \underset{-\sin\theta}{\cancel{\sin(-\theta)}} & \underset{\cos\theta}{\cancel{\cos(-\theta)}} \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \overset{0}{\cancel{\sin\theta\cos\theta - \sin\theta\cos\theta}} \\ \overset{0}{\cancel{\sin\theta\cos\theta - \sin\theta\cos\theta}} & \sin^2\theta + \cos^2\theta \end{bmatrix} = I$$

$\rightarrow A(-\theta)^{-1} = A(\theta)$

Problem set 1.4

4. Apply elimination to produce the factors L and U for

$$A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{bmatrix}.$$

(4)

c) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{bmatrix} \rightarrow E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow E_{21}A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 1 & 4 & 8 \end{bmatrix}$

$$E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 1 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 3 & 7 \end{bmatrix}$$

$$E_{32}E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{bmatrix} = U$$

$$\rightarrow E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U = A$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} U = A \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = L$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

6. Use the Gauss-Jordan method to invert

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

(6)

c)

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{\rho_1 \leftrightarrow \rho_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\rho_2 = \rho_2 - \rho_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{\rho_1 = \rho_1 - \rho_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\rho_1 = \rho_1 - \rho_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

19. Compute the symmetric LDL^{T} factorization of

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 12 & 18 \\ 5 & 18 & 30 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}.$$

b) $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \longrightarrow E_{21} = \begin{bmatrix} 1 & 0 \\ -\frac{b}{a} & 1 \end{bmatrix} \longrightarrow E_{21}A = \begin{bmatrix} a & b \\ 0 & d-\frac{b^2}{a} \end{bmatrix}$

$E_{21}^{-1} = L = \begin{bmatrix} 1 & 0 \\ \frac{b}{a} & 1 \end{bmatrix} \longrightarrow A = LDL^T \Rightarrow \begin{bmatrix} a & b \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b/a & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d-b^2/a \end{bmatrix} \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}$

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a) $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 12 & 18 \\ 5 & 18 & 30 \end{bmatrix} \longrightarrow$ symmetric

$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow E_{21}A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 5 & 18 & 30 \end{bmatrix}$ $\quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$

$E_{31}E_{21}A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 3 & 5 \end{bmatrix} \rightarrow E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \rightarrow E_{32}E_{31}E_{21}A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

$E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix} = L$

$A = LDL^T \longrightarrow \begin{bmatrix} 1 & 3 & 5 \\ 3 & 12 & 18 \\ 5 & 18 & 30 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

41. For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}.$$

(a)

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

Gaussian Elimination

$$\rightarrow E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightsquigarrow E_{21}A = \begin{bmatrix} 2 & c & c \\ 2a+c & (a+1)c & (a+1)c \\ 8 & 7 & c \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \rightsquigarrow E_{31}E_{21}A = \begin{bmatrix} 2 & c & c \\ 2a+c & (a+1)c & (a+1)c \\ 0 & 7-4c & -3c \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b & 1 \end{bmatrix} \rightsquigarrow E_{32}E_{31}E_{21}A = \begin{bmatrix} 2 & c & c \\ 2a+c & (a+1)c & (a+1)c \\ b(2a+c) & 7-4c+b(a+1)c & b(a+1)c-3c \end{bmatrix}$$

$*\quad 2a+c=0 \longrightarrow a=-\frac{c}{2} \quad ①$

$**\quad 7-4c+b(a+1)c=0 \longrightarrow b(a+1)c=4c-7 \quad ②$

$***\quad 2(a+1)c\left(b(a+1)c-3c\right)=0 \overset{①,②}{\rightsquigarrow} 2\left(1-\frac{c}{2}\right)c\left(4c-7-3c\right)=0$

$\Rightarrow 1-\frac{c}{2}=0 \longrightarrow c=1$

$\Rightarrow c=0$

$\Rightarrow 4c-7-3c=0 \longrightarrow c=7$

for $c \in \{0,1,7\}$ Matrix A is non-invertible

Thanks for your attention