Report: Short and Sparse Deconvolution

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Introduction

Deconvolution methods are widely used in signal processing and image processing to recover the original signal or image from a degraded or blurred version. These methods are typically formulated as optimization problems, where the goal is to find the best estimate of the original signal or image by minimizing a certain objective function. However, the optimization problems associated with deconvolution methods are often non-convex, meaning that they can have multiple local optima and are challenging to solve.

In this report, we will introduce two algorithms, namely the inertial Alternating Descent Method (iADM) , to solve these non-convex optimization problems in deconvolution.

Question 1

Let $a \in R^d$ and $x \in R^n$ be real vectors, and let $\lambda \in R^*$ be a positive real number. We are given the convolution $a \star x$, and we need to show that:

$$a \star x = (\lambda a) \star \left(\frac{x}{\lambda}\right)$$

The convolution of a and x is given by:

$$a \star x = \sum_{i=1}^{d} a_i x_{j-i}$$

where a_i and x_{j-i} represent the i-th entry of vectors a and x, respectively, and j denotes the index of the convolution.

Now, let's compute $(\lambda a) \star \left(\frac{x}{\lambda}\right)$:

$$(\lambda a) \star \left(\frac{x}{\lambda}\right) = \sum_{i=1}^{d} (\lambda a)_i \left(\frac{x}{\lambda}\right)_{j-i}$$

By applying the properties of scalar multiplication and division, we can simplify the above expression as follows:

$$(\lambda a) \star \left(\frac{x}{\lambda}\right) = \lambda \sum_{i=1}^{d} a_i \frac{x_{j-i}}{\lambda}$$

Notice that λ and $\frac{1}{\lambda}$ cancel out, and we obtain:

$$(\lambda a) \star \left(\frac{x}{\lambda}\right) = \sum_{i=1}^{d} a_i x_{j-i}$$

which is exactly the same as $a \star x$. Therefore, we have shown that:

$$a \star x = (\lambda a) \star \left(\frac{x}{\lambda}\right)$$

as required..

Question 2

Algorithm 1 Inertial Alternating Descent Method (iADM)

Input: Initializations $\boldsymbol{a}^{(0)} \in \mathbb{S}^{p-1}$, $\boldsymbol{x} \in \mathbb{R}^m$; observation $\boldsymbol{y} \in \mathbb{R}^m$; penalty $\lambda \geqslant 0$; momentum $\alpha \in [0, 1)$.
Output: $(\boldsymbol{a}^{(k)}, \boldsymbol{x}^{(k)})$, a local minimizer of Ψ_{BL}.

Initialize $\boldsymbol{a}^{(1)} = \boldsymbol{a}^{(0)}$, $\boldsymbol{x}^{(1)} = \boldsymbol{x}^{(0)}$.
for $k = 1, 2, \ldots$ until converged **do**
Update $\boldsymbol{x}$ with accelerated proximal gradient step:

$$\boldsymbol{w}^{(k)} \leftarrow \boldsymbol{x}^{(k)} + \alpha \cdot \left(\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}\right)$$

$$\boldsymbol{x}^{(k+1)} \leftarrow \mathrm{soft}_{\lambda t_k}\left[\boldsymbol{w}^{(k)} - t_k \cdot \nabla_{\boldsymbol{x}} \psi_\lambda\left(\boldsymbol{a}^{(k)}, \boldsymbol{w}^{(k)}\right)\right],$$

where $\mathrm{soft}_\lambda(\boldsymbol{v}) \doteq \mathrm{sign}(\boldsymbol{v}) \odot \max(|\boldsymbol{v} - \lambda|, \boldsymbol{0})$ denotes the soft-thresholding operator.
Update $\boldsymbol{a}$ with accelerated Riemannian gradient step:

$$\boldsymbol{z}^{(k)} \leftarrow \mathcal{P}_{\mathbb{S}^{p-1}}\left(\boldsymbol{a}^{(k)} + \frac{\alpha}{\langle \boldsymbol{a}^{(k)}, \boldsymbol{a}^{(k-1)} \rangle} \cdot \mathcal{P}_{\boldsymbol{a}^{(k-1)}}\left(\boldsymbol{a}^{(k)}\right)\right)$$

$$\boldsymbol{a}^{(k+1)} \leftarrow \mathcal{P}_{\mathbb{S}^{p-1}}\left(\boldsymbol{z}^{(k)} - \tau_k \cdot \mathrm{grad}_{\boldsymbol{a}}\, \psi_\lambda\left(\boldsymbol{z}^{(k)}, \boldsymbol{x}^{(k+1)}\right)\right).$$

end for

The algorithm try to solve the functions by naturally reformulating the problem into a LASSO problem:

$$\min_{\mathbf{a} \in S^{3p}, \mathbf{x} \in R^{n+2p}} \lambda \|\mathbf{x}\|_1 + \tfrac{1}{2}\|\mathbf{a} * \mathbf{x} - \mathbf{y}\|_2^2$$

The problem is a non-convex optimization problem, so in order to solve it the author's [2] proposed this algorithm which is inertial alternate gradient descent with momentum acceleration, and we update the step sizes with backtracking.

The parameters are padded to account for all possible shifts, as convolution is shift-invariant. The objective function in the minimization problem is convex near one shift of the true kernel $a0$, and exhibits good properties near multiple shifts. Therefore, the choice of initialization is crucial in order to achieve accurate recovery. Because of Convolution property, the algorithms can only recovers opposed shifts of ground truth:

$$a_0 x_0 = (\alpha s_\ell [\iota a_0]) \left(\tfrac{1}{\alpha} s_{-\ell} [x_0]\right)$$

Question 4

We've made two runs one when a is shift coherent and the other when a is incoherent, the ground truth a is generated following the project instruction, howewer x is generated with Bernouilli-Gaussian with parametres θ.

The algorithms recovers the shift incoherent a more precisely as expected, and less precisely when it is coherent.

Question 5

We created a grid of success probabilities to recover the ground truth function based on the sparsity parameter and the length of the kernel signal p. Due to computational limitations, we couldn't use high values of n. For instance, the algorithm took 48 minutes to compute for n=40 and a weak estimation of probability. We measured the success probability using the correlation between the true signal and the ground truth.

Despite the lack of data, the plot clearly showed a transition line between success and failure, indicating a relationship between and p. Specifically, when the sparsity rate decreases with the kernel, the chances of success increase. Moreover, we observed that as the length of the kernel signal increases, the kernel signal becomes more incoherent, making recovery easier. This suggests that a smaller value of θ compared to $\sqrt{p}$ leads to higher chance of success.

References

1. Kuo, Han-Wen, Yuqian Zhang, Yenson Lau, and John Wright. "Geometry and symmetry in short-and-sparse deconvolution." arXiv preprint arXiv:1901.00256 (2019).

2. Lau, Yenson, Qing Qu, Han-Wen Kuo, Pengcheng Zhou, Yuqian Zhang, and John Wright. "Short-and-Sparse Deconvolution — A Geometric Approach." Preprint (2020).

3. Qu, Qing. " Sparse Deconvolution" Available at: https://github.com/qingqu06/sparse$_d$*econvolution*.