

# GLM Factorial Designs

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## The road so far

- $Y_i = a + bX$ : Bivariate linear regression
- $Y_i = a + bX$  where  $X$  is a dummy variable: comparing two groups, aka independent samples t-test
- $Y_i = a + b_1X_1 + \dots + b_kX_k$  where  $X_{1\dots k}$  are dummy variables: comparing multiple groups, aka ANOVA
- Interaction:  $Y = a + b_1X_1 + b_2X_2 + b_3(X_1 * X_2)$

If you missed the lecture on interaction, re-watch the section on interactions between binary and continuous predictors!

# Today: Factorial ANOVA

**Factorial ANOVA:** Regression with multiple categorical predictors (and the interaction between them).

- Nothing complicated!
- We're combining the following ideas:
  - Dummy coding
  - Interactions
- ANOVA developed distinctly from regression
  - People would calculate mean value in each condition by hand, and calculate different sums of squares around these means
  - ANOVA-interface of SPSS still focuses on this type of output
- Nowadays we know that factorial ANOVA is just a regression model

## **Factorial ANOVA Applications**

- Neuroscience: Comparing the effects of two drugs administered in two ways
- Social science: Examining the effects of an empathy manipulation on prosocial behavior in young children versus teenagers
- Business and Economics: Examining the effects of advertising type and discount offered

## Throughout this Lecture:

- **Factors:** Categorical variables (usually nominal)
- Each factor has **multiple levels**
- Example: Dosage 3 levels (low, medium, high), Drug 2 levels (A vs B)

If factor A has  $a$  groups, and factor B has  $b$  groups, the total number of groups in the factorial design is  $a \times b$

## Example Data: Enjoy your meal!

- DV: How much did you enjoy your meal? 0-5 (treated as continuous)
- IV1: Type of Food (two levels)
  - Beyond sausage
  - Ben & Jerries Cookies on Cookie Dough
- IV2: Topping (two levels)
  - Mustard and onions
  - Chocolate sauce and sprinkles

Design: 2(sausage vs. ice cream) X 2(mustard vs. chocolate) = 4 groups

## Representing the Factorial Design

### Mustard    Chocolate

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Sausage	$\mu_{S,M}$	$\mu_{S,C}$
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Ice cream	$\mu_{I,M}$	$\mu_{I,C}$
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Historically, people would calculate sums of squares around these means

We are not doing this by hand!

## Regression Model

We can represent factorial ANOVA using the regression model:

- Dummy-code both factors
  - In this case, assume that the first level is the reference category
  - $D_A$  is a dummy for Factor A == 2,  $D_B$  is a dummy for Factor B == 2
- Create an interaction term
  - Multiply both dummies,  $D_A * D_B$

$$Y = b_0 + b_1 D_A + b_2 D_B + b_3 (D_A * D_B)$$



## Regression model ctd.

$$Y = b_0 + b_1 D_A + b_2 D_B + b_3 (D_A * D_B)$$

- $b_0$ : Intercept, mean value for people who score 0 on both dummies
- $b_1$ : Mean difference between intercept and people who score 1 on  $D_A$
- $b_2$ : Mean difference between intercept and people who score 1 on  $D_B$
- $b_3$ : Interaction term: additional “bump” for people who score 1 on both  $D_A$  and  $D_B$

## Regression model ctd.

$$Y = b_0 + b_1 D_A + b_2 D_B + b_3 (D_A * D_B)$$

Filling in the formula for each condition we get:

$$D_B = 0 \quad D_B = 1$$

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$$D_A = 0 \quad \mu_{1,1} = b_0 \quad \mu_{1,2} = b_0 + b_2$$

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$$D_A = 1 \quad \mu_{2,1} = b_0 + b_1 \quad \mu_{2,2} = b_0 + b_1 + b_2 + b_3$$

# Study Designs

Factorial ANOVA was initially developed for experiments where every cell had an equal number of observations / participants

- This is called a *balanced design*
  - It makes hand calculations easier
- It's fine to have an *unbalanced design*
  - We're not hand-calculating anyway!
  - For other regression models we also don't expect equal numbers of participants for every value of the predictors!
- Downsides:
  - Less power to detect a true effect
  - More susceptible to deviations from homoscedasticity

# Effects in Factorial Design

## Two Types of Effects

In a factorial design, it is common to discuss the following effects:

- Main effects of Factors A and B
  - Partial effect of the factor, controlling for the other factor (and optionally the interaction)
  - This is different from the “direct effect” of Factor A without controlling for Factor B and the interaction
- Interaction effects of Factor A  $\times$  B
  - Does the effect of Factor A differ across levels of Factor B (or vice versa?)

# Interaction Effect

Does our theory imply the presence of an interaction effect?

- If so, we have to test for an interaction **first**
  - Because we cannot straightforwardly interpret main effects in the presence of an interaction

If your theory does not imply an interaction, we can still test for it as an assumption check

- Checking the assumption that the model is correctly specified
- If the interaction is not significant, our results are easier to interpret by omitting it

## Interpreting Interactions

- Remember: Interactions are symmetrical,  $A \times B$  is the same as  $B \times A$
- You may want to interpret one of them as predictor and the other as moderator
  - E.g., the effect of the type of drug is moderated by the method of administration
  - Not: the effect of method of administration is moderated by type of drug
- That is a *theoretical decision*, not a statistical one

# Interactions for Categorical Variables

Few simple rules:

- Each factor with  $k$  levels is represented by  $k - 1$  dummy variables
- To include an interaction between factor A and B, multiply *all* dummies of factor A with all dummies of factor B
  - If factor A has 3 groups, create 2 dummies:  $D_{A1}$  and  $D_{A2}$
  - If factor B has 2 groups, create 1 dummy:  $D_{B1}$
  - To create their interaction terms, multiply all dummies:
    - $Int_1 = D_{A1} * D_{B1}$  and  $Int_2 = D_{A2} * D_{B1}$
- Dummies stick together: Always include all dummies that code for one variable in the model together
- Same goes for interactions: Always include all interaction variables and all dummy variables for the main effects of that interaction in the model together



## Example Interaction

We would expect people to prefer ice cream with chocolate sauce, and beyond sausage with mustard:

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## Main Effects

**Main Effect:** Effect of one factor, controlling for the other factor (and optionally interaction).

- Main effect of food on liking: No preference
- Main effect of topping: Slight preference for chocolate



## Hypotheses for Main and Interaction Effects

	Factor B = 1	Factor B = 2	Marginal Means
Factor A = 1	$\mu_{1,1}$	$\mu_{1,2}$	$\mu_{A=1}$
Factor A = 2	$\mu_{2,1}$	$\mu_{2,2}$	$\mu_{A=2}$
Marginal Means	$\mu_{B=1}$	$\mu_{B=2}$	$\mu_{overall}$

### Hypotheses:

- Interaction:  $H_0 : (\mu_{11} - \mu_{12}) = (\mu_{21} - \mu_{22})$ , or  
 $H_0 : (\mu_{11} - \mu_{21}) = (\mu_{12} - \mu_{22})$
- Main effect Factor A:  $H_0 : \mu_{A=1} = \mu_{A=2}$
- Main effect Factor B:  $H_0 : \mu_{B=1} = \mu_{B=2}$

## Stylized Examples

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## Rules for Interpreting Means Plots

- If lines are parallel, there is no interaction
- If lines cross (when extrapolated), there is a non-zero interaction
- Main effect of Factor A: When middle of the points on the left side is not the same as the middle of the points on the right side
- Main effect of Factor B: When middle point of one line is above or below the middle point of the other line

## Effect of Food on Liking

If people like ice cream more than sausage, irrespective of topping, we would expect a pattern of means like this:

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## Effect of Food on Liking

If people like chocolate sauce more than mustard, irrespective of food, we would expect a pattern of means like this:

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## Interactions

More likely, the preference for food depends on the topping, and vice versa:

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## Real Results

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## Results in Numbers

### Coefficients:

	B	SE	t	p
(Intercept)	3.00	0.19	15.74	0
FoodIce cream	-2.60	0.27	-9.49	0
ToppingChocolate	-2.19	0.27	-8.12	0
FoodIce cream:ToppingChocolate	5.36	0.37	14.37	0

# Testing Effects in ANOVA

## Overall Tests

**Problem:** If we have  $> 2$  categories in one variable, those coefficients and their tests do not give us an overall significance test for the main effect of each factor or for the interaction terms.

We have solved this problem previously!

## F-test for Nested Models

We solved this problem before, when performing hierarchical  $R^2$  tests for categorical variables with  $> 2$  categories:

We have Factor A (3 categories, 2 dummies), Factor B (2 categories, 1 dummy), and their interaction (2 product terms)

What is the unique effect of the interaction term?

1. Estimate a model with all dummies of Factor A and Factor B
2. Add both product terms
3. Perform F-test for  $\Delta R^2$

## Same Logic Applies Here

In ANOVA, we essentially perform such a nested model test for each variable and for the interaction terms

This gives us a test for the unique variance explained by Factor A, Factor B, and their interaction

It is customary to report the explained variance in terms of (partial) *sum of squares* explained by that variable, along with an F-test

## Example Results

For the Food/Topping analysis:

**Sums of Squares:**

	<b>Df</b>	<b>Sum Sq</b>	<b>Mean Sq</b>	<b>F value</b>	<b>p</b>
Food	1	2.00	2.00	3.45	0.07
Topping	1	6.27	6.27	10.80	0.00
Food:Topping	1	120.02	120.02	206.59	0.00
Residuals	64	37.18	0.58		

## Visualization

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- A: Sum of Squares for Factor A (0 in this example)
- B: Sum of Squares for Factor B
- C: Sum of Squares for Interaction



## Visualization

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- A: Sum of Squared Errors, SSE (or SSW)
- B+C+D+E+F+G: Corrected model Sum of Squares, SSR (or SSB)
- SSE+SSR: Corrected Total Sum of Squares (SST)
- $SST - SSE = SSR$

## Degrees of Freedom

The degrees of freedom for each Factor are equal to the number of categories  $k - 1$

- This corresponds to the number of dummies required to represent it

The degrees of freedom for the interaction are also equal to the number of dummies needed to represent it

# Effect Size Measures

## Why Effect Size?

Recall that sums of squares do not have a meaningful unit

- Their value changes depending on the scale of your dependent variable, number of observations, etc
- We standardize the sum of squares to put it on a meaningful 0-1 scale
- For regression, the explained variance (SSR/SSE) is called  $R^2$

## Eta Squared

Recall that, in ANOVA,  $R^2$  is often called “Eta squared”,  $\eta^2$

- Interpreted the same: How large is the variance explained by each Factor, relative to the total variance?
- Easy calculation from SS:

- $\eta^2_{\text{Factor A}} = \frac{SS_{\text{Factor A}}}{SST}$

- $\eta^2_{\text{Factor B}} = \frac{SS_{\text{Factor B}}}{SST}$

- $\eta^2_{\text{Interaction}} = \frac{SS_{\text{Interaction}}}{SST}$

# Partial Eta Squared

Another effect size is “partial Eta squared”,  $\eta_p^2$

- Interpretation: How large is the variance explained by each Factor, relative to the unexplained variance?
- This effect size is controlled for all other predictors
- Easy calculation from SS:

$$\blacksquare \eta_{p\text{Factor A}}^2 = \frac{SS_{\text{Factor A}}}{SS_{\text{Factor A}} + SSE}$$

$$\blacksquare \eta_{p\text{Factor B}}^2 = \frac{SS_{\text{Factor A}}}{SS_{\text{Factor B}} + SSE}$$

$$\blacksquare \eta_{p\text{Interaction}}^2 = \frac{SS_{\text{Interaction}}}{SS_{\text{Interaction}} + SSE}$$

## Visualization Eta Squared

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- Eta Squared for Factor B:  $\frac{B}{A+B+C+D+E+F+G}$
- Partial Eta Squared:  $\frac{B}{B+A}$

**Error**

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