Lecture 4 - Testing Hyptotheses

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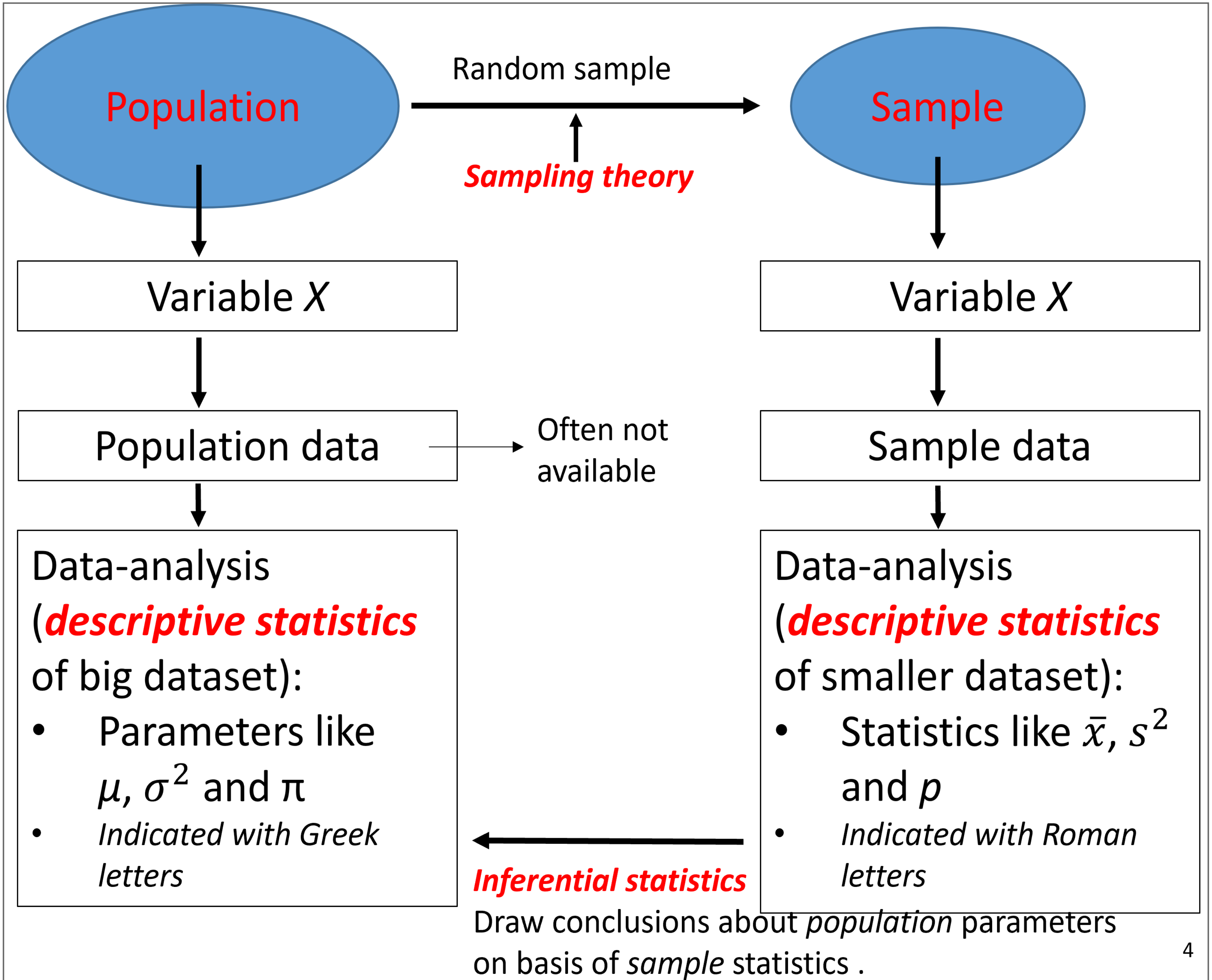
Statistical Testing

Sampling theory


Population
Random sample
Sample
Sampling theory
Variable X
Variable X
Population data
Often not available
Sample data
Data-analysis (descriptive statistics of big dataset):
• Parameters like μ, σ^2 and π
• Indicated with Greek letters
Data-analysis (descriptive statistics of smaller dataset):
• Statistics like $\bar{x}$, s^2 and p
• Indicated with Roman letters
Inferential statistics
Draw conclusions about population parameters on basis of sample statistics .

Inferential statistics

Using sample data to infer properties of the population
We previously covered estimation

- Using sample statistics as best guess of population parameters
- Using standard errors to express uncertainty about the accuracy of that guess
- M is an estimate of μ
- SE_M indicates our uncertainty about that estimate

More inferential statistics

Hypothesis testing is another inferential procedure

- Drawing conclusions about the population based on the sample
- These conclusions are based on calculated probabilities
- We use the normal distribution (as before)

Intuitive example

1. Observation: Your car won’t start.
2. Theory: The battery is dead.
3. Hypothesis: If the battery is dead, then…
    - Using jumper cables will start the car
    - Charging the battery will start the car
    - Replacing the battery will start the car
4. Experiment: You replace the battery.
5. Data: The car starts.
6. Conclusion: The battery was indeed dead.

Statistical testing

In the previous example, all you need to test the hypothesis is one piece of evidence

- Car starts or not

Statistical hypothesis tests instead use probability calculus

- This allows you to test hypotheses in the presence of uncertainty

Intuitive example, adapted

1. Observation: 6% of Brand X cars have trouble starting within the first year.
2. Theory: The battery has too low capacity.
3. Hypothesis: If the battery is dead, then…

- Increasing battery capacity will solve the problem.

4. Experiment: Brand X switches to larger batteries.
5. Data: In a sample of 1000 Brand X cars,
   4.3% have trouble starting within the first year.
6. Conclusion: ?

Is this improvement significant, relative to random variation between samples?

Steps for testing

1. Formulate hypotheses
    - Testable proposition about population parameters
2. Calculate test statistic
    - Describes how many standard errors away from the population statistic under the null hypothesis the sample statistic is
3. Calculate p-value
    - Probability of observing a value at least as extreme as the sample statistic, if H_0 were true
4. Draw conclusion about null hypothesis
    - (Act as if) we reject or fail to reject it

Hypotheses

Theory vs. Hypothesis

Theory:

- Systematic explanation for phenomena
- Can include assumptions about causality

Hypothesis:

- Proposition about the population that can be tested in a sample
- Assumption about the state of the world that is put to the test
- Must be made before seeing the results

Problem of Induction

You cannot derive general rules from specific evidence

- If you’ve only ever seen white swans, you might develop a theory that *all swans are white*
- But black swans do exist, so your theory would be wrong
- No matter how much evidence you observe

Deduction and Induction

Deduction: ‘top down’, from theory to specific observations

- Premise: All humans are mortal.
- Premise: Socrates is human.
- Conclusion: Socrates is mortal.

If the premises are true, then the conclusion must be true
Inductive: ‘bottom up’, from specific observations to general theory

- Premise: All swans I have observed were white
- Conclusion: Therefore, all swans are white

The conclusion can be (in this case, is) wrong

Problem of Induction

Can’t go from specific observations to general rules

- Because the same evidence could be explained by multiple rules

Reveal

Falsificationism

Popper: “solved” problem of induction by introducing falsificationism

- Scientific theories must be testable and conceivable proven false
- Tests cast doubt on hypotheses, not provide evidence for them
- E.g.: Observing one black swan rejects theory that “all swans are white”

Falsificationism and Testing

How Popper is applied in statistical inference:

- We formulate a “null hypothesis”, whose sole purpose is to be rejected
- This null hypothesis is the **opposite** of what the researcher believes
  - We could argue about whether this makes sense
- You will see a lot of “nil hypotheses”:
  - Hypothesis that a value / difference / effect is equal to zero
  - Rejecting this hypothesis implies acting as if there **is** a non-zero effect
  - Software often uses nil hypotheses by default
- Criticism: null hypothesis is a “straw man”, nobody believes it
- Alternatives: Bayesian statistics, informative hypothesis tests, smallest effect of interest

What kind of hypotheses?

Equality hypotheses:

- $H : \mu = 0$
- $H : \mu = 6.4$, which is the same as $H : \mu - 6.4 = 0$
- $H : \mu_1 = \mu_2$, which is the same as $H : (\mu_1 - \mu_2) = 0$

Inequality hypotheses

- Larger or smaller
- $H : \mu > 0$
- $H : \mu < 6.4$, which is the same as $H : \mu - 6.4 < 0$
- $H : \mu_1 > \mu_2$, which is the same as $H : (\mu_1 - \mu 2) > 0$

Null and alternative hypotheses

The null hypothesis exists only to be rejected,
the alternative hypothesis reflects the researcher’s true beliefs
H_0 vs. H_A
There are two philosophies of testing:

- Fisher:
  - Implicit alternative hypothesis which is mutually exclusive with H_0
  - If $H_0 : \mu = 0$, then $H_A : \mu \neq 0$
  - If $H_0 : \mu \leq 0$, then $H_A : \mu > 0$
- Neyman-Pearson:
  - Explicit alternative hypothesis for specific expected effect size
  - E.g., $H_0 : \mu = 0, H_A : \mu = 6.4$
  - By stating what effect size we expect,
    we can calculate the probability of finding an effect that really exists

Test statistics

What is a test statistic

Simple definition:

- Distance between the hypothesized population value and the sample statistic, in standard errors
- In other words: We standardize that distance by dividing by standard errors

We use the familiar formula for the Z-distribution to obtain the Z test statistic:

$$Z = \frac{M - \mu_0}{SE_\mu}$$

- M is the sample mean
- μ_0 is the expected population mean under H_0
- SE_μ is the standard error

Calculate test statistic

E.g.: $H_0 : \mu = 6.4, SE_\mu = .2$, and $M = 6.8$

- The test statistic is $Z = \frac{M-\mu_0}{SE_\mu} = \frac{6.8-6.4}{.2} = 2$

Example

Dutch people are quite tall. Worldwide distribution:
$N(\mu = 167, \sigma = 7.5)$
Hypothesis: Dutch people are significantly taller than the worldwide average.
Sample: Convenience sample of 20 students, $M = 171$

- $H_0 : \mu_0 \leq 167$: The population mean is 167 or smaller
- $H_a : \mu_0 > 167$: The population mean exceeds 167

$Z = \frac{M - \mu_0}{SE_\mu} = \frac{171 - 167}{7.5/\sqrt{2}0} = \mathbf{2.39}$
$SE_M = \frac{\sigma}{\sqrt{n}}$

P-value

The p-value

Statistical hypothesis tests use probability calculus

What probability?

Definition: Probability of observing data at least as extreme as our sample data, IF our (null) hypothesis were true.

$$P(Data|H_0)$$

Correct interpretation

- P-values give you the probability of observing certain data, ASSUMING THAT H_0 is true
- They do NOT give you the probability of H_0 being true or false!
- $p = .0001$
  - Correct: it's extremely unlikely to observe these data if we assume that H_0 is true
  - INCORRECT: there's a (1-.0001) = .9999 probability that H_0 is false
- $p = .75$:
  - Correct: it's very common to observe these data if H_0 is true
  - Incorrect: 75% probability that H_0 is true

To make probability statements about hypotheses, you need to use a different (Bayesian) definition of probability

Use the normal distribution

First, determine if you need the probability in one tail, or both

- One-sided hypothesis ($H_0 : \mu \leq / \geq$? One tailed
- Two-sided hypothesis ($H_0 : \mu =$)? Two-tailed

SPSS (and many other programs) only give two-tailed p-value

- If the effect is in the hypothesized direction, divide $p/2$
  - If not, it's non-significant

What to do with the p-value?

Two philosophies:

1. Fisher:

- Specify only a null hypothesis H_0
- p-value quantifies incompatibility of the data with H_0
- The more incompatible (i.e., smaller p-value), the more skeptical we become of H_0

What to do with the p-value?

2. Neyman-Pearson:

- Specific null- and alternative hypothesis
- Determine alpha and beta level
  - Alpha conventionally set at .05
  - Beta < .2 considered to be desirable (power analysis)
- p-value compared against alpha to make a decision
  - Binary decision; does not matter how much smaller
- Still report exact p-values so other researchers can use different approaches to inference

Conclusion

Reject null hypothesis

We reject H_0 when:

- $p < \alpha$

This means that it's very unlikely to observe data at least as extreme as we observed, assuming H_0 is true

Don’t reject null hypothesis

We don’t reject H_0 when:

- $p \geq \alpha$:

  This means that the data are unsurprising, assuming H_0 is true

Critical Values

Critical values are the Z-values corresponding to the chosen α level

- $\alpha = P(Z > Z_{\text{critical}})$
- So we also reject H_0 when the test statistic exceeds the critical value, $Z > Z_{\text{critical}}$
- These are the same thing, because you look up p using the $Z - value$

Memorize the following critical values:

- For two-sided test and $\alpha = .05$, the critical value is 1.96 (~2)
- For one-sided test and $\alpha = .05$, the critical value is 1.64

Type I and Type II errors

When we test hypotheses, we (act as if) we accept or reject H_0
You could make two errors:

- Rejecting H_0 but in reality H_0 was true
  - False-positive finding
  - Incorrectly concluding you've found something (an effect, a difference, etc)
- Accepting H_0 but in reality H_a is true
  - False negative
  - Concluding you've found nothing, but there was an effect out there

Type I and Type II errors

Decision	Ha	H0
Reject H0	Correct decision!	Type I error: α
Accept H0	Type II error: β	Correct decision!

Alpha and beta

Alpha is the risk of a false-positive finding

- Conventionally, we use $\alpha = .05$, 5% risk of Type I error

Beta is the risk of a false-negative finding

- If you specify a specific H_a, you can calculate β!
- In general, β decreases if:
  - The effect size is greater
  - The sample is larger
  - There is less “noise” (lower standard deviation)

Alpha and Beta are related

If you choose a lower α, your β will increase!

- Given a specific H_a, you can balance the risk of Type I/II Errors
- Justify your choice!
- E.g., COVID repid tests: $\alpha = .001, \beta = .50$
  - Is this the right balance?
  - Consequence of false positive: infect other people, potentially get them very sick
  - Consequence of false negative: stay home unnecessarily

Power

Defining power

Power: Probability of correctly finding a true effect

- It is simply $1 - \beta$
- If you specify an alternative hypothesis H_a, you can calculate β
  - This works well for simple tests, like Z-tests
- For complex tests, calculations can be more advanced
  - For *very* complex tests, people use "simulations": Generating many fake datasets based on the hypothesized effect, analyzing them, and calculating the % of times they find a significant effect (= power)

For more on the relationship between power and sample size, see https://doi.org/10.1525/collabra.33267

Intuitive perspective on power

It’s like looking for something in a dark basement; your chance of finding it (= power) increases if:

- The object you’re looking for is big (large effect)
- You search for longer (sample size)
- The basement is uncluttered (low noise/standard deviation)

The t-distribution

Can we always use the Z-distribution?

- To use the Z-distribution, you need M, μ_0, and σ

  Problem: We rarely know σ
  Solution: We estimate σ using our sample SD
  As explained in a previous lecture, we can then use

- $SE_M = \frac{SD}{\sqrt{n}}$ instead of $SE_\mu = \frac{\sigma}{\sqrt{n}}$

Accounting for uncertainty

Problem: Estimating σ using the sample SD introduces additional uncertainty (because SD is not EXACTLY equal to σ)

- If we don't account for this uncertainty, p-values will be too small
  - Leading us to reject the null hypothesis too easily
  - Increasing risk of false-positive findings!

Solution: We use a different distribution that gives slightly larger p-values

t-distribution

The t-distribution is similar to the Z-distribution: $\bar{X} \sim t(\mu_0, SE_M, df)$

- μ_0 is the population mean according to the null hypothesis
- SE_M is the sample estimate of the standard error
- df are the **degrees of freedom**: they control how thick the tails are
  - Lower df -> thicker tails -> higher p-value for the same test statistic
- When $df \geq 30$ the distribution converges to the Z-distribution: $t_{n \geq 30} \approx Z$

t-distribution demo

n =

0 30

2 14 26

t-test for a sample mean

Same calculation as for the Z-test!

Except you use SD instead of σ

- H_0: The population mean of height is 167, $\mu_0 \leq 167$
- H_a: The population mean of height exceeds 167, $\mu_0 > 167$

$SE_M = \frac{SD}{\sqrt{n}}$

Sample: Convenience sample of 20 students, $M = 171, SD = 8$,

- $H_0 : \mu_0 \leq 167$: The population mean is 167 or smaller
- $H_a : \mu_0 > 167$: The population mean exceeds 167

$t = \frac{M - \mu_0}{SE_M} = \frac{171 - 167}{8/\sqrt{20}} = 2.23$

P-value for t-distribution

- In t-table in the book
- Using an online calculator, e.g. https://onecompiler.com/r: `pt(tvalue, df, sd, lower.tail = TRUE)`
  - Excel (2-tailed): `=T.DIST(ABS(tvalue), df)`
  - Divide by 2 to get one-tailed

Examples

Example 1

Do warning signs near roads (e.g., photos of heavy accidents) influence the average speed (tested at the 5% level)?

Example 1 steps

1. Formulate hypotheses
   - $H_0 : \mu_0 = 50, H_a : \mu_0 \neq 50$
2. Calculate test statistic
   - Do we know population σ? NO!
   - So, calculate t: $t = -1.76 = \frac{48.5-50}{6.5/\sqrt{58}}$
3. Calculate p-value
   - Given by SPSS: $p = .084$
4. Draw conclusion about null hypothesis
   - Fail to reject null hypothesis; data are not surprising if H_0 is true

Example 1 one sided

But… we were trying to slow down cars! Doesn't this hypothesis make more sense?

1. Formulate hypotheses
   - $H_0 : \mu_0 \geq 50, H_a : \mu_0 < 50$
2. Calculate test statistic
   - Do we know population σ? NO!
   - So, calculate t: $t = -1.76 = \frac{48.5-50}{6.5/\sqrt{58}}$
3. Calculate p-value
   - Divide SPSS two-sided p-value by 2: $p = .084/2 = .042$
4. Draw conclusion about null hypothesis
   - Reject null hypothesis; data are surprising if H_0 is true

Reporting

On the road with signs, driving speed was significantly lower than the legal limit of 50km/h, $M = 48.50, t(57) = -1.76, p = .04$. The null hypothesis that driving speed would be greater than 50 was rejected.

Example 2

Let’s say one is alcoholic if one consumes > 20 units of alcohol/week

Example 2 critical value

1. Formulate hypotheses
   - $H_0 : \mu_0 \leq 20, H_a : \mu_0 > 20$
2. Calculate test statistic
   - Do we know population σ? NO!
   - So, calculate t: $t = 2.28 = \frac{26.57-20}{15.20/\sqrt{28}}$
3. Let's use the critical value this time
4. Reject H_0 because $t = 2.28$ exceeds $t_{\text{critical}} = 1.703$

Further reading

Fisher, Neyman-Pearson or NHST? A tutorial for teaching data testing

- Jose D. Perezgonzalez
- doi.org/10.3389/fpsyg.2015.00223

Improving Your Statistical Inferences

- Daniël Lakens
- https://lakens.github.io/statistical_inferences/01-pvalue.html